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# COVERINGS OF SKEW-PRODUCTS AND CROSSED PRODUCTS BY COACTIONS

**DAVID PASK, JOHN QUIGG and AIDAN SIMS** ✉

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**Abstract**

Consider a projective limit $G$ of finite groups $G_n$. Fix a compatible family $\delta^n$ of coactions of the $G_n$ on a $C^*$-algebra $A$. From this data we obtain a coaction $\delta$ of $G$ on $A$. We show that the coaction crossed product of $A$ by $\delta$ is isomorphic to a direct limit of the coaction crossed products of $A$ by the $\delta^n$. If $A = C^*(\Lambda)$ for some $k$-graph $\Lambda$, and if the coactions $\delta^n$ correspond to skew-products of $\Lambda$, then we can say more. We prove that the coaction crossed product of $C^*(\Lambda)$ by $\delta$ may be realized as a full corner of the $C^*$-algebra of a $(k+1)$-graph. We then explore connections with Yeend's topological higher-rank graphs and their $C^*$-algebras.

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## 1. Introduction

In this paper we investigate how certain coactions of discrete groups on $k$-graph $C^*$-algebras behave under inductive limits. This leads to interesting new connections between $k$-graph $C^*$-algebras, nonabelian duality, and Yeend's topological higher-rank graph $C^*$-algebras.

We consider a particularly tractable class of coactions of finite groups on $k$-graph $C^*$-algebras. A functor $c$ from a $k$-graph $\Lambda$ to a discrete group $G$ gives rise to two natural constructions. At the level of $k$-graphs, one may construct the skew-product $k$-graph $\Lambda \times_c G$; and at the level of $C^*$-algebras, $c$ induces a coaction $\delta$ of $G$ on $C^*(\Lambda)$. It is a theorem of [15] that these two constructions are compatible in the sense that the $k$-graph algebra $C^*(\Lambda \times_c G)$ is canonically isomorphic to the coaction crossed-product $C^*$-algebra $C^*(\Lambda) \times_\delta G$.

The skew-product construction is also related to discrete topology: given a regular covering map from a $k$-graph $\Gamma$ to a connected $k$-graph $\Lambda$, one obtains an isomorphism

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of $\Gamma$ with a skew-product of $\Lambda$ by a discrete group $G$ [15, Theorem 6.11]. Further results of [15] then show how to realize the $C^*$-algebra of $\Gamma$ as a coaction crossed product of the $C^*$-algebra of $\Lambda$.

The results of [12] investigate the relationship between $C^*(\Lambda)$ and $C^*(\Gamma)$ from a different point of view. Specifically, they show how a covering $p$ of a $k$-graph $\Lambda$ by a $k$-graph $\Gamma$ induces an inclusion of $C^*(\Lambda)$ into $C^*(\Gamma)$. A sequence of compatible coverings therefore gives rise to an inductive limit of $C^*$-algebras. The main results of [12] show how to realize this inductive limit as a full corner in the $C^*$-algebra of a $(k+1)$-graph.

We can combine the ideas discussed in the preceding three paragraphs as follows. Fix a $k$-graph $\Lambda$, a projective sequence of finite groups $G_n$, and a sequence of functors $c_n : \Lambda \to G_n$ which are compatible with the projective structure. We obtain from this data a sequence of skew-products $\Lambda \times_{c_n} G_n$ which form a sequence of compatible coverings of $\Lambda$. By results of [12], we therefore obtain an inductive system of $k$-graph $C^*$-algebras $C^*(\Lambda \times_{c_n} G_n)$. The results of [15] show that each $C^*(\Lambda \times_{c_n} G_n)$ is isomorphic to a coaction crossed product $C^*(\Lambda) \times_{\delta^n} G_n$. It is therefore natural to ask whether the direct limit $C^*$-algebra $\varinjlim(C^*(\Lambda \times_{c_n} G_n))$ is isomorphic to a coaction crossed product of $C^*(\Lambda)$ by the projective limit group $\varprojlim G_n$.

After summarizing in Section 2 the background needed for our results, we answer this question in the affirmative and in greater generality in Theorem 3.1. Given a $C^*$-algebra $A$, a projective limit of finite groups $G_n$ and a compatible system of coactions of the $G_n$ on $A$, we show that there is an associated coaction $\delta$ of $\varprojlim G_n$ on $A$, such that $A \times_\delta (\varprojlim G_n) \cong \varinjlim(A \times_{\delta^n} G_n)$.

In Section 4, we consider the consequences of Theorem 3.1 in the original motivating context of $k$-graph $C^*$-algebras. We consider a $k$-graph $\Lambda$ together with functors $c_n : \Lambda \to G_n$ which are consistent with the projective limit structure on the $G_n$. In Theorem 4.3, we use Theorem 3.1 to deduce that $C^*(\Lambda) \times_\delta G$ is isomorphic to $\varinjlim(C^*(\Lambda) \times_{\delta^n} G_n)$. Using results of [12], we realize $C^*(\Lambda) \times_\delta G$ as a full corner in a $(k+1)$-graph algebra (Corollary 4.5). We digress in Section 5 to investigate simplicity of $C^*(\Lambda) \times_\delta G$ via the results of [18].

We conclude in Section 6 with an investigation of the connection between our results and Yeend's notion of a topological $k$-graph [20, 21]. We construct from an infinite sequence of coverings $p_n : \Lambda_{n+1} \to \Lambda_n$ of $k$-graphs a projective limit $\Lambda$ which is a topological $k$-graph. We show that the $C^*$-algebra $C^*(\Lambda)$ of this topological $k$-graph coincides with the direct limit of the $C^*(\Lambda_n)$ under the inclusions induced by the $p_n$. In particular, the system of cocycles $c_n : \Lambda \to G_n$ discussed in the preceding paragraph yields a cocycle $c : \Lambda \to G := \varprojlim(G_n, q_n)$, the skew-product $\Lambda \times_c G$ is a topological $k$-graph, and the $C^*$-algebras $C^*(\Lambda \times_c G)$ and $C^*(\Lambda) \times_\delta G$ are isomorphic, generalizing the corresponding result [15, Theorem 7.1(ii)] for discrete groups.

## 2. Preliminaries

Throughout this paper, we regard $\mathbb{N}^k$ as a semigroup under addition with identity element 0. We denote the canonical generators of $\mathbb{N}^k$ by $e_1, \ldots, e_k$. For $n \in \mathbb{N}^k$, we denote its coordinates by $n_1, \ldots, n_k \in \mathbb{N}$ so that $n = \sum_{i=1}^{k} n_i e_i$. For $m, n \in \mathbb{N}^k$, we write $m \le n$ if $m_i \le n_i$ for all $i \in \{1, \ldots, k\}$.

We will at times need to identify $\mathbb{N}^k$ with the subsemigroup of $\mathbb{N}^{k+1}$ consisting of elements $n$ whose last coordinate is equal to zero. For $n \in \mathbb{N}^k$, we write $(n, 0)$ for the corresponding element of $\mathbb{N}^{k+1}$. When convenient, we regard $\mathbb{N}^k$ as (the morphisms of) a category with a single object in which the composition map is the usual addition operation in $\mathbb{N}^k$.

**2.1. *k*-graphs** Higher-rank graphs are defined in terms of categories. In this paper, given a category $\mathcal{C}$, we will identify the objects with the identity morphisms, and think of $\mathcal{C}$ as the collection of morphisms only. We will write composition in our categories by juxtaposition.

Fix an integer $k \ge 1$. A *k-graph* is a pair $(\Lambda, d)$ where $\Lambda$ is a countable category and $d : \Lambda \to \mathbb{N}^k$ is a functor satisfying the factorization property: whenever $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ satisfy $d(\lambda) = m + n$, there are unique $\mu, \nu \in \Lambda$ with $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. For $n \in \mathbb{N}^k$, we write $\Lambda^n$ for $d^{-1}(n)$. If $p \le q \le d(\lambda)$, we denote by $\lambda(p, q)$ the unique path in $\Lambda^{q-p}$ such that $\lambda = \lambda'\lambda(p, q)\lambda''$ for some $\lambda' \in \Lambda^p$ and $\lambda'' \in \Lambda^{d(\lambda)-q}$.

Applying the factorization property with $m = 0$, $n = d(\lambda)$ and with $m = d(\lambda)$, $n = 0$, one shows that $\Lambda^0$ is precisely the set of identity morphisms in $\Lambda$. The codomain and domain maps in $\Lambda$ therefore determine maps $r, s : \Lambda \to \Lambda^0$. We think of $\Lambda^0$ as the vertices—and $\Lambda$ as the paths—in a '$k$-dimensional directed graph'.

Given $F \subset \Lambda$ and $v \in \Lambda^0$, we write $vF$ for $F \cap r^{-1}(v)$ and $Fv$ for $F \cap s^{-1}(v)$. We say that $\Lambda$ is *row-finite* if $v\Lambda^n$ is a finite set for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and we say that $\Lambda$ has *no sources* if $v\Lambda^n$ is always nonempty.

We denote by $\Omega_k$ the $k$-graph $\Omega_k := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \le q\}$ with $r(p, q) := (p, p)$, $s(p, q) := (q, q)$ and $d(p, q) := q - p$. As a notational convenience, we will henceforth denote $(p, p) \in \Omega_k^0$ by $p$. An *infinite path* in a $k$-graph $\Lambda$ is a degree-preserving functor (otherwise known as a *k-graph morphism*) $x : \Omega_k \to \Lambda$. The collection of all infinite paths is denoted $\Lambda^\infty$. We write $r(x)$ for $x(0)$, and think of this as the range of $x$.

For $\lambda \in \Lambda$ and $x \in s(\lambda)\Lambda^\infty$, there is a unique infinite path $\lambda x \in r(\lambda)\Lambda^\infty$ satisfying $(\lambda x)\,(0, p) := \lambda x(0, p - d(\lambda))$ for all $p \ge d(\lambda)$. In particular, $r(x)x = x$ for all $x \in \Lambda^\infty$, so we denote $\{x \in \Lambda^\infty : r(x) = v\}$ by $v\Lambda^\infty$. If $\Lambda$ has no sources, then $v\Lambda^\infty$ is nonempty for all $v \in \Lambda^0$.

