2016

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Thanh Le
University of Wollongong, thanhl@uow.edu.au

Cuong Le Van

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Publication Details

Le, T. & Le Van, C. (2016). Transitional dynamics in an R&D-based growth model with natural resources. Mathematical Social Sciences, 82 1-17.

Transitional Dynamics in an R&D-based Growth Model with Natural Resources

Thanh Le[a] and Cuong Le Van[b]

[a]University of Queensland and Flinders University. Corresponding author. Tel: +61 7 33467053. Fax: + 61 7 33657299. Email: t.le2@uq.edu.au.

[b]IPAG Business School, Paris School of Economics, French National Center for Scientific Research, and Vietnam Center of Research in Economics, Management and Environment

Abstract

In this paper, we prove the existence and uniqueness of the optimal path for a resource endowed economy with R&D. This path converges to an optimal steady state, which is a saddle point, for each type of resources (renewable or non-renewable). In this steady state, a finite size resource sector coexists with other continuously growing sectors. In comparison, the corresponding decentralized equilibrium is suboptimal and there is either over- or under-investment in R&D from the social planner's perspective. At optimum, positive long-run growth will be sustained regardless type of resources used.
Keywords: R&D-based growth, natural resources, vertical innovation, transitional dynamics.
JEL classification: O13, O31, O41.

1 Introduction

There has been a growing interest in examining the pattern of economic growth when there is technological change and natural resources (e.g. Grimaud and Rouge, 2003; Lafforgue, 2008; Peretto, 2012; Peretto and Valente, 2011). This literature focuses on the interplay between economic growth and resource exploitation. In particular, it studies how the adjustment of technological change induced by purposive research and development (R&D) investment and natural resource stock affects economic sustainability and welfare. An important result of this analysis is that changes in an economy's resource endowment affects growth and welfare. However, the dynamic behaviour of these models is not well comprehended since the existing studies mostly focus on the balanced growth path.[1] In addition, a large portion of existing studies only pay attention

[1]Previous studies often consider natural resources, R&D, and growth separately; either between resource abundance and economic growth (e.g. Dasgupta and Heal, 1974, 1979;

to non-renewable resources. Given the trend of shifting from non-renewable resources which are available in fixed supply to renewable alternatives in many countries around the world (Brown, 2000), this seems inadequate not considering the impact of the latter on an economy's sustainable development process. Some authors even assume that the extraction of resources is costless (e.g. Grimaud and Rouge, 2003). This restrictive assumption rules out the possibility of analyzing the dynamic allocation of production factors across sectors.

Our main purpose in this paper is to analyze how the adjustment in resource exploitation and R&D investment changes the dynamic behaviour of the economy. To that end, we consider a general equilibrium model of endogenous growth with R&D and natural resources. Unlike existing studies in which only one type of resources (renewable or non-renewable) is considered, in this paper, both types of resources are examined. The economy under consideration consists of four productive sectors that are linked to each other: a research sector, an intermediate good sector, a final good sector, and a resource sector. Labour is the unique production input that is required for the production of knowledge, intermediate products and harvesting of resources. Resources (e.g. iron ore), after being extracted and processed into materials (e.g. iron), are used to produce intermediate capital goods which then serve as inputs for the production of a final consumption good. Vertical innovation targets at upgrading the quality of these intermediate products. This setting makes the decision on the allocation of labour across sectors become the most important one.

Given the above setting, we first determine the optimal path of the centralized economy. In doing so, we use the Euler - Lagrange equation technique to characterize the time paths of labour allocations across sectors and the dynamics of the resource stock which, in turn, help pin down the dynamics of all other variables in the economy. Given that the social planner's objective function is not concave (the constraints are not convex due to the growth of technological knowledge), we use logarithms of variables, both state and control, as tools to transform this non-convex problem into a convex one. This transformation allows us to prove the uniqueness of the optimal solution. Moreover, such a solution always exists, based on the Dunford-Pettis Criterion. This is an important contribution of the paper given that the issue of the existence and uniqueness of the solution is often neglected by the economics literature dealing with continuous time. The condition of having a sufficiently productive R&D sector ensures that positive long-run growth is sustained no matter which type of resources is used in production. Upon attaining these results, we move on to derive a unique steady state for each type of resources. Regardless of resource type, the steady state is shown to be of a saddle point where the dynamic system will converge to. Again, although the optimization problem is non-concave, we are still able to prove that the stable manifold is locally optimal.

In the socially optimal steady state, while the resource sector maintains a finite size, other sectors experience continuing growth. While an improvement

Sachs and Warner, 1995, 1997, 2001; Lederman and Maloney, 2007) or between R&D-based innovation and economic performance (e.g. Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992).

in productivity of research always increases growth and welfare regardless of the resource type, an improvement in that of resource production only results in the same outcome when resources are renewable. An economy endowed with renewable resources generally enjoys a higher growth rate than it would be in case of non-renewable resources. In addition, positive growth is always guaranteed regardless the type of resources used.

When examining an equivalent decentralized version of the economy, we find that a decentralized equilibrium path is not socially optimal. At a steady state on that equilibrium path, looking from a social optimum viewpoint, under non-renewable resources, the decentralized economy tends to over-invest in R&D in the long-run if the rate of time preference is sufficiently large and under-invest in R&D if the rate of time preference is low. By contrast, under renewable resources, the decentralized economy always under-invests in R&D.

In comparing to the related literature, a recent paper by Suphaphiphat et al (2015) contains a framework that is similar to the one used in our paper. However, our paper differs from that paper in several aspects. *Firstly*, while Suphaphiphat et al (2015) only pay attention to renewable resources and growth dynamics in different regimes of access control for this resource type, our paper studies non-renewable resources as well. By doing so, we are able to compare the dynamic time paths of the economy under alternative forms of resources. *Secondly*, in Suphaphiphat et al (2015), only the decentralized economy is presented. In our paper, the social planner's problem is also considered and this is the central focus of our study. Although the objective function of the social planner is non-convex, we are able to transform it into a convex one and prove the existence and uniqueness of the optimal time path. The analysis of the potential over-investment/under-investment in R&D of the decentralized equilibrium as compared to the socially optimal level is also conducted.[2] *Thirdly*, there is just vertical innovation in our paper instead of both types of innovation as in their paper. This simplified assumption allows us to concentrate on the more popular form of innovative activity whilst still able to explore how the interaction between technological change and resource dynamics affects economic performance. More importantly, it makes our model results more analytically tractable. *Lastly*, we introduce the resource input into the intermediate good production rather than the final good production. This modelling assumption is aimed to capture the idea that intermediate products are more resource intensive than the final goods.[3] It also permits us to better capture the dynamics of the economy as the intermediate good sector summons all important decisions of the economy: demands for resources, technological change and labour.

This paper is organized as follows. Section 2 sets out the model by describing the basic structure of the economy. Section 3 offers equilibrium concepts for this economy. Section 4 studies transitional dynamics, existence and uniqueness of

[2]The issue of under-investment/over-investment in R&D of the decentralized economy is also investigated by Grimaud and Rouge (2003). However, the context of their paper is different from that in our paper. In particular, they assume costless extraction of non-renewable resources and do not study renewable resources as well as transitional dynamics.

[3]We would like to thank an anonymous referee for this suggestion.

the solution to the social planner's problem as well as that of the optimal steady state. In addition, the optimal steady state is shown to be saddle point stable. It is then used as a benchmark for assessing the steady state of the corresponding decentralized economy in Section 5. Key properties of the optimal steady state are considered in Section 6. Finally, Section 7 concludes.

2 The economy

2.1 *The final goods sector*

This sector is assumed to be competitive with a large number of identical firms producing an homogeneous consumption good Y according to the following technology:

$$Y_t = \int_0^1 A_{it} x_{it}^{\alpha} di, \ \alpha \in (0,1) \tag{1}$$

where x_{it} is the amount of intermediate good of vintage i (indexed on a unit interval), and A_{it} is a productivity parameter attached to the latest version of that intermediate good.

The final good is taken as a *numeraire* ($P_Y = 1$). The final good producers' profit function is:

$$\pi_{Yt} = Y_t - \int_0^1 p_{xit} x_{it} di$$

where p_{xit} denotes the price of intermediate good i at time t. Profit maximization gives the (inverse) demand function for each intermediate good:

$$p_{xit} = \alpha A_{it} x_{it}^{\alpha-1}, \ \forall i \in [0,1] \tag{2}$$

2.2 *The intermediate goods sector*

This sector is assumed to be monopolistically competitive. Each intermediate producer faces the following production technology:[4]

$$x_{it} = \frac{M_{it}^{\beta} L_{it}^{1-\beta}}{A_{it}}, \ \beta \in [0,1], \ \forall i \in [0,1] \tag{3}$$

Here, L_{it} is labour employment in industry i at time t and M_{it} is the use of processed natural resource materials. The appearance of A_{it} in the denominator is aimed to capture the fact that products of higher degree of complexity cost more (in terms of labour and/or resources) to produce.

Profit function for the representative monopolist i is:

$$\pi_{xit} = p_{xit} x_{it} - p_{mt} M_{it} - w_t L_{it}$$

[4]A similar form of production function can be found in Aghion and Howitt (1998, ch.9) and Zeng (2003).

The monopolist's objective is to maximize this profit function subject to the demand equation (2) and production technology equation (3). In terms of notation, p_{mt} is the unit price of processed material and w_t is the cost of hiring one unit of labour. After taking the first order conditions with respect to M_{it} and L_{it} and then rearranging and summing over i, we obtain:

$$M_t = \frac{\alpha^2 \beta Y_t}{p_{mt}} \tag{4}$$

$$L_{xt} = \frac{\alpha^2 (1-\beta) Y_t}{w_t} \tag{5}$$

where $M_t = \int_0^1 M_{it} di$ is the aggregate stock of materials used and $L_{xt} = \int_0^1 L_{it} di$ is the total labour employment employed for producing intermediate goods. Plugging these results into equation (3) yields $x_{it} = x_t = (\frac{M_t^\beta L_{xt}^{1-\beta}}{Y_t})^{\frac{1}{1-\alpha}}$, $\forall i$. Plugging this result into the production function in (1) gives:

$$Y_t = A_t^{1-\alpha} (M_t^\beta L_{xt}^{1-\beta})^\alpha \tag{6}$$

where $A_t = \int_0^1 A_{it} di$ is the economy wide aggregate stock of knowledge.[5]

2.3 *The research sector*

This sector is assumed to be competitive with free entry. There is only one type of innovation aiming at improving the quality of existing intermediate products (vertical innovation). Each time, when an innovation is successful, a new (better) vintage of an intermediate product is introduced and replaces its older version in the final good production. Assume that designs or blue prints are protected by the patent law so that each successful innovator can charge a monopoly price over their product until the next successful innovator occurs in that industry.

With access to the stock of knowledge, research firms use labour to develop new blueprints with a Poisson arrival rate $\lambda > 0$. A successful innovation lifts up the knowledge level by a factor $\mu > 1$. Because the prospective payoff is the same in each industry, a same amount will be spent on vertical R&D in each industry. If L_{rt} is the total amount of labour devoted to doing research then the evolution of A_t can be shown as:

$$\dot{A}_t = \lambda(\mu - 1) L_{rt} A_t \tag{7}$$

With free entry, in equilibrium, marginal cost of an extra unit of labour is equal to its expected marginal benefit:

$$\lambda V_t = w_t \tag{8}$$

[5]Because the number of intermediate industries is indexed on a unit interval, A_t coincides with the economy's average technology level.

Here, V_t is the value of a vertical innovation such that:

$$V_t = \int_t^{\infty} \pi_{xt\tau} e^{-\int_t^{\tau}(r_s+I_s)ds} d\tau \tag{9}$$

where r_s is the instantaneous interest rate at date s, $I_s = \lambda L_{rs}$ is the rate of successful innovation arrival at date s, and $\pi_{xt\tau}$ is the flow of operating profit at date τ to any firm in the sector whose technology is of vintage t. In other words, as the market for design is competitive, the value of vertical innovation is equal to the expected present value of future operating profits to be earned by the incumbent intermediate monopolist until being replaced by the next innovator in the industry.

2.4 *The primary or resource sector*

Resources are extracted by resource firms. Following Gordon (1954) and Schaefer (1957), the amount of materials extracted depends on the amount of labour input used, L_{mt}, and the availability of the stock of resources, R_t:

$$M_t = BL_{mt}R_t \tag{10}$$

In this formulation, B is the productivity of resource production. The dynamics of the stock of resources are as follows:

$$\dot{R}_t = f(R_t) - M_t \tag{11}$$

Here, $f(R_t)$ is the natural growth of the resources that takes the following logistic growth form:

$$f(R_t) = \eta R_t \left(1 - \frac{R_t}{\overline{R}}\right),\ \eta \geq 0 \tag{12}$$

where $\overline{R}$ is the carrying capacity of the environment and η represents the intrinsic growth rate of resources. When $\eta > 0$, the natural resources are renewable and when $\eta = 0$, they are non-renewable. Combining (10), (11), and (12) delivers:

$$\dot{R}_t = \eta R_t \left(1 - \frac{R_t}{\overline{R}}\right) - BL_{mt}R_t \tag{13}$$

3 Equilibrium

Assume constant population and normalize the size of population to 1 ($L = 1$) for simplicity. Hence, under the assumption of full employment, the labour market equilibrium requires that:

$$L_{xt} + L_{rt} + L_{mt} = 1 \tag{14}$$

And the goods market equilibrium dictates that:

$$C_t = Y_t$$

where Y_t is given by equation (6).

