A selection of higher-order parabolic curvature flows

Scott Parkins
University of Wollongong
A Selection of Higher-Order Parabolic Curvature Flows

Scott Parkins

Supervisors:
Assoc. Prof. J. McCoy, Dr. G. Wheeler & Prof. G. Williams

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Declaration

I, Scott Parkins, declare that this thesis, submitted in partial fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged below. The results in Part I of this thesis can be viewed as a higher-order generalisation of the preprint “The geometric triharmonic heat flow of immersed surfaces near spheres” [75], which involves a sixth-order geometric flow. The work in that paper was done under the supervision of Assoc. Prof. James McCoy and Dr. Glen Wheeler, with each of the two supervisors contributing approximately 1/3. The generalisation of the results in the aforementioned paper to obtain the results in Part I were completed with significantly less input from my supervisors. The results in Part II of this thesis are an anisotropic generalisation of the publication entitled “The polyharmonic heat flow of closed plane curves” [81], which was authored by Dr. Glen Wheeler and myself, with Glen contributing approximately 20%. Again, the generalisation of that work to form Part II of this thesis was done with significantly less input from my supervisors. The document has not been submitted for qualifications at any other academic institution.

Scott Parkins
April 19, 2017
This thesis is dedicated to my family.
Abstract

This thesis is divided into two parts.

In Part I we consider closed immersed surfaces in $\mathbb{R}^3$ evolving by the geometric polyharmonic heat flow. Using local energy estimates, we prove interior estimates and a positive absolute lower bound on the lifespan of solutions depending solely on the local concentration of curvature of the initial immersion in $L^2$. We further use an $\varepsilon$-regularity type result to prove a gap lemma for stationary solutions. Using a monotonicity argument, we then prove that a blowup of the flow approaching a singular time is asymptotic to a non-umbilic embedded stationary surface. This allows us to conclude that any solution with initial $L^2$-norm of the trace-free curvature tensor smaller than an absolute positive constant converges exponentially fast to a round sphere with radius equal to $\sqrt[3]{3V_0/4\pi}$, where $V_0$ denotes the signed enclosed volume of the initial data.

In Part II we study the anisotropic polyharmonic heat flow (a flow of arbitrarily high even order) for closed curves immersed in the Minkowski plane $\mathcal{M}^2$, which is equivalent to the Euclidean plane endowed with a closed, symmetric, convex curve called an indicatrix that endows an anisotropic distance metric on vectors in $\mathcal{M}$. The indicatrix $\partial \mathcal{U}$ (where $\mathcal{U} \subset \mathbb{R}^2$ is a convex, centrally symmetric domain) induces a second convex body, the isoperimetrix $\tilde{I}$. This set is the unique convex set that minimises the isoperimetric ratio (modulo homothetic rescaling) in the Minkowski plane. We prove that under the anisotropic polyharmonic heat flow, closed curves that are initially close to a homothetic rescaling of the isoperimetrix in an averaged $L^2$ sense exists for all time and converge exponentially fast to a homothetic rescaling of the isoperimetrix that has the same enclosed area as the initial immersion.
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Part I

On the geometric polyharmonic heat flow of immersed surfaces near spheres
Chapter 1

Introduction

Mathematics has long been interested in interpreting and analysing natural phenomena in a rigorous manner. Of particular interest is predictive modelling: one would like to be able to predict how a mathematical body or object reacts to a physical process over time.

In his 1807 masterpiece, entitled *Mémoire sur la propagation de la chaleur dans les corps solides*, Joseph Fourier presented a second order parabolic differential equation which modelled the propagation of heat through a region in space. This is perhaps one of the first instances of a purely mathematical approach to heat modelling (prior fruitful attempts in this field were of a more experimental or observational nature: see, for example, Crawford [22]). His equation (aptly titled the *heat equation*), as a function of $n$ spatial variables $x_1, \ldots, x_n$ contained in an open subset $\Omega \subseteq \mathbb{R}^n$, and one time variable $t$ takes the form

$$\frac{\partial u}{\partial t} = -\Delta u = -\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \quad (H)$$

Here $u(x, 0) = \phi(x)$ is usually taken to be a sufficiently smooth function, although (quite amazingly) in the case that the initial data profile is non-smooth, it can be observed to immediately become smooth over time under the flow. One can think of $u$ as the heat profile of a body with spatial components $(x_1, \ldots, x_n)$ at time $t$. Intuitively, then, one would expect that as time gets large, the heat profile $u$ would tend to a
constant solution over our region (we will not concern ourselves at this present time with the behaviour of \( u \) near the boundary of our domain). Indeed, a particular solution \( u \) to the heat equation (H) with the prescribed initial condition \( u(x,0) = u_0(x) \) takes the form

\[
u(x,t) = \int_{\Omega} G(x - y, t) u_0(y) \, dy,
\]

where \( G \) is the fundamental solution (heat kernel)

\[
G(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.
\]

The solution \( u \) is easily observed to exhibit these aforementioned uniformisation (smoothing) properties.

Furthermore, solutions of the heat equation (H) act as flows of steepest descent in \( L^2 \) for the Dirichlet energy \( E \) over \( \Omega \):

\[
E[u] := \int_{\Omega} |Du|^2 \, dV.
\]

Here \( Du : \Omega \to \mathbb{R}^n \) denotes the ordinary (Euclidean) gradient vector field of the function \( u \). The Dirichlet energy was proposed as a solution to the following problem: find the ‘flattest’ surface \( u \) that agrees with a given function \( g \) on \( \partial\Omega \), and so it is perhaps unsurprising that the path of steepest descent for this functional should present itself so fervently within the realm of heat diffusion.

Since the days of Fourier, mathematicians have generalised the heat equation to many non-Euclidean settings: the most relevant to this thesis is the Riemannian manifold, where the solution is not longer a scalar function but a family of evolving submanifolds. Furthermore, there has been a great deal of concern over more general parabolic differential equations in these settings (the so-called ‘heat-like’ equations). Perhaps the most natural of these geometric flows is the mean curvature flow. Here we consider a one-parameter family of immersions \( f : \Sigma^n \times [0,T) \to \mathbb{R}^{n+1} \) where \( \Sigma \) is an \( n \)–dimensional Riemannian manifold and the immersion \( f \) evolves in time with
normal velocity equal to the mean curvature vector:

\[ \partial_t f = \vec{H} = -H \nu \quad \text{(MCF)} \]

(In fact the flow is defined in a more general Riemannian setting, rather than just for a smooth surface immersed in \((n + 1)\)-dimensional Euclidean space. See, for example, Huisken [48]). As an analogy to the aforementioned Dirichlet energy, the flow acts as the path of steepest descent for the area functional, and was originally proposed (with \(n = 2\)) in 1956 by Mullins as a model for the formation of grain boundaries in annealing pure metal [78]. The flow exhibits monotonically decreasing surface area for any compact initial data, and monotonically decreasing signed enclosed volume if the manifold is weakly convex (in all dimensions). In the case \(n = 2\) it has also been observed that stationary solutions of the mean curvature flow (these are called minimal surfaces in the mathematical literature, and locally extremise area) are known to model soap films. By noting that in Euclidean space the mean curvature vector \(\vec{H}\) is equal to the Laplace-Beltrami operator applied to the immersion function \(f\) (that is to say, \(\vec{H} = \Delta f\); see (C.9)), it is simple to draw parallels between the heat equation (H) and the mean curvature flow (MCF).

The mean curvature flow has attracted a huge amount of interest from within the mathematical community. Most notably, in 1984, Huisken [47] showed that any smoothly embedded closed, uniformly convex \(n\)-dimensional hypersurface \(\Sigma_0\) embedded in \(\mathbb{R}^{n+1}\) evolving under (MCF) would shrink down to a point in finite time. To see the shape of the singularity, the flow was normalised in both space and time to ‘zoom-in’ around the singularity, and Huisken was able to prove that no singularities would occur before the family of hypersurfaces contract to a round point. If \(\Sigma_0\) is not convex then this result does not hold, even if it is embedded (for example one could imagine two large approximately-round spheres connected by a very thin neck to form a ‘dumbbell’: the neck has very large mean curvature, causing it to pinch before the two spheres have enough time to shrink appreciably). In this case studying the behaviour
of the flows near singularities is a more delicate procedure. Nevertheless there have been a number of results pertaining to the possible limiting profiles of mean curvature flow singularities, beginning with the revolutionary work of Huisken [49] in which he showed singularities having a certain blow-up rate (the so-called ‘type 1’ singularities) are asymptotically self-similar. Such singularities are perhaps the easiest to examine, and concrete examples of the formation of a type 1 singularity are quite easy to obtain (for example a standard round sphere shrinks self-similarly along the mean curvature flow). Since then a large number of authors have made tremendous contributions to the subject of mean curvature flow singularities. Although this list is certainly not exhaustive, the author feels they would be being dishonest without mentioning the analysis of type 2 degenerate neckpinches by Angenant and Velázquez [9], as well as the study of singularities for rotationally symmetric surfaces by Altschuler, Angenent and Giga [3], and that of Huisken and Sinestrari for the case of mean convex surfaces [51]. For the reader who is interested in a brief introduction to the nature of singularities for the mean curvature flow (mainly concerned with dimensions one and two) the author recommends the paper of White [108].

Mean curvature flow (MCF) is a second order parabolic differential equation. Generalising to analogous higher-order systems of partial differential equations leads one to consider ‘heat-like’ flows of the form

\[ \partial_t f = (-1)^{p+1} \Delta^p H \cdot \nu, \quad p \in \mathbb{N}. \]  

(GPHF)

Here \( \Delta^p H \) represents the \( p^{th} \) repeated iteration of the Laplace-Beltrami operator applied to the mean curvature function:

\[ \Delta^p H = \Delta \Delta \cdots \Delta H, \]

where the \( \Delta \) symbol appears \( p \) times. The symbol \( \nu \) refers to the outward facing unit normal to \( \Sigma \). The author has dubbed this flow the geometric polyharmonic heat flow.
of order \((2p + 2)\). Note that \((\text{GPHF})\) is \textit{not} the same as the flow

\[
\partial_t f = (-1)^p \Delta^{p+1} f,
\]  

(1.1)

although they do share the same leading (highest order) term in \(f\). The right hand side of (1.1) also contains other terms that contribute both in tangential and normal directions. The flow \((\text{GPHF})\) is more natural than (1.1) from the curvature flow perspective in that for closed submanifolds only normal terms in the speed of a flow affect geometric quantities which are invariant under the diffeomorphism group (for example, enclosed volume and area). Hence any geometric operator which is deemed to be ‘natural’ should have its image contained in the normal bundle.

Note that \(\vec{H} = -H\nu\) is a section of the normal bundle. By considering the induced Laplacian in the normal bundle, \(\Delta^\perp\), our flow \((\text{GPHF})\) takes the form

\[
\partial_t f = (-1)^p \Delta^\perp \cdot \Delta^\perp \Delta^\perp \cdots \Delta^\perp f = (-1)^p (\Delta^\perp)^p \Delta f.
\]

Because each operator \(\Delta\) contains two derivatives, one can now easily see that the geometric flow \((\text{GPHF})\) gives rise to a system of differential equations of order \(2p + 2\). In Chapter 2 we show that in the setting of arbitrary codimensions this system is weakly parabolic, just like the mean curvature flow. Moreover, smooth flows of the form \((\text{GPHF})\) are all isoperimetrically natural in the sense that if we use \(\text{Vol}(\Sigma_t)\) and \(|\Sigma_t|\) to denote the signed enclosed volume and surface area respectively, then one has

\[
\frac{d}{dt} \text{Vol}(\Sigma_t) = 0 \quad \text{and} \quad \frac{d}{dt} |\Sigma_t| \leq 0.
\]

This contrasts with solutions of the mean curvature flow (MCF) which can only be guaranteed to be volume-reducing for mean convex initial data. Thus an advantage of these higher-order flows is that we expect not to have to rescale in order to obtain convergence results. A naïve implication of this is that geometric polyharmonic heat flows will not shrink embedded surfaces down to singular points, although this does
not rule out the possibility of other types of singularities from occurring (we show later that if a singularity of the flow does occur then they must manifest as unbounded concentrations of curvature in a small ball; see Remark 4.4). Furthermore, more exotic singularities may be possible if we replace the condition ‘embedded’ with ‘immersed’. For example, in the case $p = 1$, $\dim \Sigma = 1$ (the planar curve diffusion flow), one can think of the lemniscate (or ‘figure-eight’ curve) which has zero enclosed volume and shrinks homothetically to a point in finite time (see [28], Section 6, for example).

In the case $p = 1$, $\dim \Sigma = 2$, the flow (GPHF) becomes the surface diffusion flow, which has been quite a popular geometric flow research topic over the years (albeit not to the degree of its second order counterpart (MCF)). There are many sources available for the reader that is interested in studying the surface diffusion flow. For example, the work of Garcke, Ito and Kohsaka [35], Giga and Ito [36], Escher, Mayer, and Simonett [29], and Wheeler [104], among many others. The author and his two PhD. advisors have also studied the case $p = 2$, which is a sixth order flow they have dubbed the geometric triharmonic heat flow [75].

In this thesis we will be concerning ourselves with the geometric polyharmonic heat flow of immersed surfaces. This is the case $n = 2$ and $p \geq 1$ arbitrary in the family of flows (GPHF). A solution to the geometric polyharmonic heat flow on the interval $I$ is a family of closed, compact immersed surfaces in $\mathbb{R}^3$ which satisfy the following initial value problem:

\[
\begin{align*}
\partial_t f &= (-1)^{p+1} \Delta^p H \cdot \nu, \\
 f (\Sigma, 0) &= \Sigma_0.
\end{align*}
\]

(1.2)

Here $\Sigma_0$ is a prescribed smooth immersion of $\Sigma$ in $\mathbb{R}^3$.

Unfortunately, unlike with the mean curvature flow, when dealing with these higher-order flows, we do not have useful tools such as the maximum principle. To overcome this, we will rely heavily on local and global surface energy estimates. This is very much in the same vein as Kuwert and Schätzle [57, 58] and McCoy, Wheeler and Williams [74] in their analysis of the Willmore flow and Willmore surfaces respectively. Wheeler
also uses this method in his analysis of the surface diffusion flow near spheres [104].

These energies manifest as integrals of curvature. Of particular interest to us later on is the $L^2$-norm of the trace-free curvature,

$$\int_\Omega |A^o|^2 \, d\mu,$$

where $\Omega$ is a subset of our surface $\Sigma$. In the case $\Omega = \Sigma$, the integral (1.3) of course amounts to a global energy which can be thought of as an $L^2$ measure of the ‘sphericity’ of $\Sigma$ (in an averaged, geometric sense). Otherwise, one may wish to restrict $\Omega$ to a small ball, such as

$$\int_{f^{-1}(B_\rho(x))} |A^o|^2 \, d\mu, \quad x \in \mathbb{R}^3.$$

The local nature of this energy will allow us to establish results for more general immersions (for example, surfaces that are not necessarily closed). We later refer to (1.3) as the ‘umbilic energy’ of the immersion $f$ (see, for example, Definition 5.1).

Being scale invariant, the umbilic energy will allow us to prove $\varepsilon$-regularity results involving localised areas that are not flow-dependent. For example in Theorem 5.15 we prove a localised pointwise estimate for the trace-free curvature which holds under quite general conditions. In particular, we do not assume the presence of any geometric flow equation, nor closedness.

This integral itself is a particularly natural one for the problem at hand because it is only zero for umbilic surfaces, which are stationary solutions of flows of the form (GPHF). We show in Chapter 6 that (if initially small enough) (1.3) acts as a Lyapunov functional for a solution to (GPHF), decreasing monotonically over the duration of the flow. We refer to this as ‘preserved sphericity’.

Higher-order flows such as (GPHF) have not been studied to the degree of their lower order counterparts (namely because of the aforementioned issue involving a lack of a maximum principle). Nonetheless such flows have proven to be an effective mechanism for modelling a range of scenarios. Although a large constituent of such results
pertain to general surface and image modelling (such as the works of Bloor, Wilson and Hagen [17], Liu and Xu [66], Tosun [98], and Ugail [100]) as well as interactive design [56], the applications of such flows extend into fields of medical and material engineering modelling. For example, in [101] Ugail and Wilson utilise a sixth-order equation similar to (GPHF) with \( p = 2 \) to model the growth of venous tumors and oedemus ulcers, while in [38] Gomez and Nogueria use the phase field crystal model the microstructure evolution of two-phase systems on atomic length and diffusive time scales. The field crystal equation can easily be seen to be a sixth order partial differential equation dominated by a thrice-iterated Laplacian (along with lower order terms) which strongly resembles (GPHF) with \( p = 2 \). Although higher-order problems necessarily carry their share of analytical obstacles, when it comes to applications like computer design higher-order flows do present some advantages: for example the higher the order of the flow involved, the more freedom you have to join boundaries of patches together.

The list of applications of equations involving a thrice-iterated Laplacian is clearly expanding, and thus the author believes it is highly meaningful to study the behaviour of such systems and associated applications. The author hopes that once these sixth order flows are more well-understood, then the methods will be generalised to many higher-order equations, unlocking a barrage of associated modelling applications. Perhaps an investigation into the behaviour of such higher-order flows in more general ambient settings would also be a worthwhile and beneficial endeavour.

Of course applications of parabolic differential operators are not limited to the confines of modelling. One prime example within the field of mathematical general relativity is the Riemannian Penrose inequality which was first proven using a weak formulation of the inverse mean curvature flow (a flow that is parabolic for initial data with strictly positive mean curvature) [52]. This conjecture gives a lower bound on the ADM mass of an asymptotically flat 3-dimensional Riemannian manifold in terms of its outermost minimal surface, and has deep interpretations in general relativity that are outside the scope of this thesis. The interested reader is referred to the
following exposition of Bray which treats a number of inequalities - including the Penrose inequality - in greater generality [18].

Another is the Poincaré conjecture which claims that every simply connected closed 3–dimensional manifold is homeomorphic to the 3–sphere. The resolution of the Poincaré conjecture uses a modified Ricci flow (the so-called ‘Ricci flow with surgery’ [82]), and is widely recognised as one of the crowning achievements of modern mathematics. A fourth order flow that is particularly relevant to Part I of this thesis is the Willmore flow which gives the path of steepest descent for the Willmore (bending) energy $\int_\Sigma |\vec{H}|^2 d\mu$, where $\vec{H}$ is the mean curvature vector (for closed surfaces this energy is equal to the umbilic energy mentioned earlier, modulo the topological invariant $4\pi (1 - g)$ where $g$ is the genus of the surface). Stationary points of the Willmore flow have implications in mathematical biology: the Willmore energy appears as the leading order term in the surface energy in a popular model for lipid bilayers proposed by Helfrich [46]. The energy also appears as the dominant term in the Hawking mass in general relativity [44].

Note that the geometric polyharmonic heat flows (GPHF) are closely related to a higher-order generalisation of the Willmore flow, in that each flow shares the same leading term as the gradient flow of the energy

$$E_p[f] := \int_\Sigma |\Delta^{\frac{p+1}{2}} H|^2 d\mu, \ p \geq 1$$

(here with abuse of notation we use $\Delta^{\frac{2k+1}{2}} = \Delta^{\frac{1}{2}} \Delta^k = \nabla \Delta^k$ for $k \geq 0$). The energy $E_1$ is equal to the Willmore energy, while surfaces minimising $E_2$ are called ‘surfaces of minimal mean curvature variation’ (see, for example, [111]). It would be worthwhile to study the behaviour of these higher order ‘Willmore’ flows in generality, and the author believes that in the future some of the quantities $E_p$ will be realised as energies that describe some natural phenomena in the same way that the Willmore energy presents itself so fervently in the current scientific and mathematical literature. Since the gradient flows are the same as (GPHF) but with added lower order diffusion terms,
it would also be a worthwhile endeavour to study the geometric polyharmonic heat flows with more general lower order terms.

The ideas above lead us to some very natural questions, which the author hopes will be answered in the following years and unfortunately are outside the scope of this thesis. For instance:

1. In this thesis we look at flows that are close to spheres. What about the behaviour for flows that deviate from a sphere in a large sense?

2. To that end, what sort of singularities (if any) occur under (GPHF) for initial data with a large deviation from spheres? The analysis here is much more difficult than the lower order cases (such as the mean curvature flow) and the tools available to us are less robust.

3. What about the gradient flows for the energies $E_p$ mentioned above? Do they converge to spheres without assuming a priori that the initial data has small umbilic energy, just as in case of the Willmore flow?

4. Can an analysis of these flows (including the higher order ‘Willmore’ flows) lead to any interesting (or unexpected) geometric inequalities? For example, the planar curve shortening flow can be used to prove the classical isoperimetric inequality (10.10) – are there any analogous results for the higher order flows?

Of course it will be some time before these higher order flows are given their fair share of attention (if it ever happens at all), but it is my intention in this thesis to set the ball rolling.

We summarise the main contributions to Part I of this thesis as the following.

Chapter 1 Notation and basic geometric evolution equations. In this chapter we introduce the notation that will be used throughout the thesis. We introduce a cutoff function $\gamma$ that will be included in many of our subsequent calculations in order
to compensate for situations in which our immersion is non-compact, and go on to discuss its properties. We finish off by calculating the evolution equations for basic geometric quantities associated with our geometric flow.

Chapter 2 Well-posedness of the flow. Here we introduce the concept of parabolic operators on vector bundles, and rework the methods of Baker [12] to prove the local existence and uniqueness of solutions to the geometric polyharmonic heat flow in higher codimensions.

Chapter 3 Local estimates under a small concentration of curvature. In this chapter we prove local $L^2$ and $L^\infty$ estimates of curvature, under the assumption of a small concentration of curvature. Doing so requires for us to first prove a new estimate (Theorem 3.1) which allows for the control of integrals in which the integrand contains curvature terms of a certain algebraic structure.

Chapter 4 The Lifespan Theorem. Here we utilise our earlier results to establish an absolute lower bound on the lifespan of a geometric polyharmonic heat flow that depends solely on the concentration of curvature of our initial immersion $\Sigma_0$.

Chapter 5 A pointwise estimate for the trace-free curvature, and the Gap Lemma. In this chapter we prove some interesting geometric inequalities, along with a multiplicative Sobolev-type inequality, to establish a pointwise bound for the trace-free curvature that only depends on local $L^2$ estimates of terms of the form $\Delta^m H$, $A^o$, and the gradient of our cutoff function $\gamma$. Using these estimates, we then show that an immersion that is a weakly stationary solution to the geometric polyharmonic heat flow and satisfies a smallness condition regarding the total trace-free curvature, maps either into an embedded 2−sphere or a 2−plane.

Chapter 6 Preserved sphericity. In this short chapter we prove that if initially small enough, the global umbilic energy $||A^o||_2^2$ decreases monotonically under the geometric polyharmonic heat flow. Since the umbilic energy measures the distance from a sphere in an averaged $L^2$ sense, we refer to this phenomena as ‘preserved spheric-
Chapter 7 Construction of the blowup and long time existence. Here we prove a number of interior estimates along the flow using our local estimates from Chapter 3. This allows us to conclude that if a geometric polyharmonic heat flow encounters a finite time singularity then curvature must concentrate in a very specific way. We prove that the curvature cannot concentrate in this way if the initial concentration is sufficiently small. Via a proof by contradiction we obtain long time existence for the flow under the assumption of small initial curvature concentration.

Chapter 8 Smooth exponential convergence to spheres. In our final chapter in Part I of this thesis, we combine our previous results to prove that any geometric polyharmonic flow with initially small global umbilic energy exists for all time and converges exponentially fast in the $C^\infty$ topology to a 2-sphere with the same volume as the initial immersion.

1.1 Notation and preliminaries

We will start by introducing some of the basic ideas that will be used throughout the thesis. Some definitions in this chapter are also introduced in Appendix C (with more detail). For a more comprehensive introduction to the topic, the author recommends the classic textbooks of do Carmo [24], Peterson [83], and Kobayashi-Nomizu [55], all of which are a fantastic read for those who are new to differential geometry. The most basic notion that we will require is the idea of a differentiable manifold. This is a topological manifold (which means it is a topological space that is locally Euclidean), endowed with a differentiable structure which allows the use of calculus on the space.

Given a topological space $\Sigma$, a \emph{coordinate chart} is a subset $U \subseteq \Sigma$, along with a homeomorphism $\varphi : U \to \mathbb{R}^n$ such that $\varphi(U)$ is open in $\mathbb{R}^m$. We call $U$ an open set in $\Sigma$. A $C^k$ atlas ($k \in \mathbb{N}$) is then defined to be an indexed collection of charts $\{(U_a, \varphi_a)\}$
satisfying the following two conditions:

(a) \( \bigcup \alpha U_\alpha \) covers \( \Sigma \).

(b) If two open sets \( U_\alpha, U_\beta \subseteq \Sigma \) overlap then the map \( \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \) is \( C^k \).

Two atlases are called compatible if their union is also an atlas. The maximal atlas is defined to be the unique atlas which contains all possible coordinate charts compatible with a given atlas of \( \Sigma \). A \( C^k \) \( m \)-dimensional manifold is a set \( \Sigma \) together with a maximal atlas.

Because our differentiable manifold \( \Sigma \) allows calculus, we can now make sense of what it means for a vector to be tangent to \( \Sigma \). Indeed, if \( p \in \Sigma \) then then a vector is said to be tangent to \( \Sigma \) at \( p \) if it is the velocity vector of some curve \( \gamma \) through \( p \). We call the set

\[
T_p\Sigma := \left\{ \partial_t \gamma(0) \mid \gamma : (-\varepsilon, \varepsilon) \to \Sigma, \gamma(0) = p \right\}
\]

the tangent space to \( \Sigma \) at \( p \), and its elements are called tangent vectors at \( p \). One can easily verify that \( T_p\Sigma \) is an \( m \)-dimensional vector space for every \( p \in \Sigma \). Moreover, if a local coordinate chart around \( p \in \Sigma \) is given by \( \varphi(p) = (x^1(p), \ldots, x^m(p)) \in \mathbb{R}^m \), then the tangent space is spanned by the set \( \{ \partial_i \}_{i=1, \ldots, m} \), where \( \partial_i = \frac{\partial}{\partial x^i} \). To see this, first note that for \( \varepsilon > 0 \) sufficiently small, we can consider a curve \( \gamma : (-\varepsilon, \varepsilon) \to \Sigma, \gamma(0) = p \), as a curve from \( \mathbb{R} \) to \( \mathbb{R}^m \) via the coordinate chart \( \varphi \):

\[
\varphi^{-1} \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^m.
\]

Therefore for any \( f \in C^\infty(\Sigma) \) we can write the function \( f \) along \( \gamma \) as

\[
f \circ \gamma = (f \circ \varphi^{-1}) \circ (\varphi \circ \gamma) : (-\varepsilon, \varepsilon) \to \mathbb{R}.
\]
This implies that the directional derivative of $f$ along $\gamma$ is given by

$$\frac{df}{d\lambda} = \sum_{i=1}^{m} \frac{\partial (f \circ \varphi^{-1})}{\partial x^i} \frac{d(\varphi \circ \gamma)^i}{d\lambda} = \sum_{i=1}^{m} \frac{dx^i \partial f}{d\lambda \partial x^i}.$$ 

Since this is true for an arbitrary $f$, we have

$$\frac{d}{d\lambda} = \sum_{i=1}^{m} \frac{dx^i}{d\lambda} \frac{\partial}{\partial x^i} = \frac{d\gamma^i}{dx^i} \frac{\partial}{\partial x^i},$$

which establishes $\{\partial_i\}$ as a basis. Here we have adopted the Einstein summation convention, in which an index occurring twice on a product is to be summed from 1 up to the spatial dimension (in this case, $m$).

The tangent bundle $T\Sigma$ is the vector bundle over $\Sigma$ that is constructed by attaching the tangent space to $\Sigma$ at each point:

$$T\Sigma := \bigsqcup_{p \in \Sigma} T_p\Sigma = \bigcup_{p \in \Sigma} \left\{(p, y) \mid y \in T_p\Sigma\right\}.$$ 

It is easy to see that $\dim T\Sigma = 2m$.

Let $\Sigma$ be a differentiable manifold and $p \in \Sigma$. Let $T_p\Sigma$ be the tangent space at $p$. Then we define the cotangent space $T^*_p\Sigma$ at $p$ as the dual space of $T_p\Sigma$:

$$T^*_p\Sigma = (T_p\Sigma)^*.$$ 

Elements of the cotangent space are linear functionals $\omega : T_p\Sigma \rightarrow \mathbb{R}$ and are called cotangent vectors or one-forms. If $\{\partial_i\}_{i=1,\ldots,m}$ is a local frame for the tangent bundle $T\Sigma$, the set $\{dx^i\}_{i=1,\ldots,m}$ defined by

$$dx^i (\partial_j) = \delta^j_i$$

acts as a local basis for $T^*\Sigma$. To see that each $dx^i$ is contained in the cotangent space.
$\Sigma$ note that $x^i : \Sigma \to \mathbb{R}$ implies that the exterior derivative $dx^i$ satisfies

$$dx^i : T\Sigma \to T\mathbb{R} \cong \mathbb{R}.$$  

Furthermore the basis tangent vectors are seen to act on the basis elements of the cotangent space in a similar way:

$$\partial_i (dx^j) = \delta^j_i.$$  

This easily extends to general tangent and cotangent vectors: if $v \in T_p\Sigma$ and $\omega \in T^*_p\Sigma$ then

$$\omega(v) = \omega_i dx^i \left( v^j \partial_j \right) = \omega_i v^j dx^i \left( \partial_j \right) = \omega_i v^j \delta^i_j = \omega_i v^i.$$  

One sees similarly that $v(\omega) = \omega_i v^i$, and hence

$$\omega(v) = v(\omega). \quad (1.4)$$  

This implies a canonical isomorphism (the so-called musical isomorphism) between the tangent and cotangent space. This will be introduced in more detail after we introduce Riemannian metrics. However, the above does give us enough information to define a generalised tensor. These are most easily described as multilinear sections from Cartesian products of the tangent and cotangent bundles to $\mathbb{R}$.

First we define the tensor product. If $V$ and $W$ are vector spaces (over $\mathbb{R}$) with bases $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$, then the tensor product $V \otimes W$ is the vector space of dimension $mn$ spanned by the basis $(u_i \otimes v_j)_{i=1,\ldots,m,j=1,\ldots,n}$. There exists a canonical bilinear map

$$L : V \times W \to V \otimes W$$

which sends $(a^i u_i, b^j v_j)$ to $a^i b^j (u_i \otimes v_j)$ [53]. From this definition we can inductively define the tensor product of more than two vector spaces.
We define an \((l,m)\) tensor (field) on \(\Sigma\) to a section of\[ T^*\Sigma \otimes \cdots \otimes T^*\Sigma \otimes T\Sigma \otimes \cdots \otimes T\Sigma. \]From the above discussion, a vector \(v\) is a \((1,0)\) tensor, and a covector \(\omega\) is a \((0,1)\) tensor (as (1.4) implies that covectors act on vectors to give a real number). We denote the space of \((l,m)\) tensor fields on \(\Sigma\) by \(T^l_m(\Sigma)\), and the set of all smooth tensor fields on \(\Sigma\) by \(\chi(\Sigma)\). Hence, in this notation, \(T^1_0(\Sigma) = T\Sigma\) and \(T^0_1(\Sigma) = T^*\Sigma\). An \((l,m)\) tensor is written in terms of the local frame \(\{\partial_i\}\) and coframe \(\{dx^j\}\) in the following way:

\[ T = T^j_{i_1 \cdots i_m} \partial_{j_1} \otimes \cdots \partial_{j_l} \otimes dx^{i_1} \otimes dx^{i_m}. \]

The functions \(T^j_{i_1 \cdots i_m}\) are called the components of \(T\) with respect to the local frame and coframe. The symbol \(\otimes\) corresponds to an element of the tensor product of multiple copies of the vector spaces \(T\Sigma\) and \(T^*\Sigma\).

Next we endow our differentiable manifold with a metric structure. A Riemannian metric on a differentiable manifold \(\Sigma\) is a correspondence that associates to each point \(p \in \Sigma\) an inner product \(g\) (called the metric or first fundamental form) on the tangent space \(T_p\Sigma\) which varies smoothly with \(p\):

\[ g_p : T_p\Sigma \times T_p\Sigma \rightarrow [0, \infty). \]

The metric \(g\) is a \((0,2)\) tensor field and can be written in terms of a local coframe \(\{dx^i\}\) as

\[ g = g_{ij} dx^i \otimes dx^j. \] (1.5)

Here the \(g_{ij}\) are explicitly given by \(g_{ij} = g(\partial_i, \partial_j)\), where \(\{\partial_i\}\) is a local frame for \(T\Sigma\). As mentioned earlier, there is a canonical (‘musical’) isomorphism between the tangent and cotangent spaces on a Riemannian manifold, which is easily expressible in terms of components of the metric tensor and its inverse. If \(\omega \in T^*_p\Sigma\) is a covector, then \(\omega^i\)
(read as $\omega$ sharp) is the unique vector in $T_p\Sigma$ such that for every $v \in T_p\Sigma$ we have $\langle \omega^\sharp, v \rangle_g = \omega(v)$ at $p$. In terms of the metric, this is equivalent to

$$(\omega^\sharp)_i = g^{ij}\omega_j.$$ 

Here the $g^{ij}$ are the components of the inverse metric which is a $(2,0)$ tensor field with corresponding to the inverse of $g$ (when $g$ is viewed as an $m \times m$ matrix):

$$g^{ij} = (g^{-1})_{ij}.$$ 

Note that $g_{ij}g^{jk} = \delta_i^k$ and $g_{ij}g^{ij} = m$, where $m$ is the dimension of $\Sigma$. Similarly, for a tangent vector $v$, we define $v^\flat$ (read as $v$ flat) to be the unique covector in $T^*_p\Sigma$ such that for every $\omega \in T^*_p\Sigma$, $\langle v, \omega^\flat \rangle_p = v(\omega)$ at $p$. This is equivalent to

$$(v^\flat)_i = g_{ij}v^j.$$ 

Every paracompact Hausdorff differentiable manifold admits a Riemannian metric (this is a standard result; see, for example, [24, Proposition 2.10]). Unlike the name ‘Riemannian metric’ suggests, $g$ is not a metric on $T\Sigma$. However, if we define

$$L[\gamma] := \int_a^b \sqrt{g(\partial_t \gamma, \partial_t \gamma)} \, dt = \int_a^b \sqrt{\langle \partial_t \gamma, \partial_t \gamma \rangle_g} \, dt,$$

to be the length of a curve $\gamma : (a, b) \rightarrow \Sigma$ in a Riemannian manifold $\Sigma$, then the distance from $p$ to $q$,

$$d(p, q) := \inf \left\{ L[\gamma] \Big| \gamma : [a, b] \rightarrow \Sigma \text{ piecewise smooth, } \gamma(a) = p, \gamma(b) = q \right\},$$

(1.6)

is a metric on $\Sigma$, making $(\Sigma, d)$ a metric space [24]. Moreover the metric topology agrees with the manifold topology (see [24, 55] for example). The Riemannian metric
$g$ on $\Sigma$ extends to tensor fields. Given two $(l, m)$ tensor fields

$$S = S_{p_1 \ldots p_m}^q \partial_{q_1} \otimes \cdots \otimes \partial_{q_l} \otimes dx^{p_1} \otimes \cdots \otimes dx^{p_m}, \quad T = T_{i_1 \ldots i_m}^{j_1 \ldots j_l} \partial_{j_1} \otimes \cdots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes dx^{i_m},$$

we define

$$\langle S, T \rangle_g = g_{q_1 j_1} \cdots g_{q_l j_l} g_{p_1 i_1} \cdots g_{p_m i_m} S_{p_1 \ldots p_m}^q T_{i_1 \ldots i_m}^{j_1 \ldots j_l}.$$ 

From this we then define the norm of a tensor $S$ by $|S| = \sqrt{\langle S, S \rangle}$. For example, the norm of the metric tensor $g$ is always the square root of $m$. To see this, we simply apply the above formula:

$$|g|^2 = \langle g, g \rangle = g_{ij} g_{pq} g^{ip} g^{jq} = \delta^p_j \delta^i_p = m.$$

Let $(\Sigma, g)$, $(N, h)$ be Riemannian manifolds of dimension $m$ and $n$ respectively (where $n > m$). We call a map $f : \Sigma^m \rightarrow \mathbb{N}^n$ an immersion if the differential of $f$, $df$ is injective for all $p \in \Sigma$. If, in addition, $f$ is a homeomorphism onto its image then it is called an embedding. If $f : \Sigma \rightarrow N$ is an immersion then the tangent space of $f(\Sigma)$ is locally spanned by $\{\partial_i f\}_{i=1 \ldots m}$. For example a vector $X$ in the pushforward of the tangent bundle, $df(T\Sigma)$, can be written as

$$X \bigg|_{f(p)} = X^i \bigg|_{f(p)} \partial_i f(p),$$

which we often shorten to $X = X^i \partial_i f$ when there is no risk of confusion.