The factorization property also guarantees that for $x \in \Lambda^\infty$ and $n \in \mathbb{N}^k$ there is a unique infinite path $\sigma^n(x) \in x(n)\Lambda^\infty$ such that $\sigma^n(x)\,(p, q) = x(p + n, q + n)$. We somewhat imprecisely refer to $\sigma$ as the *shift map*. Note that $\sigma^{d(\lambda)}(\lambda x) = x$ for all $\lambda \in \Lambda$, $x \in s(\lambda)\Lambda^\infty$, and $x = x(0, n)\sigma^n(x)$ for all $x \in \Lambda^\infty$ and $n \in \mathbb{N}^k$.

We say that a row-finite $k$-graph $\Lambda$ with no sources is *cofinal* if, for every $v \in \Lambda^0$ and every $x \in \Lambda^\infty$, there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n) \ne \emptyset$. Given $m \ne n \in \mathbb{N}^k$ and $v \in \Lambda^0$, we say that $\Lambda$ has local periodicity $m, n$ at $v$ if $\sigma^m(x) = \sigma^n(x)$ for all $x \in v\Lambda^\infty$. We say that $\Lambda$ has *no local periodicity* if, for every $m, n \in \mathbb{N}^k$ and every $v \in \Lambda^0$, we have $\sigma^m(x) \neq \sigma^n(x)$ for some $x \in v\Lambda^\infty$.

**2.2. Skew-products** Let $\Lambda$ be a $k$-graph, and let $G$ be a group. A *cocycle* $c : \Lambda \to G$ is a functor from $\Lambda$ to $G$ where the latter is regarded as a category with one object. That is, $c : \Lambda \to G$ satisfies $c(\mu\nu) = c(\mu)c(\nu)$ whenever $\mu$, $\nu$ can be composed in $\Lambda$. It follows that $c(v) = e$ for all $v \in \Lambda^0$, where $e \in G$ is the identity element.

Given a cocycle $c : \Lambda \to G$, we can form the *skew-product $k$-graph* $\Lambda \times_c G$. We follow the conventions of [15, Section 6]. Note that these are different from those of [9, Section 5]. The paths in $\Lambda \times_c G$ are

$$(\Lambda \times_c G)^n := \Lambda^n \times G,$$

for each $n \in \mathbb{N}^k$. The range and source maps $r, s : \Lambda \times_c G \to (\Lambda \times_c G)^0$ are given by $r(\lambda, g) := (r(\lambda), c(\lambda)g)$ and $s(\lambda, g) := (s(\lambda), g)$. Composition is determined by $(\mu, c(\nu)g)\,(\nu, g) = (\mu\nu, g)$. It is shown in [15, Section 6] that $\Lambda \times_c G$ is a $k$-graph.

**2.3. Coverings and $(k + 1)$-graphs** We recall here some definitions and results from [12] regarding coverings of $k$-graphs. Given $k$-graphs $\Lambda$ and $\Gamma$, a $k$-graph morphism $\phi : \Lambda \to \Gamma$ is a functor which respects the degree maps. A *covering of $k$-graphs* is a triple $(\Lambda, \Gamma, p)$ where $\Lambda$ and $\Gamma$ are $k$-graphs, and $p : \Gamma \to \Lambda$ is a $k$-graph morphism which is surjective and is locally bijective in the sense that for each $v \in \Gamma^0$, the restrictions $p|_{v\Gamma} : v\Gamma \to p(v)\Lambda$ and $p|_{\Gamma v} : \Gamma v \to \Lambda p(v)$ are bijective.

REMARK 2.1. What we have called a covering of $k$-graphs is a special case of what was called a 'covering system of $k$-graphs' in [12]. In general, a covering system consists of a covering of $k$-graphs together with some extra combinatorial data. We do not need the extra generality, so we have dropped the word 'system'.

A covering $(\Lambda, \Gamma, p)$ is *row-finite* if $\Lambda$ (equivalently $\Gamma$) is row-finite, and $|p^{-1}(v)| < \infty$ for all $v \in \Lambda^0$. By [12, Proposition 2.6] we can associate to a row-finite covering $p : \Gamma \to \Lambda$ of $k$-graphs a row-finite $(k + 1)$-graph $\Lambda \overset{p}{\leftarrow} \Gamma$ containing disjoint copies $\iota(\Lambda)$ and $\jmath(\Gamma)$ of $\Lambda$ and $\Gamma$ with an edge of degree $e_{k+1}$ connecting each vertex $\jmath(v) \in \jmath(\Gamma^0)$ to its image $\iota(p(v)) \in \iota(\Lambda^0)$.

More generally, given a sequence $(\Lambda_n, \Lambda_{n+1}, p_n)$ of row-finite coverings of $k$-graphs, [12, Corollary 2.10] shows how to build a $(k + 1)$-graph $\varprojlim(\Lambda_n; p_n)$, which we sometimes refer to as a *tower graph*, containing a copy $\iota_n(\Lambda_n)$ of each individual $k$-graph in the sequence, and an edge of degree $e_{k+1}$ connecting each $\iota_{n+1}(v) \in \iota_{n+1}(\Lambda^0_{n+1})$ to its image $\iota_n(p_n(v)) \in \iota_n(\Lambda^0_n)$. The $(k + 1)$-graph $\varprojlim(\Lambda_n; p_n)$ has no sources if the $\Lambda_n$ all have no sources.

Given a covering $(\Lambda, \Gamma, p)$, [12, Proposition 3.2 and Theorem 3.8] show that the covering map $p : \Gamma \to \Lambda$ induces an inclusion $\iota_p : C^*(\Lambda) \to C^*(\Gamma)$. If $(\Lambda_n, \Lambda_{n+1}, p_n)_{n=1}^\infty$ is a sequence of coverings, the $(k + 1)$-graph algebra $C^*(\varprojlim(\Lambda_n; p_n))$ is Morita equivalent to the direct limit $\varinjlim(C^*(\Lambda_n), \iota_{p_n})$.

**2.4. Coactions and coaction crossed products** Here we give some background on group coactions on $C^*$-algebras and coaction crossed products. For a detailed treatment of coactions and coaction crossed products, see [4, Appendix A].

Given a locally compact group $G$, we write $C^*(G)$ for the full group $C^*$-algebra of $G$. We prefer to identify $G$ with its canonical image in $M(C^*(G))$, but when confusion is likely we use $s \mapsto u(s)$ for the canonical inclusion of $G$ in $M(C^*(G))$. If $A$ and $B$ are $C^*$-algebras, then $A \otimes B$ denotes the spatial tensor product. For a group $G$, we write $\delta_G$ for the natural comultiplication $\delta_G : C^*(G) \to M(C^*(G) \otimes C^*(G))$ given by the integrated form of the strictly continuous map which takes $s \in G$ to $s \otimes s \in \mathcal{U}M(C^*(G) \otimes C^*(G))$.

As in [4, Definition A.21], a *coaction* of a group $G$ on a $C^*$-algebra $A$ is an injective homomorphism $\delta : A \to M(A \otimes C^*(G))$ satisfying:

(1) the *coaction identity* $(\delta \otimes 1_G) \circ \delta = (1_A \otimes \delta_G) \circ \delta$ (as maps from $A$ to $M(A \otimes C^*(G) \otimes C^*(G))$); and
(2) the *nondegeneracy condition* $\overline{\delta(A)\,(1_A \otimes C^*(G))} = M(A \otimes C^*(G))$.

As in [7, 8], the nondegeneracy condition (2)—rather than the weaker condition that $\delta$ be a nondegenerate homomorphism—is part of our definition of a coaction (compare with [4, Definition A.21 and Remark A.22(3)]). Since we will be dealing only with coactions of compact (and hence amenable) groups, the two conditions are equivalent in our setting in any case (see [14, Lemma 3.8]).

Let $\delta : A \to M(A \otimes C^*(G))$ be a coaction of $G$ on $A$. We regard the map which takes $s \in G$ to $u(s) \in M(C^*(G))$ as an element $w_G$ of $\mathcal{U}M(C_0(G) \otimes C^*(G))$. Given a $C^*$-algebra $D$, A *covariant homomorphism* of $(A, G, \delta)$ into $M(D)$ is a pair $(\pi, \mu)$ of homomorphisms $\pi : A \to M(D)$ and $\mu : C_0(G) \to M(D)$ satisfying the covariance condition:

$$(\pi \otimes \mathrm{id}_G) \circ \delta(a) = (\mu \otimes \mathrm{id}_G)\,(w_G)\,(\pi(a) \otimes 1)(\mu \otimes \mathrm{id}_G)\,(w_G)^*,$$

for all $a \in A$.

The coaction crossed product $A \rtimes_\delta G$ is the universal $C^*$-algebra generated by the image of a universal covariant representation $(j_A, j_G)$ of $(A, G, \delta)$ (see [4, Theorem A.41]).

## 3. Continuity of coaction crossed products

In this section, we prove a general result regarding the continuity of the coaction crossed-product construction. Specifically, consider a projective system of finite groups $G_n$ and a system of compatible coactions $\delta^n$ of the $G_n$ on a fixed $C^*$-algebra $A$. We show that this determines a coaction $\delta$ of the projective limit $\varprojlim G_n$ on $A$, and that the coaction crossed product of $A$ by $\delta$ is isomorphic to a direct limit of the coaction crossed products of $A$ by the $\delta^n$.

The application we have in mind is when $A = C^*(\Lambda)$ is a $k$-graph algebra, and the $\delta^n$ arise from a system of skew-products of $\Lambda$ by the $G_n$. We consider this situation in Section 4.