The program of the social planner is to maximize the utility:

$$U = \int_0^\infty \log(Y_t).e^{-\rho t}dt$$

subject to the dynamic equations of technology and natural resources:

$$\frac{\dot{A}_t}{A_t} = \lambda(\mu - 1)L_{rt} \tag{15}$$

$$\dot{R}_t = \eta R_t(1 - \frac{R_t}{\bar{R}}) - BL_{mt}R_t \tag{16}$$

We define our equilibrium in this economy as follows:

Definition 1 *An equilibrium of this centralized economy is an infinite sequence of quantity allocations* $\{C_t,\ Y_t,\ A_t,\ R_t,\ L_{xt},\ L_{mt},\ L_{rt}\}_{t=0}^{\infty}$ *such that consumers' welfare is maximized subject to intertemporal constraints facing the social planner.*

Definition 2 *A steady state is an equilibrium path where all variables grow at a constant rate and the allocations of labour across the intermediate goods, resource, and the R&D sectors are also constant.*

Specifically, along such a steady state, $L_{xt},\ L_{mt},\ L_{rt}$ are all constant; $C_t,\ Y_t,\ A_t,\ R_t$ grow at constant rates $g_C,\ g_Y,\ g_A$, and g_R respectively. In this paper, we will first identify if there exists any socially optimal paths which lead to an optimal steady state. We then analyze transitional dynamics to and local stability around this steady state. We also contrast this optimal outcome with an equilibrium derived for the corresponding decentralized version of the economy. In the end, we conduct comparative statics analysis assuming the economy is in its optimal steady state(s).

4 Characterization of optimal path(s) and local stability of steady state(s)

4.1 *Transitional dynamics of optimal path(s)*

In this centralized economy, the key dynamic equations are given by those describing the evolution of technical knowledge and the dynamics of the stock of natural resources given in (15) and (16) respectively. From these equations, we will derive socially optimal time paths of the economy and work out conditions for achieving the convergence to the steady state.

Let us assume that L_{rt} and L_{mt} are continuous. For any t, L_{rt} and L_{mt} belong to the interval $[0, 1]$. Any solution R_t to (16) is continuously differentiable. Observe that when $R(t) \geq \bar{R}$, we have $\dot{R}_t < 0$, $\forall t$. Predicting that $R(t) \leq \bar{R}$, we can state the following:

Lemma 1 *Assume $R_0 < \bar{R}$. Then $R(t) \leq \bar{R}$ for all t. And, hence, $\log(R_t) \geq \log(R_0) - Bt$.*

Proof. See Appendix.

We now summarize our first key results in the proposition below:

Proposition 1 *To simplify notation, define $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$. Assume $\eta = 0$ and $\frac{\rho}{\varphi} + \frac{\beta\rho}{\varphi(1-\beta)+B\beta} < 1$. Then the solution to the social planner's maximization problem is an optimal steady state that is uniquely defined as follows:*

$$\begin{aligned}
\hat{L}_x &= \frac{\rho}{\varphi} \\
\hat{L}_m &= \frac{\beta\rho}{\varphi(1-\beta)+B\beta} \\
\hat{L}_r &= 1 - \hat{L}_x - \hat{L}_m \\
\hat{R}_t &= R_0 e^{-B\hat{L}_m t} \\
\hat{A}_t &= A_0 e^{\lambda(\mu-1)(1-\hat{L}_m-\hat{L}_x)t}
\end{aligned}$$

Proof. See Appendix.

This proposition gives out the optimal steady state for the case of non-renewable resources. The condition pointed out in proposition is required for having an interior solution $\hat{L}_r > 0$. For a given set of values for ρ, α, β, B, this condition is met if φ is large. This implies the necessity of having a highly productive research sector (high value of λ and/or μ) to move the economy forward. Under this condition, the allocations of labour, the essential production factor, across sectors are constant and only depend on parameters characterizing the productivities of the research sector, the resource sector, and the rate of time preference. Upon obtaining variables on labour allocations, the constant growth rates of technology, output, consumption, and natural resources can be derived.

Unless otherwise stated, from now and henceforth, it is assumed that $\eta > 0$. We can now state the following lemma:

Lemma 2 *Let L_{xt}^* and L_{mt}^* be solutions to the social planner's maximization problem. Then L_{xt}^* and L_{mt}^* satisfy the following differential equations:*

$$\frac{\dot{L}_{xt}}{L_{xt}} = \varphi L_{xt} - \rho \tag{17}$$

$$\rho\alpha\beta + \frac{\alpha\beta\dot{L}_{mt}}{L_{mt}} + \frac{\alpha\beta\eta R_t}{\bar{R}} - [\varphi\alpha(1-\beta) + B\alpha\beta] L_{mt} = \frac{\alpha(1-\beta)\eta}{L_{xt}\bar{R}} L_{mt} R_t \tag{18}$$

where $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$.

Proof. See Appendix.

We have just established the equations capturing the dynamics of the two important variables, L_{xt} and L_{mt}. These equations allow us to solve for the time paths of L_{xt} and L_{mt} explicitly as stated in the lemma below:

Lemma 3 *The solutions to the social planner's maximization problem, L^*_{xt} and L^*_{mt}, take the following forms:*

$$L^*_{xt} = \frac{1}{\frac{\varphi}{\rho} + c_x e^{\rho t}} \tag{19}$$

$$L^*_{mt} = \frac{1}{e^{\int_0^t b(u)du}\left(c_1 - \int_0^t h(x) e^{-\int_0^x b(u)du} dx\right)} \tag{20}$$

where

$$\begin{aligned} b(u) &= \rho + \eta \frac{R^*_u}{\bar{R}} \\ h(x) &= \frac{\varphi(1-\beta)}{\beta} + B + \frac{\eta(1-\beta)}{\beta L^*_{xt}} \times \frac{R^*_x}{\bar{R}} \\ \varphi &= \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)} \\ c_1 &> 0,\ c_x \geq 0 \end{aligned}$$

Proof. See Appendix.

We proceed with the analysis of the two aforementioned forces, L_{xt} and L_{mt}, that help us pin down the transition in this centralized economy. The properties of this transition will depend on the dynamic adjustments of the labour allocations across sectors along the transition.

Lemma 4 *Let $v(.)$ be a continuous function, then the following applies:*

$$\int_0^t v(x) e^{-\int_0^x v(u)du} dx = 1 - e^{-\int_0^t v(u)du}$$

Proof. See Appendix.

This lemma is important for the characterization of the dynamic adjustments of labour allocations. We continue to proceed with the following lemma:

Lemma 5 *As soon as $c_x = 0$, the following condition holds:*

$$\frac{1-\beta}{L^*_{xt}} < \frac{\beta}{L^*_{mt}}$$

Proof. See Appendix.

As shown later, this condition is required for having an interior solution to the maximization problem. Recall that the maximization problem for the social planner is to solve:

$$\max \qquad \int_0^\infty \mathcal{L}(A_t, \dot{A}_t, R_t, \dot{R}_t) e^{-\rho t} dt$$

where

$$\mathcal{L}(A_t, \dot{A}_t, R_t, \dot{R}_t) = (1-\alpha)\log(A_t) + \alpha\beta\log\left[\eta R_t\left(1-\frac{R_t}{\bar{R}}\right) - \dot{R}_t\right] \tag{21}$$

$$+\alpha(1-\beta)\log\left[1 - \frac{\dot{A}_t}{A_t\lambda(\mu-1)} - \frac{\eta}{B}\left(1-\frac{R_t}{\bar{R}}\right) + \frac{\dot{R}_t}{BR_t}\right] \tag{22}$$

Now let $z_t = \log(A_t)$ and $w_t = \log(R_t)$. From the above equation, one can define that:

$$\mathcal{M}(z_t, \dot{z}_t, w_t, \dot{w}_t) = (1-\alpha)z_t + \alpha\beta\left[w_t + \log\left(\eta(1-\frac{e^w}{\bar{R}}) - \dot{w}_t\right)\right]$$
$$+\alpha(1-\beta)\log\left[1 - \frac{\dot{z}_t}{\lambda(\mu-1)} - \frac{\eta}{B}(1-\frac{e^{w_t}}{\bar{R}}) + \frac{\dot{w}_t}{B}\right]$$

To simplify notations, we define the function:

$$G(\dot{z}_t, w_t, \dot{w}_t) = \left[1 - \frac{\dot{z}_t}{\lambda(\mu-1)} - \frac{\eta}{B}(1-\frac{e^{w_t}}{\bar{R}}) + \frac{\dot{w}_t}{B}\right]$$

Given the above notations and settings, the following lemma is the corollary of Lemma 5:

Lemma 6 *When $c_x = 0$, we have:*

$$-\frac{\alpha\beta}{\eta(1-\frac{e^{w_t}}{\bar{R}}) - \dot{w}_t} + \frac{\alpha(1-\beta)}{G(\dot{z}_t, w_t, \dot{w}_t)} \times \frac{1}{B} < 0$$

Proof. See Appendix.

Lemmas 4, 5, 6, and their corollary are crucial for our next main results below:

Proposition 2 *Assume $R_0 < \bar{R}$ and $\frac{\rho}{\varphi} < 1-\beta$. Then the social planner's optimal solutions are*

$$L^*_{xt} = \frac{\rho}{\varphi} \tag{23}$$

$$L^*_{mt} = \frac{1}{e^{\int_0^t b(u)du}\left(c_1 - \int_0^t h(x)e^{-\int_0^x b(u)du}dx\right)} \tag{24}$$

$$\frac{\dot{A}^*_t}{A^*_t} = \lambda(\mu-1)(1 - L^*_{xt} - L^*_{mt})$$

$$\frac{\dot{R}^*_t}{R^*_t} = \eta(1-\frac{R^*_t}{\bar{R}}) - BL^*_{mt}$$

R_0, A_0 are given and

$$\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$$

$$b(u) = \rho + \eta\frac{R_u^*}{R}$$

$$h(x) = \frac{(1-\alpha)\lambda(\mu-1)}{\beta\alpha} + B + \frac{\eta(1-\beta)}{\beta L_{xt}^*} \times \frac{R_x^*}{R}$$

$$c_1 = \int_0^\infty h(x)e^{-\int_0^x b(u)du}dx$$

Proof. See Appendix.

Clearly, the condition $\frac{\rho}{\varphi} < 1-\beta$ will be satisfied if φ, and hence μ and/or λ, is large, other things equal. In other words, there should be a sufficiently productive research sector in operation. In this case, $L_{rt}^* > 0$. As R&D is a worthwhile investment, there is an incentive to finance this activity from household lending. The results in the proposition imply that we can establish the transition of this economy on the values of L_{xt}^* and L_{mt}^* which jointly determine the allocations of labour, the unique production factor in the economy, and the rates of growth of interested variables (e.g. technology, output, resources, etc.).

Remark 1 *If $\eta = 0$, then L_{mt}^* takes the value of the steady state derived in Proposition 1.*

In the proof of Lemma 3, it is indicated that when resources are non-renewable ($\eta = 0$), we have:

$$L_{mt}^* = \frac{1}{\left(\frac{\varphi(1-\beta)}{\beta}+B\right).\frac{1}{\rho}+c_m e^{\rho t}}$$

where $c_m = c_1 - \left(\frac{\varphi(1-\beta)}{\beta} + B\right).\frac{1}{\rho}$ is a non-negative constant. It can be seen that if $c_m > 0$, L_{mt} approaches 0 when $t \to +\infty$. This outcome cannot be optimal since it results in zero production of resource materials, intermediate goods as well as final consumption goods (recall that welfare is increasing in the amount of final consumption goods produced). Hence, it must be that $c_m = 0$ and $L_m = \frac{\beta\rho}{\varphi(1-\beta)+B\beta}$. This leads to our next remark below.

Remark 2 *When resources are non-renewable, the economy will immediately jump to its optimal steady state derived in Proposition 1.*

Interestingly, under non-renewable resources, because stock of resources can only decrease, the optimal action for the social planner is to get the economy immediately to its steady state. However, under renewable resources, as the stock of resources can regenerate itself, the best option for the social planner is to allow the economy to gradually follow its optimal path to the steady state (which will be computed in a subsection below).

4.2 *Uniqueness and existence of the solution to the social planner's problem*

Let $z_t = \log(A_t), w_t = \log(R_t)$. The maximization problem for the social planner of this economy can be written as:

$$\max \qquad \int_0^\infty \mathcal{N}(z_t, w_t, L_{mt}, L_{xt})e^{-\rho t}dt$$

where

$$\mathcal{N}(z_t, w_t, L_{mt}, L_{xt}) = (1-\alpha)z_t + \beta\alpha\left[\log(B) + \log(L_{mt}) + w_t\right] + \alpha(1-\beta)\log(L_{xt})$$

subject to the constraints

$$\dot{z} \leq \lambda(\mu - 1)(1 - L_{mt} - L_{xt}) \quad (25)$$

$$\dot{w}_t \leq \eta(1 - \frac{e^{w_t}}{\bar{R}}) - BL_{mt} \quad (26)$$

$$L_{xt} + L_{mt} \leq 1 \quad (27)$$

Since the function $\mathcal{N}$ is strictly increasing, at the optimum the constraints (25), (26) will be binding.

Proposition 3 *If there exists a solution to the social planner's maximimization problem then that solution is unique.*

Proof. See Appendix.

This proposition highlights the uniqueness of the solution to the social planner's problem. It also stimulates our probe into the existence of such a solution. That is the content of our next proposition.

Proposition 4 *Assume $\eta > 0, R_0 < \bar{R}$ and $\frac{\rho}{\varphi} < 1 - \beta$. Then there exists a solution to the social planner's problem.*

Proof. See Appendix.

We can now sum up the results obtained in Proposition 2, Proposition 3 and Proposition 4 in the theorem below:

Theorem 1 *Assume $\eta > 0$. Further assume $R_0 < \bar{R}$ and $\frac{\rho}{\varphi} < 1 - \beta$. Then there exists a unique solution to the social planner's problem. This solution satisfies the system given in Proposition 2.*

4.3 *Long-run properties of the optimal path: convergence to the steady state*

In this subsection, we will first identify the unique socially optimal steady state. We will then show that the optimal path obtained in the previous subsection will converge to this steady state.

