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Evolution of solitary waves for a perturbed nonlinear Schrödinger equation

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Abstract

Soliton perturbation theory is used to determine the evolution of a solitary wave described by a perturbed nonlinear Schrödinger equation. Perturbation terms, which model wide classes of physically relevant perturbations, are considered. An analytical solution is found for the first-order correction of the evolving solitary wave. This solution for the solitary wave tail is in integral form and an explicit expression is found, for large time. Singularity theory, usually used for combustion problems, is applied to the large time expression for the solitary wave tail. Analytical results are obtained, such as the parameter regions in which qualitatively different types of solitary wave tails occur, the location of zeros and the location and amplitude of peaks, in the solitary wave tail. Two examples, the near-continuum limit of a discrete NLS equation and an explicit numerical scheme for the NLS equation, are considered in detail. For the discrete NLS equation it is found that three qualitatively different types of solitary wave tail can occur, while for the explicit finite-difference scheme, only one type of solitary wave tail occurs. An excellent comparison between the perturbation solution and numerical simulations, for the solitary wave tail, is found for both examples.

Keywords: soliton; nonlinear Schrödinger equation; solitary wave tails; finite-difference scheme, discrete NLS equation

1 Introduction

One of the most important model equations in nonlinear science is the nonlinear Schrödinger (NLS) equation,

$$i\eta_t + \eta_{xx} + \eta|\eta|^2 = 0. \quad (1)$$

Physically, the NLS equation describes the modulation of weakly-nonlinear wavetrains in deep water. Benjamin and Feir [Benjamin and Feir(1967)] showed that an uniform wave-train is unstable to longwave perturbations. Peregrine [Peregrine(1985)] and Yuen and Lake [Yuen and Lake(1982)] present a historical overview of fluid mechanics applications of the NLS equation and its physical origins. In the optical context, the NLS equation was derived by Hasegawa and Tappert [Hasegawa and Tappert(1973)]. It also describes the evolution of the slowly varying envelope of an optical pulse. Derived asymptotically from Maxwell's equations, it assumes slow variation in the carrier frequency and the Kerr dependence (where the nonlinear refractive index $n = n_0 + n_2|\eta|^2$). The NLS equation is central to understanding soliton propagation in optical fibres, which is of critical importance to the field of fibre-based telecommunications [Wabnitz et al.(1995)Wabnitz, Kodama, and Aceves].

Motivated by these physical applications, the evolution of a NLS soliton has been studied extensively in both the physical and mathematical communities. We consider a perturbed NLS equation of the form,

$$i\eta_t + \eta_{xx} + \eta|\eta|^2 = \alpha H(\eta), \quad \alpha \ll 1, \quad \text{where} \quad (2)$$

$$H(\eta) = c_1|\eta|^4\eta + c_2\eta^2\eta_{xx} + c_3\eta\eta_x^2 + c_4|\eta_x|^2\eta + c_5|\eta|^2\eta_{xx} + c_6\eta_{xxxx}.$$

When the higher-order coefficients in (2) are given by

$$(c_1, c_2, c_3, c_4, c_5, c_6) = (1.5, 1, 3, 2, 4, 1), \quad (3)$$

then it represents the next even member of the NLS integrable hierarchy, so (2) represents a generalisation of this hierarchy member. The perturbation terms also have a number of physical and numerical applications. Kavitha and Daniel [Kavitha and Daniel(2003)] showed that (2) is a model for classical Heisenberg ferromagnetic spin chains, with bi-quadratic exchange interactions along the spin lattice. The near-continuum limit of various forms of the discrete NLS equation are also governed by this form of perturbed NLS equation [Dmitriev et al.(2002)Dmitriev, Semagin, Sukhorukov, and Shigenari, Kivshar and Salerno(1994)]. A further application is the analysis of numerical methods for the NLS equation, as the leading-order discretization error of many schemes are described by the perturbation terms in (2).

Applying an asymptotic transformation to the perturbed NLS equation (2), Hoseini and Marchant [Hoseini and Marchant(2007)] found that the algebraic relation

$$-4c_1 + 4c_2 + 2c_3 - c_4 + c_5 - 6c_6 = 0, \quad (4)$$

describes cases for which higher-order solitary wave collisions of (2) are asymptotically elastic. They found analytical details of the solitary wave collision, including the higher-order phase shifts.

A powerful analytical solution technique is direct soliton perturbation theory. This requires that the complete set of the eigenfunctions for the linearized problem related to the nonlinear wave equation be determined. Yang [Yang(2000)] constructed this set for a large class of integrable nonlinear wave equations such as the Korteweg-de Vries (KdV), NLS and modified KdV equations. The same procedure can be exploited to find the eigenstates of the adjoint linearization operator. His finding shows that the eigenfunctions for these hierarchies are the squared Jost solutions.

Chen and Yang [Chen and Yang(2002)] developed direct soliton perturbation theory for the derivative NLS and the modified NLS equations. Using the similarity between the KdV and derivative NLS hierarchies they showed that the eigenfunctions for the linearized bright derivative NLS equation are the derivatives of the squared Jost solutions. This is in contrast to the counterpart for NLS, Hirota and mKdV hierarchies, where the eigenfunctions are just the squared Jost solutions. Suppressing the secular terms, they also found the slow evolution of soliton parameters and the perturbation-induced radiation.