THEOREM 3.1. *Let $A$ be a $C^*$-algebra, and let*

$$\cdots \xrightarrow{q_{n+1}} G_{n+1} \xrightarrow{q_n} G_n \longrightarrow \cdots \xrightarrow{q_1} G_1$$

*be surjective homomorphisms of finite groups. For each $n$ let $\delta^n$ be a coaction of $G_n$ on $A$. Suppose that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\delta^{n+1}} & M(A \otimes C^*(G_{n+1})) \\ & \searrow^{\delta^n} & \downarrow \mathrm{id}\otimes q_n \\ & & M(A \otimes C^*(G_n)) \end{array} \tag{3.1}$$

*commutes for each $n$.*

*For each $n$, write $Q_n$ for the canonical surjective homomorphism of $\varprojlim(G_m, q_m)$ onto $G_n$; write $q_n^* : C(G_n) \to C(G_{n+1})$ for the induced map $q_n^*(f) := f \circ q_n$; and write $J_n$ for the homomorphism $J_n := j_A^{\delta^{n+1}} \times (j_{G_{n+1}} \circ q_n^*)$ from $A \times_{\delta^n} G_n$ to $A \times_{\delta^{n+1}} G_{n+1}$.*

*Then there is a unique coaction $\delta$ of $\varprojlim(G_n, q_n)$ on $A$ such that:*

(i) *the diagrams*

$$\begin{array}{ccc} A & \xrightarrow{\delta} & M(A \otimes C^*(\varprojlim G_n)) \\ & \searrow^{\delta^n} & \downarrow \mathrm{id}\otimes Q_n \\ & & M(A \otimes C^*(G_n)) \end{array}$$

*commute; and*

(ii) $A \times_\delta \varprojlim(G_n, q_n) \cong \varinjlim(A \times_{\delta^n} G_n, J_n)$.

REMARK 3.2. In diagram (3.1) we could replace $M(A \otimes C^*(G_n))$ with $A \otimes C^*(G_n)$ and $M(A \otimes C^*(G_{n+1}))$ with $A \otimes C^*(G_{n+1})$ because $G_n$, $G_{n+1}$ are discrete.

PROOF OF THEOREM 3.1. Put

$$\begin{aligned} G &= \varprojlim G_n, \\ B_n &= A \times_{\delta^n} G_n, \\ J_n &= j_A^{\delta^{n+1}} \times (j_{G_{n+1}} \circ q_n^*) : B_n \to B_{n+1}, \\ B &= \varinjlim(B_n, J_n), \\ K_n &= \text{ the canonical embedding } B_n \to B. \end{aligned}$$

We aim to apply Landstad duality [17]: we will show that $B$ is of the form $C \times_\delta G$ for some coaction $(C, G, \delta)$, and then we will show that we can take $C = A$. To apply [17] we need:

- an action $\alpha$ of $G$ on $B$; and
- a nondegenerate homomorphism $\mu : C(G) \to M(B)$ which is $\mathrm{rt} - \alpha$ equivariant, where rt is the action of $G$ on $C(G)$ by right translation.

Then [17] will provide a coaction $(C, G, \delta)$ and an isomorphism

$$\theta : B \xrightarrow{\cong} C \times_\delta G,$$

such that

$$\theta \circ \mu = j_G \quad \text{and} \quad \theta(B^\alpha) = j_C(C).$$

This is simpler than the general construction of [17], because our group $G$ is compact (and then we are really using Landstad's unpublished characterization [13] of crossed products by coactions of compact groups).

We begin by constructing the action $\alpha$: for each $s \in G$, the diagrams

$$\begin{array}{ccc} B_{n+1} & \xrightarrow{\widehat{\delta^{n+1}}_{Q_{n+1}(s)}} & B_{n+1} \\ {\scriptstyle J_n}\uparrow & & \uparrow{\scriptstyle J_n} \\ B_n & \xrightarrow[\widehat{\delta^n}_{Q_n(s)}]{} & B_n \end{array}$$

commute because

$$\begin{aligned} \widehat{\delta^{n+1}}_{Q_{n+1}(s)} \circ J_n \circ j_A^{\delta^n} &= \widehat{\delta^{n+1}}_{Q_{n+1}(s)} \circ j_A^{\delta^{n+1}} \\ &= j_A^{\delta^{n+1}} \\ &= J_n \circ j_A^{\delta^n} \\ &= J_n \circ \widehat{\delta^n}_{Q_n(s)} \circ j_A^{\delta^n} \end{aligned}$$

and

$$\begin{aligned} \widehat{\delta^{n+1}}_{Q_{n+1}(s)} \circ J_n \circ j_{G_n} &= \widehat{\delta^{n+1}}_{Q_{n+1}(s)} \circ j_{G_{n+1}} \circ q_n^* \\ &= j_{G_{n+1}} \circ \mathrm{rt}_{Q_{n+1}(s)} \circ q_n^* \\ &= j_{G_{n+1}} \circ q_n^* \circ \mathrm{rt}_{q_n \circ Q_{n+1}(s)} \\ &= J_n \circ j_{G_n} \circ \mathrm{rt}_{Q_n(s)} \\ &= J_n \circ \widehat{\delta^n}_{Q_n(s)} \circ j_{G_n}. \end{aligned}$$

Thus, because the $\widehat{\delta^n}_{Q_n(s)}$ are automorphisms, by universality there is a unique automorphism $\alpha_s$ such that the diagrams

$$\begin{array}{ccc} B & \overset{\alpha_s}{\dashrightarrow} & B \\ {\scriptstyle K_n}\uparrow & & \uparrow{\scriptstyle K_n} \\ B_n & \xrightarrow[\widehat{\delta^n}_{Q_n(s)}]{} & B_n \end{array}$$

commute. It is easy to check that this gives a homomorphism $\alpha : G \to \operatorname{Aut} B$. We verify continuity: each function $s \mapsto \alpha_s(b)$ for $b \in B$ is a uniform limit of functions of the form $s \mapsto \alpha_s \circ K_n(b)$ for $b \in B_n$. But

$$\alpha_s \circ K_n(b) = K_n \circ \widehat{\delta^n}_{Q_n(s)}(b),$$

which is continuous since $K_n$, $Q_n$, and $t \mapsto \widehat{\delta^n}_t(b) : G_n \to B_n$ are.

We turn to the construction of the nondegenerate homomorphism $\mu$: first note that the increasing union $\bigcup_n Q_n^*(C(G_n))$ is dense in $C(G)$ by the Stone–Weierstrass theorem, and it follows that there is an isomorphism

$$C(G) \cong \varinjlim(C(G_n), q_n^*),$$

taking $Q_n^*$ to the canonical embedding. We have a compatible sequence of nondegenerate homomorphisms

$$\begin{array}{ccc} C(G_{n+1}) & \xrightarrow{j_{G_{n+1}}} & M(B_{n+1}) \\ {\scriptstyle q_n^*}\uparrow & & \uparrow{\scriptstyle J_n} \\ C(G_n) & \xrightarrow[j_{G_n}]{} & M(B_n), \end{array}$$

so by universality there is a unique homomorphism $\mu$ making the diagrams

$$\begin{array}{ccc} C(G) & \overset{\mu}{\dashrightarrow} & M(B) \\ {\scriptstyle Q_n^*}\uparrow & & \uparrow{\scriptstyle K_n} \\ C(G_n) & \xrightarrow[j_{G_n}]{} & M(B_n) \end{array}$$

commute. Moreover, $\mu$ is nondegenerate since $K_n$ and $j_{G_n}$ are.

We now have $\alpha$ and $\mu$, and the equivariance

$$\alpha_s \circ \mu = \mu \circ \mathrm{rt}_s$$

follows from

$$\begin{aligned} \alpha_s \circ \mu \circ Q_n^* &= \alpha_s \circ K_n \circ j_{G_n} \\ &= K_n \circ \widehat{\delta^n}_{Q_n(s)} \circ j_{G_n} \\ &= K_n \circ j_{G_n} \circ \mathrm{rt}_{Q_n(s)} \\ &= \mu \circ Q_n^* \circ \mathrm{rt}_{Q_n(s)} \\ &= \mu \circ \mathrm{rt}_s \circ Q_n^*. \end{aligned}$$

Thus we can apply [17] to obtain a coaction $(C, G, \delta)$ and an isomorphism

$$\theta : B \xrightarrow{\cong} C \times_\delta G,$$

such that

$$\theta \circ \mu = j_G \quad \text{and} \quad \theta(B^\alpha) = j_C(C).$$

We want to take $C = A$. Note that we have a compatible sequence of nondegenerate homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{j_A^{\delta^{n+1}}} & B_{n+1} \\ & \underset{j_A^{\delta^n}}{\searrow} & \uparrow J_n \\ & & B_n, \end{array}$$

so by universality there is a unique homomorphism $j$ making the diagrams

$$\begin{array}{ccc} A & \xrightarrow{j} & B \\ & \underset{j_A^{\delta^n}}{\searrow} & \uparrow K_n \\ & & B_n \end{array}$$

commute. Moreover, $j$ is injective and nondegenerate since $K_n$ and $j_A^{\delta^n}$ are. Because $j$, $j_C$, and $\theta$ are faithful, to show that we can take $C = A$ it suffices to show that

$$j(A) = B^\alpha.$$

Now

$$j(A) \subset B^\alpha,$$

because

$$\begin{aligned} \alpha_s \circ j &= \alpha_s \circ K_n \circ j_A^{\delta_n} \\ &= K_n \circ \widehat{\delta^n}_{Q_n(s)} \circ j_A^{\delta_n} \\ &= K_n \circ j_A^{\delta_n} \\ &= j. \end{aligned}$$

For the opposite containment, let $b \in B^\alpha$. There is a sequence $b_n \in B_n$ such that $K_n(b_n) \to b$. The functions $s \mapsto \alpha_s \circ K_n(b_n)$ converge uniformly to the function $s \mapsto \alpha_s(b)$, so

$$\int_G \alpha_s \circ K_n(b_n)\, ds \to \int_G \alpha_s(b)\, ds = b.$$

Also

$$\int_G \alpha_s \circ K_n(b_n)\, ds = \int_G K_n \circ \widehat{\delta^n}_{Q_n(s)}(b_n)\, ds = K_n\left(\int_G \widehat{\delta^n}_{Q_n(s)}(b_n)\, ds\right).$$