The manifold $\Sigma$ is said to be a submanifold of $N$, with codimension $\dim N - \dim \Sigma$. If $\dim N = \dim \Sigma + 1$ then we call $\Sigma$ a hypersurface immersed in $N$. Furthermore, if $\dim \Sigma = 2$ we drop the prefix and simply call $\Sigma$ a surface immersed in $N$. We often identify $\Sigma$ with $f(\Sigma)$, and nothing that is lost if we assume that $\Sigma \subset N$ (rather than $f(\Sigma) \subset N$).

Consider a smooth embedded manifold $\Sigma^m$ contained in some smooth open set $U \subseteq \mathbb{R}^n$. Let the embedding map be denoted by $f : \Omega \rightarrow \mathbb{R}^n$ with $f(\Omega) = \Sigma$ where
Ω ⊆ R^m is open. Then Σ is called a *properly embedded* surface if \( f^{-1}(K) \subset Ω \) is compact whenever \( K \subset U \) is compact.

Let \( f : \Sigma^m \rightarrow N^n \) be an immersion between differentiable manifolds. If \( (N, g^N) \) has a Riemannian structure, then \( f \) induces a Riemannian structure on \( Σ \) by setting

\[
g_Σ^p (U, V) = g^N_{f(p)} (df(U), df(V))
\]

for \( U, V \in TΣ \). Writing \( U = U^i \partial_i, V = V^j \partial_j \) respectively (as in (1.7)) and letting \( g^N_{ij} = g^N(∂_i f, ∂_j f) \), the previous equality becomes

\[
g_Σ^p (U, V) = U^i V^j g^N_{f(p)} (\partial_i f, \partial_j f).
\]

Here we have used the bilinearity of \( g^N \) and the fact that the differential of \( f \), \( df \) acts on vector fields \( V \) via

\[
df(V) = \frac{∂f}{∂x^i} dx^i (V^j \partial_j) = \frac{∂f}{∂x^i} V^j dx^i (\partial_j) = \frac{∂f}{∂x^i} V^i.
\]

We often drop the \( Σ \) completely. The metric \( f^*g^N = \langle \cdot, \cdot \rangle_Σ = g^Σ \) on \( Σ \) is called the metric *induced* by \( f \) (or sometimes the *pullback* of \( g^N \) via \( f \)). Note that the injectivity of \( df \) implies that \( \langle \cdot, \cdot \rangle_Σ \) is positive definite. In the case \( N^n = R^n \), the induced metric \( g \) takes the form

\[
g_{ij} = (∂_i f, ∂_j f) = \sum_{k=1}^n \frac{∂f^α}{∂x^i} \frac{∂f^α}{∂x^j}
\]

where \( f^α \) refers to the \( α^{th} \) component of the immersion function \( f \), and \( \langle \cdot, \cdot \rangle \) is the canonical inner product on \( R^n \).

For an immersed hypersurface \( f : Σ^α \rightarrow R^{n+1} \), the *second fundamental form* \( A = A_{ij} dx^i \otimes dx^j \) of \( Σ^α \subset R^{n+1} \) is the symmetric \((0,2)\) tensor defined by

\[
A(X,Y) = -(D_XD_Yf, ν) \text{ for } X,Y ∈ TΣ,
\]
where \( \nu \) is a chosen unit normal vector field on \( \Sigma \) and \( D \) denotes the ordinary (partial) derivative in \( \mathbb{R}^{n+1} \). Component-wise one has

\[
A_{ij} = -(\partial_{ij} f, \nu) = (\partial_i \nu, \partial_j f).
\]

The mean curvature is then defined as the trace of \( A \) with respect to the metric \( g \). That is,

\[
H = g^{ij} A_{ij}.
\]

The shape operator (or Weingarten map) \( S \) is a \((1,1)\) tensor field given by

\[
S(X) = (D_X \nu)^T,
\]

where the \( T \) denotes the tangential component. The shape operator and second fundamental form are related by the equation

\[
A(X,Y) = (S(X), Y).
\]

In the case \( A \) is diagonal we call the eigenvalues of the shape operator:

\[
\kappa_1 = A^1, \kappa_2 = A^2, \ldots, \kappa_n = A^n
\]

at a point \( p \in \Sigma \) the principal curvatures at \( p \). Locally one has

\[
H = \sum_{i=1}^n \kappa_i \quad \text{and} \quad |A|^2 = \sum_{i=1}^n \kappa_i^2.
\]

Additionally, the Gauss curvature is given by

\[
K = \kappa_1 \kappa_2 \ldots \kappa_n = \det A^i_j = \det g^{ik} A_{kj}.
\]

From the definitions of the mean curvature and second fundamental form we can defined a new \((0,2)\)-tensor field \( A^o \) by \( A^o := A - \frac{1}{n} \operatorname{tr}_g(A) \) called the trace-free second
fundamental form. Its components are explicitly given by

\[ A^o_{ij} = A_{ij} - \frac{1}{n} g_{ij} H. \]  

(1.9)

where \( g \) is the induced metric and \( n \) is the dimension of \( \Sigma \). This tensor is the trace-free symmetric part of \( A \). By definition

\[ A^o_{ij} = A_{ij} - \frac{1}{n} g_{ij} H = A_{ji} - \frac{1}{n} g_{ji} H = A^o_{ji}, \]

and so

\[ \operatorname{tr}_g (A^o) = g^{ij} A^o_{ij} = g^{ij} \left( A_{ij} - \frac{1}{n} g_{ij} H \right) = H - \frac{1}{n} \delta_i^i H = H - H = 0. \]

Moreover, for any \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) we have

\[ \nabla_{i_1...i_k} A^o_{pq} = \nabla_{i_1...i_k} A_{pq} - \frac{1}{n} g_{pq} \nabla_{i_1...i_k} H, \]

and so

\[ \left| \nabla_{(k)} A^o \right|^2 = \left| \nabla_{(k)} A \right|^2 - \frac{1}{n} \left| \nabla_{(k)} H \right|^2. \]  

(1.10)

Note that for a general symmetric \((0, 2)\) tensor \( T \), the trace-free part of \( T \) is defined by

\[ S^o (T) = T - \frac{1}{n} g \operatorname{tr}_g (T). \]

Some authors refer to the trace-free part of a \((0, 2)\) tensor \( T \) as the Einstein version of \( T \).

We will write \( S \ast T \) (using the notation of Hamilton [42]) to mean a tensor formed by a linear combination of terms, each of them obtained by contracting some indices of the pair \( S, T \) with the metric \( g_{ij} \) and/or its inverse \( g^{ij} \). For example, for two \((0, 2)\) tensors \( S, T \),

\[ \langle S, T \rangle = g^{ip} g^{jq} S_{ij} T_{pq} = S \ast T. \]  

(1.11)
A very useful property of the $\ast-$product is that

$$|S \ast T| \leq c |S| |T|$$

for some constant $c$ that depends only on the algebraic ‘structure’ of $S \ast T$ (for example, the Cauchy-Schwarz inequality allows us to take $c = 1$ in (1.11)). We shall also use what is referred to as $P-$style notation to declutter our working and help us keep track of the combinations of tensor fields. Sometimes we are not only interested in the degree of each derivative of our tensor, but rather the sum of the degrees as a whole (this will be of particular use to us later on when dealing with some inequalities that utilise Lemma A.12, for instance). For a tensor $T$, we define

$$P^j_i (T) = \sum_{r_1 + \cdots + r_j = i} c \nabla_{(r_1)} T \ast \cdots \ast \nabla_{(r_j)} T,$$

where the constant $c \in \mathbb{R}$ may vary from one term in the summation to another. As an extension to this notation, we define

$$P^{j,k}_{i} (T) = \sum_{r_1 + \cdots + r_j = i, r_j \leq k} c \nabla_{(r_1)} T \ast \cdots \ast \nabla_{(r_j)} T.$$

The area element (or volume form) induced by the metric $g$ on $\Sigma$ is given by

$$d\mu = \sqrt{\det g} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$ 

Combining these two results, we can then define integration of compactly supported functions $h : \Sigma \rightarrow \mathbb{R}$ over an immersed hypersurface:

$$\int_{\Sigma} h \, d\mu = \int_{\Sigma} h \sqrt{\det g} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$ 

We can also define the tangential gradient of a function $h : \Sigma \rightarrow \mathbb{R}$ by

$$\nabla h := g^{ij} \partial_j h \partial_i f.$$
For immersed hypersurfaces with codimension 1, $\nabla h$ is the tangential component of $Dh$ (the regular ambient derivative of $h$). Indeed

$$(Dh)^T = \operatorname{proj}_{T\Sigma} Dh = Dh^i \partial_j f = g^{ij} \partial_i h \partial_j f = \nabla h.$$  

To generalise the previous definition, an affine connection $\nabla$ on $\Sigma$ is a mapping

$$\nabla : T\Sigma \times T\Sigma \rightarrow T\Sigma$$

which satisfies the following properties:

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

(ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$.

(iii) $\nabla_X (fY) = f\nabla_X Y + X(f)Y$,

where $X, Y, Z \in T\Sigma$ and $f, g \in C^\infty(\Sigma)$. Given a Riemannian manifold $\Sigma$, there exists a unique affine connection $\nabla$ satisfying the following conditions:

(i) it preserves the metric, i.e. $\nabla g = 0$.

(ii) it is torsion free, meaning that for any vector fields $X$ and $Y$ we have $\nabla_X Y - \nabla_Y X = [X, Y]$.

We henceforth refer to this unique connection as the covariant derivative. The covariant derivative extends uniquely to higher-order covariant tensors by the formula

$$(\nabla_X S)(X_1, X_2, \ldots, X_l) = X(S(X_1, X_2, \ldots, X_l)) - \sum_{k=1}^{l} S(X_1, \ldots, \nabla_X X_k, \ldots, X_l),$$

where $X, X_1, \ldots X_l \in T\Sigma$ and $S \in T_0^l(\Sigma)$.

Next we introduce the so-called Christoffel symbols, which are coordinate-space expressions for the covariant derivative derived from the metric $g$. The Christoffel
symbols are the unique coefficients which satisfy the identity

$$\nabla_i \partial_j f = \Gamma_{ij}^k \partial_k f,$$

and are expressed by the formula

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}),$$

Applying (1.13), one can readily see that for a tangent vector field $X$ (that is, a $(1, 0)$ tensor field on $\Sigma$) described in local coordinates by $X = X^j \partial_j f$, one has

$$\nabla_i X = \nabla_i (X^j \partial_j f) = \partial_i X^j \partial_j f + X^j \Gamma_{ij}^k \partial_k f = (\partial_i X^j + \Gamma_{ij}^k X^k) \partial_j f.$$

Hence

$$\nabla_i X^j = \partial_i X^j + \Gamma_{ik}^j X^k.$$

Similarly for a one form $\omega = \omega_i \, dx^i$ (that is, a $(0, 1)$ tensor field on $\Sigma$) we have

$$\nabla_i \omega = \nabla_i (\omega_j \, dx^j) = \partial_i \omega_j \, dx^j - \omega_j \Gamma_{ik}^j \, dx^k = (\partial_i \omega_j - \Gamma_{ij}^k \omega_k) \, dx^j,$$

meaning that

$$\nabla_i \omega_j = \partial_i \omega_j - \Gamma_{ij}^k \omega_k.$$

We can concisely express the covariant derivative of a general tensor $T$ in Hamilton’s star notation as

$$\nabla T = \partial T + T \ast \Gamma.$$

We also define the tangential divergence in a way that extends naturally from the ambient case: If $X$ is a smooth tangent vector field on $\Sigma$ then

$$\text{div}_\Sigma X := \text{tr}_g (\nabla X) = \nabla_i X^i = g^{ij} \nabla_i X_j = g^{ij} \langle \nabla_i X, \partial_j f \rangle.$$
From this definition, we can then define the Laplace-Beltrami operator on a tensor $T$ on $\Sigma$. This map $\Delta_L : T^k_l (\Sigma) \rightarrow T^k_l (\Sigma)$ is given by

$$\Delta_L T^{j_1 \ldots j_k}_{i_1 \ldots i_l} := g^{ij} \nabla_i \nabla_j T^{j_1 \ldots j_k}_{i_1 \ldots i_l}.$$ 

For $h \in C^\infty (\Sigma)$ it follows that

$$\Delta_L h = \text{div}_\Sigma \nabla h = g^{ij} (\partial_{ij} h - \Gamma^k_{ij} \partial_k h) = g^{ij} \nabla_{ij} h = \nabla^i \nabla_i h.$$ 

Since there is no risk of confusion, we will drop the subscript $L$, and the Laplace-Beltrami operator will be denoted by $\Delta$, unless stated otherwise. From this one can construct a divergence theorem smooth vector fields: if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a hypersurface and $X$ is a $C^1$ vector field along $\Sigma$, then

$$\int_\Sigma (X, \vec{H}) \, d\mu = - \int_\Sigma \text{div}_\Sigma X \, d\mu + \int_{\partial \Sigma} \langle X, \nu^\partial \rangle \, d\mu^\partial,$$  

where $d\mu^\partial$ is the $(n-1)$ dimensional volume form on $\partial \Sigma$, and $\nu^\partial$ is the outer unit normal vector field to $\partial \Sigma$ that is tangent to $\Sigma$ at the boundary $\partial \Sigma$. If $X$ is tangent to $\Sigma$, then $(X, \vec{H}) \equiv 0$ and so (1.14) implies that

$$\int_\Sigma \text{div}_\Sigma X \, d\mu = \int_{\partial \Sigma} \langle X, \nu^\partial \rangle \, d\mu^\partial,$$

where we recall that $\langle \cdot, \cdot \rangle$ is the induced inner product on the tangent space. Note that here we use the induced metric $\langle \cdot, \cdot \rangle$ rather than the ambient one because we are assuming $X$ and $\nu^\partial$ are both tangent to $\Sigma$. If in addition to $X$ being tangent, we assume that either $X$ has compact support or $\partial \Sigma = \emptyset$, it follows that

$$\int_\Sigma \text{div}_\Sigma X \, d\mu = 0.$$ 

Furthermore, if $h \in C^\infty (\Sigma)$ (where $\Sigma$ is not necessarily closed) then $\text{div}_\Sigma (hX) =$
From this result and the definition tangential divergence above, we can readily see that if \( h : \Sigma \rightarrow \mathbb{R} \) is a smooth function with compact support then letting \( X = \nabla h \), we obtain
\[
\int_{\Sigma} \Delta h \, d\mu = \int_{\Sigma} \text{div}_\Sigma X \, d\mu = 0.
\]

The Codazzi equations (see Claim C.2 for a derivation) say that the the \((0, 3)\) tensor field \( \nabla A \) is completely symmetric for hypersurfaces in Euclidean space:
\[
\nabla_i A_{jk} = \nabla_j A_{ki} = \nabla_k A_{ij}.
\]

It is a consequence of the Codazzi equations the covariant derivatives of the mean curvature are completely controlled by contraction of the corresponding covariant derivative of trace-free curvature. To see this, note that by the Codazzi equations we have
\[
\nabla_j H = \nabla_j A^i_j = \nabla_i A^j_j = \nabla_i \left( (A^\alpha)^i_j + \frac{1}{n} \delta^i_j H \right),
\]
so that absorbing on the left and multiplying out by \( n/(n - 1) \) yields
\[
\nabla_j H = \frac{n}{n - 1} \nabla_i (A^\alpha)^i_j = -\frac{n}{n - 1} (\nabla^* A^\alpha)^j_j.
\]

Here \( \nabla^* \) is the formal adjoint of \( \nabla \) in \( L^2(\Sigma) \). This means that \( \nabla^* \) is the unique
operator $\nabla^* : T_j^i (\Sigma) \to T_{j-1}^i (\Sigma)$, $(j \geq 1)$ satisfying

$$
\int_{\Sigma} \langle \nabla S, T \rangle \, d\mu = \int_{\Sigma} \langle S, \nabla^* T \rangle \, d\mu
$$

for every $S \in T_{j-1}^i (\Sigma), T \in T_j^i (\Sigma)$. One readily checks that this definition agrees with (1.16). From this we can check

$$
\nabla_{i_1\ldots i_k} A_{pq} = \nabla_{i_1\ldots i_k} \left( A_o^{pq} + \frac{1}{n} g_{pq} H \right)
= \nabla_{i_1\ldots i_k} A_o^{pq} - \frac{1}{n-1} g_{pq} \nabla_{i_1\ldots i_{k-1}} (\nabla^* A_o)^i_k.
$$

Computing norms then yields

$$
|\nabla (k) A|^2
= |\nabla (k) A_o|^2 - \frac{2}{n-1} \nabla_{i_1\ldots i_{k-1}} (\nabla^* A_o)^i_k \nabla_{i_1\ldots i_k} \text{tr}_g (A_o) + \frac{n}{(n-1)^2} |\nabla (k-1) \nabla^* A_o|^2
= |\nabla (k) A_o|^2 + \frac{n}{(n-1)^2} |\nabla (k-1) \nabla^* A_o|^2
\leq \frac{n^2 - n + 1}{(n-1)^2} |\nabla (k) A_o|^2,
$$

(1.17)

because $|\nabla (k-1) \nabla^* A_o| \leq |\nabla (k) A_o|$ and $\text{tr}_g (A_o) = 0$. From this and (1.10), we obtain the estimate

$$
|\nabla (k) H|^2 = n \left( |\nabla (k) A|^2 - |\nabla (k) A_o|^2 \right) \leq \left( \frac{n}{n-1} \right)^2 |\nabla (k) A_o|^2,
$$

(1.18)

which will be heavily relied upon much later in this work when working with energy estimates (Particularly in Chapter 6).

For an immersion $f : \Sigma^n \to \mathbb{R}^{n+1}$ the Riemann curvature tensor $\text{Riem}$ is the $(0,4)$ tensor field with components

$$
R_{ijkl} = (\nabla_i \partial_j f - \nabla_j \partial_i f, \partial_k f).
$$
where $\nabla_{ij}$ is short for the Hessian operator on $\Sigma$:

$$\nabla_{ij} = \partial_{ij} - \Gamma^k_{ij} \partial_k.$$

Sometimes it is more convenient to write $\text{Riem}$ in its $(1,3)$ mixed form:

$$R^l_{ijk} := g^{lm} R_{ijkm}.$$

The Riem curvature tensor possesses various Symmetries and antisymmetries. For example,

$$R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk}.$$

This tensor appears in the formula for the interchange of covariant derivatives. For a vector $X$ and covector $Y$, the following equations hold:

$$\nabla_{ij} X^k = \nabla_{ji} X^k + R^k_{ijl} X^l \quad (1.19)$$

and

$$\nabla_{ji} Y^k = \nabla_{ji} Y^k + R^l_{ijk} Y^l. \quad (1.20)$$

These two identities are proven in Appendix C.

The Gauss equations are a famous system of equations that give a relation between the Riemann curvature tensor and the second fundamental form of a hypersurface $\Sigma$:

$$R_{ijkl} = A_{ik} A_{jl} - A_{il} A_{jk}.$$

We also have the Gauss-Weingarten relations which give the Hessian of the immersion map $f : \Sigma \rightarrow \mathbb{R}^{n+1}$ in terms of the second fundamental form:

$$\nabla_i \partial_j f^\alpha = -A_{ij} \nu^\alpha.$$
1.2 Prescribing a cutoff function

Next, we need a routine way to ‘integrate by parts’ (see (1.15)) when dealing with functions that may not have compact support. To do so we introduce a cutoff function which is in essence a function localised to a ball that can be estimated nicely in the $C^k$ norm by the second fundamental form and its derivatives.

This short section will be devoted to constructing a cutoff function $\gamma$ for our later calculation.

Many of the integrals we will be dealing with later will have a cutoff function $\gamma : \Sigma \rightarrow \mathbb{R}$ included in the integrand. The idea of this function is essentially to restrict the integrand to the support of $\gamma$ (which can be made as small as desired), thus giving us an idea of the concentration of the integrand in a small segment of $\Sigma$. Not only that, but we construct $\gamma$ with compact support so that we do not have to worry about boundary terms when performing integration by parts (see (1.14)). Since integration is only properly defined over compact sets, defining $\gamma$ with compact support allows us compensate for situations in which we do not wish to specify whether $\Sigma$ is compact or not. This is especially helpful in later chapters (such as Chapter 5) when we derive $\varepsilon$-regularity that involves concentrations of curvature on small balls. We make the assumption that $\gamma = \tilde{\gamma} \circ f$, where $f : \Sigma^n \times [0,T) \rightarrow \mathbb{R}^{n+1}$ is our immersion function and $\tilde{\gamma} \in C^{p+1}(\mathbb{R}^{n+1}; \mathbb{R})$ is a function satisfying

$$0 \leq \tilde{\gamma} \leq 1 \quad \text{and} \quad |D^k \tilde{\gamma}| \leq c_k \tilde{\gamma}^k \quad \text{for every} \quad k \geq 1,$$

(1.21)

for some universal bounded constants $c_k$ and $\tilde{c}_\gamma$. Of course, later on a specific function $\tilde{\gamma}$ will be chosen with desirable decay conditions. The existence of such a function is standard, see for example [110].

We provide some new notation before proving some estimates for this cutoff function. Firstly, for $k \in \mathbb{N}$ we let $(\partial f, \nu)^{(k)}$ denote any polynomial consisting of $k$ terms of the form $\partial_i f, \nu$ (both of which, importantly, have bounded norm). For example,
using the chain rule we have $\nabla_i \gamma = D\tilde{\gamma}(f) \partial_i f$. We write this succinctly as

$$\nabla \gamma = D\tilde{\gamma} \ast (\partial f, \nu)^{(1)}.$$  

Similarly, using the Gauss-Weingarten identities

$$\nabla_i \partial_j f = -A_{ij} \nu \text{ and } \partial_i \nu = A^i_j \partial_j f,$$

from (C.9), we have $\nabla_{ij} \gamma = D^2 \tilde{\gamma} \partial_i \partial_j f - D\tilde{\gamma} A_{ij} \nu$. Using our succinct notation from before this can be written as

$$\nabla^{(2)} \gamma = D^2 \tilde{\gamma} \ast (\partial f, \nu)^{(2)} + D\tilde{\gamma} \ast A \ast (\partial f, \nu)^{(1)}.$$  

To keep things even shorter, we set

$$\alpha_k := D^k \tilde{\gamma} \ast (\partial f, \nu)^{(k)}, \ k \in \mathbb{N}.$$  

Using (1.22) for every $k \in \mathbb{N}$ there exists a universal constant $c$ with $\alpha_k \leq c \epsilon \gamma^k$. With this notation, our earlier calculations give

$$\nabla \gamma = \alpha_1 \text{ and } \nabla^{(2)} \gamma = \alpha_2 + \alpha_1 \ast A.$$  

**Claim 1.1.** For any $k \in \mathbb{N}$ we have

$$\nabla^{(k)} \gamma = \alpha_k + \sum_{i=1}^{k-1} \sum_{j=1}^{i} P^{j-i} (A) \ast \alpha_{k-i}.$$  

Therefore

$$\nabla^{(k)} \gamma^s = \sum_{i=1}^{k} \sum_{j=1}^{k-j} \sum_{l=1}^{k} P_l^{k-(j+l)} (A) \ast \alpha_j \ast \gamma^{s-i}.$$  

The proof of Claim (1.24) is included in the appendix. The claim implies that there
is a universal, bounded constant $c_{\gamma} = c_{\gamma}(c_{\gamma}) > 0$ such that $\gamma$ satisfies

$$|\nabla(k)\gamma| \leq c \left( c_{\gamma}^k + \sum_{i=1}^{k-1} \sum_{j=1}^{i} |P_{ij}^k(A)| c_{\gamma}^{k-i} \right).$$

(1.25)

In particular, for $k = 1, 2, 3$ we have

$$|\nabla\gamma| \leq c c_{\gamma}, \quad |\nabla(2)\gamma| \leq c c_{\gamma}(c_{\gamma} + |A|),$$

and

$$|\nabla(3)\gamma| \leq c c_{\gamma}(c_{\gamma}^2 + c_{\gamma}|A| + |A|^2 + |\nabla A|),$$

respectively. For example, our earlier calculation (1.23) as well as the assumptions (1.21) together imply

$$|\nabla(2)\gamma| \leq c |D^2\gamma| + c |D\gamma||A| \leq c c_{\gamma}^2 + c c_{\gamma} |A| \leq c c_{\gamma}(c_{\gamma} + |A|),$$

which verifies the inequality for $\nabla(2)\gamma$. Observe that in contrast to [58], we include an extra factor of $c_{\gamma}$ in the first term on the right hand side to preserve scaling. That is to say, $c_{\gamma}$ scales like $|A|$.

We are now done with introducing the basic notation of what will be required for the thesis and will finish off the chapter with some basic evolution equations for geometric quantities.

### 1.3 Evolution equations

There are some geometric quantities and functions that are essential to study, if one wishes to get a grasp on how an immersion is evolving. We will start by deriving the evolution equations for $g, \nu, \Gamma^i_{jk}, A$ and $H$ for a hypersurface evolving with a normal flow speed $\varphi = (-1)^{p+1} \Delta^p H$ (that is, evolving according to the geometric polyharmonic heat flow (GPHF)). The proof of these equations is standard, and the interested reader
should feel free to read the classic paper of Huisken [47] in which similar calculations are derived for the mean curvature flow.

Lemma 1.2. Suppose that $f : \Sigma^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$ is a solution to the equation

$$\partial_t f = (-1)^{p+1} (\Delta^p H) \nu,$$

where $\nu$ is a chosen unit normal to $\Sigma$. Then the various geometric quantities evolve according to the following equations:

(a) Metric

$$\frac{\partial}{\partial t} g_{ij} = 2 (-1)^{p+1} \Delta^p H A_{ij},$$

(b) Inverse Metric

$$\frac{\partial}{\partial t} g^{ij} = 2 (-1)^p \Delta^p H A^{ij},$$

(c) Unit Normal

$$\frac{\partial}{\partial t} \nu = (-1)^p \nabla \Delta^p H,$$

(d) Second Fundamental Form

$$\frac{\partial}{\partial t} A_{ij} = (-1)^p \nabla_{ij} \Delta^p H + (-1)^{p+1} \Delta^p H A^i_s A^s_j,$$

(e) Mean Curvature

$$\frac{\partial}{\partial t} H = (-1)^p \left( \Delta^{p+1} H + \Delta^p H \mid A \mid^2 \right), \text{ and}$$

(f) Christoffel Symbols

$$\frac{\partial}{\partial t} \Gamma^k_{ij} = \nabla_{(2p)} A * \nabla A + \nabla_{(2p+1)} A * A.$$

Additionally, the enclosed surface area and enclosed volume of $\Sigma$ evolve according to the equations

$$\frac{d}{dt} |\Sigma| = (-1)^{p+1} \int_\Sigma H \Delta^p H \, d\mu \text{ and } \frac{d}{dt} \text{Vol}(\Sigma) = (-1)^{p+1} \int_\Sigma \Delta^p H \, d\mu,$$

respectively.
Proof. Since it does not complicate matters, we prove identities (a) through (f) for a general flow speed \( \varphi \). For (a), we calculate

\[
  \partial_t g_{ij} = 2 (\partial_i \partial_t f, \partial_j f) \\
  = 2 (\partial_i (\varphi \nu), \partial_j f) \\
  = 2 (\partial_i \varphi \nu + \varphi \partial_i \nu, \partial_j f) \\
  = 2 \varphi A_i^k (\partial_k f, \partial_j f) \\
  = 2 \varphi A_{ij},
\]

where we have use the fact that \( \nu \) is a section of the normal bundle \( Tf(\Sigma)^\perp \) and also the identity \( \partial_j \nu = A_j^k \partial_k f \) from (C.10). For (b), note that differentiating the formula \( g_{is} g^{sj} = \delta_i^j \) gives

\[
  0 = \frac{\partial}{\partial t} (g_{is} g^{sj}) = \frac{\partial}{\partial t} (g_{is}) g^{sj} + g_{is} \frac{\partial}{\partial t} (g^{is}) = 2 \varphi A_{is} g^{sj} + g_{is} \frac{\partial}{\partial t} (g^{is}), \quad \text{by \,(a).}
\]

Rearranging and multiplying both sides by \( g^{pi} \) then gives (b). For (c), we first note that the identity \( (\nu, \nu) \equiv 1 \) implies that both \( \partial_t \nu \) and \( \partial_i \nu \) are in the pushforward of the tangent bundle, \( df(T\Sigma) \), for \( i = 1, 2, \ldots, n \). Hence

\[
  \frac{\partial}{\partial t} \nu = \frac{\partial}{\partial t} \nu \\
  = g^{ij} (\partial_i \nu, \partial_j f) \partial_j f \\
  = -g^{ij} (\nu, \partial_i \partial_t f) \partial_j f \\
  = -g^{ij} (\nu, \partial_i (\varphi \nu)) \partial_j f \\
  = -g^{ij} (\nu, \partial_i \varphi \nu + A_i^k \partial_k f) \partial_j f \\
  = -g^{ij} \partial_i \varphi \partial_j f \\
  =: -\nabla \varphi.
\]
Here we have again used the identity \( \partial_t \nu = A^k_i \partial_k f \). To prove (d), we compute

\[
\frac{\partial}{\partial t} A_{ij} = -\frac{\partial}{\partial t} (\partial_{ij} f, \nu) = - (\partial_{ij} (\varphi \nu), \nu) - (\partial_{ij} f, \partial_t \nu) = -\partial_{ij} \varphi - \varphi (\partial_{ij} f, \partial_t f) = -\partial_{ij} \varphi - \varphi \left( \partial_i A^k_j f + A^k_i \partial_k f, \nu \right) + g^{kl} \partial_k \varphi \Gamma^s_{ij} g_{sl} = - (\partial_{ij} \varphi + \Gamma_{ij}^k \partial_k \varphi) + \varphi A^k_i A_{kj} = -\nabla_{ij} \varphi + \varphi A^k_i A_{kj}.
\]

Here we have used the fact \( (\partial_{ij} f)^T = \Gamma_{ij}^k \partial_k f \) implies that

\[
(\partial_{ij} f, \partial_t f) = \Gamma_{ij}^s (\partial_s f, \partial_t f) = \Gamma_{ij}^s g_{sl}.
\]

For (e) we use the chain rule, along with (b) and (d):

\[
\frac{\partial}{\partial t} H = \frac{\partial}{\partial t} (g^{ij}) A_{ij} + g^{ij} \frac{\partial}{\partial t} (A_{ij}) = -2 \varphi A^{ij} A_{ij} + g^{ij} \left( -\nabla_{ij} \varphi + \varphi A^k_i A_{kj} \right) = -2 \varphi |A|^2 - \Delta \varphi + \varphi |A|^2 = - (\Delta \varphi + \varphi |A|^2).
\]

For (f), we have (by the definition of the Christoffel symbols)

\[
\partial_t \Gamma_{ij}^k = \frac{1}{2} g^{kl} \left( \partial_l g_{ij} + \partial_j g_{il} - \partial_i g_{lj} \right) + \frac{1}{2} g^{kl} \left( \partial_l \dot{g}_{ij} + \partial_j \dot{g}_{il} - \partial_i \dot{g}_{lj} \right).
\]

It is worth noting that although \( \Gamma \) is not a tensor (it does not transform as one under a change of coordinates), \( \partial_t \Gamma \) is, in fact, a tensor. Hence choosing an orthonormal frame in which (at a single point) \( g_{ij} = \delta_{ij} \) and \( \partial_j g_{jk} = 0 \) for all indices \( i, j, k \), the previous
identity becomes

\[ \frac{\partial}{\partial t} \Gamma^k_{ij} = 0 + (\partial_i (\varphi A_{jk}) + \partial_j (\varphi A_{ik}) - \partial_k (\varphi A_{ij})) \]

\[ = \partial_i \varphi A_{jk} + \varphi \partial_j A_{ik} + \varphi \partial_j A_{ik} - \partial_k \varphi A_{ij} - \varphi \partial_k A_{ij} \]

\[ = \partial_i \varphi A_{jk} + \partial_j \varphi A_{ik} - \partial_k \varphi A_{ij} + \varphi \partial_k A_{ij}. \]

Here we have used that in this local frame the Codazzi identity (C.5) implies

\[ \partial_i A_{jk} = \partial_j A_{ik} = \partial_k A_{ij}. \]

Therefore in an arbitrary frame we have

\[ \frac{\partial}{\partial t} \Gamma^k_{ij} = \nabla_i \varphi A^k_j + \nabla_j \varphi A^k_i - \nabla^k \varphi A_{ij} + \varphi \nabla^k A_{ij}, \]

which proves (f).

For the penultimate assertion, recall that the induced area element on \( \Sigma \) is given by

\[ d\mu = \sqrt{\det g} dp. \]

To compute the time derivative of the right hand side, we will need a formula that describes how to differentiate the determinant of a matrix. Using Jacobi’s formula (see [67]), for a given matrix \( S = (S_{ij}) \), the following holds:

\[ \frac{\partial}{\partial t} \det S = \det S S^{ij} \frac{\partial}{\partial t} (S_{ij}). \]

Applying this to our case, we have

\[ \frac{\partial}{\partial t} d\mu = \frac{\partial}{\partial t} (\sqrt{\det g} dp) \]

\[ = \frac{1}{2 \sqrt{\det g}} \frac{\partial}{\partial t} \det g dp \]

\[ = \frac{1}{2 \sqrt{\det g}} \det g g^{ij} \frac{\partial}{\partial t} g_{ij} dp \]
\[= \sqrt{\det g \left( \varphi g^{ij} A_{ij} \right)} \, dp\]
\[= H \varphi \, d\mu\]

Differentiating under the integral then gives

\[\frac{d}{dt} |\Sigma| = \int_{\Sigma} \frac{\partial}{\partial t} \, d\mu = \int_{\Sigma} H \varphi \, d\mu = (-1)^{p+1} \int_{\Sigma} H \Delta^p H \, d\mu,\]

which is the penultimate assertion. The last assertion (regarding the evolution of the enclosed volume of \( f(\Sigma) \)) is well-established, but we will give an outline here. There are a number of different proofs (see, for example the argument of Lemma 2.1 from Barbosa, do Carmo, and Eschenburg [14]). However seeing as our ambient space is Euclidean there is a simple proof that makes use of the divergence theorem.