Hoseini and Marchant [Hoseini and Marchant(2006)] examined bright solitary wave interaction for a focusing version of the higher-order Hirota equation. A family of higher-order embedded solitons was found by using an asymptotic transformation. When embedded solitons do not exist, soliton perturbation theory was used to determine the details of a single evolving solitary wave, to first-order. In particular, an integral expression was found for the first-order correction to the solitary wave profile. They also asymptotically analysed the integral expression to derive an analytical form for the tail of the solitary wave. It was shown that for the right-moving solitary wave a steady-state tail forms, while for the left-moving soliton, some transients propagate on the steady-state tail.

In this paper soliton perturbation theory is used to describe the evolution of a single solitary wave, for the perturbed NLS equation (2). In §2 soliton perturbation theory is used to derive the details of an evolving solitary wave at first-order. The first-order correction to the solitary wave is found in integral form and an explicit expression is found for large time. The large-time solution is investigated analytically using singularity theory, a technique usually applied to combustion problems. This allows the parameter space to be identified, in which qualitatively different types of solitary wave tails occur. It also allows the location of zeros and the location and amplitude of peaks in the solitary wave tail, to be determined.

In §3 two example are considered. These are the near-continuum limit of a discrete NLS equation and an explicit finite-difference scheme for the NLS equation; in both cases the perturbed NLS equation (2) describes the leading order discretisation effects. For the discrete NLS equation it is found that three qualitatively different solitary wave tails can occur, while only one type of solitary wave tail occurs for the explicit finite-difference scheme. Excellent comparisons between the explicit large time expression and numerical solutions, for the solitary wave tails, are found. In §4 the results of the paper are reviewed and conclusions made.

2 Soliton perturbation theory

In this section the soliton perturbation theory, first developed by Yang and Kaup [Yang and Kaup(2000)], is applied to find the first-order correction to the solitary wave solution.

2.1 Preliminaries

Soliton perturbation theory has been successfully applied to the NLS equation (1) by several authors. Here we review the important aspects of this elegant theory. We refer the reader to Yang and Kaup [Yang and Kaup(2000)] for more details. The unperturbed NLS equation (1) has the soliton solution

$$\eta = \sqrt{2\kappa} e^{i\phi} \operatorname{sech} \kappa \theta, \quad \text{where} \quad (5)$$

$$\phi = ax + (\kappa^2 - a^2)t + \phi_0, \quad \theta = x - 2at - \theta_0,$$

and the parameters κ , a , ϕ_0 and θ_0 are free. To model the perturbation, the solitary wave solution is defined as

$$\eta = e^{i(\delta + \frac{V}{2}\theta)} \omega(\theta, t, T; \alpha), \quad \text{where} \quad (6)$$

$$\delta = \int_0^t (\beta + \frac{1}{4}V^2) dt - \delta_0, \quad \theta = x - \int_0^t V dt - \theta_0.$$

Here the parameters V , β , δ_0 and θ_0 are considered as functions of the slow time $T = \alpha t$. Substituting (6) in (2) gives

$$i\omega_t + \omega_{\theta\theta} - \beta\omega + \omega|\omega|^2 = \alpha G - \alpha \left(\frac{1}{2}V\theta_{0T} - \frac{1}{2}V_T\theta + \delta_{0T} \right) \omega \quad (7)$$

$$- \alpha (i\omega_T - i\omega_{\theta}\theta_{0T}), \quad \text{where } G = e^{-i(\delta + \frac{V}{2}\theta)} H(\eta).$$

The explicit form of G for the perturbation terms in (2) will be shown later. Next, we expand the solution ω as

$$\omega = \eta_0(\theta) + \alpha\eta_1 + O(\alpha^2), \quad (8)$$

and substitute (8) into (7). The $O(1)$ terms satisfy the unperturbed NLS equation, but at $O(\alpha)$ we obtain

$$i\eta_{1t} + \eta_{1\theta\theta} - \beta\eta_1 + \eta_0^2\eta_1 + 2\eta_0\eta_1^2 = w_1, \quad \text{where} \quad (9)$$

$$w_1 = G_0 - i\eta_{0T} + i\eta_{0\theta}\theta_{0T} - \left(\frac{1}{2}V\theta_{0T} - \frac{1}{2}V_T\theta + \delta_{0T} \right) \eta_0,$$

$$G_0 = e^{-i(\delta + \frac{V}{2}\theta)} H(e^{i(\delta + \frac{V}{2}\theta)} \eta_0), \quad \eta_0 = \sqrt{2\kappa} \operatorname{sech} \kappa \theta.$$

and $\eta_1|_{t=0} = 0$. By taking $U = (\eta_1, \eta_1^*)^T$ and $H = (w_1, -w_1^*)^T$, (9) can be represented in matrix form as

$$(i\partial_t + L)U = H, \quad \text{where } L = \sigma_3 \begin{pmatrix} \partial_{\theta\theta} - \beta + 2\eta_0^2 & \eta_0^2 \\ \eta_0^2 & \partial_{\theta\theta} - \beta + 2\eta_0^2 \end{pmatrix}, \quad (10)$$