Since

$$\int_G \widehat{\delta^n}_{Q_n(s)}(b_n)\, ds \in B_n^{\widehat{\delta^n}} = j_A^{\delta^n}(A),$$

we conclude that

$$b \in K_n \circ j_A^{\delta^n}(A) = j(A).$$

Therefore we can take $C = A$, so that we have a coaction $(A, G, \delta)$ and an isomorphism

$$\theta : B \xrightarrow{\cong} A \times_\delta G,$$

such that

$$\theta \circ \mu = j_G.$$

We have proved (ii). For (i), we calculate that

$$\begin{aligned}
(j_A^\delta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes Q_n) \circ \delta &= (\mathrm{id} \otimes Q_n) \circ (j_A^\delta \otimes \mathrm{id}) \circ \delta \\
&= (\mathrm{id} \otimes Q_n) \circ \mathrm{Ad}(j_G \otimes \mathrm{id})\,(w_G) \circ (j_A^\delta \otimes 1) \\
&= \mathrm{Ad}(\mathrm{id} \otimes Q_n)((j_G \otimes \mathrm{id})\,(w_G)) \circ (\mathrm{id} \otimes Q_n) \circ (j_A^\delta \otimes 1) \\
&= \mathrm{Ad}(j_G \otimes \mathrm{id})((\mathrm{id} \otimes Q_n)\,(w_G)) \circ (j_A^\delta \otimes 1) \\
&= \mathrm{Ad}(j_G \otimes \mathrm{id})((Q_n^* \otimes \mathrm{id})\,(w_{G_n})) \circ (j_A^\delta \otimes 1) \\
&= \mathrm{Ad}(j_G \circ Q_n^* \otimes \mathrm{id})\,(w_{G_n}) \circ (j_A^\delta \otimes 1) \\
&= \mathrm{Ad}(\theta \circ K_n \circ j_{G_n} \otimes \mathrm{id})\,(w_{G_n}) \circ (\theta \circ K_n \circ j_A^{\delta^n} \otimes 1) \\
&= (\theta \circ K_n \otimes \mathrm{id}) \circ \mathrm{Ad}(j_{G_n} \otimes \mathrm{id})\,(w_{G_n}) \circ (j_A^{\delta^n} \otimes 1) \\
&= (\theta \circ K_n \otimes \mathrm{id}) \circ (j_A^{\delta^n} \otimes \mathrm{id}) \circ \delta^n \\
&= (\theta \circ K_n \circ j_A^{\delta^n} \otimes \mathrm{id}) \circ \delta^n \\
&= (j_A^\delta \otimes \mathrm{id}) \circ \delta^n.
\end{aligned}$$

Since $j_A^\delta$ is faithful, we therefore have $(\mathrm{id} \otimes Q_n) \circ \delta = \delta^n$. □

The following application of Theorem 3.1 motivates the work of the following sections.

EXAMPLE 3.3. Let $A = C(\mathbb{T}) = C^*(\mathbb{Z})$, and let $z$ denote the canonical generating unitary function $z \mapsto z$. For $n \in \mathbb{N}$, let $G_n := \mathbb{Z}/2^{n-1}\mathbb{Z}$ be the cyclic group of order $2^{n-1}$. We write 1 for the canonical generator of $G_n$ and 0 for the identity element. Let $g \mapsto u_n(g)$ denote the canonical embedding of $G_n$ into $C^*(G_n)$. Define $q_n : G_{n+1} \to G_n$ by $q_n(m) := m (\mathrm{mod}\ 2^{n-1})$, and write $q_n$ also for the homomorphism $q_n : C^*(G_{n+1}) \to C^*(G_n)$ satisfying $q_n(u_{n+1}(g)) = u_n(q_n(g))$. For each $n$, let $\delta^n$ be the coaction of $G_n$ on $A$ determined by $\delta^n(z) := z \otimes u_n(1)$.

Let $g \mapsto u(g)$ denote the canonical embedding of $\varprojlim G_n$ as unitaries in the multiplier algebra of $C^*(\varprojlim G_n)$. The coaction $\delta$ of $\varprojlim G_n$ on $A$ described in Theorem 3.1 is the one determined by $\delta(z) := z \otimes u(1, 1, \ldots)$; the corresponding coaction crossed product is known to be isomorphic to the Bunce–Deddens algebra of type $2^\infty$ (see, for example, [6, 8.4.4]).

## 4. Coverings of skew-products

In this section and the next, we adopt the following notation and assumptions.

NOTATION 4.1. Let $\Lambda$ be a connected row-finite $k$-graph with no sources. Fix a vertex $v \in \Lambda^0$, and denote by $\pi\Lambda$ the fundamental group $\pi_1(\Lambda, v)$ of $\Lambda$ with respect to $v$. Fix a cocycle $c : \Lambda \to \pi\Lambda$ such that the skew-product $\Lambda \times_c \pi\Lambda$ is isomorphic to the universal covering $\Omega_\Lambda$ of $\Lambda$ (such a cocycle exists by [15, Corollary 6.5]).

Fix a descending chain of finite-index normal subgroups

$$\ldots \lhd H_{n+1} \lhd H_n \lhd \ldots \lhd H_1 := \pi\Lambda. \tag{4.1}$$

For each $n$, let $G_n := \pi\Lambda/H_n$, and let $q_n : G_{n+1} \to G_n$ be the induced homomorphism

$$q_n(gH_{n+1}) := gH_n.$$

Then

$$\cdots \xrightarrow{q_{n+1}} G_{n+1} \xrightarrow{q_n} G_n \longrightarrow \cdots \xrightarrow{q_1} G_1 := \{e\}$$

is a chain of surjective homomorphisms of finite groups. Let $G$ denote the projective limit group $\varprojlim(G_n, q_n)$.

For each $n$, let $c_n : \Lambda \to G_n$ be the induced cocycle $c_n(\lambda) = c(\lambda)H_n$, and let

$$\Lambda_n := \Lambda \times_{c_n} G_n$$

be the skew-product $k$-graph. Define covering maps $p_n : \Lambda_{n+1} \to \Lambda_n$ by $p_n(\lambda, g) := (\lambda, q_n(g))$.

As in [15, Theorem 7.1(1)], for each $n$ there is a coaction $\delta^n : C^*(\Lambda) \to C^*(\Lambda) \otimes C^*(G_n)$ determined by $\delta^n(s_\lambda) := s_\lambda \otimes c_n(\lambda)$. Denote by $J_n$ the inclusion

$$J_n := j_A^{\delta^{n+1}} \times (j_{G_{n+1}} \circ q_n^*) : C^*(\Lambda) \times_{\delta^n} G_n \to C^*(\Lambda) \times_{\delta^{n+1}} G_{n+1},$$

described in Theorem 3.1(ii).

As in [15, Theorem 7.1(ii)], for each $n$ there is an isomorphism $\phi_n$ of $C^*(\Lambda_n) = C^*(\Lambda \times_{c_n} G_n)$ onto $C^*(\Lambda) \times_{\delta^n} (G_n)$ which satisfies $\phi_n(s_{(\lambda,g)}) := (s_\lambda, g)$.

EXAMPLE 4.2 (Example 3.3 continued). Let $\Lambda$ be the path category of the directed graph $B_1$ consisting of a single vertex $v$ and a single edge $f$ with $r(f) = s(f) = v$. Note that as a category, $\Lambda$ is isomorphic to $\mathbb{N}$, and the degree functor is then the identity function from $\mathbb{N}$ to itself.

Then $\pi\Lambda$ is the free abelian group generated by the homotopy class of $f$, and so is isomorphic to $\mathbb{Z}$. We define a functor $c : \Lambda \to \mathbb{Z}$ by $c(f) = 1$.

For each $n$, let $H_n := 2^{n-1}\mathbb{Z} \subset \mathbb{Z}$, so that $\cdots \lhd H_{n+1} \lhd H_n \lhd \cdots \lhd H_1 := \pi\Lambda$ is a descending chain of finite-index normal subgroups. For each $n$, $G_n := \mathbb{Z}/H_n$ is the cyclic group of order $2^{n-1}$, and $q_n : G_{n+1} \to G_n$ is the quotient map described in

Example 3.3. The induced cocycle $c_n : \Lambda \to G_n$ obtained from $c$ is determined by $c_n(f) = 1 \in \mathbb{Z}/2^{n-1}\mathbb{Z}$.

For $p \in \mathbb{N}$, let $C_p$ denote the simple cycle graph with $p$ vertices: $C_p^0 := \{v_j^p : j \in \mathbb{Z}/p\mathbb{Z}\}$ and $C_p^1 := \{e_j^p : j \in \mathbb{Z}/p\mathbb{Z}\}$, where $r(e_i^p) = v_i^p$ and $s(e_i^p) = v_{i+1 \bmod p}^p$. For each $n$, the skew-product graph $\Lambda_n := \Lambda \times_{c_n} G_n$ is isomorphic to the path-category of $C_{2^{n-1}}$. The associated covering map $p_n : \Lambda_{n+1} \to \Lambda_n$ corresponds to the double-covering of $C_{2^{n-1}}$ by $C_{2^n}$ satisfying $v_i^{2^n} \mapsto v_{i \bmod 2^{n-1}}^{2^{n-1}}$ and $e_i^{2^n} \mapsto e_{i \bmod 2^{n-1}}^{2^{n-1}}$.

Modulo a relabelling of the generators of $\mathbb{N}^2$, the 2-graph $\varprojlim(\Lambda_n, p_n)$ obtained from this data as in [12] (see Section 2.3) is isomorphic to the 2-graph of [16, Example 6.7]. Combining this with the final observation of Example 3.3, we obtain a new proof that the $C^*$-algebra of this 2-graph is Morita equivalent to the Bunce–Deddens algebra of type $2^\infty$ (see [16, Example 6.7] for an alternative proof).

THEOREM 4.3. *Adopt Notation 4.1. Taking $A := C^*(\Lambda)$, the coactions $\delta^n$ and the quotient maps $q_n$ make the diagrams (3.1) commute. Let $\delta$ denote the coaction of $G := \varprojlim(G_n, q_n)$ on $C^*(\Lambda)$ obtained from Theorem 3.1. Let $P_0$ denote the projection $\sum_{v\in\Lambda^0} s_v$ in the multiplier algebra of $C^*(\varprojlim(\Lambda_n, p_n))$. Then $P_0$ is full and*

$$P_0 C^*(\varprojlim(\Lambda_n, p_n)) P_0 \cong C^*(\Lambda) \times_\delta G.$$

To prove this theorem, we first show that, in the setting described above, the inclusions of $k$-graph algebras induced from the coverings $p_n : \Lambda_{n+1} \to \Lambda_n$ as in [12] are compatible with the inclusions of coaction crossed products induced from the quotient maps $q_n : G_{n+1} \to G_n$.