Recall that the divergence theorem says that for a \( \mathbb{R}^{n+1} \)-valued vector field on an open set \( \Omega \subset \mathbb{R}^{n+1} \) with boundary \( \partial \Omega \), we have

\[\int_{\Omega} \text{div} \, X \, dV = \int_{\partial \Omega} (X, \nu^0) \, dA.\]

Here \((\cdot, \cdot)\) is the regular inner product in \( \mathbb{R}^{n+1} \) and \( \nu^0 \) is the outer unit normal to \( \partial \Omega \) in \( \mathbb{R}^{n+1} \). The divergence here in the left integral is the usual one in \( \mathbb{R}^3 \) given by

\[\text{div} \left( X^1, X^2, \ldots, X^{n+1} \right) = \sum_{i=1}^{n+1} \partial_i X^i.\]

Therefore if we let \( X = \frac{1}{n+1} \vec{x} \) where \( \vec{x} \) is the position vector, then \( X^i = \frac{1}{n+1} x^i \) and direct computation gives

\[\text{div} X = \sum_{i=1}^{n+1} \partial_i X^i = \frac{1}{n+1} \sum_{i=1}^{n+1} \delta^i_i = 1\]

which means that

\[\text{Vol}(\Omega) = \int_{\Omega} \text{div} X \, dV = \int_{\Omega} dV = \frac{1}{n+1} \int_{\Omega} \text{div} \vec{x} \, dV = \frac{1}{n+1} \int_{\partial \Omega} (\vec{x}, \nu^0) \, dA.\]
Therefore if \( f : \Sigma^n \to \mathbb{R}^{n+1} \) is an immersion with outer unit normal \( \nu \) then \( \vec{x} = f \), and

\[
\text{Vol}(\Sigma) = \frac{1}{n+1} \int_{\Sigma} (f, \nu) \, d\mu. \tag{1.26}
\]

We then use our other evolution equations to calculate the evolution of \( \text{Vol}(\Sigma) \):

\[
\frac{d}{dt} \text{Vol}(\Sigma) = \frac{1}{n+1} \int_{\Sigma} (\partial_t f, \nu) + (f, \partial_t \nu) \, d\mu + \frac{1}{n+1} \int_{\Sigma} (f, \nu) \partial_t \, d\mu
\]

\[
= \frac{1}{n+1} \int_{\Sigma} \varphi \, d\mu - \frac{1}{n+1} \int_{\Sigma} (f, \nabla \varphi) \, d\mu + \frac{1}{n+1} \int_{\Sigma} \varphi \, (f, \nabla \mu) \, d\mu
\]

\[
= \frac{1}{n+1} \int_{\Sigma} \varphi \, d\mu - \frac{1}{n+1} \int_{\Sigma} (f, \nabla \varphi) \, d\mu + \frac{1}{n+1} \int_{\Sigma} \varphi \, (f, \Delta f) \, d\mu
\]

\[
+ \frac{1}{n+1} \int_{\Sigma} (f, \nabla \varphi) \, d\mu
\]

\[
= \int_{\Sigma} \varphi \, d\mu
\]

\[
= (-1)^{p+1} \int_{\Sigma} \Delta^p H \, d\mu.
\]

Here we have used integration by parts, as well as the identity \( \nabla_i f^\alpha \nabla^i f^\alpha = g_{ij} g^{ij} = n \).

This proves the last assertion. \( \square \)

**Corollary 1.3.** Under the geometric polyharmonic heat flow (GPHF) the enclosed surface area is non-increasing in time and the enclosed volume is stationary in time.

**Proof.** The proof of both claims follow almost immediately from the last pair of equations in Lemma 1.2.

To see this, first observe that applying integration by parts to the evolution equation for enclosed area \( p \) times gives

\[
\frac{d}{dt} |\Sigma| = (-1)^{p+1} \int_{\Sigma} H \Delta^p H \, d\mu = \int_{\Sigma} |\Delta^{\frac{p}{2}} H|^2 \, d\mu \leq 0.
\]

Here (with abuse of notation) we have used \( \Delta^{\frac{l}{2}} := \nabla \) and \( \Delta \) acts from right to left so that for \( l \in \mathbb{N}, \Delta^{\frac{2l+1}{2}} = \Delta^{\frac{1}{2}} \circ \Delta^l = \nabla \Delta^l \).
Finally, the claim that the enclosed volume of $\Sigma$ is stationary in time follows from the corresponding evolution equation from Lemma 1.2 and the divergence theorem:

$$\frac{d}{dt} \text{Vol} (\Sigma) = (-1)^{p+1} \int_{\Sigma} \Delta^p H \, d\mu = (-1)^{p+1} \int_{\Sigma} \text{div}_\Sigma \nabla \Delta^{p-1} H \, d\mu = 0.$$  

Note that here div$_\Sigma$ refers to the tangential divergence, which is different from the ambient divergence used above.

\[\square\]

1.4 Evolution equations for higher order derivatives of curvature

Next we need to see how higher order derivatives of curvature evolve in time. It is worth noting that writing these evolution equations in terms of the $\ast-$notation is sufficient for our purposes. That is to say, we do not need to go down to the level of every precise contraction that arises.

We first note the following identity which holds for any tensor field $T$:

$$\partial_t \nabla_{(k)} T = \nabla_{(k)} \partial_t T + \sum_{i=0}^{k-1} \nabla_{(i)} \left( X \ast \nabla_{(k-1-i)} T \right) \quad (1.27)$$

Here

$$X := \partial_t \Gamma = \nabla_{(2p)} A \ast \nabla A + \nabla_{(2p+1)} A \ast A.$$  

To demonstrate how one proves (1.27) note that for the case $k = 1$ one has

$$\partial_t \nabla T = \partial_t (\partial T + \Gamma \ast T) = \nabla \partial_t T + \partial_t \Gamma \ast T. \quad (1.28)$$

Since this holds for any tensor $T$, identity (1.27) can be derived easily by mathematical induction along with (1.28).
We also use the identity
\[
\nabla (k) \Delta^{p+1} T = \nabla (k-1) \Delta^{p+1} \nabla T + \sum_{i=0}^{2p+k} \nabla (i) \left( \nabla (2p+k-i) T \ast \text{Riem} \right)
\]
\[
= \nabla (k-1) \Delta^{p+1} \nabla T + \sum_{i=0}^{2p+k} \nabla (i) \left( \nabla (2p+k-i) T \ast A \ast A \right),
\] (1.29)

which can be obtained by multiple applications of the formula for the interchange of covariant derivatives (see equations (1.19) and (1.20)). Here \text{Riem} denotes the Riemannian curvature tensor introduced earlier. The identity (1.29) can be used repeatedly to obtain
\[
\nabla (k) \Delta^{p+1} A = \Delta^{p+1} \nabla (k) A + \sum_{i=0}^{2p+m} \nabla (i) \left( \nabla (2p+m-i) A \ast A \ast A \right).
\]

We now have the ability to calculate the evolution of any covariant derivative of curvature. We do so in the following lemma.

**Lemma 1.4.** For immersions \( f : \Sigma^n \times [0,T) \to \mathbb{R}^{n+1} \) evolving according to the geometric polyharmonic heat flow (GPHF) we have
\[
\partial_t \nabla (m) A = (-1)^p \nabla^{i_{p+1} \ldots i_1} \nabla_{i_1 \ldots i_{p+1}} \nabla (m) A + \sum_{i=0}^{2p+m} \nabla (i) \left( \nabla (2p+m-i) A \ast A \ast A \right).
\] (1.30)

Therefore
\[
\frac{\partial}{\partial t} \left| \nabla (m) A \right|^2 = 2 (-1)^p \left( \nabla^{i_{p+1} \ldots i_1} \nabla_{i_1 \ldots i_{p+1}} \nabla (m) A, \nabla (m) A \right)
\]
\[
+ \sum_{i=0}^{2p+m} \nabla (i) \left( \nabla (2p+m-i) A \ast A \ast A \right) \ast \nabla (m) A.
\] (1.31)

**Proof.** To prove (1.30) we first use the identity (1.27) with \( T = A \):
\[
\partial_t \nabla (m) A = \nabla (m) \partial_t A + \sum_{i=0}^{m-1} \nabla (i) \left( X \ast \nabla (m-i-1) A \right),
\] (1.32)
where, as before,
\[ X := \partial_t \Gamma = \nabla_{(2p)} A * \nabla A + \nabla_{(2p+1)} A * A. \]

Next note that all the terms in the summation on the right hand side of (1.32) can be absorbed into the summation
\[ \sum_{i=0}^{2p+m} \nabla_{(i)} \left( \nabla_{(2p+m-1)} A * A * A \right). \]

Next by (d) from Lemma 1.2 we have
\[ \partial_t A = (-1)^p \nabla_{(2p)} \Delta^p H + \nabla_{(2p)} A * A * A, \]

where the first term on the right hand side is the exact term that appears in the evolution (and so in particular has not been condensed into $P-$style notation). Hence
\[ \partial_t \nabla_{(m)} A = (-1)^p \nabla_{(m+2)} \Delta^p H + \nabla_{(m)} \left( \nabla_{(2p)} A * A * A \right) + \sum_{j=0}^{m-1} \nabla_{(j)} \left( \partial_t \Gamma * \nabla_{(m-1-j)} A \right). \]

We next utilise the Riemann curvature tensor $\text{Riem}$ which can be written as $\text{Riem} = A * A$ in $*-$notation. Recall that by the formula for the interchange of covariant derivatives (1.20) one has
\[ \nabla_{ij} Y_k - \nabla_{ji} Y_k = R_{ijkl} g^{lm} Y_m = A * A * Y. \]

for covectors $Y$. Hence interchanging the $s^{th}$ and $(s + 1)^{th}$ covariant derivatives in $\nabla_{(N)} A$ (where $N$ is an arbitrary natural number) has the following effect:
\[
\nabla_{i_1 \ldots i_{s-1} i_{s+1} \ldots i_N} A_{pq} = \nabla_{i_1 \ldots i_{s-1}} \left( \nabla_{s} A_{pq,i_N} g^{i_{s+0} \ldots i_N} \right) = \nabla_{i_1 \ldots i_{s-1} \left( \nabla_{s+1} A_{pq,i_N} g^{i_{s+0} \ldots i_N} \right) + R_{i_s i_{s+1} \ldots i_N} g^{i_{s+1} i_{s+2}} A_{pq,i_N} g^{i_{s+0} \ldots i_N} \]
\]
We have used the brace to point out that the order of the $s^{th}$ and $(s+1)^{th}$ covariant derivatives has been reversed. Next we employ (1.34) repeatedly to deal with the first term on the right hand side of (1.33). We intend to establish an identity of the form

$$\nabla_{(m+2)} \Delta^p H = \Delta^{p+1} \nabla_{(m)} A + P_3^{i,j,k}(A).$$

Using an underbrace to highlight which covariant derivatives are being interchanged, we calculate

$$\left(\nabla_{(m+2)} \Delta^p H\right)_{\alpha_1 \ldots \alpha_{m+2}} = g^{i_1 j_1} \ldots g^{i_{p} j_{p}} \nabla_{\alpha_1 \ldots \alpha_{m+1}} \underbrace{\nabla_{\alpha_{m+2} j_1}} \nabla_{j_1 \ldots j_p} H$$

$$= g^{i_1 j_1} \ldots g^{i_{p} j_{p}} \nabla_{\alpha_1 \ldots \alpha_{m+1}} \underbrace{\nabla_{1,0 \ldots m+2}} \nabla_{j_1 \ldots j_p} H + \nabla_{(m+1)} \left(\nabla_{(2p-1)} A * A * A\right)$$

$$= g^{i_1 j_1} \ldots g^{i_{p} j_{p}} \nabla_{\alpha_1 \ldots \alpha_{m+1}} \underbrace{\nabla_{j_1 0 \ldots m+2}} \nabla_{j_1 \ldots j_p} H + \nabla_{(m+1)} \left(\nabla_{(2p-1)} A * A * A\right)$$

$$= g^{i_1 j_1} \ldots g^{i_{p} j_{p}} \nabla_{\alpha_1 \ldots \alpha_{m+1}} \underbrace{\nabla_{j_1 0 \ldots m+2}} \nabla_{j_1 \ldots j_p} H + \nabla_{(m+2)} \left(\nabla_{(2p-2)} A * A * A\right)$$

$$+ \nabla_{(m+1)} \left(\nabla_{(2p-1)} A * A * A\right)$$

$$\vdots$$

$$= g^{i_1 j_1} \ldots g^{i_{p} j_{p}} \nabla_{\alpha_1 \ldots \alpha_{m+1}} \nabla_{i_1 \ldots i_p \alpha_{m+2}} H + \sum_{j=m+1}^{2p+m} \nabla_{(j)} \left(\nabla_{(2p+m-j)} A * A * A\right)$$

$$= \left(\nabla_{(m+1)} \Delta^p \nabla H\right)_{\alpha_1 \ldots \alpha_{m+2}} + \sum_{j=m+1}^{2p+m} \nabla_{(j)} \left(\nabla_{(2p+m-j)} A * A * A\right).$$

Similarly,

$$\left(\nabla_{(m+1)} \Delta^p \nabla H\right)_{\alpha_1 \ldots \alpha_{m+2}} = \left(\nabla_{(m)} \Delta^p \nabla H\right)_{\alpha_1 \ldots \alpha_{m+2}} + \sum_{j=m}^{2p+m-1} \nabla_{(j)} \left(\nabla_{(2p+m-j)} A * A * A\right),$$
and in general, 

\[
(\nabla_{(k}) \Delta^p \nabla_{(m+2-k)} H)_{\alpha_1...\alpha_{m+2}} \\
= (\nabla_{(k-1}) \Delta^p \nabla_{(m+3-k)} H)_{\alpha_1...\alpha_{m+2}} + \sum_{j=k-1}^{2p+k-2} \nabla_{(j)} (\nabla_{(2p+m-j)} A * A * A). \tag{1.35}
\]

Applying (1.35) repeatedly, we conclude that

\[
\nabla_{(m+2)} \Delta^p H = \Delta^p \nabla_{(m+2)} H + \sum_{j=0}^{2p+m} \nabla_{(j)} (\nabla_{(2p+m-j)} A * A * A). \tag{1.36}
\]

Next note that by Gauss’ equation (1.20) and using the fact that \( H = \langle g, A \rangle = g^{ij} A_{ij} \), one obtains a ‘Bochner-type’ formula:

\[
(\nabla_{(2}) H)_{ij} = g^{st} \nabla_{is} A_{jt} \\
= g^{st} \nabla_{is} A_{jt} \\
= g^{st} (\nabla_{is} A_{jt} + Rm * A) \\
= g^{st} \nabla_{is} A_{jt} + A * A * A \\
= g^{st} \nabla_{is} A_{jt} + A * A * A \\
= (\Delta A)_{ij} + A * A * A.
\]

Here we have also used the Codazzi equations (C.5), which say that the \((0, 3)\)–tensor \( \nabla A \) is totally symmetric. Hence

\[
\nabla_{(2)} H = \Delta A + A * A * A. \tag{1.37}
\]

(We actually have more control over the exact structure of these *–terms then we are letting on, see for example Lemma 5.8. However, the above form will be enough for now). The exact formulation of (1.37) (without the *–notation) is known in the
literature as Simons’ identity [92]. Using (1.37), (1.36) then becomes

$$\nabla_{(m+2)} \Delta^p H = \Delta^p \nabla_{(m)} \Delta A + \sum_{j=0}^{2p+m} \nabla(j) \left( \nabla(2p+m-j)A \ast A \ast A \right). \quad (1.38)$$

We then proceed to interchange the order of the covariant derivatives repeatedly on the first term on the right hand side of (1.38) as we did earlier, keeping track of the $P-$style terms that arise in the process. We have

$$\Delta^p \nabla_{\alpha_1 \ldots \alpha_m} \Delta A_{\alpha_{m+1} \alpha_{m+2}}$$

$$= g^{i_p+1 \ldots i_{p+1}} \Delta^p \nabla_{i_p+1} \nabla_{\alpha_1 \ldots \alpha_m} \nabla_{i_{p+1}} A_{\alpha_{m+1} \alpha_{m+2}} + \sum_{j=2p}^{2p+m-1} \nabla(j) \left( \nabla(2p+m-j)A \ast A \ast A \right)$$

$$= \Delta^{p+1} \nabla_{\alpha_1 \ldots \alpha_m} A_{\alpha_{m+1} \alpha_{m+2}} + \sum_{j=2p}^{2p+m} \nabla(j) \left( \nabla(2p+m-j)A \ast A \ast A \right).$$

Hence (1.38) becomes

$$\nabla_{(m+2)} \Delta^p H = \Delta^{p+1} \nabla_{(m)} A + \sum_{j=0}^{2p+m} \nabla(j) \left( \nabla(2p+m-j)A \ast A \ast A \right).$$

Substituting this into (1.33),

$$\partial_t \nabla_{(m)} A = (-1)^p \Delta^{p+1} \nabla_{(m)} A + \sum_{j=0}^{2p+m} \nabla(j) \left( \nabla(2p+m-j)A \ast A \ast A \right).$$

This is the first claim of the lemma. To prove (1.31), first note that using the same method as before to interchange the order of covariant derivatives yields

$$\left( \Delta^{p+1} \nabla_{(m)} A \right)_{\alpha_1 \ldots \alpha_{m \ast \ast}}$$

$$= \nabla^{i_{p+1} \ldots i_{p+1}} \nabla_{i_1 i_2 \ldots i_{p+1}} \nabla_{\alpha_1 \ldots \alpha_m} A_{\alpha_{m+1} \alpha_{m+2}} + \sum_{j=0}^{2p+m} \nabla(j) \left( \nabla(2p+m-j)A \ast A \ast A \right).$$
1.4. Evolution Equations for Higher Order Derivatives of Curvature

Hence

$$\partial_t \nabla^{(m)} A = (-1)^p \nabla^{i_{p+1} i_p \ldots i_2 i_1} \nabla_{i_1 i_2 \ldots i_p i_{p+1}} \nabla^{(m)} A + \sum_{j=0}^{2p+m} \nabla^{(j)} \left( \nabla^{(2p+m-j)} A * A * A \right).$$  \hspace{1cm} (1.39)

We use (1.39) to calculate the evolution of $\|\nabla^{(m)} A\|^2$:

$$\frac{\partial}{\partial t} \|\nabla^{(m)} A\|^2 = \frac{\partial}{\partial t} \left( g^{i_1 j_1} \ldots g^{i_p j_p} g^{st} g^{uv} \nabla_{i_1 \ldots i_m} A_{su} \nabla_{j_1 \ldots j_m} A_{tv} \right)$$

$$= 2 \langle \nabla^{(m)} A, \partial_t \nabla^{(m)} A \rangle + \partial_t \left( g^{i_1 j_1} \ldots g^{i_p j_p} g^{st} g^{uv} \nabla_{i_1 \ldots i_m} A_{su} \nabla_{j_1 \ldots j_m} A_{tv} \right)$$

$$= 2 \langle \nabla^{(m)} A, (-1)^p \nabla^{i_{p+1} i_p \ldots i_2 i_1} \nabla_{i_1 i_2 \ldots i_p i_{p+1}} \nabla^{(m)} A \rangle$$

$$+ \nabla^{(m)} A * \sum_{j=0}^{2p+m} \nabla^{(j)} \left( \nabla^{(2p+m-j)} A * A * A \right) + \left( \nabla^{(2p)} A * A \right) * \nabla^{(m)} A * \nabla^{(m)} A$$

$$= 2 (-1)^p \langle \nabla^{(m)} A, \nabla^{i_{p+1} i_p \ldots i_2 i_1} \nabla_{i_1 i_2 \ldots i_p i_{p+1}} \nabla^{(m)} A \rangle$$

$$+ \sum_{j=0}^{2p+m} \nabla^{(j)} \left( \nabla^{(2p+m-j)} A * A * A \right) * \nabla^{(m)} A.$$

We have also used the fact that by (b) of Lemma 1.2,

$$\partial_t g^{ij} = 2 (-1)^p \Delta^p HA^{ij} = \nabla^{(2p)} A * A \text{ (in *-notation).}$$

This completes the proof. \qed
Chapter 2

Well-posedness of the flow

Before stating (and proving) our main results, which culminates in longtime exponential convergence to spheres, it is necessary for one to ask if a solution to our flow exists for positive time at all. The answer is of course a resounding \textit{yes}. We will first state the main theorem of this chapter, before laying down the groundwork for operating on sections of vector bundles over manifolds. By relying upon the earlier work of Baker in his PhD. thesis [12] we state and give an outline to the proof of an existence result pertaining to a general class of quasilinear strongly parabolic operators acting on vector bundles. Because of this, we are able to state the existence results for higher codimensions. Originally when writing this thesis, the author chose to recast (GPHF) as an equivalent parabolic scalar partial differential equation acting on some graph function and then use the known scalar existence results from Mantegazza and Martinazzi [69] to prove Theorem 2.1 for the hypersurface case. This approach at first seemed desirable mainly because it was simple. However, we noted that introducing higher codimensions into the picture does not cause any fundamental difficulty, and so have decided to include that scenario here.

After that we will show that our flow is weakly parabolic in nature, and by using the DeTurck trick show that it is equivalent to a strongly parabolic flow modulo a family of time-dependent diffeomorphisms corresponding to a polyharmonic map heat
flow. From this we conclude existence and uniqueness for the initial value problem

\[
\begin{cases}
\partial_t f = (-1)^p (\Delta^{\perp})^p \vec{H} \cdot \nu, \\
f (\Sigma, 0) = \Sigma_0.
\end{cases}
\]

Note that we have to be careful when defining the mean curvature vector in higher codimensions. In this paper we use the following convention. First note that for immersions \( f : \Sigma^n \to \mathbb{R}^{n+k} \), at any point in \( f(\Sigma) \) we can choose \( k \) orthonormal vectors \( \{\nu_\lambda\}_{\lambda=1}^k \) to span the normal bundle. Next note that the ordinary derivative in \( \mathbb{R}^{n+k} \) and the covariant derivative \( \nabla \) are related by the following formula:

\[
\nabla_X Y = D_X Y - \sum_\lambda (D_X Y, \nu_\lambda) \nu_\lambda.
\]

We define the second fundamental form in this case by the \( \mathbb{R}^{n+k} \)-valued \((0, 2)\) tensor field that satisfies the following identity:

\[
A(X, Y) = -\sum_\lambda (D_X Y, \nu_\lambda) \nu_\lambda.
\]

Component-wise one has

\[
A_{ij} = \sum_\lambda A^\lambda_{ij} \nu_\lambda,
\]

where \( A^\lambda \) is the ordinary second fundamental form corresponding to the normal vector \( \nu_\lambda \):

\[
A^\lambda_{ij} := - (\partial_{ij} f, \nu_\lambda).
\]

The mean curvature vector \( \vec{H} \) is then defined to be

\[
\vec{H} := -g^{ij} A_{ij} = -g^{ij} \sum_\lambda A^\lambda_{ij} \nu_\lambda = -H^\lambda \nu_\lambda.
\]

Here \( H^\lambda \) denotes the mean curvature of \( \Sigma \) with respect to the normal vector \( \nu_\lambda \).

In this setting, our geometric polyharmonic heat flow becomes a one-parameter
family of immersions \( f : \Sigma^n \times [0, T) \to \mathbb{R}^{n+k} \) satisfying
\[
\partial_t f = (-1)^p (\Delta^{\perp})^p \vec{H}, \quad p \in \mathbb{N}.
\] (2.1)

Here \((\Delta^{\perp})^p \vec{H}\) denotes the \(p\)th repeated iteration of the induced Laplacian in the normal bundle, \(\Delta^{\perp}\), applied to the mean curvature vector \(\vec{H}\).

Without further ado we present the main theorem for this section.

**Theorem 2.1** (Short time existence for the geometric polyharmonic heat flow). Let \( f_0 : \Sigma^n \to \mathbb{R}^{n+k} \) \((k \geq 1)\) be a smooth immersion with \( f_0(\Sigma) = \Sigma_0 \). Then there exists a maximal time \( 0 < T \leq \infty \) of existence and corresponding unique one-parameter smooth immersions \( f : \Sigma^n \times [0, T) \to \mathbb{R}^{n+k} \) satisfying the initial value problem
\[
\begin{cases}
\partial_t f = (-1)^p (\Delta^{\perp})^p \vec{H}, \\
f(\Sigma, 0) = f_0(\Sigma) = \Sigma_0,
\end{cases}
\] (2.2)

and
\[ f (\cdot, t) \to f_0 (\cdot) \text{ as } t \searrow 0, \]
where the convergence is locally smooth in the \(C^\infty\) topology.

## 2.1 Parabolic operators on vector bundles

In this section we introduce some of the basic concepts involved in vector bundles and parabolic operators. This exposition is certainly not exhaustive, and is just to introduce enough information so that the unfamiliar reader can follow along with the statement and outline of the proof of our existence theorem in the next section (Theorem 2.2).

Recall that in Chapter 1 we introduced the idea gluing together the tangent spaces at all point of our manifold \( \Sigma^n \) to form the *tangent bundle* \( T\Sigma \). Since the tangent space at each point is a vector space, we can therefore think of the tangent bundle as a local
parameterisation of a vector space (the tangent space, which is homomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$) over the base manifold $\Sigma$. The tangent bundle is a specific example of a mathematical structure called a vector bundle, which we now define.

A $k$–dimensional vector bundle consists of a total space $E$, a base $\Sigma$ (both of which are differentiable manifolds) and a differentiable surjective projection $\pi : E \to \Sigma$ satisfying the following conditions:

1. For each $x \in \Sigma$, the set $E_x := \pi^{-1}(x)$ (called the fibre of $E$ over $x$) has the structure of a vector space.

2. For each $x \in \Sigma$ there is a neighbourhood $U_x$ containing $x$ and a diffeomorphism $\varphi : \pi^{-1}(U_x) \to U_x \times \mathbb{R}^k$ called a local trivialisation of $E$, such that $(\pi_1 \circ \varphi)(x) = \pi(x)$ for every $x \in \Sigma$ (where $\pi_1 : U_x \times \mathbb{R}^k \to U_x$ is the projection onto the first factor).

3. The restriction of $\varphi$ to each fibre, $\varphi\big|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k$, is a linear isomorphism.

If $\pi : E \to \Sigma$ is a vector bundle over $\Sigma$, we define a section of $E$ to be a map $s : \Sigma \to E$ such that $\pi \circ s = \text{id}_\Sigma$, the identity map on $\Sigma$. We denote the space of smooth sections of $E$ by $\Gamma(E)$: this space is an (infinite-dimensional) vector space under pointwise addition and multiplication by scalar functions. Note that in the specific case of the tangent bundle $T\Sigma$ over the manifold $\Sigma$, the sections $s : \Sigma \to T\Sigma$ simply map points $x \in \Sigma$ into the associated tangent space at $x$, $T_x\Sigma$.

Let $(E, \pi, \Sigma)$ be a vector bundle. A bundle metric is given by a family of scalar products on the fibre $E_x = \pi^{-1}(x)$, depending smoothly on $x \in \Sigma$. For example, if $\Sigma$ is a Riemannian manifold, and $E = T\Sigma$ is the tangent space then the Riemannian metric $g$ gives the canonical example of a bundle metric.

Let $E$ and $\tilde{E}$ be two vector bundles over $\Sigma$. We associate Latin letters $i, j, k, \ldots$ with $\Sigma$ and Greek letters $\alpha, \beta, \gamma, \ldots$ with $E, \tilde{E}$. Let $\{x^i\}$ be a local coordinate system for $\Sigma$, and that $\{e_\alpha\}$ is a local frame for the bundle $E$ over $\Sigma$ with associated local
coframe \( \{ \theta^\beta \} \). A differential operator of order \( 2m \) is a map \( P : \Gamma (E) \to \Gamma (\tilde{E}) \) which in any local coordinate chart can be written in the form

\[
(P (U))^\alpha = \frac{\partial}{\partial t} U^\alpha + (-1)^m \left( A_\beta^{\alpha i_1 \cdots i_{2m}} \partial_{i_1 \cdots i_{2m}} U^\beta + \cdots + B_\beta^{\alpha i} \partial_i U^\beta + C_\beta^\alpha U^\beta \right),
\]

which we will abbreviate to

\[
P (U) := \frac{\partial}{\partial t} U + (-1)^m \sum_{|I| \leq 2m} A^I \partial_I U. \tag{2.3}
\]

Here for each \( I \leq 2m \) \( A^I \in \Gamma (T^0 (\Sigma) \otimes E \otimes E^*) \). We have also used multi-index notation, where we recall that a \( k \)-dimensional multi-index is a \( k \)-tuple \( I = (i_1, i_2, \ldots, i_k) \) and

\[
|I| = i_1 + i_2 + \cdots + i_k,
\]

so that

\[
\sum_{|I| \leq 2m} A^I \partial_I U = \sum_{i_1+i_2+\cdots+i_k} A^{i_1 i_2 \cdots i_k} \partial_{i_1 i_2 \cdots i_k} U.
\]

If the leading term \( A^{i_1 \cdots i_{2m}} \) depends at most on \( x, t, U, \ldots, \nabla (2m-1)U \), then the operator \( P \) is said to be quasilinear.

In order to define parabolicity for nonlinear operators of the form (2.3), we consider the associated linearised operator. The linearisation of an operator \( P \) about a function \( U_0 \) in the direction \( V \) is the linear operator \( \partial P [U_0] \) defined by

\[
\partial P [U_0] (V) := \frac{\partial}{\partial \lambda} P (U_0 + \lambda V) \bigg|_{\lambda = 0}.
\]

For quasilinear operators \( P \) of the form (2.3) the linearisation is

\[
\partial P [U_0] V^\alpha = \frac{\partial}{\partial \lambda} \left[ \frac{\partial}{\partial t} (U_0^\alpha + \lambda V^\alpha) \right] \bigg|_{\lambda = 0}
\]

\[
+ (-1)^m \frac{\partial}{\partial \lambda} \left[ \sum_{|I| \leq 2m} A_\beta^{\alpha I} (x, t, U_0 + \lambda V, \ldots, \nabla (2m-1)(U_0 + \lambda V)) \partial_I (U_0^\beta + \lambda V^\beta) \right] \bigg|_{\lambda = 0}
\]
2.2. SHORT-TIME EXISTENCE FOR QUASILINEAR PARABOLIC SYSTEMS

\[ \frac{\partial}{\partial t} V^\alpha + (-1)^m A^\alpha_{i_1\ldots i_{2m}} (x, t, U_0, \nabla U_0, \ldots \nabla_{(2m-1)} U_0) \nabla_{i_1\ldots i_{2m}} V^\beta + \text{lower order terms in } V'. \]

The principal symbol of the linearised operator \( \partial P[U_0] \) in the direction \( \xi \) is the vector bundle homomorphism given by

\[ \sigma(\partial P[U_0])(\xi) = A^\alpha_{i_1\ldots i_{2m}} \xi_{i_1} \ldots \xi_{i_{2m}} e_\alpha \otimes \theta^\beta \]

and is said to be weakly parabolic if the eigenvalues of the principal symbol are non-negative. If the eigenvalues are positive, then the linearised operator is instead said to be strongly parabolic. Many of the properties and behaviours of the operator \( P \) depend solely on the part of \( P \) containing the highest derivative. Since the principal symbol is a simple and invariant way to refer to this part of \( P \), it is an invaluable tool for the analysis of systems of partial differential equations.

Lastly, the linearised operator \( \partial P[U_0] \) is said to satisfy the Legendre-Hadamard condition with constant \( \lambda \) if there exists a positive constant \( \lambda \) such that

\[ A^\alpha_{i_1\ldots i_{2m}} \xi_{i_1} \ldots \xi_{i_{2m}} \eta^\alpha \eta^{2m} \geq \lambda |\xi|^{2m} |\eta|^2 \text{ for all } \xi, \eta \in \Gamma(E). \]

2.2 Short-time existence for quasilinear parabolic systems

In this section we state the main existence theorem for parabolic systems for vector bundles over a manifold \( \Sigma \). The theorem stated here was originally used by Baker in the context of proving the local existence for solutions of the mean curvature flow in
higher codimensions [12], and applied to the more general fully-nonlinear problem

\[
P(U) := \partial_t U - F(x, t, U, \nabla U, \ldots, \nabla_{2m} U) = 0 \text{ in } E \times (0, \omega), \]

\[
U(\Sigma, 0) = U_0 \in C^{2m,1,\alpha}(E \times (0, \omega)),
\]

where the leading coefficient of the linearisation of \( P \) satisfies conditions 1 – 4 in Theorem 2.2, but is not required to be quasilinear.

Here since we only require the result to hold for quasilinear operators, we have restated the results accordingly (this is not a problem since quasilinear operators are a sub-class of fully-nonlinear operators). It is stressed that the existence results here are not claimed as original, and even in the paper by Baker he mentions that his exposition closely follows the work of Tobias Lamm’s Diploma Thesis [60].

In order to state the existence theorem, we need to first introduce the appropriate function spaces. This requires some new notation. The notation used here is the same as that from the paper of Baker [12], although some of (and variations thereof) such as the parabolic Sobolev spaces are widely used in the mathematical literature, see for example [69].

If \( E \) is a vector bundle over a closed manifold \( \Sigma \), we use \( E_\omega \) and \( \Sigma_\omega \) (\( \omega > 0 \)) to denote the parabolic domains \( E \times (0, \omega) \) and \( \Sigma \times (0, \omega) \), respectively. Similarly, if \( \Omega \) is an open connected domain contained in some Euclidean space the we write \( P := \Omega \times (0, \omega) \).

Next, since we will be working with operators which are first order in time \( t \) but of order \( 2m \) in the spatial variables \( \{x^i\} \), we need a parabolic notion of distance. For \( X = (x, t), Y = (y, s) \in \Sigma_\omega \) we define the parabolic distance between \( X \) and \( Y \) to be

\[
d(X, Y) = \max \left\{ d_g(x, y), \left| t - s \right|^{1/2m} \right\},
\]

where \( d_g(x, y) \) is the geodesic distance between \( x \) and \( y \) measured by the metric \( g \) (see
If \( u : \Sigma_\omega \to \mathbb{R}^N \) and \( \alpha \in (0, 1) \) then the Hölder semi-norm is defined as
\[
[u]_{\alpha;P} := \sup_{X \neq Y \in \Sigma_\omega} \frac{|u(X) - u(Y)|}{d(X,Y)^\alpha},
\] (2.4)
and the Hölder norms as
\[
\|u\|_{2m,1;\alpha;P}^{2m,1} := \sum_{k=0}^{2m} ||\nabla^{(k)} u||_{0,P} + ||\partial_t u||_{0,P}
\] (2.5)
and
\[
\|u\|_{2m,1;\alpha;P} := \|u\|_{2m,1;P} + \|\nabla^{2m,1} u\|_{\alpha;P}.
\] (2.6)
Here \( \nabla^{2m,1} \) is shorthand for the mixed derivative \( \nabla^{(2m)} + \partial_t \). The space of functions \( \{ u \in C^{2m}(P) \mid \|u\|_{2m,1;\alpha;P} < \infty \} \) is a Banach space with respect to the norm \( \|u\|_{2m,1;\alpha;P} \) [12]. By replacing the covariant derivatives with spatial partial derivatives \( \partial_x \), one can very easily define the analogous spaces over open sets \( \Omega \times (0, \omega) \subset \mathbb{R}^N \times \mathbb{R} \).

We now introduce some relevant anisotropic Sobolev spaces. The set
\[
\{ u : \partial_x^i \partial_t^j u \in L^2(P) \text{ for } i + 2mj \leq 2m \}
\]
endowed with the norm
\[
\|u\|_{W^{2m,1}(P)} := \left( \int_{P} \int_{i+2mj \leq 2m} |\partial_x^i \partial_t^j u|^2 \, dx \, dt \right)^{1/2}
\]
is the Sobolev space denoted by \( W^{2m,1}_2(P) \). We can extend this notion to a manifold, in which the norm is exactly the same except spatial partial derivatives are replaced by covariant derivatives:
\[
\|u\|_{W^{2m,1}(E_\omega)} := \left( \int_{E_\omega} \sum_{i+2mj \leq 2m} |\nabla_x^i \partial_t^j u|^2 \, d\mu \, dt \right)^{1/2}
\]
The Sobolev in this case is denoted by \( W^{2m,1}_2(E_\omega) \). Similarly, we also use the following
spaces in which the highest-order spatial derivative is of order $m$:

$$
\|u\|_{W^{m,1}(E_\omega)} = \left( \int \int_{E_\omega} \sum_{i \leq m} |\nabla_x^i u|^2 + |\partial_t u|^2 \, d\mu \, dt \right)^{1/2}
$$

We denote the set of smooth functions that vanish near the spatial boundary

$$\{(x, t) \mid x \in \partial \Omega, t \in (0, \omega)\}$$

of $P$ by $C^\infty(\Omega_T)$, and similarly denote the set of smooth functions that vanish near the parabolic boundary

$$\{(x, t) \mid x \in \partial \Omega, t \in (0, T)\} \cup \{(x, t) : x \in \Omega, t = 0\}$$

of $P$ by $C^\infty(\Omega_\omega)$. Denote by $W_2^{0,m,1}(P)$ the closure of $C^\infty(\Omega_T)$ in $W_2^{2m,1}(P)$, and by $W_2^{2m,1}(P)$ the closure of $C^\infty(\Omega_\omega)$ in $W_2^{2m,1}(P)$.