$$\text{and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

σ_3 is the Pauli matrix. The only remaining problem is to solve (10), which uses the eigenfunctions and eigenvalues of the associated linear operator L . We denote the non-localised (continuous) eigenvectors of L , which are needed for our analysis, by

$$\Psi_1 = e^{-i\xi\theta} \begin{pmatrix} -\kappa^2 \text{sech}^2 \kappa\theta \\ (\kappa \tanh \kappa\theta + i\xi)^2 \end{pmatrix}, \Psi_2 = e^{-i\xi\theta} \begin{pmatrix} -(\kappa \tanh \kappa\theta + i\xi)^2 \\ \kappa^2 \text{sech}^2 \kappa\theta \end{pmatrix}. \quad (11)$$

The eigenfunctions of $L^\dagger$, adjoint operator of L , are also needed. These adjoint eigenfunctions are determined by $w = (-a^\dagger, b^\dagger)^T$, where $w = (a, b)^T$ is a eigenfunction of L . Note that the eigenfunctions (11) are similar to those related to the linearization problem of the Hirota equation [Hoseini and Marchant(2006)]. Lastly, the product

$$(f(\theta), g(\theta)) = \int_{-\infty}^{\infty} f(\theta)^T g(\theta) d\theta, \quad (12)$$

needs to be defined. Note, that if g is replaced by its complex conjugate in the integral, then (12) is an inner product, but for soliton perturbation theory, the product (12) need not be positive definite, see Yang [Yang(2003)] or Zhu and Yang [Zhu and Yang(2007)]. The products in (13) are real quantities. Suppressing the secular terms gives the first-order solitary wave solution of (10) as,

$$\eta_1 = \frac{\sqrt{2}}{36} \int_{-\infty}^{\infty} [g^+(t, \xi) \Psi_1 + g^-(t, \xi) \Psi_2] d\xi, \text{ where } \square = \xi^2 + \kappa^2, \quad g^\pm(t, \xi) = \pm \frac{M^\pm(\xi)}{\square} (1 - e^{i\square t}), \quad M^+(\xi) = \frac{\sqrt{18}}{2\pi\square^2} (H, \Psi_1^\dagger), \quad M^-(\xi) = -\frac{\sqrt{18}}{2\pi\square^2} (H, \Psi_2^\dagger). \quad (13)$$

2.2 The first-order perturbation solution

To study the effect of the perturbation terms in (2) on a NLS soliton envelope, we solve (10). The term G_0 , which forms part of the first-order forcing term (9), has the form

$$G_0 = H(\eta_0) + iV(-c_2 + c_3 + c_5)\eta_0^2\eta_{0\theta} + \frac{1}{4}V^2(-c_2 - c_3 + c_4 - c_5)\eta_0^3 + \frac{1}{16}V^4c_6\eta_0 - \frac{1}{2}iV^3c_6\eta_{0\theta} - \frac{3}{2}V^2c_6\eta_{0\theta\theta} + 2iVc_6\eta_{0\theta\theta\theta}. \quad (14)$$

Substituting (14) into the forcing term (9) and applying the residue theorem to $M^\pm$, yields their explicit forms

$$M^\pm(\xi) = (a_1\xi^2 \pm a_2\xi + a_3) \text{sech}(a_4\xi), \text{ where} \quad (15)$$

$$a_1 = 2c_1 - 2c_2 - c_3 - c_4 - 2c_5 + 12c_6, \quad a_2 = 6a(c_2 - c_3 - c_5 + 6c_6),$$

$$a_3 = 9[(2c_1 - c_2 - c_5 + 2c_6)\kappa^2 - (c_2 + c_3 - c_4 + c_5 - 6c_6)a^2], \quad a_4 = \frac{\pi}{2\kappa}.$$

Finally, inserting (15) in (13) gives

$$\eta_1(\theta, t) = \frac{\sqrt{2}}{36} \int_{-\infty}^{\infty} \frac{e^{-i\xi\theta}}{\square} \times [M^-(\xi)(1 - e^{-i\square t})(\kappa \tanh \kappa\theta + i\xi)^2 - \kappa^2 M^+(\xi)(1 - e^{i\square t}) \text{sech}^2 \kappa\theta] d\xi, \quad (16)$$

as an integral expression for the first-order correction to the solitary wave profile.

As the integral expression for η_1 contains no singular points, the Riemann-Lebesgue theorem implies that $\eta_1 \rightarrow 0$ as $t \rightarrow \infty$. It is to be expected that η_1 decay to zero for long time, as the higher-order NLS solitary wave (17) exists for all values of the higher-order coefficients. This is in contrast to the higher-order Hirota equation, for which a higher-order solitary wave exists only for special choices of the higher-order coefficients. As the Hirota soliton is embedded in the linear wave spectrum, in general, perturbation terms cause radiation loss and the formation of a permanent solitary wave tail, see Hoseini and Marchant [Hoseini and Marchant(2006)].