LEMMA 4.4. *With Notation 4.1, fix $n \in \mathbb{N}$, and let $\iota_{p_n}$ be the inclusion of $C^*(\Lambda_n)$ into $C^*(\Lambda_{n+1})$ obtained from* [12, Proposition 3.3(iv)]. *Then the inclusion $\iota_n$ and the isomorphisms $\phi_n, \phi_{n+1}$ of Notation 4.1 make the following diagram commute:*

$$\begin{array}{ccc} C^*(\Lambda_n) & \xrightarrow{\iota_{p_n}} & C^*(\Lambda_{n+1}) \\ \downarrow \phi_n & & \downarrow \phi_{n+1} \\ C^*(\Lambda) \times_{\delta^n} G_n & \xrightarrow{\iota_n} & C^*(\Lambda) \times_{\delta^{n+1}} G_{n+1} \end{array}$$

PROOF. By definition,

$$\iota_{p_n}(s_{(\lambda, gH_n)}) = \sum_{p(\lambda', g'H_{n+1}) = (\lambda, gH_n)} s_{(\lambda', g'H_{n+1})}.$$

By definition of $p_n$, this becomes

$$\iota_{p_n}(s_{(\lambda, gH_n)}) = \sum_{\{g'H_{n+1} \in G_{n+1} : g'H_n = gH_n\}} s_{(\lambda, g'H_{n+1})}.$$

Hence

$$\phi_{n+1} \circ \iota_{p_n}(s_{(\lambda, gH_n)}) = \sum_{\{g'H_{n+1} \in G_{n+1} : g'H_n = gH_n\}} (s_\lambda, g'H_{n+1}).$$

But this is precisely $\iota(\phi_n(s_{(\lambda, gH_n)}))$ by definition of $\iota$ and $\phi_n$. □

COROLLARY 4.5. *With Notation 4.1, let $P_0$ denote the projection $\sum_{v \in \Lambda^0} s_v$ in the multiplier algebra of $C^*(\varprojlim(\Lambda_n, p_n))$. Then $P_0$ is full and*

$$P_0 C^*(\varprojlim(\Lambda_n, p_n)) P_0 \cong \varinjlim(C^*(\Lambda) \times_{\delta^n} G_n, \iota_n).$$

PROOF. By [12, Equation (3.2)], $P_0 C^*(\varprojlim(\Lambda_n, p_n)) P_0$ is isomorphic to $\varinjlim(C^*(\Lambda_n), \iota_{p_n})$. The latter is isomorphic to $\varinjlim(C^*(\Lambda) \times_{\delta^n} G_n, \iota_n)$ by Lemma 4.4 and the universal property of the direct limit. □

PROOF OF THEOREM 4.3. It is immediate from the definitions of the maps involved that the maps $\delta^n$ and $q_n$ make the diagram (3.1) commute. The rest of the statement then follows from Corollary 4.5 and Theorem 3.1(ii). □

## 5. Simplicity

In this section we frequently embed $\mathbb{N}^k$ into $\mathbb{N}^{k+1}$ as the subset consisting of elements whose $(k+1)$th coordinate is equal to zero. For $n \in \mathbb{N}^k$, we write $(n, 0)$ for the corresponding element of $\mathbb{N}^{k+1}$.

THEOREM 5.1. *Adopt Notation 4.1. The $(k+1)$-graph $C^*$-algebra $C^*(\varprojlim(\Lambda_n, p_n))$ is simple if and only if the following two conditions are satisfied:*

(i) *each $\Lambda_n$ is cofinal;*
(ii) *whenever $v \in \Lambda^0$, $p \neq q \in \mathbb{N}^k$ satisfy $\sigma^p(x) = \sigma^q(x)$ for all $x \in v\Lambda^0$, there exist $x \in v\Lambda^\infty$, $l \in \mathbb{N}^k$ and $N \in \mathbb{N}$ such that $c_N(x(p, p+l)) \neq c_N(x(q, q+l))$.*

The idea is to prove the theorem by appealing to [18, Theorem 3.1]. To do this, we will first describe the infinite paths in $\varprojlim(\Lambda_n, p_n)$. We identify $\varprojlim(G_n, q_n)$ with the set of sequences $g = (g_n)_{n=1}^\infty$ such that $q_n(g_{n+1}) = g_n$ for all $n$.

LEMMA 5.2. *Adopt Notation 4.1. Fix $x \in \Lambda^\infty$ and $g = (g_n)_{n=1}^\infty \in \varprojlim(G_n, q_n)$. For each $n \in \mathbb{N}$ there is a unique infinite path $(x, g_n) \in \Lambda_n^\infty$ determined by $(x, g_n)\,(0, m) = (x(0, m), c_n(x(0, m))^{-1} g_n)$ for all $m \in \mathbb{N}^k$. There is a unique infinite path $x^g \in (\varprojlim(\Lambda_n, p_n))^\infty$ such that $x^g(0, (m, 0)) = x(0, m)$ for all $m \in \mathbb{N}^k$ and $x^g(ne_{k+1}) = (x(0), g_n)$ for all $n \in \mathbb{N}$; moreover, $\sigma^{ne_{k+1}}(x^g)\,(0, (m, 0)) = (x, g_n)\,(0, m)$ for all $m \in \mathbb{N}^k$. Finally, every infinite path $y \in (\varprojlim(\Lambda_n, p_n))^\infty$ is of the form $\sigma^{ne_{k+1}}(x^g)$ for some $n \in \mathbb{N}$, $x \in \Lambda^\infty$ and $g \in \varprojlim(G_n, q_n)$.*

PROOF. That the formula given determines unique infinite paths $(x, g_n)$, $n \in \mathbb{N}$, follows from [9, Remarks 2.2]. That there is a unique infinite path $x^g$ such that $x^g(0, (m, 0)) = x(0, m)$ for all $m \in \mathbb{N}^k$ and $x^g(ne_{k+1}) = (x(0), g_n)$ for all $n \in \mathbb{N}$ follows from the observation that for each $n \in \mathbb{N}$ there is a unique path

$$\alpha = \alpha_{g,n} := e(x(0), g_1)e(x(0), g_2) \cdots e(x(0), g_n),$$

with $d(\alpha_{g,n}) = ne_{k+1}$, $r(\alpha) = x(0) \in \Lambda^0$ and $s(\alpha) = (x(0), g_n) \in \Lambda_n^0$, and that for each $m \in \mathbb{N}^k$,

$$\begin{aligned}\alpha(x, g_n)\,(0, m) = x(0, m)e(x(m), c_1(x(0, m))^{-1}g_1)\\ \cdots e(x(m), c_n(x(0, m))^{-1}g_n)\end{aligned}$$

is the unique minimal common extension of $x(0, m)$ and $\alpha$. This also establishes the assertion that $\sigma^{ne_{k+1}}(x^g)\,(0, (m, 0)) = (x, g_n)\,(0, m)$ for all $m \in \mathbb{N}^k$.

For the final assertion, fix $y \in (\varprojlim(\Lambda_n, p_n))^\infty$. We must have $y(0) = (v, g_n)$ for some $v \in \Lambda^0$, $g_n \in G_n = \pi\Lambda/H_n$ and $n \in \mathbb{N}$. Let $x \in \Lambda_n^\infty$ be the infinite path determined by $x(0, m) := y(0, (m, 0))$ for all $m \in \mathbb{N}^k$. By definition of $\Lambda_n = \Lambda \times_{c_n} G_n$, we have $x(0, m) := (\alpha_m, c_n(\alpha_m)^{-1}g_n)$ where each $\alpha_m \in v\Lambda^m$ and $g$ is the element of $\pi\Lambda$ such that $y(0) = v(g_n)$ as above. There is then an infinite path in $x' \in \Lambda^\infty$ determined by $x'(0, m) = \alpha_m$ for all $m \in \mathbb{N}^k$. For $n > i \geq 1$, inductively define $g_i := q_i(g_{i+1})$, and for $n < i$ let $g_i$ be the unique element of $G_i$ such that $y((i-n)e_{k+1}) = (v, g_i)$; that such $g_i$ exist follows from the definition of $\varprojlim(\Lambda_n, p_n)$. Then $g := (g_i)_{i=1}^\infty$ is an element of $\varprojlim(G_n, q_n)$ by definition, and routine calculations using the definitions of the $\Lambda_n$ show that $x = \sigma^{ne_{k+1}}((x')^g)$. □

LEMMA 5.3. *Adopt Notation 4.1. Then the* $(k+1)$*-graph* $\varprojlim(\Lambda_n, p_n)$ *is cofinal if and only if each* $\Lambda_n$ *is cofinal.*

PROOF. Suppose that each $\Lambda_n$ is cofinal. Fix $y \in \varprojlim(\Lambda_n, p_n)$ and $w \in \varprojlim(\Lambda^0)$. By Lemma 5.2, we have $y = \sigma^{i_0 e_{k+1}}(x^g)$ for some $g = (g_n)_{n=1}^\infty \in \varprojlim(G_n, q_n)$, some $i_0 \in \mathbb{N}$ and some $x \in \Lambda^\infty$. We must show that $w(\varprojlim(\Lambda_n, p_n))y(q) \neq \emptyset$ for some $q$. We have $w \in \Lambda_m^0$ for some $m \in \mathbb{N}$, so $w = (w', h)$ for some $h \in G_m$. If $m < i_0$, fix any $h' \in \pi\Lambda$ such that $h'H_{i_0} = h$, and note that $w(\varprojlim(\Lambda_n, p_n))(w', hH_{i_0})$ is nonempty, so that it suffices to show that $(w', h'H_{i_0})(\varprojlim(\Lambda_n, p_n))y(q) \neq \emptyset$ for some $q$. That is to say, we may assume without loss of generality that $m \geq i_0$. But now $w \in \Lambda_m^0$ and $\sigma^{(0,\dots,0,m-i_0)}(y) \in (\varprojlim(\Lambda_n, p_n))^\infty$ with $r(y) \in \Lambda_{i_0}^0$. Since $\Lambda_n$ is cofinal, we have $w\Lambda_{i_0}(x, g_m)\,(q) \neq \emptyset$ for some $q \in \mathbb{N}^k$ (recall that $x$, $(g_i)_{i=1}^\infty$ are such that $y = \sigma^{i_0 e_{k+1}}(x^g)$). By definition, $(x, g_m)\,(q) = y(q_1, \dots, q_k, m - i_0)$ and this shows that $w(\varprojlim(\Lambda_n, p_n))y(q) \neq \emptyset$ for $q = (q_1, \dots, q_k, m - n)$.