In order to define analogous spaces of sections of the vector bundle $E$ there is a small caveat. The problem is that by the definition of the vector bundle, the local trivialisation (see point 2 within our definition of a vector bundle from the beginning of this section) of a vector bundle is just that: local. In general there is no guarantee that $X, Y \in E_\omega$ live in the same vector space, and so in order to compare $X$ and $Y$ we need to parallel transport $Y$, which we now explain briefly.

Recall that parallel transportation is a special way of identifying the fibres of a vector bundle $E$ above different points $x, y \in \Sigma$. The basic idea is that one first chooses a curve $\gamma$ from $x$ to $y$ (say, $\gamma(0) = x$ and $\gamma(1) = y$) with $V := \dot{\gamma}$. Next, one chooses a section $\mu(t) = \mu(\gamma(t))$ of $E$ along $\gamma$. For given initial values $\mu(0) \in E_{\gamma(0)} = E_x$, there exists a unique solution of the linear system of first order ordinary differential equations

$$\nabla_{V(t)} \mu(t) = 0.$$

This solution is called the parallel transport of $\mu(0)$ along the curve $\gamma$. We denote the
parallel translation along the curve $\gamma$ from $Y$ to $X$ by $\mathcal{P}_{Y,X,\gamma}$.

A possible problem is that there is no unique way of parallel transporting the sections because there is no unique curve that connects two points on the manifold $\Sigma$. To overcome this, recall that if we denote $i_g$ to be the injectivity radius of $\Sigma$ then every two points $x, y \in \Sigma$ with $d_g(x, y) < i_g$ can be joined by a unique geodesic $\tilde{\gamma}$. When we choose this geodesic as the path for parallel translation we will suppress the dependence of $\tilde{\gamma}$ in our notation and simply write the parallel translation along the curve $\tilde{\gamma}$ as $\mathcal{P}_{Y,X}$. For our purposes we will always choose this unique geodesic as the path for parallel translation.

Then for $U \in \Gamma(\mathcal{E}_\omega)$ we define the Hölder semi-norm as

$$[U]_{\alpha;\mathcal{E}_\omega} := \sup_{X \neq Y \in \Sigma, d_g(x, y) < i_g} \frac{|\nabla^{2m,1} U(X) - \mathcal{P}_{Y,X} \nabla^{2m,1} U(Y)|}{d(X,Y)^\alpha}$$

(compare to (2.4)). From this we define the analogous Hölder norms

$$\|U\|_{2m,1;\mathcal{E}_\omega} := \sum_{k=1}^{2m} \|\nabla^{(k)} U\|_{0;\mathcal{E}_\omega} + \|\partial_t U(X)\|_{0;\mathcal{E}_\omega}$$

and

$$\|U\|_{2m,1,\alpha;\mathcal{E}_\omega} := \|U\|_{2m,1;\mathcal{E}_\omega} + [\nabla^{2m,1} U]_{\alpha;\mathcal{E}_\omega}$$

(compare to (2.5) and (2.6) respectively). Just like with $u$ above, the space of tensor fields $\{U \in C^{2m,1}(\mathcal{E}_\omega) : \|U\|_{2m,1,\alpha;\mathcal{E}_\omega} < \infty\}$ is a Banach space with respect to the norm $\|U\|_{2m,1,\alpha;\mathcal{E}_\omega}$. We now state the main existence result of Baker.

**Theorem 2.2** (Baker [12], Main Theorem 5). Let $\mathcal{E}_\omega := E \times (0, \omega)$ be a vector bundle over $\Sigma_\omega := \Sigma \times (0, \omega)$, where $\Sigma$ is a smooth closed manifold, and let $U \in \Gamma(E \times (0, \omega))$. Consider the following initial value problem:

$$\begin{cases}
P(U) := \partial_t U - F(x, t, U, \nabla U, \ldots, \nabla^{2m} U) = 0 \text{ in } E \times (0, \omega), \\
U(\Sigma, 0) = U_0 \in C^{2m,1,\alpha}(E \times (0, \omega)),
\end{cases}$$

(2.7)
where \( P \) is a quasilinear operator of order \( 2m \). Suppose that the linearisation of \( P \) at \( U_0 \) in the direction \( V \) is given by

\[
\partial P [U_0] V = \partial_t V + (-1)^m \sum_{|I| \leq 2m} A^I \nabla I V
\]

and that the following conditions are satisfied:

1. The leading coefficient satisfies the symmetry condition

\[
A^{\alpha_{i_1} \ldots \alpha_i}_{\beta_j \ldots \beta_j} = A^{\beta_j \ldots \beta_j}_{\alpha_{i_1} \ldots \alpha_i}.
\]

2. The leading coefficient satisfies the Legendre-Hadamard condition for some constant \( \lambda > 0 \).

3. There exists a uniform constant \( \Lambda < \infty \) such that

\[
\sum_{|I| \leq 2m} |A^I|_{\alpha; E_\omega} \leq \Lambda
\]

4. \( \hat{F} := \partial_s F (U_0 + sV) \bigg|_{s=0} \) is continuous in all its arguments

Then there exists a unique solution \( U \in C^{2m,1,\beta}_{\Sigma, \omega} \), \( \beta < \alpha \), for some short time to the initial value problem (2.7). Furthermore, if \( U_0 \) and all the coefficients of the linearised operator are smooth, then \( U \) is smooth too.

We proceed with an outline of the proof of Theorem 2.2. Since the original source of the theorem, [12], contains a fantastically detailed proof already, the author feels that any attempt to improve on it here would be fruitless. As such, the outline included here will be quite terse and many of the finer details will not be discussed. The author does not claim anything new in this proof, and invites the interested reader to view the proof in its original context.

**Outline of Proof.** One begins by proving the existence of a unique solution \( U \in W^{m,1}_{2,\Sigma, \omega} \) to the associated linear initial value problem in divergence form:

\[
\begin{align*}
\partial_t U^\alpha + (-1)^{|J|} \sum_{0 \leq |I|, |J| \leq m} \nabla J (A^{\alpha J}_{\beta} \nabla I U^\beta) &= F^\alpha (X), \quad X \in \Sigma, \\
U (\Sigma, 0) &= U_0,
\end{align*}
\]
under the assumption $F, U_0 \in L^2(E_\omega)$. This is done in a rather standard fashion: one first shows that the differential operator in question admits a particular bilinear form $B : W^{k,1}_2(E_\omega) \times V(E_\omega) \to \mathbb{R}$ which satisfies certain structural properties. Here $V(E_\omega)$ is the space

$$V(E_\omega) := \left\{ U \in W^{1,1}_2(E_\omega) \mid \nabla \partial_t U \in L^2(E_\omega) \right\},$$

which is a dense subspace of $W^{1,1}_2(E_\omega)[12]$. These structural properties allow the use of the Lax-Milgram theorem (see for example [62]), which says that there exists a unique $U \in W^{m,1}_2(E_\omega)$ such that $F(W) = B(U, W)$ for every $W \in V(E_\omega)$. This $U$ is our weak solution.

One then goes on to prove interior regularity along the flow, showing that smoothness in $F$ implies smoothness in the solution $U$.

Next, interior Schauder estimates are obtained for the corresponding flow in Euclidean space on small backwards parabolic cylinders of the form

$$Q_R(X_0) := B_R(x_0) \times (t_0 - R^{2m}, t_0) \subset \mathbb{R}^N \times \mathbb{R},$$

via either an extension of Trudinger’s method of mollification [99] to higher-order systems, or Leon Simon’s method of scaling [90]. Noting that this cylinder is a subset of $\mathbb{R}^N \times \mathbb{R}$ rather than of $E_\omega$, these Euclidean estimates need to be lifted to the vector bundle.

By using the local trivialisation $\varphi$ of the vector bundle, we see that these Schauder estimates are equivalent to estimates on sections measured with respect to the bundle metric. To see this, first note that if $\tilde{x} : V \times I \to \mathbb{R}^N \times \mathbb{R}^+$ is the coordinate map for some small neighbourhood $V \times I \subset \Sigma_\omega$, then there exists a bundle trivialisation $\varphi : E_\omega \mid_{V \times I} \to V \times I \times \mathbb{R}^N$ such that

$$\tilde{x} = x \times \text{id}_I,$$
where \(x\) is the coordinate map for \(V\) and \(\text{id}_I\) is the identity map on \(I\).

Next we note that on small geodesic balls, the metric on a Riemannian manifold \(\Sigma\) is equivalent to the flat metric. Explicitly, suppose that \(p \in \Sigma\) and \(r_0 \in (0, \iota_g(p)/4)\), where \(\iota_g(p)\) denotes the injectivity radius of \(\Sigma\) at \(p\). Then if for every \(q \geq 0\) there exists constants \(A_q\) such that \(|\nabla_{(q)} \text{Riem}| \leq A_q\) in \(B_p(r_0)\), in normal coordinates in \(B_p(r_0)\) there exist constants \(C_q(n, \iota_g, A_0, \ldots, A_q)\) such that for each \(q\) the following estimates hold in \(B_p(\min\{A_1/\sqrt{A_0}, r_0\})\):

\[
\frac{1}{2} \delta_{ij} \leq g_{ij} \leq 2 \delta_{ij} \quad \text{and} \quad |\partial_{(p)} g_{ij}| \leq C_q.
\]

This is a well-known result (see [45] for example). Using this result, it is possible to show that there exists a universal constant \(C > 0\) such that

\[
C^{-1}|U \circ \tilde{x}^{-1}|_{2m,1,\alpha; E_\omega(V_\alpha \times I_\alpha)} \leq |U|_{2m,1,\alpha; E_\omega(V_\alpha \times I_\alpha)} \leq C|U \circ \tilde{x}^{-1}|_{2m,1,\alpha; E_\omega(V_\alpha \times I_\alpha)},
\]

where \(U \in \Gamma(E_\omega)\) and \((V_\alpha, \varphi_\alpha)\) is a finite covering of \(\Sigma_\omega\). Note that here each chart needs to be sufficiently small so that we may apply (2.8). Also note that the norms on the left and right hand side of the previous inequality are over a subset of Euclidean space, while the norm in the centre is over the bundle \(\Sigma_\omega\) and is measured with respect to the bundle metric. The above pair of inequalities appears as Proposition 3.1 in the original paper [12].

Next, one observes that because \(\Sigma\) is assumed to be compact, by the same logic as above \(\Sigma_\omega\) may be covered by a finite number of normal coordinate charts \(\{(U_\alpha, \varphi_\alpha)\}\) of controlled size, allowing one to patch together the aforementioned local estimates to give global estimates of the form:

\[
\|U\|_{2m,1,\alpha; E_\omega} \leq C\left(\|F\|_{\alpha; E_\omega} + \|U_0\|_{2m,\alpha; E_\omega} + \|U\|_{0; E_\omega}\right).
\]

As Baker points out in [12], the fact that we have a finite covering is essential to the argument here because in order to obtain (2.9) one must take the supremum over all
Hölder coefficients in each coordinate chart.

One then proceeds by proving unique existence to the polyharmonic heat operator

\[ \begin{cases} 
\partial_t \tilde{U} + (-1)^m \Delta^m \tilde{U} = F(X), & X \in \Sigma_\omega \\
\tilde{U}(\cdot, 0) = \tilde{U}_0 \in C^{2m,1,\alpha}(E_\omega) 
\end{cases} \]

for functions \( F \) satisfying \( F \in C^{0,0,\alpha}(E_\omega) \) (this argument appears as Proposition 3.23 in the paper by Baker [12]). Moreover, by considering a mollified problem and applying the Arzela-Ascoli theorem one can obtain the following estimate for solutions of the polyharmonic heat operator:

\[ \|\tilde{U}\|_{2m,1,\alpha;E_\omega} \leq C \|F\|_{\alpha;E_\omega} \leq C. \quad (2.10) \]

Here the last inequality holds because by assumption \( F \in C^\alpha(E_\omega) \). One then uses the *continuity method* (see for example [37]) to explain briefly. The idea behind the technique is quite simple: say one wishes to solve the problem \( F(U) = 0 \). Then consider the one-parameter family of problems \( P_t \):

\[ P_t : \quad F(U, \tau) = 0, \quad 0 \leq \tau \leq 1 \quad (2.11) \]
with \( F(U, 1) = F(U) \) and \( F(U, 0) = 0 \) being some problem that you know how to solve. If one can show that the set

\[
\mathcal{A} := \{ t \in [0, 1] \mid \text{there exists a solution to the problem } P_t \}
\]

is both open and closed, then this implies \( \mathcal{A} = [0, 1] \) and hence there exists a solution to the problem \( F(U) = 0 \).

For linear operators, the exact statement of the theorem is as follows.

**Theorem 2.3** (Continuity method [37]). Let \( B \) be a Banach space, \( V \) a normed vector space, and \( \{P_\tau\}_{\tau \in [0, T]} \) a norm-continuous family of bounded linear operators from \( B \) into \( V \). Assume that there exists a constant \( C \) such that for every \( \tau \in [0, 1] \) and every \( U \in B \)

\[
||U||_B \leq C||P_\tau(U)||_V.
\]

Then \( P_0 \) is surjective if and only if \( P_1 \) is surjective.

In our case, we define the operators

\[
L_0 U := \partial_t u^\alpha + (-1)^m \Delta^m u^\alpha,
\]

\[
L_1 U := \partial_t u^\alpha + (-1)^m \sum_{|I| \leq 2m} A^\alpha_{\beta I} (x, t) \nabla I U^\beta,
\]

and our family of problems takes the form (2.11), where

\[
P_\tau(U) = (1 - \tau) L_0 U + \tau L_1 U, \quad 0 \leq \tau \leq 1.
\]

One then uses the same techniques as for the polyharmonic heat operator and the global estimates (2.9) to obtain a priori estimate of the same form as (2.10) for a solution \( U_\tau \) to the family of problems \( P_\tau = 0 \): we find there exists a constant \( C \) that
does not depend on $\tau$ such that for every $\tau \in [0,1]$,

$$||U_\tau||_{2m,1,\alpha;E_\omega} \leq C||F||_{\alpha;E_\omega}.$$ 

Since the bound $C$ is independent of $\tau$, the continuity method applies and so the existence of solutions to the polyharmonic heat operator $P_0$ implies the existence of solutions to the problem $P_1$. But $P_1$ is exactly the problem

$$\partial_t U^\alpha + (-1)^m \sum_{|I| \leq 2m} A^{\alpha I}_\beta (x,t) \nabla I U^\beta = 0,$$

and so this establishes the existence of a solution to our general linear parabolic problem.

In order to apply the result above to our quasilinear problem, we will need to apply the inverse function theorem for Banach spaces (see, for example, [79]). To show that the inverse function theorem applies here we will need to show that the linearisation of the quasilinear operator $P$ given by (2.7) is both continuously differentiable and Fréchet differentiable around the solution to the linearised problem.

Note that the linearisation of the quasilinear operator $P$ given by (2.7) around the initial data $U_0$ in the direction $V$ is a linear system in $V$, which admits a unique solution by the above theory. We call this solution $U_I$. By linearising the operator $P$ around $U_I$ in the direction $V$ we obtain

$$\partial P[U_I]V$$

$$:= \left. \frac{\partial}{\partial \lambda} P(U_I + \lambda V) \right|_{\lambda=0}$$

$$= \left. \frac{\partial}{\partial \lambda} \left[ \frac{\partial}{\partial t}(U_I + \lambda V) - F(x,t,U_I + \lambda V, \ldots, \nabla_{(2m)}(U_I + \lambda V)) \right] \right|_{\lambda=0}$$

$$= \left. \frac{\partial}{\partial t} V - \hat{F}^{i_1 \ldots i_{2m}}(x,t,U_I, \ldots, \nabla_{(2m)}U_I) \nabla_i V \right.$$ 

$$\quad - \cdots - \hat{F}^{i}(x,t,U_I, \ldots, \nabla_{(2m)}U_I) \nabla_i V - F(x,t,U_I, \ldots, \nabla_{(2m)}U_I) V.$$
Therefore, since \( \hat{F} \) is assumed to be continuous in all its arguments, we have

\[
\lim_{s \to 0} ||\partial P[U_l + sV] - \partial P[U_l]||_{L(C^{2m,1,\alpha}(E_{\omega}),C^{0,0,\alpha}(E_{\omega}))} = 0. \tag{2.12}
\]

This tells us that \( P \) is continuously differentiable and Fréchet differentiable at \( U_l \), since (2.12) implies

\[
\|P(U_l + V) - P(V) - \partial P[U_l]\|_{0,0,\alpha;E_{\omega}} = \| \left( \int_0^1 (\partial P[U_l + sV] - \partial P[U_l]) \, ds \right) V \|_{0,0,\alpha;E_{\omega}} \\
\leq \|\partial P[U_l + sV] - \partial P[U_l]\|_{L(C^{2m,1,\alpha}(E_{\omega}),C^{0,0,\alpha}(E_{\omega}))} ||V||_{2m,1,\alpha;E_{\omega}} \\
= o(||V||_{2m,1,\alpha;E_{\omega}}).
\]

Finally, because the linearisation of \( P \) about \( U_l \) in the direction \( V \) is again a linear system in \( V \), it admits a unique solution, and we conclude that the Fréchet derivative of \( P \) is invertible at \( U_l \). This means we can apply the inverse function theorem for Banach spaces, obtaining a unique solution to the general quasilinear problem for short time, thus proving Theorem 2.2.

\[ \square \]

### 2.3 Application to geometric polyharmonic heat flows

With the previous section in mind, we need to check if our initial value problem (1.2) fits the hypothesis of Theorem 2.2 so that we may conclude short-time existence for the geometric polyharmonic heat flow. In our case the particular section that we are interested in is the position vector of the submanifold \( f(\Sigma) \subset \mathbb{R}^{n+k} \). Recall from Laplace’s identity (C.11) that the mean curvature flow (MCF) takes the following form:

\[ \partial_t f = \vec{H} = \Delta f. \]
Here \( \vec{H} \) is the mean curvature vector which in this case takes the form

\[
\vec{H} = - \sum_{\alpha=1}^{k} H_{\alpha} \nu_{\alpha},
\]

where \( \{\nu_{\alpha}\}_{\alpha=1,2,...,k} \) constitutes an orthonormal basis for the normal bundle over \( f(\Sigma) \) and \( \{H_{\alpha}\}_{\alpha=1,2,...,k} \) the corresponding mean curvature scalars. The definition of the Christoffel symbols along with the fact that the coefficients of the pullback metric on \( \Sigma \) implicitly contains derivatives of \( f \):

\[
g_{ij} := \sum_{\alpha=1}^{n+k} \partial_i f^\alpha \partial_j f^\alpha,
\]

implies that the equation is a quasilinear second order equation in \( f \). To see this more clearly, a simple calculation gives

\[
g^{ij} \Gamma^k_{ij} = \frac{1}{2} \sum_{k} \gamma(\partial_i f^\beta \partial_j f^\beta) + \partial_j(\partial_i f^\beta \partial_k f^\beta) - \partial_k(\partial_i f^\beta \partial_j f^\beta)
\]

which implies

\[
\Delta f^\alpha := g^{ij} \left( \partial_{ij} \Gamma^k_{ij} \right) f^\alpha
\]

Next, using the fact that our evolution problem in (2.2) and the system of partial differential equations given by \( \partial_t f = (-1)^p \Delta^{p+1} f \) share the same leading (highest order) term in \( f \), and by applying the Laplace operator \( p \) times to the equation (2.14) we find that the geometric polyharmonic heat flow can be recast as a quasilinear partial differential equation of order \( 2(p + 1) \) in \( f \). This equation takes the form \( P(f) = 0, \)
A similar calculation to (2.13) gives

\[
(P(f))^\alpha = \partial_i f^\alpha + (-1)^{p+1} \sum_{i=1}^{p+1} g^{ij_1} \cdots g^{ij_{p+1}} \nabla_{i_1j_1 \cdots i_{p+1}j_{p+1}} f^\alpha \\
+ \text{lower order terms in } f
\]

\[
= \partial_i f^\alpha + (-1)^{p+1} \sum_{i=1}^{p+1} g^{ij_1} \cdots g^{ij_{p+1}} \left( \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} f^\alpha - \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} \Gamma_{i_{p+1}j_{p+1}}^m \partial_m f^\alpha \right) \\
+ \text{lower order terms in } f
\]

(2.15)

A similar calculation to (2.13) gives

\[
g^{ij_1} \cdots g^{ij_{p+1}} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} \\
= \frac{1}{2} g^{ij_1} \cdots g^{ij_{p+1}} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} \left( g^{mn} \left( \partial_{j_{p+1}g_{j_{p+1}n} + \partial_{j_{p+1}g_{j_{p+1}n} - \partial_n g_{j_{p+1}n}} \right) \right) \\
= \frac{1}{2} g^{ij_1} \cdots g^{ij_{p+1}} g^{mn} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} \left( \partial_{j_{p+1}g_{j_{p+1}n} + \partial_{j_{p+1}g_{j_{p+1}n} - \partial_n g_{j_{p+1}n}} \right) \\
+ \text{lower order terms in } f
\]

\[
= g^{ij_1} \cdots g^{ij_{p+1}} g^{mn} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} (\partial_{j_{p+1}f, \partial_n f}) + \text{lower order terms in } f
\]

\[
= g^{ij_1} \cdots g^{ij_{p+1}} g^{mn} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} f^{\beta} \partial_n f^\beta + \text{lower order terms in } f
\]

Here we have used the fact that the partial derivative commutes on scalar functions, which implies

\[
\frac{1}{2} g^{ij_1} \cdots g^{ij_{p+1}} g^{mn} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} \left( \partial_{j_{p+1}g_{j_{p+1}n} + \partial_{j_{p+1}g_{j_{p+1}n} - \partial_n g_{j_{p+1}n}} \right) \\
= \frac{1}{2} g^{ij_1} \cdots g^{ij_{p+1}} g^{mn} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} \left( (\partial_{j_{p+1}f, \partial_n f}) + (\partial_{j_{p+1}f, \partial_n f}) \right) \\
+ (\partial_{j_{p+1}f, \partial_n f}) + (\partial_{j_{p+1}f, \partial_n f}) - (\partial_{j_{p+1}f, \partial_n f}) - (\partial_{j_{p+1}f, \partial_n f}) \\
= \frac{1}{2} g^{ij_1} \cdots g^{ij_{p+1}} g^{mn} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} \left( (\partial_{j_{p+1}f, \partial_n f}) + (\partial_{j_{p+1}f, \partial_n f}) \right) \\
+ (\partial_{j_{p+1}f, \partial_n f}) + (\partial_{j_{p+1}f, \partial_n f}) - (\partial_{j_{p+1}f, \partial_n f}) - (\partial_{j_{p+1}f, \partial_n f}) \\
= g^{ij_1} \cdots g^{ij_{p+1}} g^{mn} (\partial_{j_{p+1}f, \partial_n f}).
\]
2.3. APPLICATION TO GEOMETRIC POLYHARMONIC HEAT FLOWS

Therefore, from (2.15) we can see that the linearisation of $P$ at $f_0$ in the direction $V$ is given by

$$
\partial P[f_0]V^\alpha = \partial_t V^\alpha + (-1)^{p+1} \left( \delta_\beta^\alpha - g^{mn} \partial_m f_0^n \partial_n f_0^\beta \right) g^{i_1j_1} \cdots g^{i_{p+1}j_{p+1}} \partial_{i_1j_1 \cdots i_{p+1}j_{p+1}} V^\beta + \text{lower order terms in } V',
$$

(2.16)

where $g$ is metric induced by the initial condition $f_0$:

$$
g_{ij} = (\partial_i f_0, \partial_j f_0).
$$

The leading coefficient of the linearisation of $P$ obviously satisfies the symmetry condition 1 of Theorem 2.2 because of the symmetry of $g$. Additionally, because we are assuming $f_0$ is smooth, it follows that there is a uniform constant $\Lambda$ such that

$$
\sum_{|I| \leq 2(p+1)} |A_I|_{\alpha:E_\omega} \leq \Lambda
$$

and that $\tilde{F} = \partial_\lambda F(f_0 + \lambda V)\big|_{\lambda=0}$ is continuous in all its arguments. Therefore conditions 3 and 4 of Theorem 2.2 hold as well. We just need to check if condition 2 holds. By (2.16), the principal symbol of the linearised operator $\partial P[f_0]$ in the direction of $\xi \in \mathbb{T} \mathbb{R}^{k+2}$ can be seen to be equal to

$$
\sigma(\partial P[f_0])(\xi) := A_\beta^{\alpha i_1j_1 \cdots i_{p+1}j_{p+1}} \xi_{i_1} \xi_{j_1} \cdots \xi_{i_{p+1}} \xi_{j_{p+1}}
$$

$$
= (\delta_\beta^\alpha - g^{kl} \partial_k f_0^\alpha \partial_l f_0^\beta) |\xi|^{2(p+1)},
$$

which implies that for any $\eta \in \Gamma(E)$ and $\eta^* \in \Gamma(E^*)$ we have

$$
A_\beta^{\alpha i_1j_1 \cdots i_{p+1}j_{p+1}} \xi_{i_1} \xi_{j_1} \cdots \xi_{i_{p+1}} \xi_{j_{p+1}} \eta \eta^* = |\xi|^{2(p+1)} (\eta, \eta - g^{kl} (\eta, \partial_k f_0) \partial_l f_0).$$

(2.17)
Here $(\cdot, \cdot)$ is the regular inner product in $\mathbb{R}^{n+k}$. Next, we note that for any vector $\eta \in \mathbb{R}^{n+k}$

$$\eta^T := g^{kl}(\eta, \partial_k f_0) \partial_l f_0$$

is the orthogonal projection of $\eta$ onto the tangent space of $f_0(\Sigma)$. Therefore if we use $\eta^\perp$ to denote the orthogonal projection of $\eta$ onto the normal bundle $T_f^0(\Sigma)^\perp$, then (2.17) implies

$$A_{\alpha_1 j_1 \cdots i_{p+1} j_{p+1}}^{\beta_1} \xi_{i_1} \xi_{j_1} \cdots \xi_{i_{p+1}} \xi_{j_{p+1}} \eta_\alpha \eta^* \beta = |\xi|^{2(p+1)} (\eta, \eta^T) = |\xi|^{2(p+1)} |\eta^\perp|^2,$$

which is zero if $\eta \in df_0(T\Sigma)$.

Therefore the eigenvalues of the principal symbol that correspond to directions tangent to $f_0(\Sigma)$ are zero, and we conclude that the geometric polyharmonic heat flow is only a weakly parabolic system. In particular, the Legendre-Hadamard condition does not hold for any constant $\lambda > 0$. Therefore condition 2 of Theorem (2.2) fails.

We can overcome this failure by using a variant of a now-standard tool called the DeTurck trick, which was introduced by Dennis DeTurck [23] in order to provide a simple proof of the short-time existence result for the Ricci flow (the alternative proof of short-time existence at the time by Hamilton [42] relied upon the rather robust Nash-Moser inverse function theorem).

The trick is beautiful in that it is so simple and lends itself to a wide variety of different geometric flows. The basic idea is to show that a geometric flow (that is not strongly parabolic) is equivalent to a strongly parabolic initial value problem, modulo the action of some one-parameter diffeomorphism group acting on the manifold. Pulling back by this time-dependent family of diffeomorphisms then gives a solution to the original problem. The DeTurck trick has since been used to prove short time existence of other flows, such as the mean curvature flow in higher codimensions (see, for example [6, 12]), and has become part of the geometric analyst’s repertoire.

We proceed by showing that the flow \((\text{GPHF})\) is invariant under tangential diffeo-
morphisms in the sense that any flow that varies by a time-dependent smooth vector field along \( f(\Sigma) \) can be reparameterised to solve (GPHF) exactly. This result is analogous to some other results which can be found in the literature and hold for lower-order flows (for example the mean curvature flow \([27, 68]\) and the curve diffusion flow \([103]\)). This will allow us to use the DeTurck trick.

**Lemma 2.4.** Assume \( \Sigma^n \) is compact and that a smooth family of immersions \( f : \Sigma^n \times [0, T) \to \mathbb{R}^{n+k} \) satisfies the initial value problem

\[
\begin{aligned}
\partial_t f(x, t) &= (-1)^p (\Delta^\perp)^p \bar{H} + X(x, t), \\
f(x, 0) &= f_0(x),
\end{aligned}
\]

where \( X \) is a time-dependent smooth vector field along \( \Sigma_t \) such that \( X(x, t) \in df_t(T_x \Sigma) \) for each \((x, t) \in \Sigma \times [0, T)\). Then, locally around each point in space and time there exists a family of smoothly time-dependent reparameterisations of the maps \( f_t \) which solve the original initial value problem (2.2). Moreover this family of reparameterisations is unique.

**Proof.** We include a proof here based on an argument in [68]. Although the original proof was for the mean curvature flow, it is easy to follow and generalises naturally to a number of different flows, including ours here.

Since \( X(x, t) \) is assumed to be tangent to \( \Sigma_t \), we have that the vector field \( Y(x, t) = - (df_t)^{-1}(X(x, t)) \) is defined globally on \( \Sigma \). We consider a family of smooth diffeomorphisms \( \phi : \Sigma \times [0, T) \to \Sigma \) such that \( \phi(x, 0) = x \) for every \( x \) and

\[
\partial_t \phi(x, t) = Y(\phi(x, t), t).
\]

By standard results for ordinary differential equations on compact manifolds (see [64], for example), this family exists, is unique and smooth. If we consider the reparameterisation of \( f \) by \( \tilde{f}(x, t) = f(\phi(x, t), t) \), then by an application of the chain rule and
the definition of $X$, the function $\tilde{f}$ solves the equation

$$
\partial_t \tilde{f} (x,t) = \partial_t f (\phi (x,t), t) + df (\phi (x,t)) \left( \partial_t \phi (x,t) \right)
= (-1)^p (\tilde{\Delta}^\perp)^p \tilde{H} (x,t).
$$

Moreover, by the properties of $\phi$, for each $x \in \Sigma$ one has

$$
\tilde{f}_0 (x) = f (\phi (x,0), 0) = f (x, 0) = f_0 (x)
$$

and so the reparameterisation uniquely solves the original initial value problem (2.2) as claimed.

With the preceding theorem in mind, in order to use the Deturck trick and thus prove the existence of our flow (2.1) all we have to do is find a tangential vector field $X$ such that the modified flow

$$
\partial_t \tilde{f} = (-1)^p (\tilde{\Delta}^\perp)^p \tilde{H} + X
$$

satisfies property 2 of Theorem 2.2, since $\tilde{f}$ is equivalent to (2.1) modulo tangential diffeomorphisms. Let

$$
X = (-1)^{p+1} \tilde{g}^{ij} \tilde{\Delta}^{p}_{\tilde{g},\tilde{g}_0} (\tilde{\Gamma}^k_{ij} - \tilde{g}^k_{ij}) \partial_k \tilde{f} \in T \tilde{f} (\Sigma),
$$

where $\tilde{\Gamma}^k_{ij}$ denotes the Christoffel symbols associated with the initial immersion $\tilde{f}_0$, and $\Delta_{g,h}$ denotes the tension field associated with a map between manifolds with metrics $g$ and $h$.

As a brief aside, recall that if $\pi : (M, \gamma_{\alpha\beta}) \to (N, g_{ij})$ is a map of class $C^1$ then its differential,

$$
d\pi = \frac{\partial \pi^i}{\partial x^\alpha} dx^\alpha \otimes \frac{\partial}{\partial \pi^i},
$$
is a section of the bundle $T^*M \otimes \pi^{-1} (TN)$. The tension field of the map $\pi$ is given by

$$\Delta_{\gamma,g} \pi := \text{tr}_\gamma (\nabla d\pi)$$

(often written as $\tau (\pi)$), and has components

$$(\Delta_{\gamma,g} \pi)^i = \gamma^{\alpha\beta} \left( \frac{\partial^2 \pi^i}{\partial x^\alpha \partial x^\beta} - \gamma^{i\omega} \frac{\partial \pi^i}{\partial x^\omega} + g^{ij} \frac{\partial \pi^j}{\partial x^\alpha} \frac{\partial \pi^k}{\partial x^\beta} \right) = (\Delta \pi)^i + \gamma^{\alpha\beta} g^{ij} \frac{\partial \pi^j}{\partial x^\alpha} \frac{\partial \pi^k}{\partial x^\beta},$$

where $\Delta$ denotes the usual Laplace-Beltrami operator. Here $\gamma$, $g$ denote the Christoffel symbols corresponding to the metrics $\gamma$, $g$, respectively. Note that in the specific case that $M = N$ and $\pi$ is the identity map, we have

$$(\Delta_{\gamma,g} \text{id}_\Sigma)^i = -\gamma^{\alpha\beta} (\gamma^{i\alpha} - g^{i\alpha}). \tag{2.20}$$

A map that satisfies $\Delta_{\gamma,g} \pi = 0$ is called a harmonic map, and a map $\pi : M \times I \to N$ that is a solution to the equation

$$\partial_t \pi = -\Delta_{\gamma,g} \pi$$

is called a harmonic map heat flow. Similarly a map $\pi : M \times I \to N$ satisfying

$$\partial_t \pi = (-1)^q \Delta^q_{\gamma,g} \pi$$

is said to be a polyharmonic map heat flow of order $q$.

We will use the existence and properties of such flows to prove uniqueness to our initial value problem later on.

The first line of (2.15) implies that our modified flow $\tilde{f}$ satisfies $\tilde{P}(\tilde{f}) = 0$, where

$$(\tilde{P}(\tilde{f}))^\alpha := \partial_t \tilde{f}^\alpha + (-1)^p \sum_{\alpha=1}^k \hat{A}^p \hat{H}_\alpha \cdot \tilde{\nu}_\alpha + X$$
\[\begin{align*}
\partial_t \tilde{f}^\alpha &+ (-1)^{p+1} \tilde{g}^{ij} (\Delta^p (\partial_{ij} \tilde{f}^\alpha) - \tilde{\Delta}^p (\tilde{\Gamma}_{ij}^k) \partial_k \tilde{f}^\alpha) \\
&+ (-1)^{p+1} \tilde{g}^{ij} \tilde{\Delta}^p (\tilde{\Gamma}_{ij}^k) \partial_k \tilde{f}^\alpha + \text{lower order terms in } \tilde{f}' \\
&= \partial_t \tilde{f}^\alpha + (-1)^{p+1} \tilde{g}^{ij} (\Delta^p (\partial_{ij} \tilde{f}^\alpha) - \tilde{\Delta}^p (0\tilde{\Gamma}_{ij}^k) \partial_k \tilde{f}^\alpha) \\
&+ \text{lower order terms in } \tilde{f}' \\
&= \partial_t \tilde{f}^\alpha + (-1)^{p+1} \tilde{g}^{ij} \tilde{\Delta}^p \partial_{ij} \tilde{f}^\alpha + \text{lower order terms in } \tilde{f}' \\
&= \partial_t \tilde{f}^\alpha + (-1)^{p+1} \left( \delta_\beta^\alpha \tilde{g}^{i_1j_1} \cdots \tilde{g}^{i_{p+1}j_{p+1}} \partial_{i_1j_1\cdots i_{p+1}j_{p+1}} \tilde{f}^\beta + \cdots \tilde{C}_\beta^\alpha f^\beta \right) \\
&= \partial_t \tilde{f}^\alpha + (-1)^{p+1} \left( \tilde{A}_\beta^{i_1j_1\cdots i_{p+1}j_{p+1}} \partial_{i_1j_1\cdots i_{p+1}j_{p+1}} \tilde{f}^\beta + \cdots \tilde{C}_\beta^\alpha f^\beta \right),
\end{align*}\]

where we have set
\[
\tilde{A}_\beta^{i_1j_1\cdots i_{p+1}j_{p+1}} = \delta_\beta^\alpha \tilde{g}^{i_1j_1} \cdots \tilde{g}^{i_{p+1}j_{p+1}}.
\]

Here the second term in the third step has been absorbed into the lower order terms in \( \tilde{f} \) because it depends solely on the initial data. We have also used

\[\tilde{g}^{ij} \tilde{\Delta}^p (\tilde{\Gamma}_{ij}^k) \partial_k \tilde{f}^\alpha = \tilde{g}^{ij} \tilde{\Delta}^p (\tilde{\Gamma}_{ij}^k) \partial_k \tilde{f}^\alpha + \text{lower order terms in } \tilde{f}'\]

which follows for the same reason.