Proposition 5 *Assume $\eta > 0$ and*

$$\frac{2B\rho+(1-\beta)\varphi\eta-\sqrt{\triangle}}{2B(1-\beta)\varphi} < 1$$

where $\triangle = 4B^2\beta^2\rho^2 + (1-\beta)^2\varphi^2\eta^2$ and $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$ then there exists a unique socially optimal steady state that is described by

$$\hat{L}_x = \frac{\rho}{\varphi}$$

$$\hat{L}_m = \frac{2B\beta\rho+(1-\beta)\varphi\eta-\sqrt{\triangle}}{2B(1-\beta)\varphi}$$

Proof. See Appendix.

Having obtained the optimal steady state for the case of non-renewable resources in Proposition 1, this proposition only presents the optimal steady state for the case of renewable resources. The condition $\frac{2B\rho+(1-\beta)\varphi\eta-\sqrt{\triangle}}{2B(1-\beta)\varphi} < 1$ must be met for having an interior solution $\hat{L}_r > 0$. It can be checked that the left hand side of the condition is decreasing in φ, and, hence, decreasing in μ and λ, other things equal. This means that for a given set of values of parameters α, β, η, B, if the research sector is highly productive (for a large value of μ and/or λ) then this condition will be satisfied. Note that the steady state values of the two key variables depend on parameters characterizing productivity of the research sector, the resource sector, and the rate of time preference. As shown in the proof of the proposition in the Appendix, these steady state values are optimal.

We next prove the convergence of the above obtained optimal paths to the optimal steady state:

Proposition 6 *Assume parameters are such that the optimal steady state exists. Then the dynamic system is saddle point convergent.*

Proof. See Appendix.

This result indicates that in the long-run, the dynamic system will converge to the optimal steady state at which the highest level of welfare is obtained. Also at this steady state, allocations of labour across sectors are constant and key interested variables enjoy constant growth rates.

5 Optimal path versus decentralized equilibrium path

The purpose of this Section is to characterize an equilibrium path of the decentralized version of the economy and contrast it with the socially optimal path obtained above. Such an economy is described below.

For simplicity, in this decentralized economy, there is only one consumer and four productive sectors: final good, intermediate good, natural resources, and innovation. The consumer consumes the final good only and uses her saving to accumulate a financial asset in an exogenous financial market.

Final good sector
The final good Y_t is produced through a production function using an intermediate good: $Y_t = A_t x_t^\alpha$ where A_t is productivity parameter attached to the latest version of the intermediate good x_t. At period t, this sector maximizes its profit (by normalization, the price of the final good is 1 at each period), i.e.

$$\pi_{Yt} = \max_x \{A_t x_t^\alpha - p_{xt} x_t\}$$

This delivers $p_{xt} = \alpha A_t x_t^{\alpha-1}$.
Intermediate good sector
This sector is assumed to be monopolistic. Before conducting any production, the intermediate good producer is assumed to borrow an amount of money from households to finance research activity (i.e. through paying researchers a labour income wage) which aims to improve the quality of the intermediate product. Once the research is successful, the intermediate good producer acquires the patent and starts producing an amount of output x_t from using a quantity M_t of resource materials and an amount L_{xt} of labour. More precisely, at time t, $x_t = \frac{M_t^\beta L_{xt}^{1-\beta}}{A_t}$ where $\beta \in (0, 1)$. The appearance of A_t in the denominator is aimed to capture the fact that products of a higher degree of complexity cost more to produce. This sector maximizes its profit: $\pi_{xt} = \max\{p_{xt}x_t - p_{mt}M_t - w_t L_{xt}\}$. This maximization problem yields:

$$\begin{aligned} p_{mt} &= \alpha^2 \beta A_t^{1-\alpha} M_t^{\alpha\beta-1} L_{xt}^{\alpha(1-\beta)} \\ w_t &= \alpha^2 (1-\beta) A_t^{1-\alpha} M_t^{\alpha\beta} L_{xt}^{\alpha(1-\beta)-1} \end{aligned}$$

The net profit goes to the intermediate good producer will be $\hat{\pi}_{xt} = \pi_{xt} - r_t a_t$ that is the profit after making the interest payment to the households.
Research sector
This sector contains only researchers who conduct research in return for a wage income. The total wage income is $w_t L_{rt}$. There is no profit accrued to this sector.
Natural sesource sector
Resources are freely accessible. Resource firms use only labor to extract the natural resources according to the production function $M_t = BR_t L_{mt}$ where R_t is the stock of resources satisfying the following constraints:

$$\begin{aligned} R_t &\geq 0 \\ \dot{R}_t &\leq \eta R_t (1 - \frac{R_t}{\bar{R}}) - M_t \end{aligned}$$

Firms in this sector maximize profit $\pi_{mt} = \max\{p_{mt} BR_t L_{mt} - w_t L_{mt}\}$ under these resources constraints.
Consumer
The consumer maximizes her lifetime utility $\int_0^\infty \log(c_t) e^{-\rho t} dt$ under the intertemporal budget constraints:

$$c_t + \dot{a}_t = r_t a_t + (\pi_{Yt} + \hat{\pi}_{xt} + \pi_{mt}) + w_t (L_{xt} + L_{mt} + L_{rt})$$

where a_t denotes the financial asset (e.g. bond) accumulated (in an exogenous market) by the consumer and r_t is an exogenous interest rate. This maximization problem yields the Euler equation for the growth of consumption:

$$\dot{c}_t = (r_t - \rho)c_t$$

Equilibrium
At equilibrium we have:

$$\begin{aligned} c_t &= Y_t \\ L_{xt} + L_{mt} + L_{rt} &= 1 \end{aligned}$$

In addition, we have $\pi_{Yt} = (1-\alpha)Y_t$; $\hat{\pi}_{xt} = \pi_{xt} - r_t a_t = \alpha(1-\alpha)Y_t - r_t a_t$; $\pi_{mt} = \alpha^2 \beta Y_t - w_t L_{mt}$ and $w_t L_{xt} = \alpha^2(1-\beta)Y_t$.

Define $(Y_t^*, c_t^*, a_t^*, x_t^*, M_t^*, R_t^*, A_t^*, L_{xt}^*, L_{m_t}^*, L_{rt}^*)_t$ as the sequence of equilibrium allocations and $(p_{xt}^*, p_{mt}^*, w_t^*)_t$ as the associated sequence of prices and wage. Along this equilibrium path, we have:

Lemma 7 *For any t, $\pi_{mt} = \max\{p_{mt}^* B R_t^* L_m - w_t^* L_m\} = 0$. If $L_{mt}^* > 0$ then $w_t^* = B R_t^* p_{mt}^*$.*

Proof. Along an equilibrium path, the resource sector maximizes the profit $\pi_{mt} = \max_{L_m \geq 0}\{p_{mt}^* B R_t^* L_m - w_t^* L_m\}$. Hence, the profit is zero. If $L_{mt}^* > 0$ then $w_t^* = B R_t^* p_{mt}^*$. ■

We state the following propositions:

Proposition 7 *In this decentralized economy, at equilibrium, we have:*

$$\begin{aligned}
\dot{c}_t^* &= (r_t - \rho)c_t^* \\
Y_t^* &= A_t^* x_t^{*\alpha - 1} && (28) \\
x_t^* &= \frac{M_t^{*\beta} L_{xt}^{*1-\beta}}{A_t^*} && (29) \\
c_t^* &= Y_t^* \\
\dot{a}_t^* &= r_t^* a_t^* + w_t^* L_{rt}^* \\
M_t^* &= B L_{mt}^* R_t^* && (30) \\
L_{xt}^* &= \frac{\alpha^2(1-\beta)Y_t^*}{w_t^*} && (31) \\
p_{xt}^* &= \alpha A_t^* x_t^{*\alpha-1} \\
p_{mt}^* &= \alpha^2 \beta A_t^{*1-\alpha} M_t^{*\alpha\beta - 1} L_{xt}^{*\alpha(1-\beta)} && (32) \\
p_{mt}^* B R_t^* &= w_t^* && (33) \\
\dot{R}_t^* &= \eta R_t^* \left(1 - \frac{R_t^*}{\overline{R}}\right) - B L_{mt}^* R_t^* && (34) \\
\dot{A}_t^* &= A_t^* \lambda(\mu - 1) L_{rt}^* && (35) \\
L_{xt}^* + L_{mt}^* + L_{rt}^* &= 1 && (36)
\end{aligned}$$

Proof. It can be seen that these results are rather straightforward from the discussion at the beginning of the section. ■

Proposition 8 *An equilibrium of this decentralized economy does not solve the social planner's problem.*

Proof. See Appendix.

The results indicate that a decentralized equilibrium is not optimal from a social planner's point of view. Given that technological progress is the main engine of growth in this economy, this leads us to the question of comparing the rates of growth of technology in the two steady states: on the optimal path versus on the decentralized equilibrium path. We can summarize the results in the proposition below:

Proposition 9 *Denote* $\widetilde{\rho} = \frac{\varphi[\varphi(1-\beta)+B\beta]}{\varphi+B\beta}$. *If* $\eta = 0$*, over the long-run, the decentralized economy tends to under-invest in R&D if* $\rho \to 0$ *and over-invest in R&D if* $\rho \to \widetilde{\rho}$*. If* $\eta > 0$*, over the long-run, the decentralized economy always under-invests in R&D.*

Proof. See Appendix.

This result deserves some discussion. When the economy is endowed with non-renewable resources, the rate of time preference, ρ, plays an important role in determining how much long-run investment in R&D (in a decentralized steady state) deviates from its socially desirable level. If ρ is small, the cost of the trade-off between current consumption and future consumption is small so there is more incentive for households to lend money (for conducting R&D in the first place). In addition, the social planner is very patient: he prefers slower extraction of resources. As a result, the social planner directs more labour to work in the research sector than what households would do. By contrast, if ρ is large, knowing that the cost of borrowing money to finance research activity is high (because households value current consumption more highly), the social planner may want to direct labour away from the research sector. This act may result in an over-investment in R&D of the decentralized economy.

However, in case of renewable resources, given that the economy will always extract an amount of resources that is equal to the total natural regeneration of that stock, a change in ρ will not affect the resource firms' extracting strategy. Because the social planner knows technological change is a positive externality (due to knowledge spillovers), he wants to internalize this externality by directing more labour to research activity. This means that the decentralized economy will exert a smaller effort in R&D than what the social planner desires to have.

6 Properties of the socially optimal steady state: a comparative statics study

Having known that the centralized economy will converge to the socially optimal steady state in the long-run, it will be interesting to discuss the properties of this

steady state. In other words, we can do the comparative statics at this long-run equilibrium and analyze possible impacts on output growth and welfare. This section is devoted to that task.

Proposition 10 *Other things equal, along the steady state for each type of resources, output growth and welfare are increasing in parameters characterizing productivity of the R&D sector (λ and μ) but decreasing in the rate of time preference (ρ).*

Proof. See Appendix.

The results are quite intuitive. When λ or μ increases, it becomes more socially efficient to invest in the R&D sector (relatively to other sectors) so the social planner will choose a higher level of $\hat{L}_r$ which then enhances growth of technological knowledge and output. An increase in ρ means households value current consumption relatively more than future consumption. In order to produce more output to meet higher consumption demand today, the social planner will direct more labour to work in the resource sector ($\hat{L}_m$ increases) and the intermediate goods sector ($\hat{L}_x$ increases). As a result, there will be a fall in $\hat{L}_r$ meaning lower growth of technology and output. Consumption growth will also be lower because consumers increase current consumption relatively to future consumption.

Because an increase in either λ or μ raises output and consumption so welfare rises. However, an increase in ρ reduces welfare as it makes the whole path of utility fall below the one before the shock.

Proposition 11 *Other things equal, along the steady state, an improvement in the productivity of the resource sector (an increase in B):*

- *increases both welfare and output growth if resources are renewable.*
- *increases welfare but has no impact on output growth if resources are non-renewable.*

Proof. See Appendix.

The results can be explained as follows. An increase in B makes it more productive to extract natural resources. Equivalently, less labour is needed for producing resource material to meet the existing market demand. Hence, the social planner will allocate less labour to the resource sector ($\hat{L}_m$ decreases) and more into the R&D activities ($\hat{L}_r$ increases).[6] This change will increase welfare as there is more output and consumption created. It will also increase output growth for the case of renewable resources because the growth rate of technology is higher. However, it does not affect output growth under non-renewable resources. The reason is that an increase in B, on the one hand, increases $\hat{L}_r$ and, hence, technological change, will also exhaust resources at a

[6] Another way of looking at this is that as the social planner always knows the optimal level of natural resources to be $R = \bar{R}(1 - \frac{BL_m}{\eta})$, he will reduce L_m in accordance with the amount of increase in B.

faster rate on the other (the fall in $\hat{L}_m$ is less than the increase in B). These two opposing effects cancel out each other at the optimum.

Proposition 12 *Assume parameters are such that there exists a steady state for each type of resources then output growth is generally higher under renewable resources than under non-renewable resources. In addition, positive growth is always guaranteed regardless what type of resources is used.*

Proof. See Appendix.

In this economy, output growth comes from two different sources: the evolution of technological knowledge and the natural resource dynamics. Because natural resources cannot grow without bound, the best trajectory that the social planner can choose is to reach the optimal level of resources at which the rate of resource extraction is equal to the rate of natural growth. However, this policy is only achievable in case of renewable resources. With non-renewable resources, the rate of resource extraction always soften the rate of growth of output as output needs to increase to make up for the amount of natural resources that has been depleted. Given that technological progress is the key driver of the economy, it in turns requires the evolution of technological knowledge be strong enough to lift the economy up out of the stagnation trap.