Also, we note that the asymptotic expression for the higher-order steady-state NLS solitary wave of (2) (found by direct substitution), is

$$\begin{aligned} \eta(x, t) = & \sqrt{2\kappa} e^{i\Box} \operatorname{sech} \kappa \theta + \frac{\sqrt{2}}{2} \kappa \alpha e^{i\Box} \left(\frac{2}{3} \kappa^2 a \operatorname{sech}^3 \kappa \theta + \frac{1}{3} i \kappa a_2 \operatorname{sech} \kappa \theta \tanh \kappa \theta \right. \\ & \left. + a_5 \operatorname{sech} \kappa \theta \right), \quad \Box = ax + [\kappa^2 - a^2 + \alpha c_6 (\kappa^4 + a^4 - 6\kappa^2 a^2)]t, \\ & \theta = x - [2a - 4\alpha c_6 (a^2 - \kappa^2)a]t, \quad \text{where} \\ & a_5 = a^2(c_2 + c_3 - c_4 + c_5 - 6c_6) + \frac{1}{3} \kappa^2 (5c_2 - 8c_1 + c_3 + c_4 + 5c_5 - 18c_6) \end{aligned} \quad (17)$$

and the parameters a_1 and a_2 are given in (15). We note that the first-order correction, η_1 , given by (15)-(16), is zero if

$$(c_1, c_2, c_3, c_4, c_5, c_6) = (-1, 2, 14, 4, 0, 2)r + (1, 0, -2, 0, 2, 0)s, \quad (18)$$

for free parameters r and s , which implies that no $O(\alpha)$ evolution of the solitary wave profile occurs. We also see that (18) implies that the $O(\alpha)$ amplitude terms of (17) are zero, confirming that no tail evolves in this case.

2.3 Analytical results for the solitary wave tail

The leading order transient term for the tail, valid for large time, can be determined by the method of stationary phase. To obtain the large time solution we let $\theta = c_p t$ in (16) and consider c_p of either sign. The technique used here is similar to that of Hosieni and Marchant [Hoseini and Marchant(2006)] and Pelinovsky and Yang [Pelinovsky and Yang(2002)], who also obtained large time solutions. The relevant phase of (16) is $\Box_1 = -\xi c_p - \xi^2 - \kappa^2$ and the point of stationary phase, which occurs when $\frac{d\Box_1}{d\xi} = 0$, is $\xi_s = -\frac{c_p}{2}$. Using this method gives

$$\eta_1 \approx -\frac{\pi^{\frac{1}{2}} \sqrt{2} (\kappa - \frac{ic_p}{2})^2 M^+(-\frac{c_p}{2})}{36 t^{\frac{1}{2}} (\frac{c_p^2}{4} + \kappa^2)} e^{-i(\frac{c_p}{2} t + \frac{\pi}{4})}, \quad |\theta| \gg 1, \quad t \rightarrow \infty. \quad (19)$$

The term M^+ makes no contribution to the tail (16) at long times, as $\operatorname{sech}^2 \kappa \theta \approx 0$ for $|\theta| \gg 1$. The expression (19) describes the tail properties along the straight lines $\theta = c_p t$. It can be seen that the amplitude decays like $t^{-1/2}$ along these lines, which is the characteristic

decay rate associated with the NLS equation. At a fixed time t , the tail is evaluated by substituting $c_p = \frac{\theta}{t}$ into (19).

At large time the tail amplitude is given by the simple expression

$$|\eta_1| \approx \frac{\sqrt{2}}{36} \pi^{\frac{1}{2}} t^{-\frac{1}{2}} |M^-(-\frac{c_p}{2})|, \quad |\theta| \gg 1, \quad t \rightarrow \infty. \quad (20)$$

Hence analytical results for the tail amplitude $|\eta_1|$ can be obtained by examining the properties of M^- . The expression M^- is, in general, not symmetric so the left and right tails formed by the evolving soliton are also non-symmetric. This is due the non-symmetry of the phase of the initial soliton. The left and right solitary wave tails are symmetric if $a_2 = 0$ or $a = 0$. For the case of $a = 0$ the initial phase is symmetric, so the solitary wave tail for the evolving stationary soliton is also symmetric.

The expression M^- will be examined to classify all the qualitatively different profiles for the tail amplitude $|\eta_1|$. Note that the expression M^- is valid for all ξ (or θ), whilst the tail amplitude is only valid for $|\theta| \gg 1$. The classification is done by considering various degenerate parameter choices for M^- and is similar to the application of singularity theory to bifurcation problems in combustion theory [Golubitsky and Schaeffer(1985)]. The expressions

$$\begin{aligned} M^- &= (a_1 \xi^2 - a_2 \xi + a_3)S, \quad M^- = (2a_1 \xi - a_2)S - a_4(a_1 \xi^2 - a_2 \xi + a_3)ST, \\ M_{\xi\xi}^- &= 2a_1 S - 2a_4(2a_1 \xi - a_2)ST + a_4(a_1 \xi^2 - a_2 \xi + a_3)(S - 2S^3), \\ \text{where } S &= \text{sech}(a_4 \xi), \quad T = \tanh(a_4 \xi), \end{aligned} \quad (21)$$