The operator \( \tilde{P} \) is quasilinear (because \( \tilde{g} \) and therefore \( \tilde{g}^{-1} \) depends on \( \tilde{f} \)) and of order 2 \( (p + 1) \). Moreover it obviously satisfies conditions 1, 3 and 4 just as the operator \( P \) did. However, unlike \( P \), the operator \( \tilde{P} \) is strongly parabolic, as we now show.

First note that by (2.21), that the linearisation of \( \tilde{P} \) at \( \tilde{f}_0 \) in the direction \( \varphi \) is given by

\[\begin{align*}
\partial \tilde{P}[\tilde{f}_0] \varphi^\alpha &= \partial_t \varphi^\alpha + (-1)^{p+1} \left( \delta_\alpha^\beta \tilde{g}^{i_1j_1} \cdots \tilde{g}^{i_{p+1}j_{p+1}} \partial_{i_1j_1\cdots i_{p+1}j_{p+1}} \varphi^\beta + \text{lower order terms in } \varphi' \right) ,
\end{align*}\]

where \( \tilde{g} \) is the metric induced by the immersion \( \tilde{f}_0 \):

\[\tilde{g}_{ij} = \sum_\alpha \partial_i \tilde{f}_0^\alpha \partial_j \tilde{f}_0^\alpha.\]
This implies that the principal symbol of the linearised operator \( \partial \tilde{P}[\tilde{f}_0] \) in the direction of \( \xi \in T\mathbb{R}^{n+k} \) is equal to

\[
\sigma(\partial \tilde{P}[\tilde{f}_0])(\xi) := \tilde{A}_{\beta}^{\alpha i_1 j_1 \cdots i_{p+1} j_{p+1}} \xi_{i_1} \xi_{j_1} \cdots \xi_{i_{p+1}} \xi_{j_{p+1}} e_\alpha \otimes \theta^\beta
\]

\[
= \delta_\beta^\alpha \tilde{g}^{i_1 j_1} \cdots \tilde{g}^{i_{p+1} j_{p+1}} \xi_{i_1} \xi_{j_1} \cdots \xi_{i_{p+1}} \xi_{j_{p+1}} e_\alpha \otimes \theta^\beta
\]

\[
= \delta_\beta^\alpha |\xi|^{2(p+1)} e_\alpha \otimes \theta^\beta
\]

\[
= |\xi|^{2(p+1)} \text{id}_{E \otimes E^*}
\]

Therefore any eigenvalues of \( \sigma(\partial \tilde{P}[\tilde{f}_0]) \) are strictly positive which proves that \( \tilde{P} \) is strongly parabolic, as claimed.

Moreover, the linearisation of \( \tilde{P} \) satisfies the Legendre-Hadamard condition for any positive constant \( 0 < \lambda < 1 \) because

\[
\tilde{A}_{\beta}^{\alpha i_1 j_1 \cdots i_{p+1} j_{p+1}} \xi_{i_1} \xi_{j_1} \cdots \xi_{i_{p+1}} \xi_{j_{p+1}} \eta_\alpha \eta^{*\beta} = |\xi|^{2(p+1)} |\eta|^2.
\]

We conclude from Theorem 2.2 that a unique solution \( \tilde{f} \) exists to the modified initial value problem

\[
\begin{cases}
\partial_t \tilde{f} = (-1)^{p+1} \sum_{\alpha=1}^{k} \tilde{\Delta}^p \tilde{H}_\alpha \cdot \tilde{v}_\alpha + X,
\tilde{f}(\Sigma, 0) = \tilde{f}_0 \in C^{2(p+1),1,\alpha}(E \times (0, \omega)).
\end{cases}
\]

By pulling back the solution \( \tilde{f} \) by the time-dependent diffeomorphism \( \phi : \Sigma \times [0, T) \rightarrow \Sigma \) which satisfies

\[
\partial_t \phi = - (df)^{-1}(X) \quad \text{and} \quad \phi(\cdot, 0) = \text{id}_\Sigma
\]

such as in the proof of Lemma 2.4, we recover a solution \( f(x, t) := \tilde{f}(\phi(x, t), t) \) to the unmodified initial value problem (1.2). All that remains is to show that this solution is unique.
Suppose there exists another solution $F$ to the initial value problem (2.2):

$$
\begin{align*}
\partial_t F &= (-1)^{p+1} \sum_{\alpha=1}^k \Delta_{g_F}^p (H_F)_\alpha \cdot (\nu_F)_\alpha, \\
F (\cdot, 0) &= F_0.
\end{align*}
$$

Here $\Delta_{g_F}$ refers to the Laplacian associated with the metric $g_F = F^* (\cdot, \cdot)$. Then consider the following polyharmonic map heat flow $\pi : (\Sigma, g_F) \times [0, T) \to (\Sigma, g_0)$ of order $p + 1$:

$$
\begin{align*}
\partial_t \pi &= (-1)^{p+1} \Delta_{g_{\pi^\ast g_0}}^{p+1} \pi, \\
\pi (\cdot, 0) &= id_{\Sigma}.
\end{align*}
$$

A solution to (2.22) exists and is unique (see [85]). Moreover, if we define $\tilde{F} := (\pi^*)^{-1} F$, then a computation shows that $\tilde{F}$ satisfies

$$
\partial_t \tilde{F} = (-1)^{p+1} \sum_{\alpha=1}^k \Delta_{g_{\tilde{F}^* g_0}}^p (H_{\tilde{F}})_\alpha \cdot (\nu_{\tilde{F}})_\alpha - d\tilde{F} (\partial_t \pi).
$$

Consider the vector field $V = \partial_t \pi$ on $\Sigma$. Then by definition of $\pi$ and by the fact that the Laplacian commutes with isometries (the result is standard: see [63]), we find that for $x \in \Sigma$,

$$
V \bigg|_x := (-1)^{p+1} \Delta_{g_{\pi^\ast g_0}}^{p+1} \pi \bigg|_{\pi^{-1}(x)}
= (-1)^{p+1} \Delta_{(\pi^{-1})^\ast g_{\pi^\ast g_0}}^{p+1} (\pi \circ \pi^{-1}) \bigg|_x
= (-1)^{p+1} \Delta_{g_{\tilde{F}^* g_0}}^{p+1} id_{\Sigma} \bigg|_x.
$$

Here we have also used $g_{\tilde{F}} = (\pi^{-1})^* g_F$. Next note that by (2.20), we have

$$
(\Delta_{g_{\tilde{F}^* g_0}} id_{\Sigma})^k = -g_{\tilde{F}}^{ij} \left( \tilde{F} \Gamma_{ij}^k - 0 \Gamma_{ij}^k \right),
$$

and so the components of the vector $V$ from above can be calculated as

$$
V^k = (-1)^{p+1} \Delta_{g_{\pi^\ast g_0}}^p \left( -g_{\tilde{F}}^{ij} (\tilde{F} \Gamma_{ij}^k - 0 \Gamma_{ij}^k) \right) = (-1)^p g_{\tilde{F}}^{ij} \Delta_{g_{\pi^\ast g_0}}^p (\tilde{F} \Gamma_{ij}^k - 0 \Gamma_{ij}^k).
$$
which, when combined with (2.23), implies that \( \tilde{F} \) satisfies the equation

\[
\partial_t \tilde{F} = (-1)^{p+1} \sum_{\alpha=1}^{k} \Delta_{g_{\tilde{F}}}^{p}(H_{\tilde{F}})_{\alpha} \cdot (\nu_{\tilde{F}})_{\alpha} + (-1)^{p+1} g^{ij}_{\tilde{F}} \Delta_{g_{\tilde{F}, g_{0}}}^{p} (\tilde{F} \Gamma_{ij}^{k} - 0 \Gamma_{ij}^{k}) \partial_{k} \tilde{F}.
\]

Therefore \( \tilde{F} \) satisfies the same modified geometric polyharmonic heat flow as \( \tilde{f} \) from (2.18) (as the tangential vector field on the right hand side satisfies \( d\tilde{f}(V) = X \), where \( X \) is our tangent vector field from (2.19)), and by uniqueness we conclude \( \tilde{F} = \tilde{f} \).

Moreover, since \( \tilde{F} = (\pi^{*})^{-1} F \) and \( \tilde{f} = (\psi^{*})^{-1} f \) and since by Lemma 2.4 for a given vector field \( X \) the family of reparameterisations which solve the original problem (2.2) is unique, we conclude that \( \pi = \psi \) on some time interval. Therefore

\[
F = \pi^{*} \tilde{F} = \pi^{*} \tilde{f} = \psi^{*} \tilde{f} = f
\]

and uniqueness is proven.
Chapter 3

Local estimates under small concentration of curvature

Our main goal in this chapter is to establish local $L^2_g$ and $L^\infty_g$ estimates of curvature, under the assumption of small concentration of curvature. This small energy condition is quantified in terms of a cutoff function $\gamma$ satisfying the properties described in Section 1.2. We say that the curvature is $\varepsilon-$concentrated if

$$\int_{|\gamma|>0} |A|^2 \, d\mu < \varepsilon. \quad (3.1)$$

In later chapters the cutoff function $\gamma$ is chosen such that (3.1) amounts to assuming that the total curvature on a small ball is less than $\varepsilon$. We first study the evolution of

$$\|\nabla(m)^{A}\|^2_{2,\gamma^s} := \int_\Sigma |\nabla(m)^{A}|^2 \gamma^s \, d\mu$$

under the geometric polyharmonic heat flow (GPHF). Henceforth for $\varphi \in C(\Sigma)$ we also use the notation $\|\varphi\|^2_{2,\gamma^s} := \int_\Sigma \varphi^2 \gamma^s \, d\mu$ as well as $\|\varphi\|^2_{2,|\gamma|=1} := \int_{|\gamma|=1} \varphi^2 \, d\mu$ and $\|\varphi\|^2_{2,|\gamma|>0} := \int_{|\gamma|>0} \varphi^2 \, d\mu$ for brevity.

This allows us to obtain the relevant $L^2$ estimates. By utilising a multiplicative Sobolev inequality from [58] (which has been proved in the appendix as Theorem
we are able to obtain $L^\infty$ estimates for any derivative of curvature in terms of $\|\nabla^{(m)} A\|^2_{2,\gamma^s}$ under an assumption of the form (3.1), see Proposition 3.4.

Because of the high number of derivatives involved in (GPHF), there are quite a number of different $P-$style terms that arise when calculating the evolution of integral quantities (see for example the calculations contained in the proof of Proposition 19 from [75] for the case $p = 2$). To save us from having to consider each term separately, we have developed a more general way to estimate all of the $P-$style terms with a certain algebraic structural form at the same time. We present this key estimate in the following theorem.

Theorem 3.1. There exists an $\varepsilon_0$ such that the following holds. Let $f : \Sigma^2 \to \mathbb{R}^3$ be an immersion satisfying

$$\int_{\gamma > 0} |A|^2 \, d\mu \leq \varepsilon_0$$

For any $l \in \mathbb{N}$ there exists universal constants $c > 0$ such that

$$\int_{\Sigma} |A|^{2l} \gamma^{2(l-1)} \, d\mu \leq c \|A\|_{2,\gamma>0}^{2(l-1)} \left( \|\nabla (l-1) A\|^2_{2,\gamma,2(l-1)} + c_\gamma 2^{2(l-1)} \|A\|^2_{2,\gamma>0} \right). \quad (3.2)$$

Furthermore for any $l \geq 2$

$$\sum_{i=2}^{l} \int_{\Sigma} (P_i^l (A))^{#^2} \gamma^{2(l-1)} \, d\mu \leq c \|A\|_{2,\gamma>0}^{2} \left( \|\nabla (l-1) A\|^2_{2,\gamma,2(l-1)} + c_\gamma 2^{2(l-1)} \|A\|^2_{2,\gamma>0} \right). \quad (3.3)$$

Here if $T$ is a tensor then $T^{#^2}$ is shorthand for $T * T$. For example, we can write

$$|T|^2 = T * T = T^{#^2}.$$ 

Proof. The first inequality, (3.2) follows from using Theorem A.2 with $u = |A|^{l} \gamma^{l-1}$:

$$\int_{\Sigma} |A|^{2l} \gamma^{2(l-1)} \, d\mu$$
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\[ \leq c \left( \int_{\Sigma} |\nabla A| |A|^l \, d\mu + c_\gamma \int_{\Sigma} |A|^l \, d\mu + \int_{\Sigma} |A|^{l+1} \, d\mu \right)^2 \]

\[ \leq c \left( \|
abla A\|_{2,\gamma}^2 + c_\gamma^2 \|A\|_{2,\gamma}^2 \right) \int_{\Sigma} |A|^{2(l-1)} \, d\mu \]

\[ + c \|A\|_{2,\gamma}^2 \int_{\Sigma} |A|^{2(l-1)} \, d\mu. \]

Therefore if \( \epsilon_0 \) is small enough, we have

\[ \int_{\Sigma} |A|^{2(l-1)} \, d\mu \leq c \left( \|
abla A\|_{2,\gamma}^2 + c_\gamma^2 \|A\|_{2,\gamma}^2 \right) \int_{\Sigma} |A|^{2(l-1)} \, d\mu. \]

Applying this inductive inequality \( l - 1 \) times and applying the interpolative inequality from Lemma A.10 we have

\[ \int_{\Sigma} |A|^{2l \gamma^{2(l-1)}} \, d\mu \leq c \left( \|
abla A\|_{2,\gamma}^2 + c_\gamma^2 \|A\|_{2,\gamma}^2 \right) \int_{\Sigma} |A|^{2(l-1)} \, d\mu. \]

This proves the first claim.

Now let us look at the second statement, (3.3). For \( l = 2 \), (3.2) with \( n = 2 \) implies

\[ \sum_{i=2}^{2} \int_{\Sigma} (P_i^{2-i} (A)) \#^2 \gamma^{2(2-1)} \, d\mu = \int_{\Sigma} (A \ast A) \#^2 \gamma^2 \, d\mu \]

\[ \leq c \int_{\Sigma} |A|^4 \, d\mu \]

\[ \leq c \|A\|_{2,\gamma}^2 \left( \|
abla A\|_{2,\gamma}^2 + c_\gamma^2 \|A\|_{2,\gamma}^2 \right). \]

This establishes the claim for \( l = 2 \). Assume that the statement is true for \( l = 3, \ldots, m \),
that is

\[
\sum_{i=2}^{l} \int_{\Sigma} (P_{i-1}^{l-i} (A)) \#^{2} \gamma^{2(l-1)} d\mu \\
\leq c \|A\|^{2}_{2,|\gamma|>0} \left( \|\nabla(l-1)A\|^{2}_{2,|\gamma|>0} + c_{\gamma}^{2(l-1)} \|A\|^{2}_{2,|\gamma|>0} \right) \text{ for } 2 \leq l \leq m. \tag{3.4}
\]

We wish to show that the statement is then true for \( l = m + 1 \). Assume \( 2 \leq i \leq m + 1 \).

Note that we can decompose \( P_{i}^{m+1-i} (A) \) as

\[
P_{i}^{m+1-i} (A) = \sum_{j=1}^{m+1-i} \nabla(j) A * P_{i-1}^{m+1-(i+j)} (A),
\]

facilitating the application of (3.4). Using Theorem A.2 with \( u = P_{i}^{m+1-i} (A) \) \( \gamma \)

gives

\[
\int_{\Sigma} (P_{i}^{m+1-i} (A)) \#^{2} \gamma^{2m} d\mu \\
= \sum_{j=1}^{m+1-i} \int_{\Sigma} \left( \nabla(j) A * P_{i-1}^{m+1-(i+j)} (A) \right) \#^{2} \gamma^{2m} d\mu \\
\leq c \sum_{j=1}^{m+1-i} \left( \int_{\Sigma} \|\nabla(j+1)A\|^{2}_{2,|\gamma|>0} + c_{\gamma}^{2} \|\nabla(j) A\|^{2}_{2,|\gamma|>0} \right) \int_{\Sigma} \left( P_{i-1}^{m+1-(i+j)} (A) \right) \#^{2} \gamma^{2(m-(j+1))} d\mu \\
+ c \sum_{j=1}^{m+1-i} \left( \|\nabla(j+1)A\|^{2}_{2,|\gamma|>0} + c_{\gamma}^{2} \|\nabla(j) A\|^{2}_{2,|\gamma|>0} \right) \int_{\Sigma} \left( P_{i-1}^{m+1-(i+j)} (A) \right) \#^{2} \gamma^{2(m-j)} d\mu \\
+ c \sum_{j=1}^{m+1-i} \int_{\Sigma} \|\nabla(j) A\|^{2}_{2,|\gamma|>0} \int_{\Sigma} \left( P_{i-1}^{m+1-(i+j)} (A) \right) \#^{2} \gamma^{2(m-j)} d\mu \\
\leq c \sum_{j=1}^{m+1-i} \left( \|\nabla(j+1)A\|^{2}_{2,|\gamma|>0} + c_{\gamma}^{2} \|\nabla(j) A\|^{2}_{2,|\gamma|>0} \right) \int_{\Sigma} \left( P_{i-1}^{m+1-(i+j)} (A) \right) \#^{2} \gamma^{2(m-(j+1))} d\mu \\
+ c \sum_{j=1}^{m+1-i} \|\nabla(j) A\|^{2}_{2,|\gamma|>0} \int_{\Sigma} \left( P_{i-1}^{m+1-(i+j)} (A) \right) \#^{2} \gamma^{2(m-j)} d\mu 
\]
Note that applying Theorem A.2 with \( u \) interpolation inequalities of Lemma A.10 gives

\[
+ c \sum_{j=1}^{m+1-i} \int_{\Sigma} |\nabla_{(j)} A|^2 |A|^2 \gamma^{2(j+1)} \, d\mu \int_{\Sigma} (P_{i-1}^m)^{(i+j)} \gamma A^2 \, d\mu. \tag{3.5}
\]

Now if \( i = 2 \), then applying the interpolation inequalities of Lemma A.10 to the previous inequality gives

\[
\int_{\Sigma} (P_{i-1}^m)^{(2)} \gamma^2 \, d\mu = \int_{\Sigma} (P_{i-1}^m)^{(2)} \gamma^2 \, d\mu \\
\leq c \sum_{j=1}^{m-1} \left( \|\nabla_{(j+1)} A\|^2_{2,\gamma^2(j+1)} + c_\gamma^2(j+1) \|A\|^2_{2,\gamma^2(j+1)} \right) \int_{\Sigma} (P_{i-1}^m)^{(2)} \gamma^2 \, d\mu \\
+ c \sum_{j=1}^{m-1} \|\nabla_{(j)} A\|^2_{2,\gamma^2(j+1)} \int_{\Sigma} (P_{i-1}^m)^{(2)} \gamma^2 \, d\mu \\
+ c \sum_{j=1}^{m-1} \int_{\Sigma} |\nabla_{(j)} A|^2 |A|^2 \gamma^2 \, d\mu \int_{\Sigma} (P_{i-1}^m)^{(2)} \gamma^2 \, d\mu \\
\leq c \sum_{j=1}^{m-1} \left( \|\nabla_{(j+1)} A\|^2_{2,\gamma^2(j+1)} + c_\gamma^2(j+1) \|A\|^2_{2,\gamma^2(j+1)} \right) \gamma^2 \int_{\Sigma} (m)^2 \gamma^2 \, d\mu + c_\gamma^2 \|A\|^2_{2,\gamma^2(j+1)} \gamma^2 \int_{\Sigma} (m)^2 \gamma^2 \, d\mu
\tag{3.6}
\]

Note that applying Theorem A.2 with \( u = |\nabla_{(j)} A|^2 |A|^2 \gamma^2 \) along with (3.2) and the interpolation inequalities of Lemma A.10 gives

\[
\int_{\Sigma} |\nabla_{(j)} A|^2 |A|^2 \gamma^2 \, d\mu \\
\leq c \left( \int_{\Sigma} |\nabla_{(j+1)} A|^2 |A|^2 \gamma^j \, d\mu + \int_{\Sigma} |\nabla_{(j)} A|^2 \gamma^j \, d\mu \right) \\
+ c_\gamma \left( \int_{\Sigma} |\nabla_{(j)} A|^2 |A|^2 \gamma^j \, d\mu + \int_{\Sigma} |\nabla_{(j)} A|^2 \gamma^j \, d\mu \right)^2 \\
\leq c \|A\|^2_{2,\gamma^2} \left( \|\nabla_{(j+1)} A\|^2_{2,\gamma^2(j+1)} + c_\gamma \|\nabla_{(j)} A\|^2_{2,\gamma^2(j+1)} \right) \\
+ c \|\nabla_{(j)} A\|^2_{2,\gamma^2} \left( \int_{\Sigma} |\nabla A|^2 \gamma^4 \, d\mu + \int_{\Sigma} |A|^4 \gamma^4 \, d\mu \right)
\]


\[ \leq c \|A\|_{2,[\gamma > 0]}^2 \left( \|\nabla_{(j+1)} A\|_{2,\gamma^2(j+1)}^2 + c_\gamma^{2(j+1)} \|A\|_{2,[\gamma > 0]}^2 \right) \\
+ c \|\nabla_{(j)} A\|_{2,\gamma^2}^2 \left( \|\nabla A\|_{2,\gamma^4}^2 + c_\gamma^2 \|A\|_{2,[\gamma > 0]}^2 \right). \]

Substituting into (3.6) and applying the interpolation inequalities of Lemma A.10 again then gives

\[ \int_\Sigma (P_2^{m-1} (A))^2 \gamma^{2m} d\mu \leq c (m) \|A\|_{2,[\gamma > 0]}^2 \left( \|\nabla (m) A\|_{2,\gamma^2m}^2 + c_\gamma^{2m} \|A\|_{2,[\gamma > 0]}^2 \right). \] (3.7)

This establishes the claim for \( i = 2 \). Therefore we may without loss of generality assume that \( 3 \leq i \leq m + 1 \). In this case \( 2 \leq i - 1 \leq m \) and \( 2 \leq m - j \leq m \) and so we substitute the inductive assumption (3.4) into (3.5) and use the interpolation inequalities of Lemma A.10 to give

\[
\int_\Sigma (P_2^{m-i} (A))^{#2} \gamma^{2m} d\mu \\
\leq c \|A\|_{2,[\gamma > 0]}^2 \sum_{j=1}^{m+1-i} \left( \|\nabla_{(j+1)} A\|_{2,\gamma^2(j+1)}^2 + c_\gamma^{2(j+1)} \|A\|_{2,[\gamma > 0]}^2 \right) \\
\times \left( \|\nabla_{(m-j-1)} A\|_{2,\gamma^2(m-j-1)}^2 + c_\gamma^{2(m-j-1)} \|A\|_{2,[\gamma > 0]}^2 \right) \\
+ \sum_{j=1}^{m+1-i} \|\nabla_{(j)} A\|_{2,\gamma^2}^2 \left( \|\nabla_{(m-j)} A\|_{2,\gamma^2(m-j)}^2 + c_\gamma^{2(m-j)} \|A\|_{2,[\gamma > 0]}^2 \right) \\
+ \sum_{j=1}^{m+1-i} \int_\Sigma (P_2^{(j+2)-2} (A))^{#2} \gamma^{2(j+1)} d\mu \int_\Sigma (P_2^{m-i-1-(i+j)} (A))^{#2} \gamma^{2(m-j)} d\mu \\
\leq c \|A\|_{2,[\gamma > 0]}^2 \left( \|\nabla (m) A\|_{2,\gamma^2m}^2 + c_\gamma^{2m} \|A\|_{2,[\gamma > 0]}^2 \right) \\
+ \sum_{j=1}^{m+1-i} \int_\Sigma (P_2^{(j+2)-2} (A))^{#2} \gamma^{2(j+1)} d\mu \int_\Sigma (P_2^{m-i-(i-1)} (A))^{#2} \gamma^{2(m-j)} d\mu \\
\leq c \|A\|_{2,[\gamma > 0]}^2 \left( \|\nabla (m) A\|_{2,\gamma^2m}^2 + c_\gamma^{2m} \|A\|_{2,[\gamma > 0]}^2 \right) \\
+ \sum_{j=1}^{m+1-i} \left( \|\nabla_{(j+1)} A\|_{2,\gamma^2(j+1)}^2 + c_\gamma^{2(j+1)} \|A\|_{2,[\gamma > 0]}^2 \right) \\
\times \left( \|\nabla_{(m-j-1)} A\|_{2,\gamma^2(m-j-1)}^2 + c_\gamma^{2(m-j-1)} \|A\|_{2,[\gamma > 0]}^2 \right) \\
\leq c \|A\|_{2,[\gamma > 0]}^2 \sum_{j=1}^{m+1-i} \left( \|\nabla_{(j+1)} A\|_{2,\gamma^2(j+1)}^2 + c_\gamma^{2(j+1)} \|A\|_{2,[\gamma > 0]}^2 \right). \]
\[ \times \left( \| \nabla_{(m-j-1)} A \|_{2,\gamma}^2 + c_{\gamma}^2 (m-j-1) \| A \|_{2,\gamma>0}^2 \right) \]
\[ \leq c \| A \|_{2,\gamma>0}^2 \left( \| \nabla_{(m)} A \|_{2,\gamma}^2 + c_{\gamma}^2 m \| A \|_{2,\gamma>0}^2 \right). \]

Combining this with (3.7) finishes the inductive step, and therefore finishes the proof of the theorem. \( \square \)

We are now equipped with all the tools required to obtain localised \( L^2 \) estimates of curvature.

### 3.1 Local \( L^2 \) estimates for \( A \) and its derivatives

In order to obtain localised \( L^2 \) estimates for \( A \) and its derivatives, we first use Lemma 1.2 and Lemma 1.4 to obtain a formula for the evolution of \( \| \nabla_{(m)} A \|_{2,\gamma}^2 \) under the flow equation (GPHF). It is important that all integral quantities that arise in our calculation (such as the summation on the last line of (3.8)) are estimable using the tools available, such as Theorem 3.1 and the interpolation inequalities from Lemma A.10.

**Lemma 3.2.** Let \( f : \Sigma^2 \times [0, T) \rightarrow \mathbb{R}^3 \) satisfy (GPHF). Then for any \( m, s \in \mathbb{N}_0 \):

\[
\frac{d}{dt} \| \nabla_{(m)} A \|_{2,\gamma}^2 + 2 \| \nabla_{(m+p+1)} A \|_{2,\gamma}^2 = -2 \sum_{j=1}^{p+1} \binom{p+1}{j} \int_{\Sigma} \langle \nabla_{(j)} \gamma^s, \nabla_{(m+p+1-j)} A \rangle \nabla_{(m+p+1)} A \ d\mu \\
+ \int_{\Sigma} |\nabla_{(m)} A|^2 \partial_t \gamma^s \ d\mu + \sum_{i=0}^{2p+m} \int_{\Sigma} \nabla_{(i)} \left( \nabla_{(2p+m-i)} A \ast A \ast A \right) \nabla_{(m)} A \gamma^s \ d\mu. \tag{3.8}
\]

**Proof.** Using the chain rule along with the results of Lemma 1.2 and Lemma 1.4 yields

\[
\frac{d}{dt} \| \nabla_{(m)} A \|_{2,\gamma}^2 \\
= \int_{\Sigma} \partial_t |\nabla_{(m)} A|^2 \gamma^s \ d\mu + \int_{\Sigma} |\nabla_{(m)} A|^2 \partial_t \gamma^s \ d\mu + \int_{\Sigma} |\nabla_{(m)} A|^2 \gamma^s \frac{\partial}{\partial t} \ d\mu
\]
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\[
\begin{align*}
= & \ 2 \ (-1)^p \int_{\Sigma} \left\langle \nabla_{(m)} A, \nabla_i \nabla_{i_1 i_2 \ldots i_{p+1}} \nabla_{(m)} A \right\rangle \gamma^s d\mu \\
+ & \sum_{i=0}^{2p+m} \int_{\Sigma} \nabla_{(i)} \left( \nabla_{(2p+m-i)} A \star A \star A \right) \nabla_{(m)} A \gamma^s d\mu \\
+ & \int_{\Sigma} |\nabla_{(m)} A|^2 \partial_t \gamma^s d\mu + (-1)^{p+1} \int_{\Sigma} |\nabla_{(m)} A|^2 H \Delta^p H \gamma^s d\mu \\
= & \ 2 \ (-1)^p \int_{\Sigma} \left\langle \nabla_{(m)} A, \nabla_i \nabla_{i_1 i_2 \ldots i_{p+1}} \nabla_{(m)} A \right\rangle \gamma^s d\mu \\
+ & \sum_{i=0}^{2p+m} \int_{\Sigma} \nabla_{(i)} \left( \nabla_{(2p+m-i)} A \star A \star A \right) \nabla_{(m)} A \gamma^s d\mu \\
+ & \int_{\Sigma} |\nabla_{(m)} A|^2 \partial_t \gamma^s d\mu + \sum_{i=0}^{2p+m} \int_{\Sigma} \nabla_{(i)} \left( \nabla_{(2p+m-i)} A \star A \star A \right) \nabla_{(m)} A \gamma^s d\mu.
\end{align*}
\]

(3.9)

We can then perform integration by parts \((p+1)\) times on the first term on the right hand side of (3.9) to give

\[
2(-1)^p \int_{\Sigma} \left\langle \nabla_{(m)} A, \nabla_i \nabla_{i_1 i_2 \ldots i_{p+1}} \nabla_{(m)} A \right\rangle \gamma^s d\mu \\
= 2(-1)^{2p+1} \sum_{j=0}^{p+1} \binom{p+1}{j} \int_{\Sigma} \left\langle \nabla_{(j)} \gamma^s \right\rangle \left( \nabla_{(m+p+1-j)} A \right), \nabla_{(m+p+1)} A \right\rangle d\mu \\
= -2 \left\| \nabla_{(m+p+1)} A \right\|_{2,\gamma^s}^2 - 2 \sum_{j=1}^{p+1} \binom{p+1}{j} \int_{\Sigma} \left\langle \nabla_{(j)} \gamma^s \right\rangle \left( \nabla_{(m+p+1-j)} A \right), \nabla_{(m+p+1)} A \right\rangle d\mu.
\]

We leave the proof by induction to the interested reader. Substitution into (3.9) then gives the desired result. \(\square\)

To deal with the extraneous intermediate terms above, our main tool will be Theorem 3.1 and interpolation inequalities (for example, Lemma A.10). We obtain the following estimate.

**Proposition 3.3.** There exists an \( \varepsilon_0 \) such that the following holds. Let \( f : \Sigma \times [0,T) \rightarrow \mathbb{R}^3 \) satisfy (GPHF) and \( s \geq 2 (k + p + 1) \). For any \( \eta > 0 \), \( k \in \mathbb{N}_0 \), and \( s \geq 2 (k + p + 1) \)
there exists a constant \( c > 0 \) such that if
\[
\sup_{[0,T^*]} \int_{[\gamma > 0]} |A|^2 \, d\mu \leq \varepsilon_0
\]
then for \( t \in [0, T^*] \)
\[
\frac{d}{dt} \| \nabla^(k)A \|_2,\gamma, s^{(k+p+1)} + \left( 2 - \eta - c \|A\|_2,\gamma, s^{(k+p+1)} \right) \| \nabla^(k+p+1)A \|_2,\gamma, s^{(k+p+1)} \leq c(k, p) \eta^{-1} c_\gamma^{2(k+p+1)} \|A\|^2_{2,\gamma > 0}. \tag{3.10}
\]
for any \( \eta > 0 \). Furthermore during this time there is another absolute constant \( C_k \) such that
\[
\| \nabla^(k)A \|_2,\gamma = 1 + \int_0^t \left( \| \nabla^(k+p+1)A \|_2,\gamma = 1 + \sum_{i=2}^{k+p+2} \int_{[\gamma = 1]} (P_i^{k+p+2-i} (A)) \|A\|_2,\gamma > 0 \right) d\tau \leq \| \nabla^(k)A \|_2,\gamma > 0 \bigg|_{t=0} + C_k c_\gamma^{2(k+p+1)} \varepsilon_0 t. \tag{3.11}
\]

Proof. By using the interpolation inequalities from Lemma A.10 as well as (1.25) we estimate each of the extraneous terms from Lemma 3.2.

Firstly we estimate the penultimate term: for each \( j \in \{1, 2, \ldots, p + 1\} \) we have
\[
\int_{\Sigma} \left( \nabla^(j)A \right)^{2,\gamma} \otimes \nabla^(k+p+1-j)A, \nabla^(k+p+1)A \right) \, d\mu
\]
\[
= \sum_{i=1}^{j} \sum_{l=1}^{j} \sum_{m=1}^{j-i} \int_{\Sigma} P_{m}^{j-l(m)}(A) \ast \alpha_l \ast \nabla^(k+p+1-j)A \ast \nabla^(k+p+1)A \right) \gamma^{s-i} \, d\mu
\]
\[
\leq \eta \| \nabla^(k+p+1)A \|_2,\gamma \leq c \eta^{-1} \sum_{i,l,m} C_\gamma^{2l} \int_{\Sigma} \left( P_{m}^{j-l(m)}(A) \ast \nabla^(k+p+1-j)A \right) \right\} \right] \gamma^{s-2l} \, d\mu
\]
\[
= \eta \| \nabla^(k+p+1)A \|_2,\gamma \leq c \eta^{-1} \sum_{i,l,m} C_\gamma^{2l} \left( \| \nabla^(k+p+1-j)A \|_2,\gamma \right)^{s-2l} \|A\|^2_{2,\gamma > 0} + C_\gamma^{2(k+p+1)} \|A\|^2_{2,\gamma > 0}
\]
\[
\leq 2\eta \| \nabla^(k+p+1)A \|_2,\gamma \leq c(s, k, p) \eta^{-1} C_\gamma^{2(k+p+1)} \|A\|^2_{2,\gamma > 0}
\]
for any \( \eta > 0 \). Here we have used (1.24), Lemma 3.1 and Lemma A.10, as well as Young’s inequality. The summations in the third line onwards are over the same range as in line two. Next we estimate the last sum of terms on the right hand side of (3.8).