Now suppose that $\varprojlim(\Lambda_n, p_n)$ is cofinal. Fix $n \in \mathbb{N}$ and a vertex $w$ and an infinite path $x$ in $\Lambda_n$. Then $x(0) = (v, gH_n)$ for some $v \in \Lambda^0$, $g \in \pi\Lambda$. There are paths $\alpha_m \in \Lambda_n^m$, $m \in \mathbb{N}^k$, determined by $x(0, m) = (\alpha_m, c_n(\alpha_m)^{-1} g H_n)$; there is then an infinite path $x' \in \Lambda^\infty$ such that $x'(0, m) = \alpha_m$ for all $m$. Let $g_i := gH_i$ for all $i \in \mathbb{N}$. In an abuse of notation we denote by $g$ the element $(gH_i)_{i=1}^\infty$ of $\varprojlim(G_n, q_n)$. Let $y = \sigma^n((x')^g)$ be the infinite path of $\varprojlim(\Lambda_n, p_n)$ provided by Lemma 5.2. As $\varprojlim(\Lambda_n, p_n)$ is cofinal, we may fix a path $\lambda \in \varprojlim(\Lambda_n, p_n)$ such that $r(\lambda) = w$ and $s(\lambda)$ lies on $y$. By definition of $y$, there exist $n' \geq n$ and $m \in \mathbb{N}^k$ such that $s(\lambda) = (x'(m), c_{n'}(\alpha_m)^{-1} g_{n'})$. We then have $d(\lambda)_{k+1} = n' - n$, and we may factorize $\lambda = \lambda'\lambda''$ where $d(\lambda') = d(\lambda) - (n' - n)e_{k+1}$ and $d(\lambda'') = (n' - n)e_{k+1}$. By construction of $\varprojlim(\Lambda_n, p_n)$, if $d(\mu) = je_{k+1}$ and $s(\mu) = (v, gH_n) \in \Lambda_n^0$ then $n \geq j$ and $r(\mu) = (v, gH_{n-j}) \in \Lambda_{n-j}^0$. In particular,

$$s(\lambda') = r(\lambda'') = (x'(m), c_n(\alpha_m)^{-1} g_n) = x(m),$$

so $w\Lambda_n x(m) \neq \emptyset$. □

LEMMA 5.4. *Adopt Notation 4.1. Then the* $(k + 1)$*-graph* $\varprojlim(\Lambda_n, p_n)$ *has no local periodicity if and only if it satisfies condition (ii) of Theorem 5.1.*

PROOF. First suppose that condition (ii) of Theorem 5.1 holds. Fix a vertex $v \in (\varprojlim(\Lambda_n, p_n))^0$ and $p \neq q \in \mathbb{N}^{k+1}$. So $v \in \Lambda_n^0$ for some $n$, and $v$ therefore has the form $v = (w, gH_n)$ for some $w \in \Lambda^0$ and $g \in \pi\Lambda$. We must show that there exists $x \in v(\varprojlim(\Lambda_n, p_n))^\infty$ such that $\sigma^p(x) \neq \sigma^q(x)$.

We first consider the case where $p_{k+1} \neq q_{k+1}$. By construction of the tower graph $\varprojlim(\Lambda_n, p_n)$, this forces the vertices $x(p)$ and $x(q)$ to lie in distinct $\Lambda_n$ for any $x \in v(\varprojlim(\Lambda_n, p_n))^\infty$; in particular, they cannot be equal.

Now suppose that $p_{k+1} = q_{k+1}$. If every $x \in v(\varprojlim(\Lambda_n, p_n))^\infty$ satisfies $\sigma^p(x) = \sigma^q(x)$, then for any $\alpha \in v(\varprojlim(\Lambda_n, p_n))^{p_{k+1}e_{k+1}}$ and any $y \in s(\alpha)\,(\varprojlim(\Lambda_n, p_n))^\infty$, we have $\sigma^p(\alpha y) = \sigma^q(\alpha y)$; that is,

$$\sigma^{p - p_{k+1}e_{k+1}}(y) = \sigma^{q - q_{k+1}e_{k+1}}(y) \quad \text{for all } y \in s(\alpha)\,(\varprojlim(\Lambda_n, p_n))^\infty.$$

So we may assume without loss of generality that $p_{k+1} = q_{k+1} = 0$. Write $p'$ and $q'$ for the elements of $\mathbb{N}^k$ whose entries are the first $k$ entries of $p$ and $q$.

We have $v \in \Lambda_n$ for some $n$, so there exist $w \in \Lambda^0$ and $g \in \pi\Lambda$ such that $v = (w, gH_n)$. Suppose first that there exists $x \in w\Lambda^\infty$ such that $\sigma^{p'}(x) \neq \sigma^{q'}(x)$. Then the infinite path $(x, gH_n) \in v\Lambda_n^\infty$ such that

$$(x, gH_n)\,(0, m) := (x(0, m), c_n(x(0, m))^{-1} gH_n) \quad \text{for all } m \in \mathbb{N}^k,$$

also satisfies $\sigma^{p'}((x, gH_n)) \neq \sigma^{q'}((x, gH_n))$. By Lemma 5.2 we may choose an infinite path $y$ such that $y|_{\mathbb{N}^k \times \{0\}} = (x, gH_n)$, and then $y \in v(\varprojlim(\Lambda_n, p_n))^\infty$ satisfies $\sigma^p(y) \neq \sigma^q(y)$.

Now suppose that every path $x \in w\Lambda^\infty$ satisfies $\sigma^{p'}(x) = \sigma^{q'}(x)$. Then by condition (ii) of Theorem 5.1, we may fix $x \in w\Lambda^\infty$ and $N \in \mathbb{N}$ such that $c_N(x(0, p')) \neq c_N(x(0, q'))$. It then follows from the definition of the $c_j$ that $c_j(x(0, p')) \neq c_j(x(0, q'))$ whenever $j \geq N$. So with $j := \max\{N, n\}$,

$$\begin{aligned}(x, gH_j)\,(p') &= (x(p'), c_j(x(0, p'))^{-1} gH_j) \neq (x(q'), c_j(x(0, q'))^{-1} gH_j)\\ &= (x, gH_j)\,(q').\end{aligned}$$

There is an element $g = (g_i)_{i=1}^\infty$ of $\varprojlim(G_n, q_n)$ determined by $g_i := gH_i$ for all $i$. Let $x^g$ be the element of $(\varprojlim(\Lambda_n, p_n))^\infty$ determined by $x$ and $g$ as in Lemma 5.2. Then $(x, gH_n)\,((j-n)e_{k+1} + p) \neq (x, gH_n)\,((j-n)e_{k+1} + q)$, and therefore $x^g$ satisfies $\sigma^p(x^g) \neq \sigma^q(x^g)$ as required. Hence condition (ii) of Theorem 5.1 implies that $\varprojlim(\Lambda_n, p_n)$ has no local periodicity.

To show that if $\varprojlim(\Lambda_n, p_n)$ has no local periodicity then condition (ii) of Theorem 5.1 holds, we prove the contrapositive statement. Suppose that condition (ii) of Theorem 5.1 does not hold. Fix $v \in \Lambda^0$ and $p, q \in \mathbb{N}^k$ such that $\sigma^p(x) = \sigma^q(x)$ for all $x \in v\Lambda^\infty$ and $c_n(x(p, p+l)) = c_n(x(q, q+l))$ for all $n \in \mathbb{N}$, $l \in \mathbb{N}^k$. Then for each $x \in v\Lambda^\infty$ and each $g = (g_n)_{n=1}^\infty \in \varprojlim(G_n, p_n)$, we have $\sigma^p(x, g_n)\,(0, l) = \sigma^q(x, g_n)\,(0, l)$ for all $n \in \mathbb{N}$ and $l \in \mathbb{N}^k$. Hence Lemma 5.2 implies that every $y \in v(\varprojlim(\Lambda_n, p_n))^\infty$ satisfies $\sigma^{(p,0)}(y) = \sigma^{(q,0)}(y)$. □

PROOF OF THEOREM 5.1. From [18, Theorem 3.1] we see that $C^*(\varprojlim(\Lambda_n, p_n))$ is simple if and only if $\varprojlim(\Lambda_n, p_n)$ is cofinal and has no local periodicity. The result then follows directly from Lemmas 5.3 and 5.4. □

## 6. Projective limit *k*-graphs

Let $(\Lambda_n, \Lambda_{n+1}, p_n)_{n=1}^\infty$ be a sequence of row-finite coverings of $k$-graphs with no sources as in Section 2.3. We aim to show that the sets $(\varprojlim \Lambda_i)^m := \varprojlim(\Lambda_i^m, p_i)$ under the projective limit topology with the natural (coordinate-wise) range and source maps specify a topological $k$-graph (in the sense of Yeend). Moreover, we show that the associated topological $k$-graph $C^*$-algebra is isomorphic to the full corner $P_0 C^*(\varprojlim(\Lambda_n; p_n)) P_0$ determined by $P_0 := \sum_{v \in \Lambda_1^0} s_v$. In particular, when the $\Lambda_n$ and $p_n$ are as in Notation 4.1, the $C^*$-algebra of the projective limit topological $k$-graph is isomorphic to the crossed product of $C^*(\Lambda)$ by the coaction of the projective limit of the groups $G_i$ obtained from Theorem 3.1.