7 Concluding remarks

This paper has introduced a resource sector into an endogenous growth model with R&D investment. We have shown how the dynamic equilibrium could be represented by a dynamic system characterized by the sectoral allocation of labour, the evolution of technology and the dynamics of the resource stock. We show that under plausible assumptions, the social planner can achieve a stable transitional dynamics to a unique socially optimal steady state. In this steady state, the stock of resources remains in finite size while other sectors carry continuous growth.

Comparing to this optimal path, in the long-run, the equivalent decentralized economy always under-invests in R&D in case of renewable resources. Meanwhile, it tends to over-invest in R&D if the rate of time preference is large and under-invest in R&D if the rate of time preference is low in case of non-renewable resources.

The socially optimal steady state has the following features. Equilibrium growth rate in an economy endowed with renewable resources is higher than it would be in the case of non-renewable resources. As soon as the research sector is highly productive, positive growth is always sustained regardless type of resources used.

We have also examined the long-run reaction of the economy to a number of changes regarding innovative production capacity, rate of time preference, and resource sector productivity. While an improvement in productivity of the R&D is always growth and welfare enhancing, that of the resource sector is subject to the type of resources that is considered.

Results of this paper convey several important messages to policy makers. If there is an option between alternative resources, a decision maker may wish to use more of renewable resources in production as that policy is growth enhancing. He may also channel more investment towards enhancing R&D efficiency in order to achieve higher permanent positive growth. Given that the decentralized economy equilibrum is suboptimal, this raises the question of designing an appropriate R&D tax/subsidy policy that helps make this equilibrium optimal. Investing this issue would be an interesting direction for future research. Another future research avenue would be to examine the extent to which empirical evidence is consistent with theoretical predictions set out in the paper.

Appendix

Proof of Lemma 1

We will prove this lemma by method of contradiction. To that end, assume there exists t_0 such that $R(t_0) > \bar{R}$. For any $\varepsilon > 0$ which satisfies $\bar{R} + \varepsilon < R(t_0)$, there exists $t \in (0,\, t_0)$ such that $R(t) = \bar{R} + \varepsilon$. Now define:

$$I_\varepsilon = \left\{ t \in [0,\, t_0] :\, R(t) = \bar{R} + \varepsilon \right\}$$

Since R_t is continuous, the set I_ε is compact. Let $t_1 = \max\{t :\; t \in I_\varepsilon\}$ then $t_1 < t_0$. Evaluating the dynamics of resources at time t_1, we have:

$$\left(\frac{\dot{R}(t)}{R(t)}\right)_{t_1} = -\frac{\eta\varepsilon}{\bar{R}} - BL_m < 0$$

Hence, for $t' \in (t_1,\, t_0)$ that is close enough to t_1, we have $R(t') < R(t_1) = \bar{R} + \varepsilon < R(t_0)$. In this case, there must be $t_2 \in (t',\, t_0)$ such that $R(t_2) = \bar{R} + \varepsilon$. This implies $t_2 \in I_\varepsilon$ and $t_2 \leq t_1 < t'$ which is a contradiction. Therefore, $R(t) \leq \bar{R}$ for any t.

From equation (16), we have:

$$\frac{d\log R}{dt} \geq -BL_{mt} \geq -B$$

since $L_{mt} \leq 1$. By integrating this inequality we get $\log(R_t) \geq \log(R_0) - Bt$.

Proof of Proposition 1

We will prove this proposition in two parts. In the first part, we prove that there exists a unique steady state that solves the social planner's maximization problem. In the second part, we show that this steady state is optimal.

Using (6) and (10), the utility function can be rewritten as:

$$U = \int_0^\infty \log(A_t^{1-\alpha} B^{\alpha\beta} L_{mt}^{\alpha\beta} R_t^{\alpha\beta} L_{xt}^{\alpha(1-\beta)}).e^{-\rho t} dt$$

On the steady state, we now have $R_t = R_0 e^{-tBL_m}$ where R_0 is the initial stock of natural resources. With a note that $\int_0^\infty te^{-\rho t}dt = \frac{1}{\rho^2}$ and $\int_0^\infty e^{-\rho t}dt = \frac{1}{\rho}$, the utility function on the stedy state is:

$$
\begin{aligned}
\rho U &= (1-\alpha)\log(A_0) + \alpha\beta\log(B) + \alpha\beta\log(L_m) + \alpha\beta\log(R_0) \\
&+ \alpha(1-\beta)\log(L_x) - \frac{\alpha\beta B L_m}{\rho} + \frac{(1-\alpha)\lambda(\mu-1)(1-L_x-L_m)}{\rho}
\end{aligned}
$$

L_x and L_m will be chosen to maximize this utility functions. The first order conditions give:

$$\hat{L}_x = \frac{\alpha\rho(1-\beta)}{\lambda(\mu-1)(1-\alpha)} = \frac{\rho}{\varphi} \tag{37}$$

$$\hat{L}_m = \frac{\alpha\beta\rho}{\lambda(\mu-1)(1-\alpha) + B\alpha\beta} = \frac{\beta\rho}{\varphi(1-\beta) + B\beta} \tag{38}$$

where $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$. Clearly, $\hat{L}_x > 0$ and $\hat{L}_m > 0$. Hence, the value of $\hat{L}_r$ is:

$$\hat{L}_r = 1 - \left[\frac{\rho}{\varphi} + \frac{\beta\rho}{\varphi(1-\beta) + B\beta}\right] \tag{39}$$

When $\hat{L}_r > 0$ or $\frac{\rho}{\varphi} + \frac{\beta\rho}{\varphi(1-\beta)+B\beta} < 1$ we automatically have $0 < \hat{L}_x, \hat{L}_m, \hat{L}_r < 1$. With these obtained results, we can calculate the growth rates of technology and natural resources along the steady state as follows:

$$
\begin{aligned}
\hat{R}_t &= R_0 e^{-B\hat{L}_m t} \\
\hat{A}_t &= A_0 e^{\lambda(\mu-1)(1-\hat{L}_m-\hat{L}_x)t}
\end{aligned}
$$

To prove that this solution is optimal, we compute the following:

$$\triangle U = \lim_{T\to\infty}\left[\int_0^T \log\left(\hat{A}_t^{1-\alpha} B^{\alpha\beta} \hat{L}_{mt}^{\alpha\beta} \hat{R}_t^{\alpha\beta} \hat{L}_{xt}^{\alpha(1-\beta)}\right).e^{-\rho t}dt - \int_0^T \log\left(A_t^{1-\alpha} B^{\alpha\beta} L_{mt}^{\alpha\beta} R_t^{\alpha\beta} L_{xt}^{\alpha(1-\beta)}\right).e^{-\rho t}dt\right]$$

Using integration by parts we have:

$$\int_0^T \log\left(\hat{A}_t^{1-\alpha}\right).e^{-\rho t}dt = \left[-\tfrac{1}{\rho}e^{-\rho t}\log\left(\hat{A}_t^{1-\alpha}\right)\right]_0^T + \int_0^T \tfrac{1}{\rho}e^{-\rho t}(1-\alpha)\tfrac{\dot{\hat{A}}_t}{\hat{A}_t}dt$$

$$\int_0^T \log\left(\hat{R}_t^{\alpha\beta}\right).e^{-\rho t}dt = \left[-\tfrac{1}{\rho}e^{-\rho t}\log\left(\hat{R}_t^{\alpha\beta}\right)\right]_0^T + \int_0^T \tfrac{1}{\rho}e^{-\rho t}\alpha\beta\tfrac{\dot{\hat{R}}_t}{\hat{R}_t}dt$$

$$\int_0^T \log\left(A_t^{1-\alpha}\right).e^{-\rho t}dt = \left[-\tfrac{1}{\rho}e^{-\rho t}\log\left(A_t^{1-\alpha}\right)\right]_0^T + \int_0^T \tfrac{1}{\rho}e^{-\rho t}(1-\alpha)\tfrac{\dot{A}_t}{A_t}dt$$

$$\int_0^T \log\left(R_t^{\alpha\beta}\right).e^{-\rho t}dt = \left[-\tfrac{1}{\rho}e^{-\rho t}\log\left(R_t^{\alpha\beta}\right)\right]_0^T + \int_0^T \tfrac{1}{\rho}e^{-\rho t}\alpha\beta\tfrac{\dot{R}_t}{R_t}dt$$

In addition, applying the inequality $x \geq \log(1+x)$ we have:

$$\log(\frac{L_{mt}}{\hat{L}_{mt}}) = \log(1 + \frac{L_{mt}-\hat{L}_{mt}}{\hat{L}_{mt}}) \leq \frac{L_{mt}-\hat{L}_{mt}}{\hat{L}_{mt}}$$

Hence

$$-\alpha\beta \log(\frac{L_{mt}}{\hat{L}_{mt}}) \geq -\alpha\beta . \frac{L_{mt}-\hat{L}_{mt}}{\hat{L}_{mt}}$$

or

$$\alpha\beta \left[\log(\hat{L}_{mt}) - \log(L_{mt})\right] \geq \alpha\beta . \frac{\hat{L}_{mt}-L_{mt}}{\hat{L}_{mt}}$$

Similarly, the following holds:

$$\alpha(1-\beta) \left[\log(\hat{L}_{xt}) - \log(L_{xt})\right] \geq \alpha(1-\beta) . \frac{\hat{L}_{xt}-L_{xt}}{\hat{L}_{xt}}$$

Inserting these results into the equation for $\triangle U$ and observing that:

$$\left[-\frac{1}{\rho}e^{-\rho t} \log\left(\hat{A}_t^{1-\alpha}\right)\right]_0^{+\infty} = \left[-\frac{1}{\rho}e^{-\rho t} \log\left(A_t^{1-\alpha}\right)\right]_0^{+\infty} = \frac{1}{\rho}\log\left(A_0^{1-\alpha}\right)$$

$$\left[-\frac{1}{\rho}e^{-\rho t} \log\left(\hat{R}_t^{\alpha\beta}\right)\right]_0^{T} = \left[-\frac{1}{\rho}e^{-\rho t} \log\left(R_t^{\alpha\beta}\right)\right]_0^{T} = \frac{1}{\rho}\log\left(R_0^{\alpha\beta}\right)$$

we get:

$$\triangle U \geq \int_0^{+\infty} \frac{1}{\rho}e^{-\rho t}(1-\alpha)\frac{\dot{\hat{A}}_t}{\hat{A}_t}dt + \int_0^{+\infty} \frac{1}{\rho}e^{-\rho t}\alpha\beta\frac{\dot{\hat{R}}_t}{\hat{R}_t}dt + \int_0^{+\infty} e^{-\rho t}\alpha\beta . \frac{\hat{L}_{mt}-L_{mt}}{\hat{L}_{mt}}dt +$$
$$\int_0^{+\infty} e^{-\rho t}\alpha(1-\beta) . \frac{\hat{L}_{xt}-L_{xt}}{\hat{L}_{xt}}dt - \int_0^{+\infty} \frac{1}{\rho}e^{-\rho t}(1-\alpha)\frac{\dot{A}_t}{A_t}dt - \int_0^{+\infty} \frac{1}{\rho}e^{-\rho t}\alpha\beta\frac{\dot{R}_t}{R_t}dt$$

Using (37) and (38) then:

$$\alpha\beta . \frac{\hat{L}_{mt}-L_{mt}}{\hat{L}_{mt}} = \frac{\lambda(\mu-1)(1-\alpha)+B\alpha\beta}{\rho} . (\hat{L}_{mt} - L_{mt})$$

$$\alpha(1-\beta) . \frac{\hat{L}_{xt}-L_{xt}}{\hat{L}_{xt}} = \frac{\lambda(\mu-1)(1-\alpha)+B\alpha\beta}{\rho} . (\hat{L}_{xt} - L_{xt})$$

Plugging these results in, we can figure out that $\triangle U \geq 0$.

Proof of Lemma 2

The maximization problem for the social planner of this economy is to solve:

$$\max \qquad \int_0^{\infty} \mathcal{L}(A_t, \dot{A}_t, R_t, \dot{R}_t)e^{-\rho t}dt$$

where

$$\mathcal{L}(A_t, \dot{A}_t, R_t, \dot{R}_t) = (1-\alpha)\log(A_t) + \alpha\beta \log\left[\eta R_t\left(1 - \frac{R_t}{\bar{R}}\right) - \dot{R}_t\right] \quad (40)$$

$$+\alpha(1-\beta)\log\left[1 - \frac{\dot{A}_t}{A_t\lambda(\mu-1)} - \frac{\eta}{B}\left(1 - \frac{R_t}{\bar{R}}\right) + \frac{\dot{R}_t}{BR_t}\right] \quad (41)$$

Considering interior solutions, we have the following Euler-Lagrange equations:

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{A}_t}e^{-\rho t}\right] = \frac{\partial \mathcal{L}}{\partial A_t}e^{-\rho t} \tag{42}$$

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{R}_t}e^{-\rho t}\right] = \frac{\partial \mathcal{L}}{\partial R_t}e^{-\rho t} \tag{43}$$

The LHS of (42) is given by:

$$\rho e^{-\rho t}\frac{\alpha(1-\beta)}{\lambda(\mu-1)}\frac{1}{A_t L_{xt}} + \frac{e^{-\rho t}}{A_t L_{xt}}\alpha(1-\beta)\left[L_{rt} + \frac{\dot{L}_{xt}}{\lambda(\mu-1)L_{xt}}\right]$$

while its RHS is equal to:

$$e^{-\rho t}\frac{1-\alpha}{A_t} + \frac{\alpha(1-\beta)}{L_{xt}}\frac{L_{rt}}{A_t}e^{-\rho t}$$

Equating the LHS with the RHS gives:

$$\rho\frac{\alpha(1-\beta)}{\lambda(\mu-1)}\frac{1}{L_{xt}} + \frac{1}{L_{xt}}\alpha(1-\beta)\frac{\dot{L}_{xt}}{\lambda(\mu-1)L_{xt}} = 1-\alpha$$

or

$$\rho\alpha(1-\beta) + \alpha(1-\beta)\frac{\dot{L}_{xt}}{L_x} = (1-\alpha)\lambda(\mu-1)L_{xt}$$