We define \( X_i \) by

\[
X_i := \int_\Sigma \nabla_{(i)} \left( \nabla_{(2p+k-i)} A \ast A \ast A \right) \ast \nabla_{(k)} A \gamma^s \, d\mu
\]

for \( i \in \{0, 1, \ldots, 2p + k\} \). If \( i \leq p - 1 \) by integrating by parts repeatedly we may write

\[
X_i = \int_\Sigma \nabla_{(k+p+1)} A \ast \nabla_{(p-(i+1))} \left( A \ast A \ast \nabla_{(i)} \left( \nabla_{(k)} A \gamma^s \right) \right) \, d\mu
\]

\[
= \sum_{u=0}^{p-(i+1)} \int_\Sigma \nabla_{(k+p+1)} A \ast \nabla_{(p-(i+1)-u)} \left( A \ast A \ast \nabla_{(i+u)} \left( \nabla_{(k)} A \gamma^s \right) \right) \, d\mu
\]

\[
= \sum_{u=0}^{p-(i+1)} \sum_{v=0}^{i+u} \int_\Sigma \nabla_{(k+p+1)} A \ast \nabla_{(p-(i+1)-u)} \left( A \ast A \ast \nabla_{(i+u-v+k)} A \ast \nabla_{(v)} \gamma^s \right) \, d\mu
\]

\[
= \sum_{u,v,l,m,n} \int_\Sigma \nabla_{(k+p+1)} A \ast \nabla_{(p-(i+1)-u)} \left( A \ast A \ast \nabla_{(i+u-v+k)} A \ast P_n^{v-(m+n)} (A) \ast \alpha_m \gamma^{s-1} \right) \, d\mu
\]

\[
\leq \eta \| \nabla_{(k+p+1)} A \|_{2, \gamma^s}^2 + c \eta^{-1} \sum_{u,v,l,m,n} c_{2m}^2 \int_\Sigma \left( \nabla_{(p-(i+1)-u)} \left( A \ast A \ast \nabla_{(i+u-v+k)} A \ast P_n^{v-(m+n)} (A) \right) \right) \#^2 \gamma^{s-2l} \, d\mu
\]

\[
\leq \eta \| \nabla_{(k+p+1)} A \|_{2, \gamma^s}^2 + c \eta^{-1} \sum_{u,v,l,m,n} c_{2m}^2 \int_\Sigma \left( P_{n+2}^{k+p-(m+n)} (A) \right) \#^2 \gamma^{s-2l} \, d\mu
\]

\[
\leq \eta \| \nabla_{(k+p+1)} A \|_{2, \gamma^s}^2 + c \eta^{-1} \sum_{u,v,l,m,n} c_{2m}^2 \left( \| \nabla_{(k+p+1-m)} A \|_{2, \gamma^{s-2l}}^2 + c_{2 \gamma}^2 (k+p+1-m) \| A \|_{2, \gamma > 0}^2 \right)
\]

\[
\leq 2\eta \| \nabla_{(k+p+1)} A \|_{2, \gamma^s}^2 + c \eta^{-1} \sum_{u,v,l,m,n} c_{2m}^2 (k+p+1) \| A \|_{2, \gamma > 0}^2
\]

for any \( \eta > 0 \). Here we have used Theorem 3.1, Lemma A.10, and Young’s inequality.
We have also used the shorthand
\[
\sum_{u,v,l,m,n} := \sum_{u=0}^{p-(i+1)} \sum_{v=0}^{i+u} \sum_{l=1}^{v} \sum_{m=l}^{v-m} \sum_{n=1}^{v-m}
\]
in the fourth line onwards for brevity. If instead \( p + 1 \leq i \leq 2p + k \) then integrating by parts \( p + 1 \) times gives

\[
X_i := \int_{\Sigma} \nabla_{(i)} \left( \nabla_((2p+k-i)A * A * A) * \nabla_{(k)} A \gamma^s \right) d\mu
\]

\[
= \int_{\Sigma} \nabla_{(i-(p+1))} \left( \nabla_((2p+k-i)A * A * A) * \nabla_{(p+1)}(\nabla_{(k)} A \gamma^s) \right) d\mu
\]

\[
= \sum_{u,v,l,m,n} \int_{\Sigma} \nabla_{(k+p-1-u)} A * \nabla_{(u)} (A * A) * \nabla_{(k+p+1-v)} A * P_n^{-(m+n)}(A) * \alpha_m \gamma^{s-l} d\mu,
\]

(3.12)

where we have used the shorthand

\[
\sum_{u,v,l,m,n} = \sum_{u=0}^{i-(p+1)} \sum_{v=0}^{p+1} \sum_{l=1}^{v} \sum_{m=l}^{v-m} \sum_{n=1}^{v-m}
\]
as before, and we have also used Claim 1.1 to get to the last step. Noting that

\[
\nabla_{(k+p-1-u)} A * \nabla_{(u)} (A * A) = P_3^{k+p-1}(A)
\]

and

\[
\nabla_{(k+p+1-v)} A * P_n^{-(m+n)}(A) = P_{n+1}^{k+p+1-m-n}(A),
\]

we may use Young’s inequality on (3.12) to obtain

\[
X_i \leq c \int_{\Sigma} (P_3^{k+p-1}(A))^{#2} \gamma^s d\mu + c \sum_{u,v,l,m,n} \epsilon_{\gamma}^{2m} \int_{\Sigma} (P_{n+1}^{k+p+1-m-n}(A))^{#2} \gamma^{s-2l} d\mu
\]

\[
= c \int_{\Sigma} (P_3^{k+p+2-3}(A))^{#2} \gamma^s d\mu + c \sum_{u,v,l,m,n} \epsilon_{\gamma}^{2m} \int_{\Sigma} (P_{n+1}^{k+p+2-m-(n+1)}(A))^{#2} \gamma^{s-2l} d\mu
\]
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\[ \leq c \| A \|^2_{2, [\gamma > 0]} \left( \| \nabla (k + p + 1) A \|_{2, \gamma^s}^2 + c_\gamma^{k+p+1} \| A \|^2_{2, [\gamma > 0]} \right) \]

\[ + c \| A \|^2_{2, [\gamma > 0]} \sum_{u,v,l,m,n} c_\gamma^{2m} \left( \| \nabla (k + p + 2 - m) A \|_{2, \gamma^{s-2l}}^2 + c_\gamma^{2(k+p+2-m)} \| A \|^2_{2, [\gamma > 0]} \right) \]

\[ \leq c \| A \|^2_{2, [\gamma > 0]} \left( \| \nabla (k + p + 1) A \|_{2, \gamma^s}^2 + c_\gamma^{k+p+1} \| A \|^2_{2, [\gamma > 0]} \right). \]

Here we have used (3.3) from Theorem 3.1 and Lemma A.10.

Lastly, we estimate the first term on the right hand side of the evolution equation of Lemma 3.2. Using the definition of $\gamma$, we have

\[ \partial_t \gamma^s = s \gamma^{s-1} \partial_t \gamma \]

\[ = s \gamma^{s-1} \partial_t (\tilde{\gamma} \circ f) \]

\[ = s \gamma^{s-1} (D \tilde{\gamma}, \partial_t f) \]

\[ = (-1)^{p+1} s \gamma^{s-1} (D \tilde{\gamma}, \Delta^p H \nu) \]

\[ = (-1)^{p+1} s \gamma^{s-1} D_{\nu} \tilde{\gamma} (\Delta^p H), \]

where $D_{\nu} \tilde{\gamma}$ denotes the normal projection of the ordinary derivative of $D \tilde{\gamma}$, $(D \tilde{\gamma}, \nu)$. Therefore

\[ \int_{\Sigma} | \nabla (k) A |^2 \partial_t \gamma^s d\mu = (-1)^{p+1} s \int_{\Sigma} | \nabla (k) A |^2 (D_{\nu} \tilde{\gamma}) (\Delta^p H) \gamma^{s-1} d\mu. \] (3.13)

We also use the identity

\[ \nabla (j) \nu = \left( \sum_{i=1}^{j} P_i^{j-i} (A) \right) * (\partial f, \nu)^{\#(1)}, \quad j \in \mathbb{N} \]

which can be proven inductively by utilising the identities (1.22). Note that we are abusing notation here slightly in that the covariant derivative is defined as a mapping

\[ \nabla : df(T\Sigma) \times df(T\Sigma) \to df(T\Sigma), \]
where \( \nu \in df(T\Sigma)^\perp \). However, we set

\[
\nabla \nu = (D\nu)^T,
\]

where \( D \) is the ambient (partial) derivative in \( \mathbb{R}^3 \). Note that with this notation, the shape operator (see (1.8)) is given by

\[
S(X) = \nabla_X \nu.
\]

We have the estimate

\[
||\nabla (k) D\nu \tilde{\gamma}|| \leq c \left( c\gamma^{k+1} + \sum_{i=1}^k \sum_{j=1}^i |P^{i-j}_j(A) c\gamma^{k+1-i} \right), \quad k \in \mathbb{N}_0. \tag{3.14}
\]

We now estimate (3.13), first assuming that \( p \geq k + 1 \). Integrating by parts \( p - k - 1 \) times and applying Young’s inequality gives

\[
\int_\Sigma |\nabla (k) A|^2 \partial \gamma^s \, d\mu \\
= \int_\Sigma \nabla (k+p+1) A \ast \nabla (p-k-1) \left( |\nabla (k) A|^2 D\nu \tilde{\gamma} \gamma^{s-1} \right) \, d\mu \\
= \sum_{u+v+1=p-k-1} \int_\Sigma \nabla (k+p+1) A \ast \nabla (u) |\nabla (k) A|^2 \ast \nabla (v) D\nu \tilde{\gamma} \ast \nabla (w) \gamma^{s-1} \, d\mu \\
= \sum_{u+v+w=p-k-1} \int_\Sigma \nabla (k+p+1) A \ast P^{2k+u}_2(A) \ast \nabla (v) D\nu \tilde{\gamma} \ast \nabla (w) \gamma^{s-1} \, d\mu \\
\leq c \sum_{u,v,w,\alpha,\beta,\pi} c\gamma^{2\beta} \int_\Sigma |\nabla (k+p+1) A| |P^{2k+u}_2(A) | \nabla (v) D\nu \tilde{\gamma} ||P^{w-(\beta+\pi)}_\pi(A) \gamma^{s-(\alpha+1)} \, d\mu \\
\leq \eta \|\nabla (k+p+1) A\|_{2, \gamma^s}^2 \\
+ c\eta^{-1} \sum_{u,v,w,\alpha,\beta,\pi} c\gamma^{2\beta} \int_\Sigma |P^{2k+u}_2(A) | \nabla (v) D\nu \tilde{\gamma} ||P^{w-(\beta+\pi)}_\pi(A) \gamma^{s-(\alpha+1)} \, d\mu \\
= \eta \|\nabla (k+p+1) A\|_{2, \gamma^s}^2 \\
+ c\eta^{-1} \sum_{u,v,w,\alpha,\beta,\pi} c\gamma^{2\beta} \int_\Sigma \nabla (v) D\nu \tilde{\gamma} ||P^{w+(2k-(\beta+\pi)}_\pi(A) \gamma^{s-2(\alpha+1)} \, d\mu \\
\leq \eta \|\nabla (k+p+1) A\|_{2, \gamma^s}^2
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$$+ c \eta^{-1} \sum_{u,v,w,\alpha,\beta,\pi} c_{\gamma}^{2(\beta+v+1)} \int_{\Sigma} \left( P_{\pi+2}^{u+w+2k-(\beta+\pi)} (A) \right)^{\#2} \gamma^{s-2(\alpha+1)} d\mu$$

$$+ c \eta^{-1} \sum_{u,v,w,\alpha,\beta,\pi,i,j} c_{\gamma}^{2(\beta+v+1-i)} \int_{\Sigma} \left( P_{\pi+j+2}^{u+w+2k-i-j-(\beta+\pi)} (A) \right)^{\#2} \gamma^{s-2(\alpha+1)} d\mu$$

$$= \eta \| \nabla_{(k+p+1)} A\|_{2,\gamma}^2$$

$$+ c \eta^{-1} \sum_{u,v,w,\alpha,\beta} c_{\gamma}^{2(\beta+v+1)} \int_{\Sigma} \left( P_{\pi+2}^{u+w+2k-(\beta+\pi)} (A) \right)^{\#2} \gamma^{s-2(\alpha+1)} d\mu$$

$$+ c \eta^{-1} \sum_{u,v,w,\alpha,\beta,\pi,i,j} c_{\gamma}^{2(\beta+v+1-i)} \int_{\Sigma} \left( P_{\pi+j+2}^{u+w+2k-i-j-(\beta+\pi)} (A) \right)^{\#2} \gamma^{s-2(\alpha+1)} d\mu$$

(3.15)

Here we have used (3.14). We have also used the shorthand

$$\sum_{u,v,w,\alpha,\beta,\pi} := \sum_{u+v+1=p-k-1} \sum_{\alpha=1} \sum_{\beta=\alpha} \sum_{\pi=1}$$

and

$$\sum_{u,v,w,\alpha,\beta,\pi,i,j} := \sum_{u+v+1=p-k-1} \sum_{\alpha=1} \sum_{\beta=\alpha} \sum_{\pi=1} \sum_{i=1} \sum_{j=1}$$

throughout the equation. Next, using the identity $u + w + 2k = k + p - (v + 1)$, we have

$$u + w + 2k - (\beta + \pi) + \pi + 2 = k + p + 1 - (\beta + v)$$

and

$$u + w + 2k + i - j - (\beta + \pi) + \pi + j + 2 = k + p + 1 - (\beta + v - i).$$

Therefore using Theorem 3.1,

$$\int_{\Sigma} \left( P_{\pi+2}^{u+w+2k-(\beta+\pi)} (A) \right)^{\#2} \gamma^{s-2(\alpha+1)} d\mu$$

$$\leq c \| A \|_{2,\gamma}^2 \left( \| \nabla_{(k+p-(\beta+v)} A \|_{2,\gamma}^2 + c_{\gamma}^{2(k+p-(\beta+v))} \| A \|_{2,\gamma}^2 \right)$$

and

$$\int_{\Sigma} \left( P_{\pi+j+2}^{u+w+2k-i-j-(\beta+\pi)} (A) \right)^{\#2} \gamma^{s-2(\alpha+1)} d\mu$$
≤ c \|A\|_{2,\gamma>0}^2 \left( \| \nabla_{(k+p−(β+ν+i))}A\|_{2,\gamma^s−2(α+1)}^2 + c_{\gamma} 2^{(k+p−(β+ν+i))} \|A\|_{2,\gamma>0}^2 \right).

Substituting these into (3.15) and using the interpolation inequalities from Lemma A.10, we then have

$$\int_{\Sigma} |\nabla_{(k)}A|^2 \partial_t \gamma^s d\mu$$

$$\leq \left( \eta + c \|A\|_{2,\gamma>0}^2 \right) \| \nabla_{(k+p+1)}A\|_{2,\gamma^s}^2 + c(k, p, s) \eta^{-1} c_{\gamma} 2^{(k+p+1)} \|A\|_{2,\gamma>0}^4 \right) \|\nabla_{(k)}A\|_{2,\gamma>0}^2 \right).$$

for any $\eta > 0$. If instead $p \leq k \leq 2p$ we apply integration by parts $k − p + 1$ times on (3.13) to give

$$\int_{\Sigma} |\nabla_{(k)}A|^2 \partial_t \gamma^s d\mu = \int_{\Sigma} \nabla_{(p−1)}A \ast \nabla_{(k−p+1)} \left( \nabla_{(k)}A \ast D_{\nu} \gamma_s \ast \nabla_{(2p)}A \gamma^{s−1} \right) d\mu$$

and then proceed as before, noting that in this scenario no more than $k+p+1$ derivatives are applied to any single curvature term. Lastly, if $k \geq 2p+1$ then we apply integration by parts $p + 1$ times, giving

$$\int_{\Sigma} |\nabla_{(k)}A|^2 \partial_t \gamma^s d\mu = \int_{\Sigma} \nabla_{(k−(p+1))}A \ast \nabla_{(p+1)} \left( \nabla_{(k)}A \ast D_{\nu} \gamma_s \ast \nabla_{(2p)}A \gamma^{s−1} \right) d\mu$$

and then proceed as before once again. Here we note that in this case $3p+1 \leq k+p$ and so once again no more than $k+p+1$ derivatives are applied to any single curvature term.

Combining our earlier estimates for $X_i$ and (3.16), combining all of the small $\eta$ coefficients into one single small coefficient, and substituting into (3.8) gives

$$\frac{d}{dt} \| \nabla_{(k)}A\|_{2,\gamma>0}^2 + (2−\eta−c \|A\|_{2,\gamma>0}^2) \| \nabla_{(k+p+1)}A\|_{2,\gamma^s}^2 \leq c(k, p, s) \eta^{-1} c_{\gamma} 2^{(k+p+1)} \|A\|_{2,\gamma>0}^2 \right).$$

$$\sum_{i=2}^{k+p+2} \int_{\Sigma} \left( P_i^{k+p+2−i} (A) \right)^2 \gamma^{2(k+p+1)} d\mu$$

(3.17)
for some absolute constant $c$, and so combining this with (3.17) and setting $s = 2(k + p + 1)$ gives

$$
\frac{d}{dt} \left\| \nabla (k) A \right\|_{2, \gamma^2}^2 + \sum_{i=2}^{k+p+2} \int_{\Sigma} \left( P^i_{k+p+2-i} (A) \right)^{\#2} \gamma^{2(k+p+1)} d\mu 
+ \left( 2 - \eta - c \left\| A \right\|_{2, \gamma^0}^2 \right) \left\| \nabla (k+p+1) A \right\|_{2, \gamma^2}^2 
\leq c(k,p) \eta^{-1} \gamma^{2(k+p+1)} \left\| A \right\|_{2, \gamma^0}^2
$$

for any $\eta > 0$, which proves (3.10). Therefore if $\varepsilon_0 > 0$ is sufficiently small, then by choosing $\eta > 0$ small enough it follows that

$$
\frac{d}{dt} \left\| \nabla (k) A \right\|_{2, \gamma^2}^2 + \sum_{i=2}^{k+p+2} \int_{\Sigma} \left( P^i_{k+p+2-i} (A) \right)^{\#2} \gamma^{2(k+p+1)} d\mu + \left\| \nabla (k+p+1) A \right\|_{2, \gamma^2}^2 
\leq C_k \gamma^{2(k+p+1)} \varepsilon_0
$$

for some absolute constant $C_k$. Integrating over $[0,t]$ and using the fundamental theorem of calculus proves (3.11). \qed

### 3.2 Local $L^\infty$ estimates for $A$ and its derivatives

We conclude this chapter by customising two $\varepsilon$-regularity style inequalities from Kuwert-Schätzle [58], the second of which is time dependent. This is not only interesting in its own right, but is used later on in Theorem 7.1 of Chapter 7 to prove the interior estimates along the flow. (Note that the author and his advisors have proven a similar result for a general tensor $T$ in the paper [75]).

**Proposition 3.4.** There exists an $\varepsilon$ such that the following holds. Let $f : \Sigma^2 \to \mathbb{R}^3$ be
a smooth immersion satisfying
\[
\int_{\gamma > 0} |A|^2 d\mu \leq \varepsilon,
\]
where \( \gamma \) is a cutoff function as in Section 1.2. For every \( m \in \mathbb{N} \) there exists a universal constant \( c > 0 \) such that
\[
\| \nabla_{(m)} A \|_{\infty, \gamma = 1} \leq c \| A \|_{2, \gamma > 0}^{2} \left( \| \nabla_{(m+2)} A \|_{2, \gamma > 0}^{2(m+1)} + c_{\gamma}^{2(m+1)(m+2)} \| A \|_{2, \gamma > 0}^{2(m+1)} \right).
\]

Proof. Using the multiplicative Sobolev inequality from Theorem A.6 with \( \varphi = \| \nabla_{(m)} A \|_{2} 2^{m+3} \gamma^{2m+3} \) gives
\[
\| \nabla_{(m)} A \|_{\infty, \gamma = 1}^6 \leq c \| \nabla_{(m)} A \|_{2, \gamma = 2^{m+3}}^2 \left( \int_{\Sigma} | \nabla_{(m+1)} A |^4 \gamma^{2(2m+3)} d\mu + c_{\gamma}^4 \int_{\Sigma} | \nabla_{(m)} A |^4 \gamma^{2(2m+1)} d\mu \right.
\]
\[
+ \int_{\Sigma} | \nabla_{(m)} A |^4 H^4 \gamma^{2(2m+3)} d\mu \right)
\]
\[
\leq c \| \nabla_{(m)} A \|_{2, \gamma = 2^{m+3}}^2 \left( \int_{\Sigma} | \nabla_{(m+1)} A |^4 \gamma^{2(2m+3)} d\mu + c_{\gamma}^4 \int_{\Sigma} | \nabla_{(m)} A |^4 \gamma^{2(2m+1)} d\mu \right.
\]
\[
+ \int_{\Sigma} | \nabla_{(m)} A |^4 H^4 \gamma^{2(2m+3)} d\mu \right).
\]
(3.18)

We estimate each of these terms separately. Firstly, we use Theorem A.2 with \( u = | \nabla_{(m+1)} A | \gamma^{2m+3} \) to estimate the second term:
\[
\int_{\Sigma} | \nabla_{(m+1)} A |^4 \gamma^{2(2m+3)} d\mu
\]
\[
\leq c \left( \int_{\Sigma} | \nabla_{(m+2)} A | | \nabla_{(m+1)} A | \gamma^{2m+3} d\mu + c_{\gamma} \| \nabla_{(m+1)} A \|_{2, \gamma = 2^{m+1}}^2 \right.
\]
\[
+ \int_{\Sigma} | \nabla_{(m+1)} A |^2 | H | \gamma^{2m+3} d\mu \right)^2
\]
\[
\leq c \| \nabla_{(m+1)} A \|_{2, \gamma = 2^{m+1}}^2 \left( \| \nabla_{(m+2)} A \|_{2, \gamma = 2^{m+2}}^2 \right.
\]
\[
+ c_{\gamma}^2 \| \nabla_{(m+1)} A \|_{2, \gamma = 2^{m+1}}^2 + \int_{\Sigma} | \nabla_{(m+1)} A |^2 | A |^2 \gamma^{2(m+2)} d\mu \right)
\[ \leq c \| \nabla (m+1) A \|^2_{2, \gamma^2(m+1)} \left( \| \nabla (m) A \|^2_{2, \gamma^2(m+2)} + c_\gamma^2 \| A \|^2_{2, \gamma^2(0)} \right) \]
\[ + \int \left( P^{m+1}_2 (A) \right)^2 \gamma^2 \| A \|^2 \ d\mu \]
\[ \leq c \| \nabla (m+1) A \|^2_{2, \gamma^2(m+1)} \left( \| \nabla (m) A \|^2_{2, \gamma^2(m+2)} + c_\gamma^2 \| A \|^2_{2, \gamma^2(0)} \right). \quad (3.19) \]

Here we have used Theorem 3.1 to estimate the P-style terms. Similarly, using Theorem A.2 with \( u = \| \nabla (m) A \|^2 \gamma^{2m+1} \) gives

\[ c_\gamma^4 \int \| \nabla (m) A \|^4 \gamma^{2(2m+1)} \ d\mu \]
\[ \leq c_\gamma^4 \int \| \nabla (m+1) A \| \| \nabla (m) A \| \gamma^{2m+1} \ d\mu + c_\gamma \| \nabla (m) A \|^2_{2, \gamma^2(m+1)} \]
\[ + \int \| \nabla (m) A \|^2 |H| \gamma^{2m+1} \ d\mu \]
\[ \leq c_\gamma^4 \| \nabla (m) A \|^2_{2, \gamma^2} \left( \| \nabla (m+1) A \|^2_{2, \gamma^2(m+1)} + c_\gamma^2 \| \nabla (m) A \|^2_{2, \gamma^2(m+1)} \right) \]
\[ + \int \| \nabla (m) A \|^2 |A|^2 \gamma^{2(m+1)} \ d\mu \]
\[ \leq c_\gamma^4 \| \nabla (m) A \|^2_{2, \gamma^2(m+1)} \left( \| \nabla (m+1) A \|^2_{2, \gamma^2(m+1)} + c_\gamma^2 \| A \|^2_{2, \gamma^2(0)} \right) \]
\[ + \int \left( P^{m+1}_2 (A) \right)^2 \gamma^2 \| A \|^2 \ d\mu \]
\[ \leq c \left( \| \nabla (m+2) A \|^2_{2, \gamma^2(m+2)} + c_\gamma^2 \| A \|^2_{2, \gamma^2(0)} \right) \]
\[ \times \left( \| \nabla (m+1) A \|^2_{2, \gamma^2(m+1)} + c_\gamma^2 \| A \|^2_{2, \gamma^2(0)} \right). \quad (3.20) \]

We have again used Theorem 3.1 here to estimate the P-style terms. For the last term in (3.18) we use Theorem A.2 again, this time with \( u = \| \nabla (m) A \|^2 H^2 \gamma^{2m+3} \):

\[ \int \| \nabla (m) A \|^4 H^4 \gamma^{2m+3} \ d\mu \]
\[ \leq c \left( \int \| \nabla (m+1) A \| \| \nabla (m) A \| \gamma^{2m+3} \ d\mu + \int \| \nabla (m) A \|^2 |\nabla H| \|H| \gamma^{2m+3} \ d\mu \right) \]
\[ + c_\gamma \int \| \nabla (m) A \|^2 H^2 \gamma^{2(m+1)} \ d\mu \]
\[ \leq c \int \| \nabla (m) A \|^2 |A|^2 \gamma^{2(m+1)} \ d\mu \left( \int \| \nabla (m+1) A \|^2 |A|^2 \gamma^{2(m+2)} \ d\mu \right) \]
+ \int \Sigma |\nabla (m)A|^2 |\nabla A|^2 \gamma^2(m+2) d\mu + c_\gamma^2 \int \Sigma |\nabla (m)A|^2 |A|^2 \gamma^2(m+1) d\mu
\leq c \int \Sigma (P^{m}_2 (A))^{#2} \gamma^2(m+1) d\mu \left( \sum_{i=2}^{3} \int \Sigma (P^{m+3-i}_i (A))^{#2} \gamma^2(m+2) d\mu \right)
\leq c |A|^{4}_{2,|\gamma|>0} \left( \int \Sigma |\nabla (m+1)A|^2 \gamma^2(m+1) d\mu + c_\gamma^2 \sum_{i=2}^{3} \gamma^2(m+2) |A|^{2}_{2,|\gamma|>0} \right)
\leq c |A|^{4}_{2,|\gamma|>0} \left( \int \Sigma |\nabla (m+1)A|^2 \gamma^2(m+1) \gamma^2(m+2) |A|^{2}_{2,|\gamma|>0} \right)
\leq c |A|^{4}_{2,|\gamma|>0} \left( \int \Sigma |\nabla (m+1)A|^2 \gamma^2(m+1) \gamma^2(m+2) |A|^{2}_{2,|\gamma|>0} \right)
= \left( \int \Sigma |\nabla (m+2)A|^2 \gamma^2 \gamma^2(m+2) + c_\gamma^2 \gamma^2(m+1) |A|^{2}_{2,|\gamma|>0} \right).

(3.21)
\[ \times \left( \| \nabla_{(m+2)} A \|^2_{2, \gamma^{2(m+2)}} + c^2_{\gamma} \| A \|^2_{2, [\gamma > 0]} \right) \]
\[ \leq c \| A \|^6_{2, [\gamma > 0]} \left( \| \nabla_{(m+2)} A \|^6_{2, \gamma^{2(m+2)}} + c^3_{\gamma} \| A \|^3_{2, [\gamma > 0]} \| \nabla_{(m+2)} A \|_{2, \gamma^{2(m+2)}}^{2(2m+1)} + c^{6(m+1)}_{\gamma} \| A \|_{2, [\gamma > 0]}^{6(m+1)} \right) \]
\[ \leq c \| A \|^6_{2, [\gamma > 0]} \left( \| \nabla_{(m+2)} A \|^6_{2, \gamma^{2(m+2)}} + c^3_{\gamma} \| A \|^3_{2, [\gamma > 0]} \| \nabla_{(m+2)} A \|_{2, \gamma^{2(m+2)}}^{2(2m+1)} + c^{6(m+1)}_{\gamma} \| A \|_{2, [\gamma > 0]}^{6(m+1)} \right) . \]

Raising each side to the power \((m + 2) / 3\) then finishes the proof. \(\square\)

The preceding proposition has the following corollary that allows us to estimate any derivative of curvature along the flow in terms of initial datum, the cutoff function \(\gamma\) and final time \(T^\ast\).

We will first need to introduce some new notation. Given a choice of cutoff function \(\gamma\) as in Section 1.2, we introduce an associated two-parameter family of smooth cutoff functions \(\gamma_{\sigma, \tau}\) for \(0 \leq \sigma < \tau \leq 1\) that satisfy the properties

\[ \gamma_{\sigma, \tau} = 1 \text{ for } \gamma \geq \tau \text{ and } \gamma_{\sigma, \tau} = 0 \text{ for } \gamma \leq \sigma. \quad (3.22) \]

We observe that increasing \(\tau - \sigma\) has the affect of “flattening out” the function, while decreasing the quantity acts to sharpen it. A similar family of cutoff functions is used by Kuwert-Schätzle in [58].

**Proposition 3.5.** There exists an \(\varepsilon_0\) such that the following holds. Let \(f : \Sigma \times [0, T) \to \mathbb{R}^3\) satisfy (GPHF), as well as

\[ \sup_{[0,T]} \int_{[\gamma > 0]} |A|^2 \, d\mu \leq \varepsilon_0. \]

Then for every \(k \in \mathbb{N}_0\) there are constants \(\tilde{c}_k\) depending only on \(k, T^\ast, c, \varepsilon_0\) and

\[ \alpha_0 (k + 2) := \sum_{j=0}^{k+2} \| \nabla(j) A \|^2_{2, [\gamma > 0]} \bigg|_{t=0} \]
such that
\[
\| \nabla (k) A \|^2_{\infty, |\gamma| = 1} \leq \tilde{c}_k.
\]

**Proof.** Applying (3.11) with \( \gamma_{0,1/2} \) gives
\[
\left\| \nabla (k) A \right\|^2_{2,|\gamma| \geq 1/2} \leq \left\| \nabla (k) A \right\|^2_{2,|\gamma| > 0} \bigg|_{t=0} + C_k c_2^{(k+p+1)} \varepsilon_0 t \leq c_k (\alpha_0 (k), k, T^*, c_\gamma, \varepsilon_0)
\]
for any \( k \in \mathbb{N}_0 \). Here we have used (3.22) which implies the identities
\[
|\gamma| \geq 1/2 \subseteq [\gamma_{0,1/2} = 1] \text{ and } |\gamma_{0,1/2} > 0 | \subseteq [\gamma > 0].
\]

Combining (3.23) and Proposition 3.4 with \( \gamma_{1/2,1} \) gives
\[
\left\| \nabla (k) A \right\|^2_{\infty, |\gamma| = 1} \leq c \left\| A \right\|^2_{2,|\gamma| > 0} \left( \left\| \nabla (k+2) A \right\|^2_{2,|\gamma| > 0} + c_\gamma^{2(k+1)(k+2)} \| A \|^2_{2,|\gamma| > 0} \right)
\]
\[
\leq c \left\| A \right\|^2_{2,|\gamma| > 0} \left( \alpha_0 (k+2)^{k+2} + c_k^{k+2} + c_\gamma^{2(k+1)(k+2)} \| A \|^2_{2,|\gamma| > 0} \right)
\]
\[
\leq c (\varepsilon_0, \alpha_0 (k+2), k, T^*, c_\gamma, \varepsilon_0),
\]
which finishes the proof. \( \square \)

We improve on this inequality later on in Theorem 7.1.
Chapter 4

The Lifespan Theorem

In this chapter, we primarily utilise the results of Proposition 3.3 to establish an absolute lower bound on the lifespan of a geometric polyharmonic heat flow that depends solely on the concentration of curvature for our initial immersion $\Sigma_0$. This idea of concentration of curvature presents itself in a very precise way: we ask that at time $t = 0$ the amount of curvature inside $f^{-1}(B_\rho(x))$ is small for any $x \in \mathbb{R}^3$, where $\rho > 0$ is some constant. As we mention in Remark 4.2, the underlying concepts of the theorem (both in statement and proof) are not original and are customised from a few different examples of lower order geometric flows (see, for example, Kuwert-Schätzle [59], and McCoy-Wheeler [74]).

We present a statement of the theorem right away, and then spend the rest of the chapter working at it, presenting and proving the necessary supporting results when they appear.

**Theorem 4.1** (Lifespan Theorem). Suppose $f : \Sigma^2 \times [0, T) \rightarrow \mathbb{R}^3$ is a one-parameter family of closed immersions with smooth initial data that evolves via the geometric polyharmonic heat flow

$$\frac{\partial}{\partial t} f = (-1)^{p+1} (\Delta^p H) \nu.$$  

(GTHF)
Then there are constants $\rho > 0, \varepsilon_0 > 0$ and $c < \infty$ such that if $\rho$ is chosen with

$$\int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \bigg|_{t=0} = \varepsilon(x) \leq \varepsilon_0 \text{ for any } x \in \mathbb{R}^3,$$

then the maximal time $T$ of smooth existence for the flow with initial data $f_0 = f(\cdot, 0)$ satisfies

$$T \geq \frac{1}{c} \rho^{2(p+1)},$$

and we have the estimate

$$\int_{f^{-1}(B_\rho(x))} |A|^2 d\mu \leq c \varepsilon_0 \text{ for } 0 \leq t \leq \frac{1}{c} \rho^{2(p+1)}.$$

**Remark 4.2.** The argument presented for the Lifespan Theorem is essentially re-crafted from similar established results for lower order flows. Namely the analogous proof presented in a paper by McCoy, Wheeler and Williams [73] regarding the constrained surface diffusion flow (which is described by Wheeler as being inspired by the work of Kuwert-Schätzle [57]). The details have been changed to fit our higher order scenario, but the crux of the argument (in particular the nature of our argument by contradiction) can not be claimed to be original at its core. However (as far as the writer is aware) this is the first application of such arguments to a geometric flow of general order. Additionally, the idea of exploiting the scale invariance of $\|A\|_2^2$ (in particular, in order to obtain an absolute bound on a lifespan) is far from new. The interested reader is prompted to read, for example, the work of Struwe [93], Kuwert-Schätzle [57, 58], and Wheeler [74].

**Remark 4.3.** If the initial immersion $\Sigma_0$ has finite total curvature then for an $\varepsilon_0 > 0$ it is always possible to find a positive $\rho = \rho(\varepsilon_0, \Sigma_0)$ such that assumption (4.1) is satisfied. This is always the case in the setting we consider here.

**Remark 4.4.** A consequence of the Lifespan Theorem is that a geometric polyharmonic heat flow can only cease to exist and lose regularity in finite time if we encounter a curvature singularity at time $T$. That is, we must be in a situation where curvature
has concentrated in some small ball in $\mathbb{R}^3$. Specifically, if we define $\rho(t)$ by

$$\rho(t) = \sup \left\{ r > 0 : \int_{f^{-1}(B_r(x))} |A|^2 \, d\mu \leq \varepsilon_0 \text{ for all } x \in \mathbb{R}^3 \right\}, \quad (4.4)$$

then

$$\lim_{t \uparrow T} \int_{f^{-1}(B_{\rho(t)}(x(t)))} |A|^2 \, d\mu \geq \varepsilon_0, \quad (4.5)$$

where $x(t)$ is taken to be the centre of the ball where the integral is maximised at time $t$. Inequality (4.5) is equivalent to saying that at least $\varepsilon_0$ of curvature is concentrated in the ball $f^{-1}(B_{\rho(t)}(x(t)))$. Moreover, the results of Theorem 4.1 would then imply that $\rho(t) \leq \langle c(T-t) \rangle^{\frac{1}{2p+1}}$, and so if $T < \infty$ then

$$\rho(t) \to 0 \text{ as } t \uparrow T.$$

Hence if the geometric polyharmonic heat flow is extinguished in finite time $T$, then the maximal radius $\rho(t)$ must shrink down to a point as we approach $T$. This tells us that the curvature of our family of immersions must concentrate in a very precise manner.

We now consider a rescaling of $f$ that is equivalent (that is, satisfies a version of the geometric polyharmonic heat flow on a modified time interval) that allows us to assume $\rho = 1$. This will allow us to simplify calculations in our proof of Theorem 4.1.

**Claim 4.5 (Rescaling the flow).** If $f$ is a geometric polyharmonic heat flow, then scaled flow

$$\tilde{f}(x,t) = \frac{1}{\rho} f(x, \rho^{2(p+1)} t). \quad (4.6)$$

is as well. Moreover, we have the identity

$$\int_{f^{-1}(B_r(x))} |A|^2 d\mu = \int_{f^{-1}(B_1(x))} |\tilde{A}|^2 d\tilde{\mu}. \quad (4.7)$$

**Proof.** To see the motivation for this particular choice of scaling, we first consider the more general scaling

$$\tilde{f}(x, \tilde{t}(t)) = \alpha f(x, t). \quad (4.8)$$
We wish the equation (4.8) to satisfy the geometric polyharmonic heat flow equation.

First we must see what the modified geometric polyharmonic heat flow is for the scaling. To do so, we will need to first look at how the basic geometric quantities vary with our scaling (4.8). For the metric, we have

\[ \tilde{g}_{ij} = (\partial_i\tilde{f}, \partial_j\tilde{f}) = (\alpha\partial_i f, \alpha\partial_j f) = \alpha^2 g_{ij}, \]

meaning that \( \tilde{g} = \alpha^2 g \). We use this to conclude that \( \tilde{g}^{-1} = \alpha^{-2} g^{-1} \). We now use these two identities to show that the Christoffel symbols \( \Gamma \) vary with the scaling (4.8). We calculate:

\[ \tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{kl}(\partial_i\tilde{g}_{jl} + \partial_j\tilde{g}_{il} - \partial_l\tilde{g}_{ij}) = \frac{1}{2}\alpha^{-2}g^{kl}(\alpha^2\partial_i g_{jl} + \alpha^2\partial_j g_{il} - \alpha^2\partial_l g_{ij}) = \Gamma^k_{ij}. \]

This tells us that \( \tilde{\Gamma} = \Gamma \). It then follows that \( \tilde{\nabla} = \nabla \). Scaling an immersion does not affect its unit normal because Euclidean space is self-similar, and so obviously \( \tilde{\nu} = \nu \).