are needed. The hysteresis and zero degenerate points are given by the relations

$$M_{\xi}^- = M_{\xi\xi}^- = 0, \quad (22)$$

$$M^- = M_{\xi}^- = 0, \quad (23)$$

respectively. (22) represents a condition for the occurrence of a hysteresis region in the tail profile while (23) is the condition for the generation of a point of zero amplitude in the tail profile. As the zeros of M^- are governed by a quadratic, the condition (23) simplifies to

$$a_2^2 - 4a_1 a_3 = 0. \quad (24)$$

The degenerate hysteresis condition (22) is solved numerically by a root finding routine from the IMSL library, while the degenerate zero condition (24) can be solved explicitly. In the special case of $a = 0$ an explicit condition can be found for (22) as the hysteresis point represents a bifurcation from a symmetric tail profile, at $\xi = 0$. Substituting $a = \xi = 0$ into (22) gives the condition

$$2a_1 - a_3 a_4^2 = 0. \quad (25)$$

3 Examples and numerical results

3.1 Evolution of discrete solitary waves

We consider the discrete NLS equation

$$i\frac{d}{dt}\eta_j + |\eta_j|^2(\lambda_2(\eta_{j+1} + \eta_{j-1}) + \lambda_3\eta_j) + D(\eta_{j+1} + \eta_{j-1} - 2\eta_j) + \alpha\lambda_1|\eta_j|^4\eta_j, \text{ where } \eta_j = \eta(j\Delta x, t), \quad j = 1 \dots n, \quad \alpha \ll 1. \quad (26)$$

$D = \Delta x^{-2}$, is the discretisation parameter and the coefficients of the nonlinear terms are related by $\lambda_3 + 2\lambda_2 = 1$. Versions of the discrete NLS equation (26) have been considered by Dimitriev et al. [Dimitriev et al.(2002)Dimitriev, Semagin, Sukhorukov, and Shigenari] and Kivshar and Salerno [Kivshar and Salerno(1994)]. In particular, if $\lambda_1 = \lambda_3 = 0$ the integrable AL model is obtained. We assume that D is large and consider the near-continuum limit of (26). Expanding $\eta_{j\pm 1}$ in a Taylor series and substituting into (26) gives

$$i\eta_t + \eta_{xx} + \eta|\eta|^2 + \alpha\lambda_1|\eta|^4\eta + \frac{\Delta x^2}{12}\eta_{xxxx} + \Delta x^2\lambda_2|\eta|^2\eta_{xx} + O(\Delta x^4), \quad (27)$$

which shows that, in the near-continuum limit, (26) is equivalent to the perturbed NLS equation (2) with higher-order coefficients

$$c_1 = -\lambda_1, \quad c_5 = -\lambda_2, \quad c_6 = -\frac{1}{12}, \quad \alpha = \Delta x^2. \quad (28)$$

Hence the evolution of a discrete solitary wave can be analytically described by the integral expression (16), the first-order correction to the solitary wave. The theory of §2.3 is used to determine the types of qualitatively different solitary wave tails that can occur, at large time. The coefficients of the quadratic associated with M^- are

$$a_1 = -2\lambda_1 + 2\lambda_2 - 1, \quad a_2 = a(6\lambda_2 - 3), \quad a_3 = 9\kappa^2(\lambda_2 - 2\lambda_1 - \frac{1}{6}) + 9a^2(\lambda_2 - \frac{1}{2}). \quad (29)$$

The parameter values $\lambda_2 = 2$ and $\kappa = 1$ are taken and the occurrence of qualitatively different tails in the (a, λ_1) plane is considered. The degenerate zero curves can be described, analytically, by the quadratic equation

$$\lambda_1^2 - \lambda_1(\frac{29}{12} + \frac{3}{4}a^2) + \frac{11}{8} + \frac{9}{16}a^2 = 0. \quad (30)$$

The degenerate hysteresis curve (22) is solved numerically.

Figure 1 shows the division of the (a, λ_1) plane into regions describing the qualitatively different tail profiles. The parameters are $\kappa = 1$ and $\lambda_2 = 2$. The degenerate hysteresis (22) and zero curves (30) are shown. The degenerate curves are symmetric about the λ_1 axis so only $a > 0$ is shown. There are four different regions in the plane corresponding to three qualitatively different kinds of solitary wave tail. The degenerate hysteresis curve

Figure 1: The parameter regions in which qualitatively different solitary wave tails occur, for the discrete NLS equation (26). The degenerate curves (22) (dashed line) and (24) (solid lines) are shown, for parameters $\lambda_2 = 2$ and $\kappa = 1$.

crosses the λ_1 axis (where $a = 0$) at $\lambda_1 = 0.820$ (given by (25)), and approaches the lower degenerate zero curve as $a \rightarrow \infty$. The lower and upper degenerate zero curves, solutions of (30), cross the λ_1 axis at $\lambda_1 = 0.917$ and $\lambda_1 = 1.5$, respectively.