Further simplifying leads to:

$$\rho + \frac{\dot{L}_{xt}}{L_{xt}} = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}L_{xt}$$

Using the notation that is previously defined $\frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)} = \varphi$ then:

$$\frac{\dot{L}_{xt}}{L_{xt}} = \varphi L_{xt} - \rho$$

Similarly, the LHS of (43) is:

$$\begin{aligned} & -\rho e^{-\rho t}\left[-\frac{\alpha\beta}{BR_t L_{mt}} + \frac{\alpha(1-\beta)}{BR_t L_{xt}}\right] \\ + \quad & e^{-\rho t}\left[\frac{\alpha\beta}{B}\left(\frac{\dot{L}_{mt}}{R_t L_{mt}^2} + \frac{1}{L_{mt}R_t^2}\dot{R}_t\right) - \frac{\alpha(1-\beta)\dot{L}_{xt}}{BR_t L_{xt}^2} - \frac{\alpha(1-\beta)}{BR_t^2 L_{xt}}\dot{R}_t\right] \end{aligned}$$

and its RHS is:

$$e^{-\rho t}\left[\frac{\alpha\beta}{BL_{mt}R_t}\eta\left(1-\frac{2R_t}{\bar{R}}\right) + \frac{\alpha(1-\beta)}{L_{xt}}\left(\frac{\eta}{B\bar{R}} - \frac{\dot{R}_t}{BR_t^2}\right)\right]$$

Now equating LHS with RHS noting $\frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)} = \varphi$ to get:

$$\rho\alpha\beta + \frac{\alpha\beta\dot{L}_{mt}}{L_{mt}} + \frac{\alpha\beta\eta R_t}{\bar{R}} - [\varphi\alpha(1-\beta) + B\alpha\beta]\, L_{mt} = \frac{\alpha(1-\beta)\eta}{L_{xt}\bar{R}} L_{mt}R_t$$

Proof of Lemma 3

Consider equation (17), we write $\Upsilon = \frac{1}{L_{xt}}$ to obtain $\dot{\Upsilon} = \rho\Upsilon - \varphi$. After sloving this simplified differential equation we get:

$$L_{xt}^{*} = \frac{1}{\frac{\varphi}{\rho} + c_x e^{\rho t}}$$

where c_x is a constant such that $c_x \geq 0$.

To get the functional form of L_{mt}^{*} we transform equation (18) to obtain:

$$\dot{L_{mt}} + \left(\rho + \frac{\eta R_t}{\bar{R}}\right) L_{mt} = \left(\frac{\varphi(1-\beta)}{\beta} + B + \frac{\eta(1-\beta)R_t}{\beta\bar{R}L_{xt}}\right) L_{mt}^2$$

To simplify notations, define $b(t) = \rho + \frac{\eta R_t}{\bar{R}}$ and $h(t) = \frac{\varphi(1-\beta)}{\beta} + B + \frac{\eta(1-\beta)R_t}{\beta\bar{R}L_{xt}}$. The above equation now becomes:

$$\dot{L_{mt}} + b(t)L_{mt} = h(t)L_{mt}^2$$

Let $z_t = \frac{1}{L_{mt}}$ (noting that $L_{mt} \neq 0$) then $\dot{z}_t = -\frac{\dot{L_{mt}}}{L_{mt}^2}$. This is equivalent to $\dot{L_{mt}} = -\dot{z}_t L_{mt}^2$. Substituting results into the above equation and simplifying gives:

$$\dot{z}_t - b(t)z_t = -h(t)$$

The homogeneous equation takes the form:

$$\dot{z}_t - b(t)z_t = 0$$

This equation yields the homogeneous solution as (noting that $z_t \neq 0$)

$$z_h = c_1 e^{\int_0^t b(u)du}$$

Returning to the original non-homogeneous equation given above ($h(t) \neq 0$), assume that a particular solution exists and takes the following form:

$$z_p = v(t).e^{\int_0^t b(u)du}$$

where $v(t)$ will need to be determined. Substituting this into the LHS of the non-homogeneous equation yields:

$$\dot{z}_p - b(t)z_p = v'(t)e^{e^{\int_0^t b(u)du}} + v(t)e^{\int_0^t b(u)du}b(t) - b(t)v(t)e^{\int_0^t b(u)du} = v'(t)e^{\int_0^t b(u)du}$$

The function $v(t)$ must be chosen so that:

$$v'(t)e^{\int_0^t b(u)du} = -h(t)$$

or equivalently

$$v'(t) = \frac{-h(t)}{e^{\int_0^t b(u)du}}$$

Upon taking integral we get:

$$v(t) = -\int_0^t \frac{h(x)}{e^{\int_0^x b(u)du}} dx + c_2$$

For simplicity, set $c_2 = 0$ then $z_p = -e^{\int_0^t b(u)du} \int_0^t \frac{h(x)}{e^{\int_0^x b(u)du}} dx = -e^{\int_0^t b(t)dt} \int_0^t h(x) e^{-\int_0^x b(u)du} dx$. The general solution to the non-homegeneous equation will be:

$$z_t = z_h + z_p = c_1 e^{\int_0^t b(u)du} - e^{\int_0^t b(u)du} \int_0^t h(x) e^{-\int_0^x b(u)du} dx$$

Thus,

$$L^*_{mt} = \frac{1}{e^{\int_0^t b(u)du} \left(c_1 - \int_0^t h(x) e^{-\int_0^x b(u)du} dx\right)}$$

where $c_1 > 0$ is a constant. It can be shown that in the special case when $\eta = 0$, we have:

$$L^*_{mt} = \frac{1}{\left(\frac{\varphi(1-\beta)}{\beta} + B\right) . \frac{1}{\rho} + c_m e^{\rho t}}$$

where $c_m = c_1 - \left(\frac{\varphi(1-\beta)}{\beta} + B\right) . \frac{1}{\rho}$ is a non-negative constant.

Proof of Lemma 4
We have:

$$v(x) e^{-\int_0^x v(u)du} = -\frac{d}{dx}\left(e^{-\int_0^x v(u)du}\right)$$

Hence, it follows that:

$$\begin{aligned} \int_0^t v(x) e^{-\int_0^x v(u)du} dx &= -\left[e^{-\int_0^x v(u)du}\right]_{x=0}^{x=t} \\ &= 1 - e^{-\int_0^t v(u)du} \end{aligned}$$

Proof of Lemma 5
Let $c_x = 0$. First, since $L^*_{mt} > 0$, $\forall t$ we must have $c_1 \geq \int_0^\infty h(x) e^{-\int_0^x b(u)du} dx$. Let $\triangle = \frac{1-\beta}{L^*_{xt}} - \frac{\beta}{L^*_{mt}}$, we have $\frac{1-\beta}{L^*_{xt}} = (1-\beta)\frac{\varphi}{\rho}$. Tedious computations lead to:

$$h(t) e^{-\int_0^t b(u)du} = \left[\frac{\varphi(1-\beta)}{\beta} + B + \frac{\eta(1-\beta)\varphi R(t)}{\beta\rho\bar{R}}\right] e^{-\rho t - \int_0^t \frac{\eta R(x)}{\bar{R}} dx}$$

Using Lemma 4 we obtain:

$$\int_0^t h(x) e^{-\int_0^x b(u)du} dx = B \int_0^t e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}\left(1 - e^{-\int_0^t b(u)du}\right)$$

so that

$$\frac{\beta}{L^*_{mt}} =$$

$$\beta c_1 e^{\int_0^t b(u)du} - \beta B e^{\int_0^t b(u)du} \int_0^t e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho} e^{\int_0^t b(u)du} + \frac{\varphi(1-\beta)}{\rho}$$

Therefore

$$\begin{aligned}\triangle &= -\beta c_1 e^{\int_0^t b(u)du} + \beta B e^{\int_0^t b(u)du} \int_0^t e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho} e^{\int_0^t b(u)du} \\ &= -\beta e^{\int_0^t b(u)du} \left(c_1 - B \int_0^t e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho\beta} \right)\end{aligned}$$

Since

$$\begin{aligned}c_1 &\geq \int_0^\infty h(x) e^{-\int_0^x b(u)du} dx \\ &= B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta} \left[1 - e^{-\int_0^t b(u)du} \right]_0^\infty\end{aligned}$$

we get

$$\begin{aligned}&c_1 - B \int_0^t e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho\beta} \\ &\geq B \int_t^\infty e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho\beta} e^{-\int_0^\infty b(u)du} \\ &\geq B \int_t^\infty e^{-\int_0^x b(u)du} dx > 0\end{aligned}$$

since $e^{-\int_0^\infty b(u)du} = 0$. Thus, $\triangle < 0$.

Proof of Lemma 6
We have $\eta(1 - \frac{e^{w_t}}{R}) - \dot{w}_t = BL_{mt}$ and $G(\dot{z}_t, w_t, \dot{w}_t) = L_{xt}$. Hence

$$\begin{aligned}-\frac{\alpha\beta}{\eta(1-\frac{e^{w_t}}{R}) - \dot{w}_t} + \frac{\alpha(1-\beta)}{G(\dot{z}_t, w_t, \dot{w}_t)} \times \frac{1}{B} &= \frac{-\alpha\beta}{BL_{mt}} + \frac{\alpha(1-\beta)}{BL_{xt}} \\ &= \frac{\alpha}{B} \left(\frac{1-\beta}{L_{xt}} - \frac{\beta}{L_{mt}} \right) < 0\end{aligned}$$

where the result obtained in the second line comes from Lemma 5.

Proof of Proposition 2
Part of the proof for this proposition has been done in Lemma 3. Here we only need to show that the solutions to the system of differential equations of Proposition 2 are optimal. We have:

$$\begin{aligned}c_1 &= B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta} \left[1 - e^{-\int_0^\infty b(u)du} \right] \\ &= B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}\end{aligned}$$

since $e^{-\int_0^\infty b(u)du} = 0$. Hence, it follows that

$$
\begin{aligned}
c_1 - \int_0^t h(x) e^{-\int_0^x b(u)du} dx &= c_1 - B\int_0^t e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho\beta} + \frac{\varphi(1-\beta)}{\rho\beta} e^{-\int_0^t b(u)du} \\
&= B\int_t^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta} e^{-\int_0^t b(u)du}
\end{aligned}
$$

Therefore

$$
L_{mt}^* = \frac{1}{Be^{\int_0^t b(u)du}\int_t^\infty e^{-\int_0^x b(u)du}dx + \frac{\varphi(1-\beta)}{\rho\beta}}
$$

Obviously, $L_{mt}^* \leq \frac{\rho\beta}{\varphi(1-\beta)}$. Now observe that

$$
\begin{aligned}
e^{\int_0^t b(u)du}\int_t^\infty e^{-\int_0^x b(u)du}dx &= \int_t^\infty e^{-\int_t^x b(u)du}dx \\
&\leq \int_t^\infty e^{-\rho(x-t)}dx = \frac{1}{\rho}
\end{aligned}
$$

Thus, $L_{mt}^* \geq \frac{1}{\frac{B}{\rho}+\frac{\varphi(1-\beta)}{\rho\beta}} = \frac{\rho\beta}{B\beta+\varphi(1-\beta)}$. Since $L_{xt}^* + L_{mt}^* < 1$, we need to impose that:

$$
\frac{\rho}{\varphi} + \frac{\rho\beta}{\varphi(1-\beta)} < 1
$$

or equivalently

$$
\frac{\rho}{\varphi} < 1-\beta
$$

Again, let $z_t = \log(A_t)$ and $w_t = \log(R_t)$

$$
\mathcal{M}(z,\dot{z},w,\dot{w}) = (1-\alpha)z + \alpha\beta\left[w + \log\left(\eta(1-\frac{e^w}{\bar{R}})\right) - \dot{w}\right] \tag{44}
$$

$$
+ \quad \alpha(1-\beta)\log(1-\frac{\dot{z}}{\lambda(\mu-1)} - \frac{\eta}{B}(1-\frac{e^w}{\bar{R}}) + \frac{\dot{w}}{B}) \tag{45}
$$

The maximization problem is to solve:

$$
\max \quad \int_0^\infty \mathcal{M}(z_t,\dot{z}_t,w_t,\dot{w}_t)e^{-\rho t}dt
$$

Considering interior solutions, we have the following Euler-Lagrange equations:

$$
\frac{d}{dt}\left[\frac{\partial\mathcal{M}}{\partial\dot{z}_t}e^{-\rho t}\right] = \frac{\partial\mathcal{M}}{\partial z_t}e^{-\rho t} \tag{46}
$$

$$
\frac{d}{dt}\left[\frac{\partial\mathcal{M}}{\partial\dot{w}_t}e^{-\rho t}\right] = \frac{\partial\mathcal{M}}{\partial w_t}e^{-\rho t} \tag{47}
$$

Let

$$
G(\dot{z},w,\dot{w}) = \left(1-\frac{\dot{z}}{\lambda(\mu-1)} - \frac{\eta}{B}(1-\frac{e^w}{\bar{R}}) + \frac{\dot{w}}{B}\right)
$$

One can check that (46) can be written as

$$\frac{d}{dt}\left(-\frac{\alpha(1-\beta)}{\lambda(\mu-1)}\times\frac{1}{G(\dot{z},w,\dot{w})}e^{-\rho t}\right)=(1-\alpha)e^{-\rho t} \tag{48}$$

while (47) can be written as:

$$\frac{d}{dt}\left[\left(-\frac{\alpha\beta}{\eta(1-\frac{e^w}{\bar{R}})-\dot{w}}+\frac{\alpha(1-\beta)}{G(\dot{z},w,\dot{w})}\times\frac{1}{B}\right)e^{-\rho t}\right] \tag{49}$$

$$=\left(\alpha\beta-\frac{\alpha\beta}{\eta(1-\frac{e^w}{\bar{R}})-\dot{w}}\times\frac{\eta}{\bar{R}}e^w+\frac{\alpha(1-\beta)}{G(\dot{z},w,\dot{w})}\times\frac{\eta}{\bar{R}B}e^w\right)e^{-\rho t} \tag{50}$$

We will show that

$$\lim_{T\to+\infty}\int_0^\infty\left[\mathcal{M}(z*,\dot{z}*,w*,\dot{w}*)-\mathcal{M}(z,\dot{z},w,\dot{w})\right]e^{-\rho t}dt\geq 0$$

where $z_t=\log(A_t^*)$, $w_t=\log(R_t^*)$, $z_t=\log(A_t)$, $w_t=\log(R_t)$. The variables A_t^*, R_t^* are given in the statement of Proposition (2). The variables A_t, R_t satisfy the dynamic equations (15) and (16).