We now calculate the second fundamental form for \( \tilde{f} \):

\[ \tilde{A}_{ij} = -(\partial_{ij}\tilde{f}, \tilde{\nu}) = -\alpha(\partial_{ij} f, \nu) = \alpha A_{ij}, \]

meaning that \( \tilde{A} = \alpha A \). This implies

\[ \tilde{H} = \tilde{g}^{ij}\tilde{A}_{ij} = (\alpha^{-2}g^{ij})(\alpha A_{ij}) = \alpha^{-1}g^{ij}A_{ij} = \alpha^{-1}H. \]

We are almost ready to prove the motivation for our choice of scaling. We finally calculate

\[ \tilde{\Delta}^p\tilde{H} = (\alpha^{-2}g^{i_1j_1})\cdots(\alpha^{-2}g^{i_pj_p})\nabla_{i_1j_1\ldots i_pj_p}(\alpha^{-1}H) = \alpha^{-(2p+1)}\Delta^p H. \]

We want \( \tilde{f} \) from (4.8) to satisfy the corresponding geometric polyharmonic heat flow:

\[ \frac{\partial \tilde{f}}{\partial \tilde{t}} = (-1)^{p+1}(\tilde{\Delta}^p\tilde{H})\tilde{\nu}. \] (4.9)
We will need to look at the condition on our scaled time variable $\hat{t}$ which will allow $\hat{f}$ to do so. Using the chain rule on (4.8), we have

$$\frac{\partial \hat{f}}{\partial \hat{t}} = \alpha \frac{\partial f}{\partial t} \frac{1}{\partial \hat{t}} = \alpha^{2(p+1)} \frac{dt}{dt} \left( (-1)^{p+1} \tilde{\nabla}^p \tilde{H} \right) \tilde{\nu}.$$ 

It is clear that in order for $\hat{f}$ to satisfy (4.9), we require

$$\frac{dt}{dt} = \alpha^{-2(p+1)}.$$ 

We conclude that a natural choice is $\tilde{t} = \alpha^{2(p+1)} t$. We substitute this into (4.8) and make the substitution $t \mapsto \alpha^{-2(p+1)} t$. This tells us that for rescaling a solution of the geometric polyharmonic heat flow equation, the most natural choice is of the form

$$\hat{f}(x, t) = \alpha f(x, \alpha^{-2(p+1)} t).$$ 

We note that making the substitution $\alpha \mapsto \rho^{-1}$ gives us equation (4.6). To prove (4.7), we have to first check how scaling affects the induced measure. We calculate

$$\tilde{\mu} = \sqrt{\det g} d^2 x = \sqrt{\det (\alpha^2 g)} d^2 x = \alpha^4 \sqrt{\det g} d^2 x = \alpha^2 \sqrt{\det g} d^2 x = \alpha^2 d\mu.$$ 

We have to check how the scaled function $\hat{f}$ affects the domain of the integral in question. Assuming that our balls are centred at the origin, we have ($y \in \mathbb{R}^3$)

$$y \in \hat{f}^{-1}(B_1) \iff |\tilde{f}(y)| < 1 \iff |f(y)| < \rho \iff y \in f^{-1}(B_\rho).$$ 

Combining this with our scaled geometric quantities above, we have

$$\int_{f^{-1}(B_\rho)} |A|^2 \, d\mu = \int_{\hat{f}^{-1}(B_1)} \hat{g}^{ij} \hat{g}^{jq} A_{ij} A_{pq} \, d\mu$$

$$= \int_{\hat{f}^{-1}(B_1)} (\rho^{-2} \hat{g}^{ij})(\rho^{-2} \hat{g}^{jq})(\rho \hat{A}_{ij})(\rho \hat{A}_{pq})(\rho^2 \hat{d}\mu)$$

$$= \int_{\hat{f}^{-1}(B_1)} \hat{g}^{ij} \hat{g}^{jq} \hat{A}_{ij} \hat{A}_{pq} \hat{d}\mu.$$
This completes the claim. □

Hence for any \( \rho > 0 \) the domain \( f^{-1}(B_\rho(x)) \) of our integral can be transformed into \( f^{-1}(B_1(x)) \) with an appropriately rescaled geometric polyharmonic heat flow \( \tilde{f} \). Note that under the assumption \( \rho = 1 \), to prove Theorem 4.1 it is enough to show that (4.2) is equivalent to the rescaled maximal time satisfying \( \tilde{T} \geq c^{-1} \), because

\[
\tilde{T} \geq c^{-1} \iff T \geq c^{-1} \rho^{2(p+1)}.
\]

We define

\[
\kappa(t) := \sup_{x \in \mathbb{R}^3} \|A\|_{2;f^{-1}(B_1(x))}^2.
\]

Then, by covering the ball \( B_1 \) with a number of translated copies of \( B_{1/2} \), it is possible to find a universal constant \( C_\kappa > 1 \) (\( C_\kappa = 64 \) is sufficient [103]) such that

\[
\kappa(t) \leq C_\kappa \sup_{x \in \mathbb{R}^3} \|A\|_{2,f^{-1}(B_{1/2}(x))}^2.
\]

(4.10)

By short time existence, we have \( f(\Sigma \times [0,t]) \) compact for \( t < T \), and hence \( \kappa : [0,T) \to \mathbb{R} \) is continuous. We wish to define 3 more constants. Firstly we define \( C_0 \) as the largest of the constants on the right hand side of the inequality from Proposition 3.3. From this, we make the definition \( \lambda = 1/(C_0C_\gamma^{2(p+1)}) \), and

\[
t_0 = \sup \{ 0 \leq t \leq \min \{ T, \lambda \} : \kappa(\tau) \leq 3C_\kappa \varepsilon_0 \text{ for } 0 \leq \tau \leq t \}.
\]

(The reason for this parameter \( \lambda \) will become apparent later when we try to establish a contradiction).

**Proof of the Lifespan Theorem.** We will go through this proof in 3 steps, labelled (4.11)—
Show that $t_0 = \min \{T, \lambda\}$, \hspace{1cm} (4.11)

Show that if $t_0 = \lambda$ then we have the Lifespan Theorem, \hspace{1cm} (4.12)

and

Show that if $T \neq \infty$ then $t_0 \neq T$. \hspace{1cm} (4.13)

Note that (4.13) is equivalent to showing that $t_0 = T \implies T = \infty$. We claim that the three statements (4.11), (4.12) and (4.13) together prove the Lifespan Theorem. To see this, note that if (4.11) holds then we must have either $t_0 = T$ or $t_0 = \lambda$. If $t_0 = T$, then (4.13) implies that $T = \infty$, meaning that the flow exists for all time, which proves the Lifespan Theorem. If instead $t_0 = \lambda$, then (4.12) will directly give us the Lifespan Theorem.

To begin, we note that by one of the assumptions of the theorem we have

$$\kappa(0) = \sup_{x \in \mathbb{R}^3} \|A\|_{L^2, f^{-1}(B_1(x))} \leq \varepsilon_0 < 3C_\kappa \varepsilon_0,$$

because $C_\kappa > 1$ and so we must have $t_0$ strictly positive. To see this, we note that the continuity of $\kappa$ and the fact that $\kappa(0) < 3C_\kappa \varepsilon_0$ strictly forces $\kappa(t) < 3C_\kappa \varepsilon_0$ for some strictly positive time period. The definition of $t_0$ then guarantees $t_0 > 0$.

Also, by the definition of $t_0$ and by continuity of $\kappa$, we must have

$$\kappa(t_0) = 3C_\kappa \varepsilon_0.$$ \hspace{1cm} (4.14)

Let us assume that $t_0 < \min \{\lambda, T\}$ and aim for a contradiction. We choose a cutoff function $\gamma$ such that

$$\chi_{B_{1/2}}(x) \leq \tilde{\gamma} \leq \chi_{B_1(x)} \text{ for any } x \in \Sigma_t.$$
By the initial smoothness of $|A|^2$, we can always find a $\rho^* > 0$ in assumption (4.1) that is small enough to guarantee the smallness condition

$$\sup_{[0,T^*]} \int_{[\gamma>0]} |A|^2 d\mu \leq \varepsilon_0$$

holds for every $T^* < t_0$. (Note that this is not sufficient to show (4.3) because we do not have the result for every $T^* \leq c^{-1}$). Also note that this is equivalent to assuming the hypothesis of Proposition 3.3 on the interval $[0,t_0)$.

The results of Proposition 3.3 then allow us to establish the inequality

$$\int_{[\gamma=1]} |A|^2 \leq \int_{[\gamma>0]} |A|^2 d\mu \bigg|_{t=0} + C_0 c_\gamma^2(p+1) \int_{[\gamma>0]} |A|^2 d\mu.$$  

We note that the identities

$$\gamma(y) > 0 \implies |f(y) - x| < 1 \implies y \in f^{-1}(B_1(x))$$

and

$$y \in f^{-1}\left(B_{1/2}(x)\right) \implies \gamma(y) = 1$$

imply that

$$[\gamma > 0] \subseteq f^{-1}(B_1(x)) \text{ and } f^{-1}(B_{1/2}(x)) \subseteq [\gamma = 1].$$

Combining this with the results above give us for $t \in [0,t_0)$ and $x \in \mathbb{R}^3$:

$$\int_{f^{-1}(B_{1/2}(x))} |A|^2 d\mu \leq \int_{f^{-1}(B_1(x))} |A|^2 d\mu \bigg|_{t=0} + C_0 c_\gamma^2(p+1) \int_{f^{-1}(B_1(x))} |A|^2 d\mu \leq \varepsilon_0 + C_0 c_\gamma^2(p+1) \varepsilon_0 t_0$$

It follows from the assumption that $t_0 < \lambda$ and the definition of $\lambda$ that for $t \in [0,t_0)$ and $x \in \mathbb{R}^3$ we have

$$\int_{f^{-1}(B_{1/2}(x))} |A|^2 d\mu \leq \varepsilon_0 + C_0 c_\gamma^2(p+1) \varepsilon_0 t_0$$
\[< \varepsilon_0 + C_0 c_1^{2(p+1)} \varepsilon_0 \lambda\]
\[= 2\varepsilon_0.\]

We deduce from identity (4.10) that

\[\kappa(t) \leq C_\kappa \sup_{x \in \mathbb{R}^3} \|A\|_2 f^{-1}(B_{1/2}(x)) \leq 2C_\kappa \varepsilon_0 \text{ for } t \in [0, t_0). \tag{4.15}\]

By the continuity of \(\kappa\) this means that

\[\lim_{t \to t_0} \kappa(t) \leq 2C_\kappa \varepsilon_0,\]

which contradicts (4.14), because \(C_k > 0\). This contradiction means that the assumption that \(t_0 < \min\{T, \lambda\}\) must have been incorrect. Thus we have proven (4.11).

We now note that under assumption (4.11) (which was just proven) we must have either \(t_0 = T\) or \(t_0 = \lambda\). If \(t_0 = \lambda\) then the already-proven identity \(t_0 = \min\{\lambda, T\}\) forces

\[T \geq t_0 = \lambda = (C_0 c_1^{2(p+1)})^{-1},\]

and the definition of \(t_0\) tells us that

\[\int_{f^{-1}(B_1(x))} |A|^2 d\mu \leq 3C_\kappa \varepsilon_0\]

for \(x \in \Sigma_t\) and \(t \in [0, \lambda]\). Together these two statements give us both statements (4.2) and (4.3) of the theorem with \(c = \max\{C_0 c_1^{2(p+1)}, 3C_\kappa\}\). That is, assuming that \(t_0 = \lambda\), we have the Lifespan Theorem. This proves (4.12).

Finally, we turn our attention to (4.13). We will assume

\[t_0 = T \neq \infty\]

and then aim to contradict the maximality of \(T\). We note that we have not included
the case $T = \infty$ because in that case we automatically have $T \geq \frac{1}{c} \rho^{2(p+1)}$ for any $c > 0$, meaning that the second part of the Lifespan Theorem 4.2 would hold automatically. Additionally, we would have $\lambda \leq T$ so that $t_0 = \lambda = T$ and then statement (4.3) with $c = 2C_\kappa$ would directly follow from our earlier estimate (4.15). We can also exclude the case $T < \lambda$ for the sake of our argument, because by (4.11) it would follows that $t_0 = T$, from which we could conclude (using (4.12)) the Lifespan Theorem. To clarify, the assumption $t_0 = T \neq \infty$ is logically equivalent to the negation of the statement

$$t_0 = T \implies T = \infty,$$

and so reaching a contradiction from such an assumption will lead us to the conclusion that

$$T \neq \infty \implies t_0 \neq T,$$

which is statement (4.13).

The following argument is originally from Hamilton [42], and has become rather standard. The idea is to show that for the geometric polyharmonic heat flow, our time-dependent metrics $g(t)$ are equivalent for all $t$ up until the maximal existence time of the flow, and converge uniformly to a positive-definite metric tensor which is continuous. From this we can establish that our flow exists smoothly up to and including time $T$, meaning that our immersion $f_T : \Sigma \rightarrow \mathbb{R}^3$ is smooth and compact. We are then able to consider the flow $h : \Sigma \times [0, \delta] \rightarrow \mathbb{R}^3 (\delta \geq 0)$ given by $h(\cdot, t) = f(\cdot, t + T)$. Since $f_T(\Sigma)$ is an initial smooth and compact surface of the immersion $h_0$, our short time existence theorem, Theorem 2.1, allows us to guarantee the existence of a flow on some positive time interval (that is, we are allowed to assume $\delta > 0$). As a result we have existence of the flow on $[0, T + \delta]$, an interval properly containing $[0, T]$, contradicting the maximality of $T$.

**Lemma 4.6** (Hamilton [42], Lemma 14.2). *Let $g_{ij}$ be a time-dependent metric on a*
compact manifold $\Sigma$ for $0 \leq t < T \leq \infty$. Suppose

$$\int_0^T \max_{\Sigma_t} |\partial_t g| \, dt \leq C < \infty.$$ 

Then the metrics $g_{ij}(t)$ are equivalent for all times, and they converge uniformly as $t \to T$ to a positive definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent such that

$$e^{-C} g(x, 0) \leq g(x, T) \leq e^C g(x, 0).$$

Here

$$|\partial_t g|^2 := g^{ik} g^{jl} \partial_t g_{ij} \partial_t g_{kl}.$$ 

Proof. Let $x \in \Sigma, t_* \in [0, T)$, and $v \in T_x \Sigma$ be arbitrary. Then define

$$|v|^2_t = g_{ij}(x, t) v^i v^j$$

to be the squared norm of the vector $v$, with respect to the metric $g$ at time $t$. It follows that

$$\frac{d}{dt} |v|^2_t = (\partial_t g_{ij}) v^i v^j \leq |\partial_t g| |v|^2_t,$$

where we have used the Cauchy-Schwarz inequality in the last step. Hence

$$\frac{d}{dt} \ln |v|^2_t = \frac{1}{|v|^2_t} \partial_t |v|^2_t \leq |\partial_t g|.$$ 

Applying the fundamental theorem of calculus then yields

$$\ln \left( \frac{|v|^2_\theta}{|v|^2_\tau} \right) = \int_\tau^\theta \frac{\partial}{\partial t} (\ln |v|^2_t) \, dt \leq \int_\tau^\theta |\partial_t g| \, dt,$$

for any $0 \leq \tau \leq \theta < T$. Therefore if the integral on the right hand side is finite with bound $0 < C < \infty$ (as in the statement of the Lemma), we can exponentiate the
previous inequality to obtain

\[ e^{-C} |v|^2_\tau \leq |v|^2_{\theta} \leq e^C |v|^2_\tau \]

for any \( 0 \leq \tau \leq \theta < T \). This means that all the time-dependent metrics are equivalent. Moreover \( |v|^2_\tau \) converges uniformly to a continuous function \( |v|^2_T \) as \( t \to T \) and \( v \neq 0 \implies |v|_T \neq 0 \). The parallelogram law

\[ |u + v|^2 + |u - v|^2 = 2 (|u|^2 + |v|^2) \]

still holds in the limiting case. That is,

\[ |u + v|^2_T + |u - v|^2_T = 2 (|u|^2_T + |v|^2_T) \]

and we may use this to establish the positive-definite structure of \( g_{ij}(T) \). The limiting norm comes from an inner product \( g_{ij}(T) \) using the (standard) rule

\[ g(u,v) \bigg|_T = \frac{1}{4} \left( |u + v|^2_T - |u - v|^2_T \right) . \]

From our equivalence relation on the metrics, for every \( \tau \in [0,T) \) and every \( v \in T\Sigma \) we may establish the inequality

\[ |v|^2_\tau \geq e^{-C} |v|^2_0 . \]

for some positive constant \( C > 0 \). The positive-definiteness of the metric \( g(0) \) and the continuity of the convergence \( |v|_\tau \to |v|_T \) then forces \( g(T) \) to be positive-definite as well. To see this, we note that for any non-zero \( v \in T\Sigma \) we must have \( |v|^2_0 > 0 \) (because \( g(0) \) is positive-definite). Then by our rule above we have

\[ g(u,u) \bigg|_T = \frac{1}{4} |2u|^2_T = |u|^2_T \geq e^{-C} |v|^2_0 > 0. \]
Hence \( g(u, u) \bigg|_T > 0 \) for every non-zero \( u \in T\Sigma \). This completes the proof.

We wish to apply this result to the previously-established bounds for our metric and second fundamental form so that we may establish regularity for the immersion at time \( T \) (that is, of \( \Sigma_T \)). Indeed, our aim is to show that \( \Sigma_T \) is smooth. To do this we will need to look at establishing a uniform bound on the derivatives of our immersion function \( f \). This will be enough to establish the uniform convergence of the map \( \Sigma_t \rightarrow \Sigma_T \). Importantly, this uniform convergence (along with Lemma 4.6) is enough to ensconce the uniqueness of our \( \Sigma_T \). Once we have established a uniform bound on the derivatives of \( f \), we will consider the induced metric topology induced by \( g(t) \) on the evolving surface \( \Sigma_t \) to show that the topology of the evolving surfaces is equivalent to the topology of \( \Sigma_T \).

We proceed with a number of general tensorial gradient estimates. As mentioned earlier, our aim is to show that our immersion \( f \) if \( C^\infty \) right to and including time \( T \). To do so we first establish pointwise estimates for the repeated covariant derivatives of general tensors, which we can apply to the second fundamental form \( A \). This gives us bounds on any covariant derivative of \( A \) in terms of a constant that only depends upon the cutoff function \( \gamma \), the initial immersion \( \Sigma_0 \), and the maximal time \( T \). Under the assumption that \( T < \infty \) this bounding constant is universal and finite. Converting the norms of the partial derivatives of \( f \) into covariant derivatives of \( A \) (see Claim 4.9) then allows us to conclude that our one-parameter family of immersions are smooth right up until time \( T \). Now let us look back at our evolution equations from Lemma 1.2, in which we established that for a surface evolving via the geometric polyharmonic heat flow the following identities hold:

\[
\frac{\partial}{\partial t} g = \nabla_{(2p)} A \ast A \quad \text{and} \quad \frac{\partial}{\partial t} \Gamma^k_{ij} = \nabla_{(2p)} A \ast \nabla A + \nabla_{(2p+1)} A \ast A.
\]

(4.16)

Also, recall that we are still under the assumption that \( T = t_0 \), and so by Proposition
3.5 for every \( m \in \mathbb{N}_0 \) we may establish the bound

\[
\| \nabla_{(m)} A \|_{\infty, f^{-1}(B_{1/2}(x))}^2 \leq C_m (\alpha_0 (m + 2), m, \varepsilon_0, c_\gamma, T)
\]

for \( t \in [0, T) \) and for any \( x \in \Sigma_t \). Moreover, since \( T \) is assumed to be finite, this bound is absolute. From this we can drop the redundant addition of the mention of the localised ball \( B_{1/2}(x) \): as the definition of \( \kappa \) allows us the extend this pointwise bound to the whole of \( \Sigma_t \). That is, we can establish the bound

\[
\| \nabla_{(m)} A \|_{\infty, \Sigma_t}^2 \leq C_m (\alpha_0 (m + 2), m, \varepsilon_0, c_\gamma, T)
\]

for \( t \in [0, T) \). Taking the square root of both sides, we have

\[
\| \nabla_{(m)} A \|_{\infty, \Sigma_t} \leq \sqrt{C_m} < \infty
\]

for any \( t \in [0, T) \). Combining this result with the second identity in (4.16) we have

\[
\nabla_{(m)} \partial_t \Gamma^k_{ij} = \nabla_{(m)} \left( \nabla_{(2p)} A \ast \nabla A + \nabla_{(2p+1)} A \ast A \right)
\]

\[
= \sum_{i=0}^{m} \nabla_{(2p+1)} A \ast \nabla_{(m+1-i)} A + \sum_{i=0}^{m} \nabla_{(2p+1+i)} A \ast \nabla_{(m-i)} A,
\]

meaning that

\[
\| \nabla_{(m)} \partial_t \Gamma^k_{ij} \|_{\infty} \leq c \sum_{i=0}^{m} \left( \sqrt{C_{2p+i}} \sqrt{C_{m+1-i}} + \sqrt{C_{2p+1+i}} \sqrt{C_{m-i}} \right) < \infty.
\]

Now looking at the first identity in (4.16), we have

\[
\nabla_{(m)} \partial_t g = \nabla_{(m)} \left( \nabla_{(2p)} A \ast A \right) = \sum_{i=0}^{m} \nabla_{(2p+i)} A \ast \nabla_{m-i} A,
\]

so that

\[
\| \nabla_{(m)} \partial_t g \|_{\infty} \leq c \sum_{i=0}^{m} \sqrt{C_{2p+i}} \sqrt{C_{m-i}} < \infty.
\]
for some universal constant $c > 0$. This clearly implies that our metric $g$ satisfies the hypothesis of Lemma 4.6, and so it follows from the Lemma that the metrics $g(t)$ are all equivalent for $t \in [0, T)$. Thus we can choose a local chart such that

$$\frac{1}{C^*} \delta_{ij} \leq g_{ij}(t) \leq C^* \delta_{ij}$$

on some neighbourhood $U \subset \Sigma, t \in [0, T)$ for some fixed constant $C^*$.

We now wish to establish another identity that allows us to bound the ordinary directional derivatives of tensors in terms of the norms of their covariant derivatives. We will use it to find pointwise bounds on the immersion map (and its derivative) so that we may guarantee the convergence $f(\cdot, t) \to f(\cdot, T)$ is in the smooth topology.

**Claim 4.7.** For any tensor $T$ we have the formula

$$\nabla_{(m)} T = \partial_{(m)} T + \sum_{j=1}^{m} \sum_{i_1 + \cdots + i_j = m-j} \partial_{(i_1)} T * \partial_{(i_1)} \Gamma * \cdots * \partial_{(i_j)} \Gamma$$

where $\Gamma$ are the usual Christoffel symbols. Here (and henceforth) $\partial_{(m)}$ will be used to denote the $m^{th}$ iterated ordinary coordinate derivative.

**Proof.** The case $m = 1$ follows directly from the definition of the covariant derivative. Assume that the claim holds true for some $m \in \mathbb{N}$. That is to say, assume

$$\nabla_{(m)} T = \partial_{(m)} T + \sum_{j=1}^{m} \sum_{i_1 + \cdots + i_j = m-j} \partial_{(i_1)} T * \partial_{(i_1)} \Gamma * \cdots * \partial_{(i_j)} \Gamma.$$

Then taking the covariant derivative of each side yields

$$\nabla_{(m+1)} T = \nabla \left( \partial_{(m)} T + \sum_{j=1}^{m} \sum_{i_1 + \cdots + i_j = m-j} \partial_{(i_1)} T * \partial_{(i_1)} \Gamma * \cdots * \partial_{(i_j)} \Gamma \right)$$

$$= \partial_{(m+1)} T + \partial_{(m)} T * \Gamma$$
We will now use Claim 4.7 to establish a pointwise bound on the iterated ordinary derivative of a general tensor $T$. This will almost take us to the crux the proof of the Lifespan Theorem.

Claim 4.8. For any tensor $T$ we have the estimate

$$
\|\partial_{(m)} T\|_\infty \leq c \sum_{\alpha=0}^{m} \omega_{\alpha,m} \|\nabla_{(\alpha)} T\|_\infty
$$

for some universal constant $c > 0$. Here the coefficients $\omega_{\alpha,m}$ are given specifically by $\omega_{m,m} = 1$ and

$$
\omega_{\alpha,m} = \sum_{j=1}^{m-\alpha} \sum_{i_1 + \cdots + i_j = m - \alpha - j} |\partial_{(i_1)} \Gamma| \cdots |\partial_{(i_j)} \Gamma| \text{ for } 0 \leq \alpha \leq m - 1.
$$

Proof. To give a better idea of what is going on, let us look at the cases $m = 1, 2$...
specifically. For the case \( m = 1 \) we have \( \nabla T = \partial T + T * \Gamma \), meaning that

\[
|\partial T| \leq c \left( |\nabla T| + |\Gamma| |T| \right) = c \sum_{\alpha=0}^{1} \omega_{\alpha,1} |\nabla_{(\alpha)} T|.
\]

For the case \( m = 2 \) it follows from Claim 4.7 that

\[
\nabla_{(2)} T = \partial_{(2)} T + \partial T * \Gamma + T * \partial \Gamma + T * \Gamma * \Gamma.
\]

Using our bound above for \( \partial T \), we then have

\[
|\partial_{(2)} T| \leq c \left( |\nabla_{(2)} T| + |\Gamma| |\nabla T| + |T| |\partial \Gamma| + |T| |\Gamma|^2 \right)
\]

\[
\leq c \left( |\nabla_{(2)} T| + |\Gamma| (|\nabla T| + |T| |\Gamma|) + |T| |\partial \Gamma| + |T| |\Gamma|^2 \right)
\]

\[
\leq c \left( |\nabla_{(2)} T| + |\Gamma| |\nabla T| + (|\Gamma|^2 + |\partial \Gamma|) |T| \right)
\]

\[
= c \sum_{\alpha=0}^{2} \omega_{\alpha,2} |\nabla_{(\alpha)} T|.
\]

It is clear that the statement of our claim holds true for the cases \( m = 1, 2 \). Assume that the statement is true for some general \( m \in \mathbb{N} \). That is, assume that

\[
|\partial_{(m)} T| \leq c \left( |\nabla_{(m)} T| + \sum_{\alpha=0}^{m-1} \omega_{\alpha,m} |\nabla_{(\alpha)} T| \right),
\]

where each \( \omega_{\alpha,m} \) is as previously described. Then, using Claim 4.7 we have

\[
\partial_{(m+1)} T = \nabla_{(m+1)} T + \sum_{j=1}^{m} \sum_{\substack{i_1+i_2+\cdots+i_j \overset{m+1-j}{=} \beta}} \partial_{(i_1)} T * \partial_{(i_2)} T * \cdots * \partial_{(i_j)} T \Gamma
\]

\[
= \nabla_{(m+1)} T + \sum_{\beta=0}^{m} \eta_{m+1}^{\beta} (\Gamma) * \partial_{(\beta)} T,
\]

where the coefficients \( \eta_{m+1}^{\beta} (\Gamma) \) are explicitly given by

\[
\eta_{m+1}^{\beta} (\Gamma) = \sum_{j=1}^{m+1-\beta} \sum_{\substack{i_1+i_2+\cdots+i_j \overset{m+1-\beta-j}{=} \beta}} \partial_{(i_1)} T * \cdots * \partial_{(i_j)} T \Gamma.
\]
Taking the norm of each side, and utilising our inductive assumptions on $|\partial_m T|$, it then follows that

$$|\partial_{m+1} T| \leq c \left( |\nabla_m T| + \sum_{\beta=0}^m |\eta^{m+1}_\beta (\Gamma)| |\partial_\beta T| \right)$$

$$\leq c \left( |\nabla_m T| + \sum_{\beta=0}^m |\eta^{m+1}_\beta (\Gamma)| \left( \sum_{\alpha=0}^\beta \omega_{\alpha,\beta} |\nabla_\alpha T| \right) \right)$$

$$\leq c \left( |\nabla_m T| + \sum_{\beta=0}^m \omega_{\beta,m+1} \left( \sum_{\alpha=0}^\beta \omega_{\alpha,\beta} |\nabla_\alpha T| \right) \right), \quad (4.19)$$

where we have used the trivial identity $|\eta^{m+1}_\beta (\Gamma)| \leq c \omega_{\beta,m+1}$. We now note that for any $\alpha, \beta$ satisfying $0 \leq \beta \leq m$, $0 \leq \alpha \leq \beta$ we have

$$\omega_{\beta,m+1} \omega_{\alpha,\beta}$$

$$= \left( \sum_{j=1}^{m+1-\beta} \sum_{i_1+\cdots+i_j = m+1-\beta-j} |\partial_{(i_1)} \Gamma| \cdots |\partial_{(i_j)} \Gamma| \right) \left( \sum_{j=1}^{\beta-\alpha} \sum_{j_1+\cdots+j_j = \beta-\alpha-j} |\partial_{(i_1)} \Gamma| \cdots |\partial_{(i_j)} \Gamma| \right).$$

A quick look at the line above shows that every element produced in the multiplication is of the form

$$|\partial_{(i_1)} \Gamma| |\partial_{(i_2)} \Gamma| \cdots |\partial_{(i_j)} \Gamma|,$$

with $i_1 + i_2 + \cdots + i_j = K$ for some $j$, $K$ satisfying $2 \leq j \leq m+1-\alpha$, $0 \leq K \leq m - \alpha - 1$. (In fact, these quite are loose bounds on our $j,K$ but are tight enough for the argument). Hence some of our terms in the inequality (4.19) can be absorbed into the others, so that the inequality reads:

$$|\partial_{m+1} T| \leq c \left( |\nabla_{m+1} T| + \sum_{\beta=0}^{m+1} \omega_{\beta,m+1} |\nabla_\beta T| \right) \leq c \sum_{\beta=0}^{m+1} \omega_{\beta,m+1} |\nabla_\beta T|,$$

completing the inductive step for the $m + 1$ case. The statement of the claim then follows by induction.

Now, by applying the fundamental theorem of calculus on the bounded interval
[0, T), we have

\[ |\Gamma^k_{ij}(t)| \leq |\Gamma^k_{ij}(0)| + \int_0^t |\partial_t \Gamma^k_{ij}(\xi)| \, d\tau \]

\[ \leq |\Gamma^k_{ij}(0)| + \int_0^t |(\nabla_{(2p)} A \ast \nabla A + \nabla_{(2p+1)} A \ast A)| \, d\xi \]

\[ \leq c \left( \Gamma^k_{ij}(0), C_0, C_1, C_{2p}, C_{2p+1}, T \right) \]

\[ \leq c \left( \Gamma(0), \alpha_0 (2p + 3), \varepsilon_0, c, T \right) \]

\[ < \infty, \]

where we have used the notation of Proposition 3.5.

Combining the result of Claim 4.8 with \( T = \partial_t \Gamma \) and (4.18) we obtain:

\[ \| \partial_{(m)} \partial_t \Gamma \|_\infty \leq c \sum_{\alpha=0}^{m} \omega_{\alpha,m} \left| \nabla_{(\alpha)} \partial_t \Gamma \right| \]

\[ \leq c \sum_{\alpha=0}^{m} \sum_{i=0}^{\alpha} \omega_{\alpha,m} \left( \sqrt{C_{2p+i}} \sqrt{C_{\alpha+1-i}} + \sqrt{C_{2p+1+i}} \sqrt{C_{a-i}} \right). \] (4.20)

It is slightly nontrivial to check that this sum is in fact bounded (because the summation on the right contains \( \omega_{\alpha,m} \) factors, which depend on the norms of the spatial derivatives of \( \Gamma \)). However, for the sum \( \sum_{\alpha=0}^{m} \omega_{\alpha,m} \), the highest order spatial derivative of \( \Gamma \) that is encountered is \( m - 1 \) (which occurs when \( \alpha = 0, i = 1 \) in the summation on the right hand side of (4.20)). So, because \( |\Gamma| \) is bounded, it then follows by an inductive argument that

\[ \| \partial_{(m)} \Gamma \|_\infty \leq c \left( \Gamma(0), C_0, C_1, \ldots, C_{2p+m+1}, T \right) \]

\[ \leq c \left( \Gamma(0), \alpha_0 (2p + m + 3), \varepsilon_0, c, T \right) \]

\[ < \infty. \] (4.21)

We just have one more claim to make and prove before we finalise our argument.
Claim 4.9. For $k, l, m \in \mathbb{N}_0$ with $k + l = m \geq 0$ we have
\[
\|\partial_{(k)}\nabla_{(l)} A\|_\infty, \|\partial_{(m+1)} f\|_\infty \leq c \left( \Gamma (0), \alpha_0 (2p + m + 2), \varepsilon_0, c_\gamma, T \right). \tag{4.22}
\]

Proof. For the case $m = 0$ we have to show that
\[
\|A\|_\infty, \|\partial f\|_\infty \leq c \left( \Gamma (0), \alpha_0 (2p + 2), \varepsilon_0, c_\gamma, T \right).
\]
The first inequality follows directly from (4.17), in which we found that
\[
\|A\|_\infty \leq \sqrt{C_0} \leq c \left( \alpha_0 (2), \varepsilon_0, c_\gamma, T \right) < \infty.
\]
For the second inequality for any $x \in \Sigma_t$ we have
\[
|\partial f| = \sqrt{g_{ij}g^{ij}} = \sqrt{2},
\]
which is obviously bounded. So, the case $m = 0$ is trivial. Consider the case $m = 1$. If $k = 0$ and $l = 1$ we have the obvious bound
\[
\|\nabla A\|_\infty \leq \sqrt{C_1} \leq c \left( \alpha_0 (3), \varepsilon_0, c_\gamma, T \right).
\]
Meanwhile, if $k = 1$ and $l = 0$ we have
\[
\|\partial A\|_\infty \leq c \left( \|\nabla A\|_\infty + \|\Gamma\|_\infty \|A\|_\infty \right)
\leq c \left( \sqrt{C_1} + c \left( \Gamma (0), \alpha_0 (2p + 3), c_\gamma, T \right) \sqrt{C_0} \right)
\leq c \left( \Gamma (0), \alpha_0 (2p + 3), \varepsilon_0, c_\gamma, T \right) < \infty,
\]
where we have used (4.21).

To give an idea of where the dependence of $\|\partial_{(m+1)} f\|$ on $C_i$ comes from, let us look
at $\partial_{(2)}f$. By the Gauss equation (C.9) we have

$$
\partial_{(2)}f = A * \nu + \Gamma * \partial f.
$$

This implies the estimate

$$
\|\partial_{(2)}f\|_\infty \leq c \left(\|A\|_\infty + \|\Gamma\|_\infty \|\partial f\|_\infty\right)
\leq c \left(\sqrt{C_0} + \sqrt{2} c (\Gamma (0), \alpha_0 (2p + 3), \varepsilon_0, c_\gamma, T)\right)
\leq c (\Gamma (0), \alpha_0 (2p + 3), \varepsilon_0, c_\gamma, T).
$$

So the estimate in the hypothesis at least \textit{seems} to be correct. Let us now prepare the inductive step.

Assume the inequality (4.22) holds true for $k + l = 1, 2, \ldots, m$, where $m \in \mathbb{N}_0$. That is, assume

$$
\|\partial_{(k)} \nabla_{(l)} A\|_\infty, \|\partial_{(u+1)} f\|_\infty \leq c (\Gamma (0), \alpha_0 (2p + u + 2), \varepsilon_0, c_\gamma, T)
$$

for $k + l = u$ and $u = 0, 1, 2, \ldots, m$. Next, set $k + l = m + 1$.