Below the degenerate hysteresis curve the tail profile has a single peak and decays monotonically to zero as $\theta \rightarrow \pm\infty$. The peak can occur in the left or right tail, depending on the value of a . For $a = 0$, M^- has a peak, located at $\theta = 0$. As the long time solution for $|\eta_1|$ is not valid for $\theta = 0$, this peak is not numerically realised; the left and right tails, for $|\theta| \gg 1$, undergo monotonic decay. In the region bounded by the degenerate hysteresis and lower zero curves multiple peaks occur, in the tail profile. In this parameter region the tail amplitude, $|\eta_1|$, has continuous slope. In the region bounded by the two degenerate zero curves multiple peaks also occur, but the tail amplitude, $|\eta_1|$ has discontinuous slope (at the zeros, where $\eta_1 = 0$). Note that this discontinuity occurs in the slope of the amplitude $|\eta|$, not η itself, and the derivative term in the NLS equation, η_{xx} is continuous. In the parameter region above the upper degenerate zero curve the tail has a single peak.

Figure 2 shows the evolution of a discrete NLS solitary wave. Shown is the solitary wave tail amplitude, $|\eta_1|$ versus θ , for $t = 40$. The other parameters are $\lambda_1 = 0.88$, $\lambda_2 = 2$, $a = 0$ and $\kappa = 1$. Shown is the analytical tail amplitude (20) and the numerical solution of the discrete NLS equation (26). Only the right tail, $\theta > 0$, is shown, due to the symmetry of the tail solution for the $a = 0$ case. The discrete NLS equation (26) was solved numerically using a fourth-order Runge-Kutta method with discretisation

Figure 2: The amplitude of the solitary wave tail, $|\eta_1|$ versus θ , at $t = 40$, for the discrete NLS equation (26). The parameters are $\lambda_1 = 0.88$, $\lambda_2 = 2$, $\kappa = 1$ and $a = 0$. Shown are the numerical (dashed line) and the analytical (solid line) solutions.

parameters $\Delta x = 1.525 \times 10^{-2}$ and $\Delta t = 1.25 \times 10^{-4}$. For this choice of Δt the temporal discretisation error is negligible. For the numerical solution, the quantity $\alpha^{-1}|\eta - \eta_0|$ is plotted, where η_0 is the NLS soliton solution (5). This quantity represents the appropriate comparison with the perturbation solution η_1 in the tail regions, away from the solitary wave, located at $\theta = 0$.

This choice of parameters is in the region of parameter space corresponding to multiple peaks, where the tail amplitude has continuous slope. It can be seen that the perturbation and numerical solutions are the same to graphical accuracy, except near $\theta = 0$, where the soliton is located. Hence the analytical solution (16), with parameters (28), is an excellent description of the evolving tail, for the discrete NLS equation. As the solution is symmetric about $\theta = 0$ the second peak occurs for $\theta < 0$ and is not shown in the figure. Analytically, the tail peak moves with velocity $c_p = 1.67$ and $|M^-| = 0.766$. At $t = 40$ the analytical solution predicts the peak location as $\theta = 66.8$ and its amplitude as $|\eta_1| = 8.43 \times 10^{-3}$. Numerically, the peak is located at $\theta = 67.9$ and its amplitude is $|\eta_1| = 8.48 \times 10^{-3}$. The numerical values for the peak were found with the aid of a curve fitting procedure. The differences between the numerical and analytical values for the peak are less than 2%, confirming the excellent comparison. At larger times the tail peak moves away from the solitary wave (located at $\theta = 0$) with velocity $c_p = 1.67$ and its amplitude decays like $t^{-1/2}$.

Figures 3 and 4 show the evolution of a discrete NLS solitary wave. Shown is the solitary wave tail amplitude, $|\eta_1|$ versus θ , for $t = 40$. The other parameters are $\lambda_1 = 1$,

Figure 3: The amplitude, $|\eta_1|$, of the left tail of solitary wave, versus θ , for the discrete NLS equation(26), at $t = 40$. The parameters are $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa = 1$ and $a = 0.15$. Shown are the numerical (dashed line) and the analytical (solid line) solutions.

$\lambda_2 = 2$, $a = 0.15$ and $\kappa = 1$. Shown is the analytical tail amplitude (20) and the numerical solution of the discrete NLS equation (26). The numerical scheme and the discretisation parameters used are the same as for the example described in figure 2.

This example is in the region of parameter space corresponding to multiple peaks, where the tail amplitude $|\eta_1|$, has discontinuous slope. It can be seen that the perturbation and numerical solutions are the same to graphical accuracy, except near $\theta = 0$, where the soliton is located. The tail profile has both peaks and zeros (where $|\eta_1|$ has discontinuous slope) occurring. Solving the quadratic associated with M^- gives the analytical location of the zeros as $c_p = -3.921$ and 1.221 . Solving $M_\varepsilon = 0$ gives the location of the peaks as $c_p = -5.492$, -0.4951 and 2.835 . Hence the left tail has one zero and two peaks while the right tail has one zero and one peak.

At $t = 40$ the analytical location of the zeros are $\theta = -156.8$ and 48.84 . Numerically their locations are $\theta = -157.4$ and 50.37 , respectively. At this time the analytical location of the peaks are $\theta = -219.7$, -19.80 and 113.4 whilst the amplitudes of the peaks are $|\eta_1| = 7.77 \times 10^{-4}$, 1.50×10^{-2} and 6.40×10^{-3} , respectively. Numerically, the peaks are located at $\theta = -219.5$, -22.13 and 113.9 and their amplitudes are 7.77×10^{-4} , 1.51×10^{-2} and 6.42×10^{-3} , respectively. The variation between the analytical and numerical peak location at $\theta = -22.13$ is about 10%, but the variations in the location of the rest of the zeros and peaks is much smaller. The analytical and numerical peak amplitudes are the same to at least two significant figures.