Let $(\Lambda_n, \Lambda_{n+1}, p_n)_{n=1}^\infty$ be a sequence of row-finite coverings of $k$-graphs with no sources. Let $\varprojlim(\Lambda_i, p_i)$ be the projective limit category, equipped with the projective limit topology. That is, $\varprojlim(\Lambda_i, p_i)$ consists of all sequences $(\lambda_i)_{i=1}^\infty$ such that each $\lambda_i \in \Lambda_i$ and $p_i(\lambda_{i+1}) = \lambda_i$; the structure maps $\tilde{r}$, $\tilde{s}$, $\tilde{\circ}$ and $\widetilde{\mathrm{id}}$ on $\varprojlim(\Lambda_i, p_i)$ are obtained by pointwise application of the corresponding structure maps for $\Lambda$. The cylinder sets $Z(\lambda_1, \ldots, \lambda_j) := \{(\mu_i)_{i=1}^\infty \in \varprojlim(\Lambda_i, p_i) : \mu_i = \lambda_i \text{ for } 1 \leq i \leq j\}$ form a basis of compact open sets for a locally compact Hausdorff topology.

Define $\tilde{d} : \varprojlim(\Lambda_i, p_i) \to \mathbb{N}^k$ by $\tilde{d}((\lambda_i)_{i=1}^\infty) := d(\lambda_1)$. Since the $p_i$ are degree-preserving,

$$\tilde{d}((\lambda_i)_{i=1}^\infty) = d(\lambda_i) \quad \text{for all } i \geq 1.$$

For fixed $\lambda = (\lambda_i)_{i=1}^\infty \in \varprojlim(\Lambda_i, p_i)^{m+n}$, the unique factorization property for each $\lambda_i$ produces unique elements $\lambda(0, m) := (\lambda_i(0, m))_{i=1}^\infty \in \varprojlim(\Lambda_i, p_i)^m$ and $\lambda(m, n) := (\lambda_i(m, n))_{i=1}^\infty \in \varprojlim(\Lambda_i, p_i)^n$ such that $\lambda = \lambda(0, m)\lambda(m, n)$; that is, $(\varprojlim(\Lambda_i, p_i), \tilde{d})$ is a second-countable small category with a degree functor satisfying the factorization property.

The identity $\tilde{d}((\lambda_i)_{i=1}^\infty) = d(\lambda_i)$ for all $i \geq 1$ implies that $Z(\lambda_1, \ldots, \lambda_j)$ is empty unless $d(\lambda_1) = \cdots = d(\lambda_j)$, and it follows that $\tilde{d}$ is continuous.

We claim that $\tilde{r}$ and $\tilde{s}$ are local homeomorphisms. To see this, fix a cylinder set $Z(v_1, \ldots, v_j) \subset \varprojlim(\Lambda_i, p_i)^0$ and, for $\lambda \in v_1\Lambda_1$ and $2 \leq l \leq j$, let $v_l p_{1,l}^{-1}(\lambda)$ be the unique element of $v_l\Lambda_l$ such that $p_1 \circ p_2 \circ \cdots \circ p_{l-1}(v_l p_{1,l}^{-1}(\lambda)) = \lambda$. Then

$$\tilde{r}^{-1}(Z(v_1, \ldots, v_j)) \cap \varprojlim(\Lambda_i, p_i)^n := \bigsqcup_{\lambda \in v_1\Lambda_1^n} Z(\lambda, v_2 p_{1,2}^{-1}(\lambda), \ldots, v_j p_{1,j}^{-1}(\lambda)),$$

which is clearly open, showing that $\tilde{r}$ is continuous. Moreover, this same formula shows that for $\lambda = (\lambda_i)_{i=1}^\infty \in \varprojlim(\Lambda_i, p_i)$, the restriction of $\tilde{r}$ to $Z(\lambda_1)$ is a homeomorphism, and $\tilde{r}$ is a local homeomorphism as claimed. A similar argument shows that $\tilde{s}$ is also a local homeomorphism.

It is easy to see that the inverse image under composition of the cylinder set $Z(\lambda_1, \ldots, \lambda_j) \in \varprojlim(\Lambda_i, p_i)^n$ is equal to the disjoint union

$$\bigsqcup_{p+q=n} Z(\lambda_1(0, p), \ldots, \lambda_j(0, p)) \times Z(\lambda_1(p, q), \ldots, \lambda_j(p, q)),$$

of cartesian products of cylinder sets and hence is open, so that composition is continuous, and it follows that $(\varprojlim(\Lambda_i, p_i), \tilde{d})$ is a topological $k$-graph in the sense of Yeend [20, 21].

Let $\varprojlim(\Lambda_n;\, p_n)$ be as described in Section 2.3, and let $P_0$ denote the full projection $\sum_{v\in\Lambda_1^0} s_v \in M(C^*(\varprojlim(\Lambda_n;\, p_n)))$. For the following proposition, we need to describe $P_0 C^*(\varprojlim(\Lambda_n;\, p_n))P_0$ in detail. For $n \geq m \geq 1$, we write $p_{m,n} : \Lambda_n \to \Lambda_m$ for the covering map $p_{m,n} := p_m \circ \cdots \circ p_{n-1}$, with the convention that $p_{n,n}$ is the identity map on $\Lambda_n$. For $v \in \Lambda_m^0$, and $l \leq m$, we denote by $\alpha_{l,m}(v)$ the unique path in $\varprojlim(\Lambda_n;\, p_n)^{(m-l)e_{k+1}}$ whose source is $v$ (and whose range is $p_{l,m}(v)$). In particular, $\alpha_{1,m}(v)$ the unique path in $\varprojlim(\Lambda_n;\, p_n)^{(m-1)e_{k+1}}$ whose source is $v$ with range in $\Lambda_1$. For $\lambda \in \Lambda_m$,

$$\begin{aligned} s_{\alpha_{1,m}(r(\lambda))} s^*_{\alpha_{1,m}(r(\lambda))} s_{p_{1,m}(\lambda)} &= s_{\alpha_{1,m}(r(\lambda))} s_\lambda s^*_{\alpha_{1,m}(s(\lambda))} \\ &= s_{p_{1,m}(\lambda)} s_{\alpha_{1,m}(s(\lambda))} s^*_{\alpha_{1,m}(s(\lambda))}. \end{aligned}$$

Furthermore, $P_0 C^*(\varprojlim(\Lambda_n, p_n))P_0$ is equal to the closed span

$$P_0 C^*(\varprojlim(\Lambda_n, p_n))P_0 = \overline{\operatorname{span}}\{s_{\alpha_{1,m}(r(\lambda))} s_\lambda s^*_{\alpha_{1,m}(s(\lambda))} : m \geq 1, \lambda \in \Lambda_m\}.$$

PROPOSITION 6.1. *Let $(\Lambda_n, \Lambda_{n+1}, p_n)_{n=1}^\infty$ be a sequence of row-finite coverings of $k$-graphs with no sources, and let $\varprojlim(\Lambda_n; p_n)$ be the associated $(k+1)$-graph as in* [12]. *Let $P_0 := \sum_{v \in \Lambda_1^0} s_v \in MC^*(\varprojlim(\Lambda_n; p_n))$. Let $(\varprojlim(\Lambda_i, p_i), \tilde{d})$ be the topological $k$-graph defined above. Then there is a unique isomorphism*

$$\pi : P_0 C^*(\varprojlim(\Lambda_n, p_n))P_0 \to C^*(\varprojlim(\Lambda_i, p_i)),$$

*such that for $\lambda \in \Lambda_m$,*

$$\pi(s_{\alpha_{1,m}(r(\lambda))} s_\lambda s^*_{\alpha_{1,m}(s(\lambda))}) = \chi_{Z(p_{1,m}(\lambda), p_{2,m}(\lambda), \ldots, p_{m-1,m}(\lambda), \lambda)}. \tag{6.1}$$

*In particular, with Notation 4.1, there is an isomorphism of the $C^*$-algebra $C^*(\varprojlim(\Lambda_i, p_i))$ of the topological $k$-graph $\varprojlim(\Lambda_i, p_i)$ with the coaction crossed product $C^*(\Lambda) \times_\delta G$.*

PROOF. The final statement will follow from Theorem 4.3 once we establish the first statement.

To prove the first statement we will use Allen's gauge-invariant uniqueness theorem for corners in $k$-graph algebras [1]. We adopt Allen's notation: for $\mu, \nu \in \Lambda_1^0 \varprojlim(\Lambda_n; p_n)$, we let $t_{\mu,\nu} := s_\mu s_\nu^* \in P_0 C^*(\varprojlim(\Lambda_n; p_n))P_0$. The factorization property guarantees that for $\mu, \nu \in \Lambda_1^0 \varprojlim(\Lambda_n; p_n)$, we can rewrite $\mu = \alpha_{1,m}(r(\mu'))\mu'$ and $\nu = \alpha_{1,m}(r(\nu'))\nu'$ for some $m \geq 1$ and $\mu', \nu' \in \Lambda_m$ with $s(\mu') = s(\nu')$. By [1, Corollary 3.7], there is an isomorphism $\theta$ of $P_0 C^*(\varprojlim(\Lambda_n; p_n))P_0$ onto Allen's universal algebra $C^*(\varprojlim(\Lambda_n; p_n), \Lambda_1^0)$ (see [1, Definition 3.1 and the following paragraphs]) which satisfies $\theta(t_{\mu,\nu}) = T_{\mu,\nu}$ for all $\mu, \nu$. It therefore suffices to show that there is an isomorphism $\psi : C^*(\varprojlim(\Lambda_n; p_n), \Lambda_1^0) \to C^*(\varprojlim(\Lambda_i, p_i))$ such that $\psi(T_{\alpha_{1,m}(r(\mu))\mu, \alpha_{1,m}(r(\nu))\nu}) = \chi_{Z(p_{1,m}(\mu), \ldots, \mu) *_s Z(p_{1,m}(\nu), \ldots, \nu)}$ for all $m \geq 1$ and $\mu, \nu \in \Lambda_m$ with $s(\mu) = s(\nu)$; the composition $\pi := \psi \circ \theta$ clearly satisfies (6.1), and it is uniquely specified by (6.1) because the elements $\{t_{\alpha_{1,m}(r(\lambda))\lambda, \alpha_{1,m}(s(\lambda))} : m \geq 1, \lambda \in \Lambda_m\}$ generate $P_0 C^*(\varprojlim(\Lambda_n; p_n))P_0$ as a $C^*$-algebra.