We have

$$\mathcal{M}(z*,\dot{z}*,w*,\dot{w}*)-\mathcal{M}(z,\dot{z},w,\dot{w})\geq$$

$$(1-\alpha)(z_t^*-z_t)+\alpha\beta(w_t^*-w_t)+\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}\left[-\frac{\eta}{\bar{R}}(e^{w^*}-e^w)-(\dot{w}^*-\dot{w})\right]$$

$$+\frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)}\left[-\frac{1}{\lambda(\mu-1)}(\dot{z}^*-\dot{z})+\frac{\eta}{B\bar{R}}(e^{w^*}-e^w)+\frac{\dot{w}^*-\dot{w}}{B}\right]-\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}(\dot{w}_t^*-\dot{w}_t)$$

$$=(1-\alpha)(z_t^*-z_t)+\alpha\beta(w_t^*-w_t)+\left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}-\frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}\right)\frac{\eta}{\bar{R}}(-e^{w^*}+e^w)$$

$$-\frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)}\times\frac{1}{\lambda(\mu-1)}(\dot{z}^*-\dot{z})+\frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}(\dot{w}^*-\dot{w})-\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}(\dot{w}_t^*-\dot{w}_t)$$

On the one hand, we have:

$$-e^{w^*}+e^w\geq -e^{w^*}(w^*-w)$$

On the other hand, from Lemma 6, we have:

$$\left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}-\frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}\right)>0$$

Hence

$$\left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}-\frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}\right)\times\frac{\eta}{\bar{R}}(-e^{w^*}+e^w)$$

$$> -\left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}\right) \times \frac{\eta}{\bar{R}} e^{w^*}(w^*-w)$$

and

$$\mathcal{M}(z*,\dot{z}*,w*,\dot{w}*) - \mathcal{M}(z,\dot{z},w,\dot{w}) \geq$$

$$\left[(1-\alpha)(z_t^*-z_t) - \frac{\alpha(1-\beta)}{\lambda(\mu-1)}\,\frac{1}{G(\dot{z}^*,w^*,\dot{w}^*)}(\dot{z}^*-\dot{z})\right]$$

$$+\alpha\beta(w_t^*-w_t) - \left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}\right) \times \frac{\eta}{\bar{R}} e^{w^*}(w^*-w) + \frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}(\dot{w}^*-\dot{w})$$

$$-\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}(\dot{w}_t^*-\dot{w}_t)$$

Let

$$\triangle_T = \int_0^T \left[\mathcal{M}(z*,\dot{z}*,w*,\dot{w}*) - \mathcal{M}(z,\dot{z},w,\dot{w})\right] e^{-\rho t} dt \geq I_T + J_T$$

where

$$I_T = \int_0^T \left[(1-\alpha)(z_t^*-z_t) - \frac{\alpha(1-\beta)}{\lambda(\mu-1)}\frac{1}{G(\dot{z}^*,w^*,\dot{w}^*)}(\dot{z}^*-\dot{z})\right] e^{-\rho t} dt$$

$$J_T = \int_0^T \left[\alpha\beta(w_t^*-w_t) - \left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}\right) \times \frac{\eta}{\bar{R}} e^{w^*}(w^*-w)\right] e^{-\rho t} dt$$

$$+\int_0^T \frac{\alpha(1-\beta)}{G(\dot{z}^*,w^*,\dot{w}^*)B}(\dot{w}^*-\dot{w})e^{-\rho t}dt - \int_0^T \frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}(\dot{w}_t^*-\dot{w}_t)e^{-\rho t}dt$$

Computing I_T, we get:

$$I_T = \int_0^T \left[(1-\alpha)e^{-\rho t} + \frac{\alpha(1-\beta)}{\lambda(\mu-1)}\frac{d}{dt}\left(\frac{1}{G(\dot{z}^*,w^*,\dot{w}^*)}e^{-\rho t}\right)\right](z_t^*-z_t)dt$$

$$-\frac{\alpha(1-\beta)}{\lambda(\mu-1)}\left[\frac{1}{G(\dot{z}^*,w^*,\dot{w}^*)}(z_t^*-z_t)e^{-\rho t}\right]_0^T$$

Using (48), we have:

$$I_T = -\frac{\alpha(1-\beta)}{\lambda(\mu-1)}\left[\frac{1}{G(\dot{z}^*,w^*,\dot{w}^*)}(z_t^*-z_t)e^{-\rho t}\right]_0^T = -\frac{\alpha(1-\beta)}{\lambda(\mu-1)}\frac{z_T^*-z_T}{G(\dot{z}_T^*,w_T^*,\dot{w}_T^*)}e^{-\rho T}$$

since $z_0^* = z_0 = \log(A_0)$ and $G(\dot{z}_0^*,w_0^*,\dot{w}_0^*) = L_{x0}^* = \frac{\rho}{\varphi}$. Observe that we have:

$$G(\dot{z}_T^*,w_T^*,\dot{w}_T^*) = L_{xT}^* = \frac{\rho}{\varphi}$$

$$\log(A_0) \leq z_T^* \leq \log(A_0) + \lambda(\mu-1)T$$

and

$$\log(A_0) \leq z_T \leq \log(A_0) + \lambda(\mu - 1)T$$

Therefore, we obtain:

$$\lim_{T\to+\infty} I_T = 0$$

Now consider J_T. We have:

$$J_T = \int_0^T \alpha\beta(w_t^* - w_t)e^{-\rho t}dt + \frac{\alpha(1-\beta)}{B}\left\{\left[\frac{(w_t^* - w_t)e^{-\rho t}}{G(\dot{z}^*, w^*, \dot{w}^*)}\right]_0^T - \int_0^T (w_t^* - w_t)\frac{d}{dt}\left(\frac{e^{-\rho t}}{G(\dot{z}^*, w^*, \dot{w}^*)}\right)dt\right\}$$

$$-\int_0^T \left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}}) - \dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B}\right) \times \frac{\eta}{\bar{R}}e^{w^*}(w^* - w)$$

$$-\left[\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}(w_t^* - w_t)e^{-\rho t}\right]_0^T + \int_0^T \alpha\beta(w_t^* - w_t)\frac{d}{dt}\left(\frac{e^{-\rho t}}{\eta(1-\frac{e^{w^*}}{\bar{R}})-\dot{w}^*}\right)dt$$

Now using (50), and observing that $G(\dot{z}^*, w^*, \dot{w}^*) = L_x^*$, $\eta(1-\frac{e^{w^*}}{\bar{R}}) - \dot{w}^* = BL_m^*$, and $w = w_0$ we get

$$J_T = \left[\frac{\alpha(1-\beta)}{BL_x^*} - \frac{\alpha\beta}{BL_m^*}\right](w_T^* - w_T)e^{-\rho T}$$

We have:

$$\log R_0 - bT \leq \log R_T^* = w_T^* \leq \log \bar{R}$$
$$\log R_0 - bT \leq \log R_T = w_T \leq \log \bar{R}$$

which implies

$$(\log R_0 - bT)e^{-\rho T} \leq w_T^* e^{-\rho T} \leq \log \bar{R}e^{-\rho T}$$
$$(\log R_0 - bT)e^{-\rho T} \leq w_T e^{-\rho T} \leq \log \bar{R}e^{-\rho T}$$

Hence, we can induce that:

$$\lim_{T\to\infty} w_T^* e^{-\rho T} = \lim_{T\to\infty} w_T e^{-\rho T} = 0$$

These imply:

$$\lim_{T\to+\infty} J_T = 0$$

And finally, we obtain:

$$\lim_{T\to+\infty} \triangle_T \geq 0$$

That is the end of the proof.

Proof of Proposition 3

Observe that $\mathcal{N}$ is concave and the constraints (25), (26), (27) are convex. Assume that $(z_t, w_t, L_{mt}, L_{xt}), (z'_t, w'_t, L'_{mt}, L'_{xt})$ are two solutions with $z_0 = z'_0, w_0 = w'_0$. We have

$$\begin{aligned}
\dot{z}_t &= \lambda(\mu-1)(1-L_{mt}-L_{xt}) \\
\dot{z}'_t &= \lambda(\mu-1)(1-L'_{mt}-L'_{xt}) \\
\dot{w}_t &= \eta(1-\frac{e^{w_t}}{\bar{R}}) - BL_{mt} \\
\dot{w}'_t &= \eta(1-\frac{e^{w_t}}{\bar{R}}) - BL_{mt}
\end{aligned}$$

and

$$\nu = \int_0^\infty \mathcal{N}(z_t, w_t, L_{mt}, L_{xt})e^{-\rho t}dt = \int_0^\infty \mathcal{N}(z'_t, w'_t, L'_{mt}, L'_{xt})e^{-\rho t}dt$$

Since $\log(L_{mt}), \log(L_{xt})$ are strictly concave we must have $L_{mt} = L'_{mt}, L_{xt} = L'_{xt}$ and, hence, $\dot{z}_t = \dot{z}'_t, \dot{w}_t = \dot{w}'_t$. Since $z_0 = z'_0, w_0 = w'_0$ we obtain $z_t = z'_t, w_t = w'_t$.

Proof of Proposition 4

We have $\dot{z} = \lambda(\mu-1)L_r \geq 0$. Hence $z(t) \geq z_0 = \log(A_0)$. We also have $0 \leq \dot{z} \leq \lambda(\mu-1)$ which implies $z(t) \leq z_0 + \lambda(\mu-1)$ and hence

$$|z(t)| \leq |z_0| + \lambda(\mu-1)t \tag{51}$$

Thus

$$\int_0^{+\infty} |z(t)|e^{-\rho t}dt \leq \int_0^{+\infty} (|z_0| + \lambda(\mu-1)t)\, e^{-\rho t}dt \equiv \mu_1 \tag{52}$$

and

$$\int_0^{+\infty} |\dot{z}(t)|e^{-\rho t}dt \leq \int_0^{+\infty} \lambda(\mu-1)e^{-\rho t}dt \equiv \mu_2 \tag{53}$$

Since $R_0 < \bar{R}$ we have $-B \leq \dot{w} \leq \eta$. Hence

$$|\dot{w}| \leq B + \eta \tag{54}$$

$$w_0 - Bt \leq w_t \leq w_0 + \eta t$$

This implies

$$\int_0^{\infty} |w(t)|e^{-\rho t}dt \leq \int_0^{\infty} [|w_0| + (\eta+B)t]\, e^{-\rho t}dt \equiv \mu_3 \tag{55}$$

$$\int_0^{\infty} |\dot{w}(t)|e^{-\rho t}dt \leq (B+\eta)\int_0^{\infty} e^{-\rho t}dt \equiv \mu_4 \tag{56}$$

It is obvious that

$$\int_0^\infty L_m e^{-\rho t} dt \le \frac{1}{\rho} \equiv \mu_5, \tag{57}$$

$$\int_0^\infty L_x e^{-\rho t} dt \le \frac{1}{\rho} \equiv \mu_5 \tag{58}$$

The Criterion $\int_0^\infty \mathcal{N}(z, w, L_m, L_x)e^{-\rho t}dt$ is concave in (z, w, L_m, L_x) and is bounded above by a constant, which is equal to $\frac{1}{\rho}\times[(1-\alpha)\mu_1 + \alpha\beta\mu_3 + \alpha\beta|\log(B)|]$, when (z, w, L_m, L_x) satisfy (25), (26), and (27). Denote ν as its supremum.

Now define $(z^n, w^n, L_m^n, L_x^n)_n$ as a sequence such that $\lim_{n\to+\infty}\int_0^\infty \mathcal{N}(z^n, w^n, L_m^n, L_x^n) = \nu$. From (51)-(58) and since $L_m \in [0,1]$, $L_x \in [0,1]$, by using Dunford-Pettis Criterion (Dunford and Schwartz, 1966) one obtains that the sequence $(z^n, w^n, \dot{z}^n, \dot{w}^n, L_m^n, L_x^n)_n$ is in a relatively compact set for the weak topology $\sigma(L^1(e^{-\rho t}dt), L^\infty)$. Hence, it weakly converges to $(z^*, w^*, \phi^*, \psi^*, L_m^*, L_x^*)$. Using the same argument as in d'Albis et al (2008), we obtain that $\phi^* = \dot{z}^*, \psi^* = \dot{w}^*$. Since weak convergence implies pointwise convergence we have:

$$\begin{aligned} \dot{z}^* &= \lambda(\mu-1)(1 - l_m^* - L_x^*) \\ \dot{w}^* &= \eta(1 - \frac{e^{w^*}}{\bar{R}}) - BL_m^* \end{aligned}$$

That means (z^*, w^*, L_m^*, L_x^*) is feasible.