Figure 4: The amplitude, $|\eta_1|$, of the right tail of the solitary wave, versus θ , for the discrete NLS equation (26), at $t = 40$. The parameters are $\lambda_1 = 1$, $\lambda_2 = 2$, $\kappa = 1$ and $a = 0.15$. Shown are the numerical (dashed line) and the analytical (solid line) solutions.

3.2 Evolution of numerical solitary waves

When an NLS soliton propagates on a numerical grid, evolution of the wave will occur due to the discretisation error. To illustrate this the classical explicit finite-difference method for the NLS equation is considered. The numerical method is

$$\eta_j^{k+1} = \eta_j^k + s(\eta_{j+1}^k + \eta_{j-1}^k - 2\eta_j^k) + i\rho|\eta_j^k|\eta_j^k, \quad \text{where} \quad (31)$$

$$\eta_j^k = \eta(k\Delta t, j\Delta x), \quad s = \frac{2\Delta t}{\Delta x^2}, \quad \rho = 2\Delta t.$$

The scheme is stable if $s \leq \frac{1}{4}$ and the truncation error is $O(\Delta t^2, \Delta x^2)$. As $\Delta t = O(\Delta x^2)$ for stability the scheme (31) is described, to leading-order, by the perturbed NLS equation (2) with

$$c_6 = -\frac{1}{12}, \quad \alpha = \Delta x^2. \quad (32)$$

Hence the solitary wave tail due to the numerical evolution of the soliton, is analytically described by the integral expression (16), the first-order correction to the solitary wave. The coefficients of the quadratic associated with M^- are

$$a_1 = -1, \quad a_2 = -3a, \quad a_3 = -\frac{3}{2}\kappa^2 - \frac{9}{2}a^2. \quad (33)$$

Figure 5: The velocity c_p , of the peak of the solitary wave tail, versus a , for the NLS finite-difference scheme (31). The other parameter is $\kappa = 1$.

For these coefficients degenerate hysteresis or zero points do not occur and the solitary wave tail always has a single peak with monotonic decay.

Figure 5 shows the velocity, c_p , of the tail peak, versus a , for the evolution of a numerical solitary wave, using the finite-difference scheme (31). The other parameter is $\kappa = 1$. The analytical location of the tail peak is the solution of $M_{\xi}^- = 0$. The tail peak with the maximum value of c_p , is found by differentiating $M_{\xi}^- = 0$ wrt to a and letting $\frac{d\xi}{da} = 0$ in the resulting expression. For positive a the tail peak occurs on the right, reaching a maximum velocity, at $c_p = 0.582$, for $a = 0.4$. For larger a the velocity of the tail peak is reduced, and $c_p \rightarrow 0$ as $a \rightarrow \infty$. For $a < 0$ the figure is not shown, as it is anti-symmetric about the c_p axis; the tail peak moves with negative velocity for negative a . For $a = 0$ M^- has a peak at $\theta = 0$, at which the long time solution for $|\eta_1|$ is not valid. For this case the numerically realized tail decays monotonically, for $|\theta| \gg 1$, with no peak. Solutions for other values of κ are not shown as they are self similar to the $\kappa = 1$ case shown here. If the parameter set (a, κ, c_p) describes the peak of a solitary wave tail, then self-similarity implies that the parameter set $(pa, p\kappa, pc_p)$ also describes a peak, where p is an arbitrary scale factor.

Figures 6 and 7 shows the solitary wave tail amplitudes, $|\eta_1|$, versus θ , for a solitary wave evolving due to the discretisation error of the finite difference scheme (31). Shown is the first-order correction from the perturbation solution and numerical solutions of (31) at $t = 40$, for both the left and right tails. The parameters are $\Delta x = 6.25 \times 10^{-3}$, $\Delta t = 6.25 \times 10^{-6}$, $\kappa = 1$ and $a = 0.4$. For the numerical solution, the quantity $\alpha^{-1}|\eta - \eta_0|$

Figure 6: The amplitude, $|\eta_1|$, of the left tail of the solitary wave, versus θ , for the NLS finite-difference scheme (31), at $t = 40$. The other parameters are $\kappa = 1$ and $a = 0.4$. Shown are the analytical (dashed line) and numerical (solid line) solutions.

is plotted, where η_0 is the NLS soliton solution (5). It can be seen that there is an excellent comparison between the perturbation and numerical solutions, except in the region near $\theta = 0$. The tail has a single peak located on the right tail. Analytically the peak moves with velocity $c_p = 0.582$ and $|M^-| = 2.40$. At $t = 40$ the analytical peak location is $\theta = 23.3$ and its amplitude is $|\eta_1| = 2.66 \times 10^{-2}$. Numerically, the peak location is $\theta = 26$ with amplitude $|\eta_1| = 2.66 \times 10^{-2}$. The difference between the numerical and analytical peak locations is about 10%, while the peak amplitudes are the same to three significant figures. It can be seen that some numerical oscillations occur on the left tail. These are due to the effect of the neglected, $O(\Delta t^2)$, terms, on the evolving tail.