Let $\Gamma$ denote the topological $k$-graph $\varprojlim(\Lambda_i, p_i)$. Since $\Gamma$ is row-finite and has no sources, $\partial\Gamma = \Gamma^\infty$. As in [21], for open subsets $U, V \subset \Gamma$, let $Z_{\mathcal{G}_\Gamma}(U *_s V, m)$ denote the set $\{(\mu x, m, \nu x) : \mu \in U, \nu \in V, x \in \Gamma^\infty, s(\mu) = s(\nu) = r(x)\}$. Then $\mathcal{G}_\Gamma$ is the locally compact Hausdorff topological groupoid

$$\mathcal{G}_\Gamma = \{(x, m - n, y) : x, y \in \Gamma^\infty, m, n \in \mathbb{N}^k, \sigma^m(x) = \sigma^n(y)\},$$

where the $Z_{\mathcal{G}_\Gamma}(U *_s V, m)$ form a basis of compact open sets for the topology.

For $m \geq 1$ and $\lambda \in \Lambda_m$, let $U_{m,\lambda} := Z(p_{1,m}(\lambda), \ldots, \lambda) \subset \Gamma$. So the $U_{m,\lambda}$ are a basis for the topology on $\Gamma = \varprojlim(\Lambda_i, p_i)$. Now for $m \geq 1$ and $\mu, \nu \in \Lambda_m$ with $s(\mu) = s(\nu)$, let

$$u_{\alpha_{1,m}(r(\mu))\mu, \alpha_{1,m}(r(\nu))\nu} := \chi_{Z(U_{m,\mu} *_s U_{m,\nu}, d(\mu)-d(\nu))} \in C_c(\mathcal{G}_\Gamma).$$

Tedious but routine calculations using the definition of the convolution product and involution on $C_c(\mathcal{G}_\Gamma) \subset C^*(\mathcal{G}_\Gamma)$ show that

$$\{u_{\alpha_{1,m}(r(\mu))\mu, \alpha_{1,m}(r(\nu))\nu} : m \geq 1,\ \mu, \nu \in \Lambda_m,\ s(\mu) = s(\nu)\},$$

is a Cuntz–Krieger $(\varprojlim(\Lambda_n; p_n), \Lambda_1^0)$-family in $C^*(\mathcal{G}_\Gamma)$. By the universal property of $C^*(\varprojlim(\Lambda_n; p_n), \Lambda_1^0)$ (see [1, Section 3]), there is a homomorphism $\psi : C^*(\varprojlim(\Lambda_n; p_n), \Lambda_1^0) \to C^*(\mathcal{G}_\Gamma)$ such that

$$\psi(T_{\alpha_{1,m}(r(\mu))\mu, \alpha_{1,m}(r(\nu))\nu}) = u_{\alpha_{1,m}(r(\mu))\mu, \alpha_{1,m}(r(\nu))\nu},$$

for each $m, \mu, \nu$. The canonical gauge action $\beta : \mathbb{T}^k \to \mathrm{Aut}(C^*(\mathcal{G}_\Gamma))$ determined by $\beta_z(f)\,(x, m, y) := z^m f(x, m, y)$ satisfies $\psi \circ \gamma_z = \beta_z \circ \psi$ for all $z \in \mathbb{T}^k$, where $\gamma$ is the gauge action on $C^*(\varprojlim(\Lambda_n; p_n), \Lambda_1^0)$. By [21, Proposition 4.3], each $u_{\alpha_{1,m}(r(\mu))\mu, \alpha_{1,m}(r(\mu))\mu}$ is nonzero, and it follows from the gauge-invariant uniqueness theorem [1, Theorem 3.5] that $\psi$ is injective. The topology on $\mathcal{G}_\Gamma^{(0)}$ is generated by the collection of compact open sets $\{U_{m,\lambda} : m \geq 1, \lambda \in \Lambda_m\}$, and the topology on $\mathcal{G}_\Gamma$ is generated by the collection of compact open sets $\{U_{m,\mu} *_s U_{m,\nu} : m \geq 1,\ \mu, \nu \in \Lambda_m,\ s(\mu) = s(\nu)\}$. Since $C^*(\{u_{\alpha_{1,m}(r(\mu))\mu, \alpha_{1,m}(r(\nu))\nu} : m \geq 1,\ \mu, \nu \in \Lambda_m,\ s(\mu) = s(\nu)\}) \subset C^*(\mathcal{G}_\Gamma)$ contains the characteristic functions of these sets, it follows that $\psi$ is also onto, and this completes the proof. □

REMARK 6.2. The final statement of Proposition 6.1 suggests that we can regard $\varprojlim(\Lambda_i, p_i)$ as a skew-product of $\Lambda$ by $G$.

To make this precise, note that for $\lambda \in \Lambda$, $c(\lambda) := (c_n(\lambda))_{n=1}^{\infty}$ belongs to $G$, and $c : \Lambda \to G$ is then a cocycle. There is a natural bijection between the cartesian product $\Lambda \times G$ and the topological $k$-graph $\varprojlim(\Lambda_i, p_i)$, so we may view $\Lambda \times G$ as a topological $k$-graph by pulling back the structure maps from $\varprojlim(\Lambda_i, p_i)$. What we obtain coincides with the natural definition of the skew-product $\overrightarrow{\Lambda} \times_c G$.

With this point of view, we can regard Proposition 6.1 as a generalization of [15, Theorem 7.1(ii)] to profinite groups and topological $k$-graphs: $C^*(\Lambda \times_c G) \cong C^*(\Lambda) \times_\delta G$.

EXAMPLE 6.3 (Example 3.3 continued). Resume the notation of Examples 3.3 and 4.2. The resulting projective limit $\varprojlim(\Lambda_n, p_n)$ is the topological 1-graph $E$ associated to the odometer action of $\mathbb{Z}$ on the Cantor set as in [21, Example 2.5(3)]. That is, $E$ can be realized as the skew-product of $B_1^*$ by the 2-adic integers $\mathbb{Z}_2$ with respect to the functor $c : B_1^* \to \mathbb{Z}_2$ determined by $c(f) = (1, 1, 1, \ldots)$, where $f$ is the loop edge generating $B_1^*$.

## References


[1] S. Allen, 'A gauge invariant uniqueness theorem for corners of higher rank graph algebras', *Rocky Mountain J. Math.* **38** (2008), 1887–1907.

[2] T. Bates, J. Hong, I. Raeburn and W. Szymański, 'The ideal structure of the $C^*$-algebras of infinite graphs', *Illinois J. Math.* **46** (2002), 1159–1176.

[3] D. Drinen and M. Tomforde, 'The $C^*$-algebras of arbitrary graphs', *Rocky Mountain J. Math.* **35** (2005), 105–135.

[4] S. Echterhoff, S. Kaliszewski, J. Quigg and I. Raeburn, 'A categorical approach to imprimitivity theorems for $C^*$-dynamical systems', *Mem. Amer. Math. Soc.* **180** (2006), viii+169.

[5] M. Enomoto and Y. Watatani, 'A graph theory for $C^*$-algebras', *Math. Japon.* **25** (1980), 435–442.

[6] P. A. Fillmore, *A User's Guide to Operator Algebras*, Canadian Mathematical Society Series of Monographs and Advanced Texts (John Wiley & Sons, New York, 1996), pp. xiv+223.

[7] S. Kaliszewski and J. Quigg, 'Mansfield's imprimitivity theorem for full crossed products', *Trans. Amer. Math. Soc.* **357** (2005), 2021–2042.

[8] ———, 'Landstad's characterisation for full crossed-products', *New York J. Math.* **13** (2007), 1–10.

[9] A. Kumjian and D. Pask, 'Higher rank graph $C^*$-algebras', *New York J. Math.* **6** (2000), 1–20.

[10] A. Kumjian, D. Pask and I. Raeburn, 'Cuntz–Krieger algebras of directed graphs', *Pacific J. Math.* **184** (1998), 161–174.

[11] A. Kumjian, D. Pask, I. Raeburn and J. Renault, 'Graphs, groupoids and Cuntz–Krieger algebras', *J. Funct. Anal.* **144** (1997), 505–541.

[12] A. Kumjian, D. Pask and A. Sims, '$C^*$-algebras associated to coverings of $k$-graphs', *Documenta Math.* **13** (2008), 161–205.

[13] M. B. Landstad, 'Duality for dual $C^*$-covariance algebras over compact groups', Preprint, 1978.

[14] ———, 'Duality theory for covariant systems'', *Trans. Amer. Math. Soc.* **248** (1979), 223–267.

[15] D. Pask, J. Quigg and I. Raeburn, 'Coverings of $k$-graphs', *J. Algebra* **289** (2005), 161–191.

[16] D. Pask, I. Raeburn, M. Rørdam and A. Sims, 'Rank-2 graphs whose $C^*$-algebras are direct limits of circle algebras', *J. Funct. Anal.* **239** (2006), 137–178.

[17] J. Quigg, 'Landstad duality for $C^*$-coactions', *Math. Scand.* **71** (1992), 277–294.

[18] D. I. Robertson and A. Sims, 'Simplicity of $C^*$-algebras associated to higher-rank graphs', *Bull. London Math. Soc.* **39** (2007), 337–344.

[19] G. Robertson and T. Steger, 'Affine buildings, tiling systems and higher rank Cuntz–Krieger algebras', *J. Reine Angew. Math.* **513** (1999), 115–144.

[20] T. Yeend, 'Topological higher-rank graphs and the $C^*$-algebras of topological 1-graphs', *Contemp. Math.* **414** (2006), 231–244.

[21] ———, 'Groupoid models for the $C^*$-algebras of topological higher-rank graphs', *J. Operator Theory* **57** (2007), 95–120.



DAVID PASK, School of Mathematics and Applied Statistics,
University of Wollongong, NSW, 2522, Australia
e-mail: dpask@uow.edu.au

JOHN QUIGG, Department of Mathematics and Statistics, Arizona State University,
Tempe, Arizona, 85287, USA
e-mail: quigg@asu.edu

AIDAN SIMS, School of Mathematics and Applied Statistics,
University of Wollongong, NSW, 2522, Australia
e-mail: asims@uow.edu.au