One can easily check that $\int_0^\infty \mathcal{N}(z, w, L_m, L_n)e^{-\rho t}dt$ is continuous with respect to (z, w, L_m, L_x) for the strong topology $L^1(e^{-\rho t}dt)$. Since this function is concave, it is upper semi-continuous for $\sigma(L^1(e^{-\rho t}dt), L^\infty)$. Thus, $\nu \le \int_0^\infty \mathcal{N}(z^*, w^*, L_m^*, L_n^*)e^{-\rho t}dt$. This actually proves $\nu = \int_0^\infty \mathcal{N}(z^*, w^*, L_m^*, L_n^*)e^{-\rho t}dt$ and (z^*, w^*, L_m^*, L_n^*) is the optimal solution.

Proof of Proposition 5

In the optimal steady state, we have $A_t = A_0 e^{tg_A}$ where A_0 is the initial level of technology and g_A is the constant rate of growth of technology. Given that the rate of growth of resources $g_R = \eta\left(1 - \frac{\hat{R}_t}{\bar{R}}\right) - B\hat{L}_m$ is constant on the steady state then $\hat{R} = \bar{R}\left(1 - \frac{B\hat{L}_m}{\eta}\right)$ is also constant. Note that the condition $0 \le \hat{L}_m \le \frac{\eta}{B}$ must hold for $R \ge 0$.

The utility in the steady state is:

$$\begin{aligned} U &= (1-\alpha)\int_0^\infty [\log(A_0) + tg_A]e^{-\rho t}dt + \alpha\beta\int_0^\infty \log(B)e^{-\rho t}dt \\ &+ \alpha\beta\int_0^\infty \log(\hat{L}_m)e^{-\rho t}dt + \alpha\beta\int_0^\infty \log(\hat{R})e^{-\rho t}dt \\ &+ \alpha(1-\beta)\int_0^\infty \log(\hat{L}_x)e^{-\rho t}dt \end{aligned}$$

Using $\int_0^\infty te^{-\rho t}dt = \frac{1}{\rho^2}$, $\int_0^\infty e^{-\rho t}dt = \frac{1}{\rho}$ and noting $g_A = \lambda(\mu-1)\hat{L}_r = \lambda(\mu-1)(1-\hat{L}_x-\hat{L}_m)$ the above utility function becomes:

$$\begin{aligned}\rho U &= (1-\alpha)\log(A_0) + \alpha\beta\log(B) + \alpha\beta\log(\hat{L}_m) \\ &+ \alpha\beta\log(\bar{R}) + \alpha\beta\log(1-\frac{B\hat{L}_m}{\eta}) + \alpha(1-\beta)\log(\hat{L}_x) \\ &+ \frac{(1-\alpha)\lambda(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho}\end{aligned}$$

We now maximize U with respect to $\hat{L}_x$ and $\hat{L}_m$. The first order conditions with respect to these choice variables give:

$$\frac{B}{\eta - B\hat{L}_m} = \frac{1}{\hat{L}_m} - \frac{(1-\alpha)\lambda(\mu-1)}{\alpha\beta\rho} \tag{59}$$

and

$$\hat{L}_x = \frac{\alpha\rho(1-\beta)}{(1-\alpha)\lambda(\mu-1)} = \frac{\rho}{\varphi} \tag{60}$$

The left hand side (LHS) of equation (59) is increasing in $\hat{L}_m$, equal $\frac{B}{\eta}$ when $\hat{L}_m = 0$ and approaching $+\infty$ when $\hat{L}_m \to \frac{\eta}{B}$. The right hand side (RHS) is, in contrast, decreasing in $\hat{L}_m$, approaching $+\infty$ when $\hat{L}_m \to 0$ and equal $\frac{B}{\eta} - \frac{(1-\alpha)\lambda(\mu-1)}{\alpha\beta\rho}$ when $\hat{L}_m = \frac{\eta}{B}$. Hence, this equation has a unique solution $\hat{L}_m \in (0, \frac{\eta}{B})$.

Actually, equation (59) can be solved explicitly to get the expression of $\hat{L}_m$. Indeed, rearranging (59) gives a quadratic equation of $\hat{L}_m$:

$$B\lambda(\mu-1)(1-\alpha)\hat{L}_m^2 - [2B\alpha\beta\rho + \lambda(\mu-1)(1-\alpha)\eta]\,\hat{L}_m + \alpha\beta\rho\eta = 0$$

Noting that $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$, this equation can be rewritten as:

$$B(1-\beta)\varphi\hat{L}_m^2 - [2B\beta\rho + (1-\beta)\varphi\eta]\,\hat{L}_m + \beta\rho\eta = 0$$

This equation has two distinct real roots but one of them has to be ruled out due to violating $\hat{L}_m < \frac{\eta}{B}$.[7] The accepted root is $\hat{L}_m = \frac{2B\beta\rho+(1-\beta)\varphi\eta-\sqrt{\triangle}}{2B(1-\beta)\varphi}$ where $\triangle = 4B^2\alpha^2\beta^2\rho^2 + (1-\beta)^2\varphi^2\eta^2$.

Having obtained $\hat{L}_x$ and $\hat{L}_m$, we can calculate $\hat{L}_r = 1-\hat{L}_x-\hat{L}_m = 1 - \frac{2B\rho+(1-\beta)\varphi\eta-\sqrt{\triangle}}{2B(1-\beta)\varphi}$. Since $\hat{L}_x \geq 0$, $\hat{L}_m \geq 0$, the condition $\hat{L}_r \geq 0$ or $\frac{2B\rho+(1-\beta)\varphi\eta-\sqrt{\triangle}}{2B(1-\beta)\varphi} \leq 1$ is sufficient for L_x, L_m, $L_r \leq 1$. Using these results, the growth rates of technology, natural resources, output, and consumption are calculated as follows:

$$g_A = \lambda(\mu-1)\hat{L}_r$$

[7] The ruled out root is $L_m = \frac{2B\alpha\beta\rho+\lambda(\mu-1)(1-\alpha)\eta+\sqrt{\triangle}}{2B\lambda(\mu-1)(1-\alpha)}$.

$$g_R = 0$$

$$g_Y = g_C = (1-\alpha)g_A = (1-\alpha)\lambda(\mu-1)\hat{L}_r$$

Proof of Proposition 6
We consider dynamic equations (16) and (18). We can write them as follows

$$\begin{aligned}\frac{\dot{R}}{R_t} &= G(L_{mt}, R_t)\\ \frac{\dot{L}_{mt}}{L_{mt}} &= H(L_{mt}, R_t)\end{aligned}$$

where

$$G(L_{mt}, R_t) = \eta(1 - \tfrac{R_t}{\bar{R}}) - BL_{mt}$$

$$H(L_{mt}, R_t) = \tfrac{\varphi\alpha(1-\beta)+B\alpha\beta}{\alpha\beta}L_{mt} + \tfrac{\alpha(1-\beta)\eta}{\alpha\beta L_{xt}\bar{R}}L_{mt}R_t - \rho - \tfrac{\eta R_t}{\bar{R}}$$

The steady state $(\hat{L}_m, \hat{R})$ is solutions to the system:

$$G(L_{mt}, R_t) = 0,\ H(L_{mt}, R_t) = 0$$

However, we can introduce the variables $w_t = \log(R_t)$, $v_t = \log(L_{mt})$. To simplify notations, we drop the time subscript t unless there is a confusion. The dynamic system becomes:

$$\dot{w} = \eta(1 - \tfrac{e^w}{\bar{R}}) - Be^v$$

$$\dot{v} = \left[\tfrac{\varphi(1-\beta)+B\beta}{\beta}\right]e^v + \tfrac{(1-\beta)\eta}{\beta L_x \bar{R}}e^v e^w - \rho - \tfrac{\eta}{\bar{R}}e^w$$

The steady state satisfies:

$$\eta(1 - \tfrac{e^w}{\bar{R}}) - Be^v = 0$$

$$\left[\tfrac{\varphi(1-\beta)+B\beta}{\beta}\right]e^v + \tfrac{(1-\beta)\eta}{\beta L_x \bar{R}}e^v e^w - \rho - \tfrac{\eta}{\bar{R}}e^w = 0$$

The Jacobian matrix of the system is:

$$J = \begin{bmatrix} -\frac{\eta e^w}{\bar{R}} & -Be^v \\ \frac{(1-\beta)\eta}{\beta L_x \bar{R}}e^v e^w - \frac{\eta e^w}{\bar{R}} & \left[\frac{\varphi(1-\beta)+B\beta}{\beta}\right]e^v + \frac{(1-\beta)\eta}{\beta L_x \bar{R}}e^v e^w \end{bmatrix}$$

Let λ_1 and λ_2 be eigen values of vector J then they will be solutions to the following equation:

$$\lambda^2 - tr(J)\lambda + \det(J) = 0$$

It can be seen that λ_1 and λ_2 will satisfy:

$$\lambda_1 + \lambda_2 = -tr(J)$$

$$\lambda_1 \lambda_2 = \det(J)$$

Evaluating at the steady state value, we obtain:

$$tr(J) = \rho$$

$$\det(J) = \frac{\eta e^w}{\bar{R}} \left[Be^v \left(\frac{(1-\beta)L_m^*}{\beta L_x^*} - 1 \right) - \rho - \frac{\eta e^w}{\bar{R}} \right]$$

Because $\left(\frac{(1-\beta)L_m^*}{\beta L_x^*} - 1 \right) < 0$ according to Lemma 5, $\det(J) < 0$. This means that λ_1 and λ_2 take real value with $\lambda_1 > 0$ and $\lambda_2 < 0$. The steady state is a saddle point. Since the problem is not concave, we will prove that the stable manifold is actually optimal.

Now take a feasible path on the stable manifold and close to the steady state. Denoting it by (z^*, w^*), we have:

$$L_{mt}^* = \frac{1}{e^{\int_0^t b(u)du} \left(c_1 - \int_0^t h(x) e^{-\int_0^x b(u)du} dx \right)}$$

in which $c_1 \geq \int_0^\infty h(x) e^{-\int_0^x b(u)du} dx$ and

$$\begin{aligned} h(x) &= \frac{(1-\alpha)\lambda(\mu-1)}{\beta\alpha} + B + \frac{\eta(1-\beta)}{\beta L_{xt}^*} \times \frac{R_x^*}{\bar{R}} \\ &= \frac{(1-\alpha)\lambda(\mu-1)}{\beta\alpha} + B + \frac{\eta(1-\beta)\varphi}{\beta\rho} \times \frac{R_x^*}{\bar{R}} \end{aligned}$$

We claim that $c_1 = \int_0^\infty h(x) e^{-\int_0^x b(u)du} dx$. Indeed, we have

$$\int_0^\infty h(x) e^{-\int_0^x b(u)du} dx = B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}$$

Assume $c_1 = a + B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}$ with $a > 0$ then

$$\begin{aligned} e^{\int_0^t b(u)du} \left(c_1 - \int_0^t h(x) e^{-\int_0^x b(u)du} dx \right) &= a e^{\int_0^t b(u)du} + B e^{\int_0^t b(u)du} \int_t^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta} \\ &\geq a e^{\int_0^t b(u)du} dx \to +\infty \text{ when } t \to +\infty \end{aligned}$$

Therefore $L_{mt}^* \to 0$ when $t \to +\infty$. Because we are on the stable manifold at which L_{mt}^* converges to a strictly positive value, we have a contradiction. Thus $c_1 = \int_0^\infty h(x) e^{-\int_0^x b(u)du} dx$ and

$$L_{mt}^* = \frac{1}{B e^{\int_0^t b(u)du} \int_t^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}}$$

We compute again

$$\triangle_T = \int_0^T \left[\mathcal{M}(z*, \dot{z}*, w*, \dot{w}*) - \mathcal{M}(z, \dot{z}, w, \dot{w})\right] e^{-\rho t} dt$$

where (z, w) is a feasible path. The technique in the proof of Proposition 2 can be used again to obtain:

$$\triangle_T \geq I_T + J_T$$

where

$$I_T = -\frac{\alpha(1-\beta)}{\lambda(\mu-1)} \left[\frac{1}{G(\dot{z}^*, w^*, \dot{w}^*)}(z_t^* - z_t)e^{-\rho t}\right]_0^T = -\frac{\alpha(1-\beta)}{\lambda(\mu-1)} \frac{z_T^* - z_T}{L_{xt}^*} e^{-\rho T}$$

and

$$J_T = \left[\frac{\alpha(1-\beta)}{BL_x^*} - \frac{\alpha\beta}{BL_m^*}\right](w_T^* - w_T)e^{-\rho T}$$

We still have:

$$z_0^* = z_0 = \log(A_0)$$

$$L_{x0}^* = \tfrac{\rho}{\varphi}$$

and

$$\lim_{t\to\infty} I_T = 0$$

We also have:

$$\log R_0 - bT \leq \log R_T^* = w_T^* \leq \log \bar{R}$$
$$\log R_0 - bT \leq \log R_T = w_T \leq \log \bar{R}$$

which implies

$$(\log R_0 - bT)e^{-\rho T} \leq w_T^* e^{-\rho T} \leq \log \bar{R} e^{-\rho T}$$
$$(\log R_0 - bT)e^{-\rho T} \leq w_T e^{-\rho T} \leq \log \bar{R} e^{-\rho T}$$

Because L_{mt}^* converges to a steady state which is strictly positive, we get:

$$\lim_{T\to+\infty} J_T = 0$$

This means that the optimal path locally converges to the steady state.