4 Conclusion

Soliton perturbation theory has been applied to the perturbed NLS equation (2). An analytical solution in integral form is obtained for the first-order correction to the solitary wave profile. The method of stationary phase is used to obtain an explicit solution, valid for large time. The simple form of the large time solution allows analytical results to be obtained, such as the classification of different qualitative forms for the solitary wave tail and the amplitude and location of the solitary wave tail zeros and peaks. The application of singularity theory, usually used for combustion problems, to analytically describe the

Figure 7: The amplitude, $|\eta_1|$, of the right tail of the solitary wave, versus θ , for the NLS finite-difference scheme (31), at $t = 40$. The other parameters are $\kappa = 1$ and $a = 0.4$. Shown are the analytical (dashed line) and numerical (solid line) solutions.

solitary wave tail, is a novel feature of the paper.

Two example are considered, of a discrete NLS equation and an explicit numerical scheme for the NLS equation. It is shown that for the evolution of a discrete soliton, three different types of solitary wave tail are possible while for the numerically evolving soliton, only one type of tail can occur. Examples of the three different types of solitary wave tail are illustrated and an excellent agreement between numerical and perturbation solutions is obtained.

Wide applicability exists in applying soliton perturbation theory to other applications for the NLS equation and for other model equations, such as the Hirota equation. Other possible applications include the examination of alternate numerical schemes and the evolution of dark solitons.

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References

[Benjamin and Feir(1967)] T. Benjamin, J. F. Feir, The disintegration of wavetrains on deep water, J. Fluid Mech. 27 (1967) 417–430.

- [Peregrine(1985)] D. H. Peregrine, Water waves and their development in space and time, Proc. Roy. Soc. Lond. A 400 (1985) 1–18.
- [Yuen and Lake(1982)] H. C. Yuen, B. M. Lake, Nonlinear dynamics of deep-water gravity waves, Adv. Appl. Mech. 22 (1982) 67–229.
- [Hasegawa and Tappert(1973)] A. Hasegawa, F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibres. I. Anomalous dispersion, Appl. Phys. Lett. 23 (1973) 142–144.
- [Wabnitz et al.(1995)] Wabnitz, Kodama, and Aceves] S. Wabnitz, Y. Kodama, A. B. Aceves, Control of optical soliton interactions, Opt. Fibr. Tech. 1 (1995) 187–217.
- [Kavitha and Daniel(2003)] L. Kavitha, M. Daniel, Integrability and soliton in a classical one dimensional site dependent biquadratic Heisenberg spin chain and the effect of nonlinear inhomogeneity, J. Phys. A: Math. Gen. 36 (2003) 10471–10492.
- [Dmitriev et al.(2002)] Dmitriev, Semagin, Sukhorukov, and Shigenari] S. V. Dmitriev, D. A. Semagin, A. A. Sukhorukov, T. Shigenari, Chaotic character of two-soliton collisions in the weakly perturbed nonlinear Schrödinger equation, Phys. Rev. E 66 (2002) 046609.
- [Kivshar and Salerno(1994)] Y. S. Kivshar, M. Salerno, Modulational instabilities in the discrete deformable nonlinear Schrödinger equation, Phys. Rev. E 49 (1994) 3543–3546.
- [Hoseini and Marchant(2007)] S. Hoseini, T. R. Marchant, Solitary wave interaction for a higher-order nonlinear Schrödinger equation, IMA J. Applied Math. 72 (2007) 206–222.
- [Yang(2000)] J. Yang, Complete eigenfunctions of linearized integrable equation expanded around a soliton solution, J. Math. Phys. 41 (2000) 6614–6638.
- [Chen and Yang(2002)] X.-J. Chen, J. Yang, Direct perturbation theory for solitons of the derivative nonlinear Schrödinger equation and the modified nonlinear Schrödinger equation, Phys. Rev. E 65 (2002) 066608.
- [Hoseini and Marchant(2006)] S. Hoseini, T. R. Marchant, Solitary wave interaction and evolution for a higher-order Hirota equation, Wave Motion 44 (2006) 92–106.
- [Yang and Kaup(2000)] J. Yang, D. J. Kaup, Stability and evolution of solitary waves in perturbed generalized nonlinear Schrödinger equations, SIAM J. Appl. Math. 60 (2000) 967–989.
- [Yang(2003)] J. Yang, Stable embedded solitons, Phys. Rev. Lett. 91 (2003) 143903.

- [Zhu and Yang(2007)] Y. Zhu, J. Yang, Universal fractal structure in the weak interaction of solitary waves in generalized nonlinear Schrödinger equations, *Phy. Rev. E.* 75 (2007) 036605.
- [Pelinovsky and Yang(2002)] D. E. Pelinovsky, J. Yang, A normal form for nonlinear resonance of embedded solitons, *Proc. Roy. Soc. Lond. A* 458 (2002) 1469–1497.
- [Golubitsky and Schaeffer(1985)] M. Golubitsky, D. G. Schaeffer, *Singularities and groups in bifurcation theory*, Springer-Verlag, New York, 1985.