Proof of Proposition 8

Consider an equilibrium of the decentralized economy. Since $p_{mt}^* = \frac{\alpha^2\beta Y_t^*}{M_t^*}$ and $L_{xt}^* = \frac{\alpha^2(1-\beta)Y_t^*}{w_t^*}$, on the one hand, we have:

$$\frac{w_t^*}{p_{mt}^*} = \frac{1-\beta}{\beta} \times \frac{M_t^*}{L_{xt}^*} = \frac{1-\beta}{\beta} \times \frac{BL_{mt}^* R_t^*}{L_{xt}^*}$$

and on the other hand from Lemma 7, we have:

$$\frac{w_t^*}{p_{mt}^*} = BR_t^*$$

These imply:

$$L^*_{mt} = \frac{\beta}{1-\beta} L^*_{xt}$$

Now we check if this equilibrium satisfies the social planner's maximization problem or not. In doing so, assume that this equilibrium solves the social planner's maximization problem. In that case, we should have $L^*_{xt} = \frac{\rho}{\varphi}$ and $L^*_{mt} = \frac{\beta}{1-\beta} L^*_{xt} = \frac{\beta}{1-\beta}\frac{\rho}{\varphi}$.

Using relation (18) in Lemma 2, we get:

$$\rho\alpha\beta + \frac{\alpha\beta\eta R^*_t}{\bar{R}} - [\varphi\alpha(1-\beta) + B\alpha\beta]\frac{\beta}{1-\beta}\frac{\rho}{\varphi} = \frac{\alpha(1-\beta)\eta}{L^*_{xt}\bar{R}} L^*_{mt} R^*_t$$

or

$$\rho\alpha\beta + \frac{\alpha\beta\eta R^*_t}{\bar{R}} - [\varphi\alpha(1-\beta) + B\alpha\beta]\frac{\beta}{1-\beta}\frac{\rho}{\varphi} = \alpha(1-\beta)\eta\frac{\beta}{1-\beta}\frac{R^*_t}{\bar{R}_t}$$

This implies $\frac{B\alpha\beta^2\rho}{(1-\beta)\varphi} = 0$ which is a contradiction.

Proof of Proposition 9

We first consider the case of non-renewable resources where $\eta = 0$. Denote L^*_x, L^*_m, L^*_r as allocations of labour across sectors at decentralized equilibrium steady state. Because these labour allocations are constant in steady state, we drop the time subscript for simplicity.

From (34) we have:

$$R_t = R_0 e^{-BL^*_m t} \tag{61}$$

and

$$A_t = A_0 e^{\lambda(\mu-1)L^*_r t} \tag{62}$$

By successive substitution, we have:

$$\begin{aligned}
L^*_x &= \frac{\alpha^2(1-\beta)A_t x_t^{\alpha-1}}{w_t}\text{from(31)and(28)} \\
&= \frac{\alpha^2(1-\beta)A_t}{w_t}\left[\frac{M_t^{\beta}L_x^{*1-\beta}}{A_t}\right]^{\alpha-1}\text{ from(29)} \\
&= \frac{\alpha^2(1-\beta)A_t^{2-\alpha}}{w_t} M_t^{\beta(\alpha-1)} L^{*(\alpha-1)(1-\beta)} \\
&= \frac{\alpha^2(1-\beta)A_t^{2-\alpha}}{p_{mt}BR_t} M_t^{\beta(\alpha-1)} L_x^{*(\alpha-1)(1-\beta)}\text{from(33)} \\
&= \frac{\alpha^2(1-\beta)A_t^{2-\alpha}M_t^{\beta\alpha-\beta}L_x^{*(\alpha-1)(1-\beta)}}{\alpha^2\beta A_t^{*1-\alpha}M_t^{*\alpha\beta-1}L_x^{*\alpha(1-\beta)}BR_t}\text{from(32)} \\
&= \frac{(1-\beta)A_t M_t^{1-\beta}L_x^{*-(1-\beta)}}{\beta BR_t}
\end{aligned}$$

Hence

$$\begin{aligned} L_x^{*2-\beta} &= \frac{(1-\beta)A_t B^{1-\beta} R_t^{1-\beta} L_m^{*1-\beta}}{\beta B R_t} \text{from(30)} \\ &= \frac{1}{\beta}(1-\beta)A_t B^{-\beta} R_t^{-\beta} L_m^{*1-\beta} \end{aligned}$$

Now using (61) and (62) then this becomes:

$$L_x^{*2-\beta} = \frac{1}{\beta}(1-\beta)A_0 e^{\lambda(\mu-1)L_r^* t} B^{-\beta} R_0^{-\beta} e^{-\beta B L_m^* t} L_m^{*1-\beta} \tag{63}$$

Since L_x^* is constant, we must have:

$$\lambda(\mu-1)L_r^* = \beta B L_m^* \tag{64}$$

Relation (63) becomes:

$$L_x^{*2-\beta} = \frac{1}{\beta}(1-\beta)A_0 B^{-\beta} R_0^{-\beta} L_m^{*1-\beta} \tag{65}$$

Combining (64), (65) and (36) we get an equation for L_r^*:

$$\frac{\lambda(\mu-1)}{\beta B} L_r^* + \left[\frac{1}{\beta}(1-\beta)A_0 B^{-\beta} R_0^{-\beta}\right]^{\frac{1}{2-\beta}} \left[\frac{\lambda(\mu-1)}{\beta B}\right]^{\frac{1-\beta}{2-\beta}} L_r^{*\frac{1-\beta}{2-\beta}} + L_r^* = 1 \tag{66}$$

The LHS is strictly increasing in L_r^*, equals to 0 if $L_r^* = 0$, and greater than 1 if $L_r^* = 1$. Hence, there exists a unique solution $L_r^* \in (0,1)$.

Proposition 1 gives the values of labors on the steady state of the optimal path. In particular, we have:

$$\hat{L}_r = 1 - \left(\frac{\rho}{\varphi} + \frac{\beta\rho}{\varphi(1-\beta) + B\beta}\right)$$

Let $\tilde{\rho}$ satisfy $0 = 1 - \left(\frac{\tilde{\rho}}{\varphi} + \frac{\beta\tilde{\rho}}{\varphi(1-\beta)+B\beta}\right)$ or $\widetilde{\rho} = \frac{\varphi[\varphi(1-\beta)+B\beta]}{\varphi+B\beta}$. Then $1 - \left(\frac{\rho}{\varphi} + \frac{\beta\rho}{\varphi(1-\beta)+B\beta}\right) > 0 \Leftrightarrow \rho < \tilde{\rho}$. If ρ is close to 0 then $\hat{L}_r$ is close to 1 and tends to be larger than L_r^*. In this case, an under-investment in R&D will happen in the decentralized economy in the long-run and technology will grow below the socially desirable rate. By contrast, if ρ is close to $\tilde{\rho}$ then $\hat{L}_r$ is close to 0 and tends to be smaller than L_r^*. This implies an over-investment in R&D of the decentralized economy in the long-run and technology grows faster than what the society desires to have.

For the case of renewable resources, everything stays the same except now $\eta > 0$ and $\dot{R}_t = \eta R_t \left(1 - \frac{R_t}{\bar{R}}\right) - B L_{mt} R_t$. In steady state, given that L_m^* is constant and the growth rate of R_t^* is also constant, we can deduce that $R^* = \left(1 - \frac{B L_m^*}{\eta}\right)\bar{R}$. Using (30), we get:

$$L_x^{*2-\beta} = \frac{1}{\beta}(1-\beta)A_t B^{-\beta} R^{*-\beta} L_m^{*1-\beta}$$

Note that this implies A_t constant in steady state or $L_r^* = 0$.

Recall that under the socially optimal steady state (presented in Proposition 5), we have:

$$\hat{L}_r = 1 - \frac{\rho}{\varphi} - \frac{2B\beta\rho + (1-\beta)\varphi\eta - \sqrt{\triangle}}{2B(1-\beta)\varphi} > 0$$

This means that the decentralized steady state entails an under-investment in R&D in the long-run in the case of renewable resources.

Proof of Proposition 10

(i) When resources are renewable:

When λ (or μ) increases, the RHS of (59) decreases while its LHS does not change. The graph of the RHS shifts down implying that $\hat{L}_m$ decreases. It is obvious that in this case $\hat{L}_x$ decreases. Hence, $\hat{L}_r$ increases since $\hat{L}_r = 1 - \hat{L}_x - \hat{L}_m$. As a result, g_Y increases.

When ρ increases, $\hat{L}_x$ increases as per (60). The graph of the RHS of (59) shifts up while the LHS does not change implying an increase in $\hat{L}_m$. Therefore, $\hat{L}_r$ decreases and, thus, g_Y decreases as well.

As for the welfare effects, we have:

$$\begin{aligned}
\frac{\partial U}{\partial \lambda} &= \frac{1}{\rho}.\frac{\partial \hat{L}_m}{\partial \lambda}\left[\frac{\alpha\beta}{\hat{L}_m} - \frac{B\alpha\beta}{\eta - B\hat{L}_m} - \frac{(1-\alpha)\lambda(\mu-1)}{\rho}\right] \\
&+ \frac{1}{\rho}.\frac{\partial \hat{L}_x}{\partial \lambda}\left[\frac{\alpha(1-\beta)}{\hat{L}_x} - \frac{(1-\alpha)\lambda(\mu-1)}{\rho}\right] \\
&+ \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^2}
\end{aligned}$$

Observe that the first two terms are equal to zero along the optimal steady state as per (59) and (60). Hence, $\frac{\partial U}{\partial \lambda} = \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^2} > 0$. Similarly, we have $\frac{\partial U}{\partial \mu} = \frac{(1-\alpha)\lambda(1-\hat{L}_x-\hat{L}_m)}{\rho^2} > 0$ and $\frac{\partial U}{\partial \rho} = -\frac{1}{\rho}U - \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^3} < 0$.

(ii) When resources are non-renewable:

It is obvious from (37) and (38) that when λ (or μ) increases, $\hat{L}_x$ and $\hat{L}_m$ both decrease meaning $\hat{L}_r$ increases and g_Y increases (because $g_Y = (1-\alpha)\lambda(\mu-1)\hat{L}_r - \alpha\beta B\hat{L}_m$). By contrast, when ρ increases, $\hat{L}_x$ and $\hat{L}_m$ both increase implying $\hat{L}_r$ decreases and g_Y decreases.

Regarding the welfare effect, we have:

$$\begin{aligned}\frac{\partial U}{\partial \lambda} &= \frac{1}{\rho}.\frac{\partial \hat{L}_m}{\partial \lambda}\left[\frac{\alpha\beta}{\hat{L}_m} - \frac{B\alpha\beta}{\rho} - \frac{(1-\alpha)\lambda(\mu-1)}{\rho}\right] \\ &+ \frac{1}{\rho}.\frac{\partial \hat{L}_x}{\partial \lambda}\left[\frac{\alpha(1-\beta)}{\hat{L}_x} - \frac{(1-\alpha)\lambda(\mu-1)}{\rho}\right] \\ &+ \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^2}\end{aligned}$$

With a note that the first two terms are zero according to (37) and (38) then $\frac{\partial U}{\partial \lambda} = \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^2} > 0$. A similar result applies for μ as $\frac{\partial U}{\partial \mu} = \frac{(1-\alpha)\lambda(1-\hat{L}_x-\hat{L}_m)}{\rho^2} > 0$. However, $\frac{\partial U}{\partial \rho} = -\frac{1}{\rho}U - \frac{\alpha\beta B\hat{L}_m}{\rho^3} - \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^3} < 0$.

Proof of Proposition 11
(i) When resources are renewable:
From (59) and (60), it can be seen that an increase in B does not affect $\hat{L}_x$. However, it reduces $\hat{L}_m$ as the graph of the LHS of (59), which is increasing in $\hat{L}_m$, shifts up while the RHS of that equation, which is decreasing in $\hat{L}_m$, stays the same. Hence, $\hat{L}_r$ rises and so does g_Y.

With respect to welfare, using (59), we have $\frac{\partial U}{\partial B} = \frac{\alpha\beta\hat{L}_m}{\rho B}\left[\frac{1}{\hat{L}_m} - \frac{B}{\eta - B\hat{L}_m}\right] = \frac{(1-\alpha)\lambda(\mu-1)\hat{L}_m}{\rho^2 B} > 0$ meaning welfare rises with B.

(ii) When resources are non-renewable:
Plugging the values of $\hat{L}_m$ and $\hat{L}_r$ from (38) and (39) into the equation for the growth rate of output along the BGP we obtain:

$$g_Y = \lambda(\mu-1)(1-\alpha) - \alpha\rho$$

It can be seen that B does not appear in the result for g_Y. Hence, an increase in B does not have any impact on long-run output growth.

As for the welfare, we have $\frac{\partial U}{\partial B} = \frac{\alpha\beta}{\rho}\left[\frac{1}{B} - \frac{\hat{L}_m}{\rho}\right]$. From (38), it can be figured out that $\hat{L}_m < \frac{\rho}{B}$. Therefore, $\frac{\partial U}{\partial B} > 0$ or U is increasing in B.

Proof of Proposition 12
Under renewable resources, output growth is:

$$g_Y = \alpha(1-\beta)\varphi - \alpha\rho + \frac{\alpha\sqrt{\triangle} - \alpha(1-\beta)\varphi\eta}{2B}$$

where $\triangle = 4B^2\alpha^2\beta^2\rho^2 + (1-\beta)^2\varphi^2\eta^2$. Under non-renewable resources, output growth is:

$$g_Y = \alpha(1-\beta)\varphi - \alpha\rho$$

Clearly, $\sqrt{\triangle} - (1-\beta)\varphi\eta \geq 0$ implying a generally higher output growth for renewable resource case. The two rates are equal only when $\sqrt{\triangle} - (1-\beta)\varphi\eta = 0$ or $\beta = 0$ meaning there is absolutely no utilization of natural resources in intermediate good production.

Obviously, along the optimal steady state for renewable resources, output growth is non-negative. Along the optimal steady state for non-renewable resources, output growth may be negative if:

$$(1-\beta)\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha} < \rho$$

However, this condition violates what stated in Lemma 5 so negative growth will not occur. Hence, we always have $\lambda(\mu-1) > \frac{\alpha\rho}{1-\alpha}$ implying that as soon as the R&D sector is sufficiently productive, positive growth will be sustained no matter what type of resources is employed for production.

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