We will start by trying to show the second part of the claim, by looking at the $\partial_{(i)}f$ terms. From our $*$--representation of $\partial_{(2)}f$ above, we can establish the generalised identity inductively:

$$
\partial_{(m+2)}f = \partial_{(m)}A + \sum_{i_1 + i_2 = m} \partial_{(i_1)} \Gamma * \partial_{(i_2+1)} f \text{ for } m \in \mathbb{N}.
$$

Computing the norms of both sides we find that

$$
\|\partial_{(m+2)}f\|_\infty
\leq c \left(\|\partial_{(m)}A\|_\infty + \sum_{i_1 + i_2 = m} \|\partial_{(i_1)} \Gamma\|_\infty \|\partial_{(i_2+1)} f\|_\infty\right).
$$
\[ \leq c \left( c(\Gamma(0), \alpha_0 (2p + m + 2), \varepsilon_0, c_\gamma, t) + c(\Gamma(0), \alpha_0 (2p + m + 3), \varepsilon_0, c_\gamma, T) \sum_{i=1}^{m+1} \| \partial(i)f \|_\infty \right) \]

\[ \leq c(\Gamma(0) , \alpha_0 (2p + m + 3), \varepsilon_0, c_\gamma, T) \left( 1 + \sum_{i=1}^{m+1} \| \partial(i)f \|_\infty \right) \]

\[ \leq c(\Gamma(0) , \alpha_0 (2p + m + 3), \varepsilon_0, c_\gamma, T) (1 + c(\Gamma(0) , \alpha_0 (2p + m + 2), \varepsilon_0, c_\gamma, T)) \]

\[ \leq c(\Gamma(0) , \alpha_0 (2p + m + 3), \varepsilon_0, c_\gamma, T). \]

Here we have used the inductive hypothesis, along with the previously established bounds for \( \| \partial(i)\Gamma \|_\infty \) from (4.21) to get to the last step. This gives us the second inequality from (4.22) for \( \| \partial(m+2)f \|_\infty \) and hence completes half of the inductive step. Let us return to the first part of the inductive step. Note that because

\[ \partial(i) \nabla(k+l-i)A \bigg|_{i=\alpha} = \partial(i-1) \nabla(k+l+1-i)A \bigg|_{i=\alpha+1}, \]

and because \( k + l = m + 1 \implies \nabla(k) \nabla(l)A = \nabla_{m+1}A \), we can form a telescoping series as follows:

\[ \partial(k) \nabla(l)A - \nabla_{(m+1)}A = \sum_{i=1}^{k} \left( \partial(i) \nabla_{(k+1-i)}A - \partial(i-1) \nabla_{(k+l+1-i)}A \right) \]

\[ = \sum_{i=1}^{k} \left( \partial(i) \nabla_{(k+1-i)} - \partial(i-1) \nabla_{(k+l+1-i)} \right) A = \sum_{i=1}^{k} \partial(i-1) \left( \partial \nabla_{(k+l-i)} - \nabla_{(k+l+1-i)} \right) A \]

\[ = \sum_{i=1}^{k} \partial(i-1) \left( \partial - \nabla \right) \nabla_{(k+l-i)}A. \tag{4.23} \]

We want to look at simplifying the term \( \left( \partial - \nabla \right) \nabla_{(k+l-i)}A \) in the last line of equation (4.23). To do so, we note that the covariant derivative is a tangential tensor, we will analyse the behaviour of the operation \( \left( \partial - \nabla \right) \) on an arbitrary tangential tensor \( T \).
By the definition of the covariant derivative we have $\nabla T = \partial T + T \ast \Gamma$, meaning that

$$(\partial - \nabla) T = T \ast \Gamma.$$  

Applying this identity to $T = \nabla (k + l - i) A$ and substituting into (4.23) then gives us

$$\partial(k) \nabla(l) A - \nabla(m+1) A = \sum_{i=1}^{k} \partial(i-1) (\partial - \nabla) \nabla(k+l-i) A$$

$$= \sum_{i=1}^{k} \partial(i-1) (\nabla(k+l-i) A \ast \Gamma)$$

$$= \sum_{i=1}^{k} \sum_{i_1+i_2=i-1} \partial(i_1) \nabla(k+l-i) A \ast \partial(i_2) \Gamma.$$  

Rearranging and taking norms then gives us the inequality

$$\|\partial(k) \nabla(l) A\|_\infty$$

$$\leq c \left( \|\nabla(m+1) A\|_\infty + \sum_{i=1}^{k} \sum_{i_1+i_2=i-1} \|\partial(i_1) \nabla(k+l-i) A\|_\infty \|\partial(i_2) \Gamma\|_\infty \right)$$

$$\leq c \left( \sqrt{C_{m+1}} + c (\Gamma(0), \alpha_0 (2p + m + 3), \varepsilon_0, c_\gamma, T) \right)$$

$$\leq c (\Gamma(0), \alpha_0 (2p + m + 3), \varepsilon_0, c_\gamma, T).$$  

Here we have utilised the inductive hypothesis, as well as (4.21). Thus we have proven the first part of the inductive step. Hence the claim follows by induction on our results.  

The results of Claim 4.9 together with the Gauss-Weingarten equations from Claim C.6 tell us that for a one parameter family of solutions to the geometric polyharmonic heat flow, the estimates

$$\|\partial(m+1) f\|_\infty \leq c (\Gamma(0), \alpha_0 (2p + m + 2), \varepsilon_0, c_\gamma, T)$$
and
\[ \left\| \partial_{(m)} \partial_t f \right\|_\infty \leq c (\Gamma (0) , \alpha_0 (4p + m + 2) , \varepsilon_0 , c_\gamma , T) \]
hold for \( m \in \mathbb{N}_0 \).

Note that both of these constants depend upon the initial immersion implicitly (because the \( \alpha_0 (i) \) terms depend explicitly on \( \Sigma_0 \)). Thus \( \Sigma_t \) converges to \( \Sigma_T \) in the \( C^\infty \) topology, implying the uniqueness (up to reparametrisation) of \( \Sigma_T \). Moreover, by Lemma 4.6 our time-dependent metrics at each time \( t \) are uniformly equivalent, and the continuous convergence \( g (t) \to g (T) \) guarantees the smoothness of the immersion \( f (\cdot , T) \). Hence we can define a new immersion \( \hat{f} : \Sigma \times [0, \delta) \to \mathbb{R}^3 \) given by \( \hat{f} (\cdot , t) = f (\cdot , t + T) \). Since \( f (\cdot , T) \) is smooth, then by our results on short-time existence from Chapter 2, we can guarantee that \( \hat{f} \) exists smoothly as a geometric polyharmonic heat flow for some positive maximal time \( \delta \), meaning that (by the construction of \( \hat{f} \)) we can extend the original solution of the geometric polyharmonic heat flow to the interval \( [0, T + \delta) \) (which contains \( [0, T) \) as a proper subset), contradicting the maximality of \( T \). Hence by contradiction we have established the step (4.13). This means we have collectively proven steps (4.11), (4.12) and (4.13), which together gives us the Lifespan Theorem. \( \square \)
Chapter 5

A pointwise estimate for the trace-free curvature, and the Gap Lemma

In this chapter we shall aim to prove that for a given immersion $f : \Sigma^2 \to \mathbb{R}^3$ that is a weakly stationary solution to the geometric polyharmonic heat flow (GPHF) and satisfies some smallness condition regarding the global umbilic energy $\tilde{W}(f)$ (defined later in Definition 5.1) of our immersion, we may conclude that $f$ either maps into an embedded $2-$sphere, or a $2-$plane. We refer to this theorem (Theorem 5.4) as the ‘Gap Lemma’, the reason for which is contained below.

In the compact case, this is relatively straightforward (as we will later see in the proof). In the non-compact case, we proceed by establishing a localised $L^\infty$ bound on the trace-free curvature that depends solely on the localised norms of $\Delta^p H$ and the trace-free curvature in $L^2$, Theorem 5.15. This can be viewed as the higher-order analogue of $\varepsilon-$regularity (compare to, for example, [57, 75, 86, 104]).

Combining this new result with previous results (particularly those contained in Chapters 4, 5 and 6) will almost put us in a position to perform the blowup argument in Chapter 7 that will then yield convergence to spheres for our geometric polyharmonic
heat flow.

Since second order flows such as the mean curvature flow (MCF) are often more straightforward to deal with and admit a maximum principle, the methods that we employ here are usually reserved for higher order flows. Energy-based gap lemmata have been observed in the study of a few fourth-order flows, including by Wheeler with his study of the surface diffusion flow [104] and Kuwert-Schätzle with their study of the Willmore flow under initial energy constraints [57]. The author and his PhD. advisors have previously obtained a similar result for the sixth order ‘geometric triharmonic flow’ in [75]. The results in this section can be thought of as an extension to those findings.

The results within this chapter are time-independent. That is to say, they apply to any immersion satisfying the specified geometric properties (namely $\Delta^p H(f) \equiv 0$ and small total umbilic energy $\tilde{W}(f)$), and hold free-standing of any geometric flow. If desired, one can think of these time-independent results as being a time-slice (that is, a single moment in time) of a one-parameter family of surfaces evolving via the geometric polyharmonic heat flow (GPHF), although this is not strictly necessary. If one is taking this route, it is best to think of the results pertaining to a time-slice at the final time $T$ of the flow. This will make sense in Chapters 7 and 8 when we use the results of this chapter to analyse the long-term behaviour of our flow. We first state the main theorem for this section, before providing some supporting lemmata which we then use to prove the theorem. A lemma of note is our $\varepsilon-$regularity result, Theorem 5.15.

Let us first elaborate on the trace-free curvature $A^o$, which is a $(0, 2)$-tensor field introduced in (1.9). As mentioned earlier, it is written explicitly as $A^o = A - \frac{1}{2} gH$, and is trace-free. Note that (at a point $p \in \Sigma$) $A$ can be diagonalised and we can choose the vector fields $\partial_1 f, \partial_2 f$ to be a orthonormal basis of $T_p \Sigma$. In this frame, $A$, $g$ and $H$
can be written at $p$ as

$$A = \kappa_1 \, dx^1 \otimes dx^1 + \kappa_2 \, dx^2 \otimes dx^2, \quad g = I_2, \quad \text{and} \quad H = \kappa_1 + \kappa_2.$$  

Here $\kappa_1$ and $\kappa_2$ are the \textit{principal curvatures} of $\Sigma$ at $p$, and are also the eigenvalues of the Weingarten map. Additionally, the scalar curvature $K$ can be written at $p$ as

$$K = \kappa_1 \kappa_2.$$  

Note that we have used the same notation as (1.5) to write $A$. Hence $A^o$ can be written as

$$A^o = \frac{1}{2} \begin{pmatrix} \kappa_1 - \kappa_2 & 0 \\ 0 & \kappa_2 - \kappa_1 \end{pmatrix},$$

which implies that at $p$ we have $|A^o|^2 = \frac{1}{2} (\kappa_1 - \kappa_2)^2$ and demonstrates that the quantity $|A^o|^2$ is a measure of local sphericity.

\textbf{Definition 5.1 (Umbilic Energy).} Let $f : \Sigma^2 \to \mathbb{R}^3$ be an immersion. Then we define the \textit{umbilic energy} of $f$ by

$$\widetilde{W}(f) := \int_{\Sigma} |A^o|^2 \, d\mu = \frac{1}{2} \int_{\Sigma} (\kappa_1 - \kappa_2)^2 \, d\mu.$$  

Moreover, if $\Omega \subset \Sigma$ then we define the \textit{local umbilic energy} of $f$ restricted to $\Omega$ by

$$\widetilde{W}_\Omega(f) := \int_{\Omega} |A^o|^2 \, d\mu = \frac{1}{2} \int_{\Omega} (\kappa_1 - \kappa_2)^2 \, d\mu.$$  

\textbf{Remark 5.2.} If $f : \Sigma^2 \to \mathbb{R}^3$ is a closed (compact without boundary) embedding with $\widetilde{W}(f) = 0$ then $\Sigma$ is a round sphere (this follows directly from a theorem of Willmore [109]). If we drop the ‘closed’ requirement then $\Sigma$ may also have planar components.

The umbilic energy $\widetilde{W}(f)$ is closely related to the \textit{Willmore energy} (see, for example, [57, 58]):

$$W(f) = \frac{1}{4} \int_{\Sigma} H^2 \, d\mu.$$
Indeed, for a closed immersion \( f : \Sigma^2 \to \mathbb{R}^3 \), the Gauss-Bonnet theorem allows us to relate the two energies via the following formula:

\[
\tilde{W}(f) = 2W(f) - 2\int_{\Sigma} K \, d\mu = 2W(f) - 4\pi \chi_{\Sigma}.
\]

Here \( \chi_{\Sigma} \) is the Euler characteristic associated to \( \Sigma \). The energy is scale invariant. Theorem 5.16 of Li-Yau [65] guarantees that an immersion \( f : \Sigma^2 \to \mathbb{R}^3 \) is an embedding if

\[
\tilde{W}(f) < 8\pi.
\]

However, for the purposes of this thesis, in the case where \( \Sigma \) is non-compact, the bound \( 8\pi \) is quite loose. For many of the results in Chapters 5, and 6 we will require that the condition

\[
\tilde{W}_{[\gamma>0]}(f) \leq \varepsilon_0 < 8\pi \tag{5.1}
\]

holds on our immersion for some sufficiently small \( \varepsilon_0 > 0 \). Here \( \gamma \) is a cutoff function satisfying the properties described in Section 1.2, and so in particular \( \text{supp}(\gamma) \subseteq \Sigma \).

The local smallness condition (5.1) can essentially be globalised by choosing a nice cutoff function and taking limits, (see, for example, the proof of Theorem 5.4). The \( \varepsilon_0 \) in (5.1) will have to be small enough such that it allows for positive quantities to be absorbed into the left hand side of integral energy calculations. The critical value for \( \varepsilon_0 \) in the non-compact case is not calculated in this thesis, but the author wants to make it clear that it is in fact calculable.

Next we give a concrete example that demonstrates how the total trace-free curvature measures sphericity.

**Example 5.3** (Umbilic energy for a family of ellipsoids). We begin with the standard parameterisation of the ellipsoid \( f : \Sigma^2 \to \mathbb{R}^3 \) given by:

\[
f(u, v) = (R \cos u \cos v, R \cos u \sin v, r \sin u)^T, \quad (u, v) \in (-\pi/2, \pi/2) \times [-\pi, \pi].
\]

Here the superscript ‘\( T \)’ refers to the matrix transpose rather than the tangential com-
ponent. One can easily calculate the induced metric

\[ g = (R^2 + (r^2 - R^2) \cos^2 u) \, du^2 + R^2 \cos^2 u \, dv^2, \]

A lengthy calculation then gives

\[ H = \frac{r \left(2R^2 + (r^2 - R^2) \cos^2 u\right)}{R \left(R^2 + (r^2 - R^2) \cos^2 u\right)^{3/2}}. \]

Therefore by the Gauss-Bonnet theorem,

\[ \tilde{W}(f) = \frac{1}{2} \int_{\Sigma} H^2 \, d\mu - 4\pi = \pi \left(\frac{1}{\hat{r}\sqrt{1-\hat{r}^2}} \ln \left|\frac{2 + 2\sqrt{1-\hat{r}^2} - \hat{r}}{\hat{r}}\right| + \frac{2}{3} \left(1 + \frac{2}{\hat{r}}\right) - 4\right), \]

where \( \hat{r} := (r/R)^2 \).

From figures 5.1 and 5.2, one can clearly see that as \( \hat{r} \nearrow 1 \) (that is, as \( \Sigma \) becomes more spherical), the total trace-free curvature goes to zero monotonically. Because \( \tilde{W}(f) \) is a scale invariant energy (see Claim 4.5), we may assume without loss of generality that \( R = 1 \) (forcing \( 0 < r \leq 1 \)). The above calculation reduces to

\[ \tilde{W}(f)(r) = \pi \left(\frac{1}{r^2\sqrt{1-r^2}} \ln \left|\frac{2 + 2\sqrt{1-r^2} - r^2}{r^2}\right| + \frac{2}{3} \left(1 + \frac{2}{r^2}\right) - 4\right). \]

Figure 5.1: Here we see three examples of ellipsoids as described in Example 5.3. Each is progressively more spherical. From left to right we have \( \hat{r} = 0.3182, 0.5455, 1 \), respectively.

Figure 5.2: This graph displays the total umbilic energy for a one-parameter family of ellipsoids as described in Example 5.3. The energy tends to zero monotonically as the ellipsoids become more spherical (as \( \hat{r} \nearrow 1 \)). The three examples from figure 5.1 are marked.
Now that we have successfully established an absolute lower bound on the lifespan of a geometric polyharmonic heat flow with initially smooth datum, our next major result will be a gap lemma (Theorem 5.4) which is proved at the end of this chapter. This will eventually be combined with the Lifespan Theorem and some additional minor results when we perform a blowup analysis to prove long time existence of the geometric polyharmonic heat flow.

**Theorem 5.4** (Gap Lemma for Geometric Polyharmonic Immersions). Suppose \( f : \Sigma^2 \rightarrow \mathbb{R}^3 \) is a locally \( C^{2(p+1)} \) immersion with \( \Delta^p H = 0 \) (in which case we will define \( f(\Sigma) \) as a \( p \)-polyharmonic immersion) and that \( f \) is proper. Suppose first that \( f(\Sigma) \) is compact. Then if
\[
\tilde{W}(f) = \int_{\Sigma} |A^o|^2 \, d\mu < 8\pi,
\]
we have
\[
f(\Sigma) = \mathbb{S}^2,
\]
where \( \mathbb{S}^2 \) denotes an embedded 2–sphere in \( \mathbb{R}^3 \).

If instead \( f(\Sigma) \) is non-compact and connected, there exists an \( \varepsilon_0 > 0 \) such that if
\[
\tilde{W}(f) = \int_{\Sigma} |A^o|^2 \, d\mu < \varepsilon_0 \quad (5.2)
\]
then
\[
f(\Sigma) = \mathbb{P}^2,
\]
where \( \mathbb{P}^2 \) denotes an embedded 2–plane in \( \mathbb{R}^3 \).

**Remark 5.5.** In the compact case, the value \( \varepsilon_0 = 8\pi \) is a critical value. This is because by Theorem 5.16 and the Gauss-Bonnet Theorem, a surface with \( \|A^o\|_2^2 > 8\pi \) need not be embedded, and there are many examples of closed constant mean curvature surfaces (which in particular satisfy \( \Delta^p H \equiv 0 \)), such as the Wente torus [102] if we allow non-embedded immersions.
Remark 5.6. The critical value of $\varepsilon_0$ does not depend on $p$. In the compact case this is easy to see, since integration by parts gives

$$\int_\Sigma |\Delta^p H|^2 d\mu = \int_\Sigma \Delta^{p-1} H \Delta^{p+1} H d\mu.$$  

Hence any $(p+1)$--polyharmonic immersion is also a $p$--polyharmonic immersion (the other direction is trivially true). Therefore all polyharmonic immersions are equivalent in the sense that

$$\Delta^p H \equiv 0 \iff \Delta^q H \equiv 0 \quad (p, q \in \mathbb{N}),$$

and the result holds for the same critical value of $\varepsilon_0$.

The non-compact case is non-trivial but nevertheless the result is true. Note in particular, from Proposition A.13 we have that for any $p \in \mathbb{N}$

$$\int_\Sigma |\Delta H|^2 \gamma^4 d\mu \leq c \left( \int_{[\gamma>0]} |A^p|^2 d\mu \right)^{\frac{p-1}{p}} \left( \int_\Sigma |\Delta^p H|^2 \gamma^{4p} d\mu \right)^{\frac{1}{p}} + c c_4^4 \int_{[\gamma>0]} |A^p|^2 d\mu. \quad (5.3)$$

By considering a particular cutoff function $\gamma = \tilde{\gamma} \circ f$ like in the proof of Theorem 4.1 and taking $\rho \to \infty$ then shows that the implication

$$\Delta^p H \equiv 0 \implies \Delta H \equiv 0$$

holds if $\varepsilon_0 > 0$ is small enough. However, since the proof of Proposition A.13 relies upon the the estimates contained within this chapter, it will not be enough to simply state (5.3) as a proof of the gap lemma.

Remark 5.7. Often when dealing with gap lemmata, an additional growth hypothesis is assumed at infinity. For example, both the works of Kuwert and Schätzle [57], and Wheeler [104] (both of which involve fourth-order parabolic evolution equations) also require (in addition to assumption (5.2)) that

$$\liminf_{\rho \to \infty} \frac{1}{\rho^4} \int_{f^{-1}(B_\rho(0))} |A|^2 d\mu = 0.$$
That is, in order to prove their Gap Lemma the authors have required that they rule out the possibility of many small loops contracting faster than radius $\rho$ to the fourth power at infinity (remembering that circles of radius $\rho$ have curvature $\rho^{-1}$). The fact that one should require such a ‘fourth power’ requirement is no coincidence: it is a result of the fourth-order nature of the flows being studied. Indeed, when first trying to prove Theorem 5.4, the author was under the impression that we would require the assumption that

$$\liminf_{\rho \to \infty} \frac{1}{\rho^{2(p+1)}} \int_{f^{-1}(B_{\rho}(0))} |A|^2 \, d\mu = 0.$$  

Fortunately, we have managed (by taking advantage of the interplay between some very nice negative terms which crop up when utilising integration by parts and Lemma 5.8, along with the small energy assumption (5.2)) to avoid requiring this extra assumption.

Some other examples of gap lemmata not requiring this extra growth assumption include that of McCoy-Wheeler in their study of Helfrich surfaces [72], the second author’s investigation into the phenomena for a class of fourth-order operators on a surface with boundary [106], and that of Bernard-Riviere in the case of Willmore Spheres [15].

Note that as mentioned earlier in Remark 5.2, the prescribed regularity of $f$ (we are assuming that $f \in C^{2(p+1)}$) helps to rule out the possibility of geometric ‘anomaly’, such as a plane with a spherical cap affixed to it (see figure 5.3). This surface is umbilic almost everywhere in the sense that

$$\int_{\Sigma} (\kappa_1 - \kappa_2)^2 \, d\mu = 0$$

where $\kappa_1, \kappa_2$ are the principal curvatures of $f$. It also satisfies $\Delta^p H = 0$ almost everywhere. It is definitely not $C^{2(p+1)}$, however.

Recall that the umbilic energy given on the left hand side of (5.2) gives a measure of the ‘average’ distance of our immersion from a round sphere in $L^2$. The theorem then says that, given a geometric polyharmonic immersion $f$ ($\Sigma$) with small umbilic energy, $f$ ($\Sigma$) can only exist in one form (either an embedded 2-sphere or plane). The name
‘gap lemma’ is akin to the energy (or band) gap which arises when studying solid state physics (see [54], for example), and says for any given solid, there is an energy range in which no electron states can exist. In our analogy, the integral \( \int_{\Sigma} |A^o|^2 \, d\mu \) takes the place of the ‘energy’ of our immersion, and thus the theorem tells us that there is only one ‘state’ of existence that a geometric polyharmonic immersion with small energy can take.

We proceed by using some interesting geometric inequalities, along with a multiplicative Sobolev-type inequality, to establish a pointwise bound for the trace-free curvature \( A^o \) that only depends on local \( L^2 \) estimates of terms of the form \( \Delta^m H, A^o \), as well as \( ||\nabla \gamma||_{\infty} \). This culminates in Theorem 5.15.

This is not only interesting in its own right, but will be used at the very end of the chapter to prove Theorem 5.4.

**Lemma 5.8.** The following identities hold for any surface immersed in \( \mathbb{R}^3 \):

\[
\Delta \nabla H = \nabla \Delta H + \frac{1}{4} \nabla H \left( H^2 - 2 |A^o|^2 \right) \tag{5.4}
\]

and

\[
\Delta A^o = S^o \left( \nabla_{(2)} H \right) + \frac{1}{2} H^2 A^o - |A^o|^2 A^o. \tag{5.5}
\]
Here $S^\alpha(T)$ represents the symmetric trace-free part of a bilinear form $T$:

$$S^\alpha(T) = T - \frac{1}{2} g \text{trace}_g(T).$$

**Proof.** The first statement is a result of the interchange of covariant derivatives. We first note that by (C.8), a 2-dimensional Riemannian manifold (surface) satisfies the identity

$$R_{ijkl} = K (g_{ik}g_{jl} - g_{il}g_{jk}) = \frac{1}{2} R (g_{ik}g_{jl} - g_{il}g_{jk}),$$

(5.6)

where $R = 2K = \frac{1}{2} H^2 - |A^\alpha|^2$ is the scalar curvature. Tracing this equation over the second and fourth index then gives

$$R_{ik} = \frac{1}{2} R g_{ik},$$

(5.7)

(which is equivalent to saying every surface is an Einstein manifold). We will use these facts below to simplify our working. A quick calculation gives

$$\Delta \nabla_k H = g^{ij} \nabla_{ijk} H$$
$$= g^{ij} \nabla_{skj} H$$
$$= g^{ij} \left( \nabla_{kij} H + g^{\lambda\eta} R_{skj\lambda} \nabla_\eta H \right)$$
$$= \nabla_k \Delta H + R_{k\lambda} \nabla^\lambda H$$
$$= \nabla_k \Delta H + \frac{1}{2} \nabla_k H R$$
$$= \nabla_k \Delta H + \frac{1}{4} \nabla_k H \left( H^2 - 2 |A^\alpha|^2 \right),$$

proving (5.4).

To prove (5.5) we utilise interchange of covariant derivatives and the Codazzi equations, as well as (5.6) and (5.7). A calculation then gives

$$\Delta A_{ij} = g^{kl} \nabla_{kl} A_{ij}$$
$$= g^{kl} \nabla_{ki} A_{lj}$$

\[ g^{kl} \left( \nabla_{ik}A_{lj} + R_{kils}g^{st}A_{lj} + R_{kijs}g^{st}A_{lt} \right) = \nabla_{ij}H + R_{is}A_{sj} + R_{kijs}g^{st}A_{st} \]
\[ = \nabla_{ij}H + \frac{1}{2} R \left( g_{is}A_{sj} + (g_{kj}g_{is} - g_{ks}g_{ij}) A_{ks} \right) \]
\[ = \nabla_{ij}H + \frac{1}{2} R \left( 2A_{ij} - g_{ij}H \right) \]
\[ = \nabla_{ij}H + R A^o_{ij} \]
\[ = \nabla_{ij}H + \frac{1}{2} H^2 A^o_{ij} - |A^o|^2 A^o_{ij}. \]

Subtracting \( \frac{1}{2} g_{ij} \Delta H \) from both sides then gives (5.5).

\[ \square \]

We proceed with a sequence of energy-based integral inequalities, all of which take place under the assumption of small umbilic energy (5.2). Similar inequalities can be found in [72, 73], as well as [75].

**Lemma 5.9.** There exists an \( \varepsilon_0 \) such that the following holds. If \( f : \Sigma^2 \to \mathbb{R}^3 \) is an immersion satisfying

\[ \int_{|\gamma| > 0} |A^o|^2 \, d\mu \leq \varepsilon_0 \]  \hspace{1cm} (5.8)

then there exists a universal constant \( c > 0 \) such that the following inequalities hold:

\[ \int_{\Sigma} |\nabla A^o|^2 \gamma^2 \, d\mu + \int_{\Sigma} |A^o|^2 H^2 \gamma^2 \, d\mu \]
\[ \leq c \left( \int_{|\gamma| > 0} |A^o|^2 \, d\mu \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \right)^{\frac{1}{2}} + c c^2 \int_{|\gamma| > 0} |A^o|^2 \, d\mu \]  \hspace{1cm} (5.9)

and

\[ \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu + \int_{\Sigma} |\nabla H|^2 H^2 \gamma^4 \, d\mu \]
\[ \leq c \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + c c^2 \int_{|\gamma| > 0} |A^o|^2 \, d\mu \int_{\Sigma} |\nabla A^o|^2 \gamma^2 \, d\mu + c \left( \int_{\Sigma} |\nabla A^o|^2 \gamma^2 \, d\mu \right)^2. \]  \hspace{1cm} (5.10)
Proof. To prove (5.9) we use the identity (5.5) as well as integration by parts. This gives

\[
\int_{\Sigma} |\nabla A^0|^2 \gamma^2 \, d\mu = -\int_{\Sigma} \langle A^0, \Delta A^0 \rangle \gamma^2 \, d\mu + \int_{\Sigma} \nabla A^0 \ast A^0 \ast \nabla \gamma \gamma \, d\mu
\]

\[
= -\int_{\Sigma} \langle A^0, \nabla (2) H \rangle \gamma^2 \, d\mu - \frac{1}{2} \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu + \int_{\Sigma} |A^0|^4 \gamma^2 \, d\mu + c \gamma \int_{\Sigma} |\nabla A^0| |A^0| \gamma \, d\mu.
\]

Rearranging and applying Young’s inequality as well as Theorem A.2 with \( u = |A^0|^2 \gamma \) gives

\[
(1 - \eta) \int_{\Sigma} |\nabla A^0|^2 \gamma^2 \, d\mu + \frac{1}{2} \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu \leq \left( \int_{\gamma > 0} |A^0|^2 \, d\mu \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \right)^{1/2} + \int_{\Sigma} |A^0|^4 \gamma^2 \, d\mu + c \eta^{-1} c^2 \gamma \int_{\gamma > 0} |A^0|^2 \, d\mu
\]

\[
\leq \left( \int_{\gamma > 0} |A^0|^2 \, d\mu \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \right)^{1/2} + c \eta^{-1} c^2 \gamma \int_{\gamma > 0} |A^0|^2 \, d\mu
\]

\[
+ c \left( \int_{\Sigma} |\nabla A^0| |A^0| \gamma \, d\mu + c \gamma \int_{\gamma > 0} |A^0|^2 \, d\mu \right) \leq \left( \int_{\gamma > 0} |A^0|^2 \, d\mu \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \right)^{1/2} + \int_{\Sigma} |A^0|^2 \gamma^2 \, d\mu + \int_{\Sigma} |A^0|^2 |H| \gamma \, d\mu \leq \left( \int_{\gamma > 0} |A^0|^2 \, d\mu \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \right)^{1/2} + c \eta^{-1} c^2 \gamma \int_{\gamma > 0} |A^0|^2 \, d\mu
\]

\[
+ c \int_{\gamma > 0} |A^0|^2 \, d\mu \left( \int_{\Sigma} |\nabla A^0|^2 \gamma^2 \, d\mu + \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu + c^2 \gamma \int_{\gamma > 0} |A^0|^2 \, d\mu \right)
\]

for any \( \eta > 0 \). Therefore if \( \varepsilon_0 > 0 \) is sufficiently small, and by choosing \( \eta > 0 \) small enough, we may absorb and multiply out to give

\[
\int_{\Sigma} |\nabla A^0|^2 \gamma^2 \, d\mu + \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu \leq c \left( \int_{\gamma > 0} |A^0|^2 \, d\mu \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \right)^{1/2} + c c^2 \gamma \int_{\gamma > 0} |A^0|^2 \, d\mu,
\]

proving (5.9).

To prove (5.10) we first state an inequality which holds for any \( \varphi \in C^3 \) if \( \varepsilon_0 \) is
sufficiently small:

\[
\int_\Sigma |\nabla (2)\varphi|^2 \gamma^4 d\mu + \int_\Sigma |\nabla \varphi|^2 H^2 \gamma^4 d\mu
\leq c \int_\Sigma |\Delta \varphi|^2 \gamma^4 d\mu + c c_2^4 \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu + c \int_\Sigma |\nabla A^0|^2 \gamma^2 d\mu \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu. \quad (5.11)
\]

The proof of (5.11) is proved at the very beginning of Appendix A. Using (5.11) with \( \varphi = H \) and using the identity (1.18) proves (5.10).

**Corollary 5.10.** There exists an \( \epsilon_0 \) such that the following holds. If \( f : \Sigma^2 \to \mathbb{R}^3 \) is an immersion satisfying (5.8), then there exists a universal constant \( c > 0 \) such that

\[
\int_\Sigma |\nabla A^0|^2 \gamma^2 d\mu + \int_\Sigma |A^0|^2 H^2 \gamma^2 d\mu
\leq c \left( \int_{[\gamma > 0]} |A^0|^2 d\mu \int_\Sigma |\Delta H|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} + c c_2^2 \int_{[\gamma > 0]} |A^0|^2 d\mu. \quad (5.12)
\]

**Proof.** Combining (5.9) and (5.10) and using Young’s inequality gives

\[
\int_\Sigma |\nabla A^0|^2 \gamma^2 d\mu + \int_\Sigma |A^0|^2 H^2 \gamma^2 d\mu
\leq c \left( \int_{[\gamma > 0]} |A^0|^2 d\mu \int_\Sigma |\Delta H|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} + c c_2^2 \int_{[\gamma > 0]} |A^0|^2 d\mu
+ \left( \int_\Sigma |\nabla A^0|^2 \gamma^2 d\mu \right)^{\frac{1}{2}} + c c_2^2 \int_{[\gamma > 0]} |A^0|^2 d\mu
\leq c \left( \int_{[\gamma > 0]} |A^0|^2 d\mu \int_\Sigma |\Delta H|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} + c c_2 \int_{[\gamma > 0]} |A^0|^2 d\mu
+ c \left( \int_{[\gamma > 0]} |A^0|^2 d\mu \right)^{\frac{1}{2}} \int_\Sigma |\nabla A^0|^2 \gamma^2 d\mu + c c_2^2 \int_{[\gamma > 0]} |A^0|^2 d\mu
\leq c \left( \int_{[\gamma > 0]} |A^0|^2 d\mu \int_\Sigma |\Delta H|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} + \left( \eta + c \left( \int_{[\gamma > 0]} |A^0|^2 d\mu \right)^{\frac{1}{2}} \right) \int_\Sigma |\nabla A^0|^2 \gamma^2 d\mu + c c_2 \int_{[\gamma > 0]} |A^0|^2 d\mu
\]

for any \( \eta > 0 \). Therefore if \( \epsilon_0 \) and \( \eta > 0 \) are small enough, absorbing and multiplying through gives the result. \( \square \)
Lemma 5.11. There exists an \( \varepsilon_0 \) such that the following holds. If \( f : \Sigma^2 \rightarrow \mathbb{R}^3 \) is an immersion satisfying (5.8), then there exists a universal constant \( c > 0 \) such that

\[
\int_{\Sigma} |\Delta A^0| \, \gamma^4 \, d\mu + \int_{\Sigma} |\nabla A^0| \, H^2 \gamma^4 \, d\mu \\
\leq c \int_{\Sigma} |\Delta H|^2 \, \gamma^4 \, d\mu + c \int_{\gamma > 0} |A^0|^2 \, d\mu \int_{\Sigma} |\nabla (2) A^0|^2 \, \gamma^4 \, d\mu + c c_4 \int_{\gamma > 0} |A^0|^2 \, d\mu.
\]

Proof. Using identity (5.5) and integration by parts gives

\[
\int_{\Sigma} |\Delta A^0| \, \gamma^4 \, d\mu \\
= \int_{\Sigma} \left\langle \Delta A^0, \nabla (2) H + \frac{1}{2} H^2 A^0 - |A^0|^2 A^0 \right\rangle \gamma^4 \, d\mu \\
= \int_{\Sigma} \left\langle \Delta A^0, \nabla (2) H \right\rangle \gamma^4 \, d\mu - \frac{1}{2} \int_{\Sigma} |\nabla A^0|^2 \, H^2 \gamma^4 \, d\mu + \int_{\Sigma} H \nabla A^0 \ast \nabla H \ast A^0 \, \gamma^4 \, d\mu \\
+ \int_{\Sigma} \nabla A^0 \ast \nabla A^0 \ast A^0 \ast A^0 \, \gamma^4 \, d\mu + \int_{\Sigma} H^2 \nabla A^0 \ast A^0 \ast A^0 \ast A^0 \ast \nabla \gamma \, \gamma^3 \, d\mu \\
\leq \int_{\Sigma} \left\langle \Delta A^0, \nabla (2) H \right\rangle \gamma^4 \, d\mu - \frac{1}{2} \int_{\Sigma} |\nabla A^0|^2 \, H^2 \gamma^4 \, d\mu + c \int_{\Sigma} |\nabla A^0| \, |\nabla H| \, |A^0| \, |H| \, \gamma^4 \, d\mu \\
+ c \int_{\Sigma} |\nabla A^0|^2 \, |A^0|^2 \, \gamma^4 \, d\mu + c c_\gamma \int_{\Sigma} |\nabla A^0| \, |A^0| \, H^2 \, \gamma^3 \, d\mu \\
+ c c_\gamma \int_{\Sigma} |\nabla A^0| \, |A^0|^3 \, \gamma^3 \, d\mu.
\]

Next, using Young’s inequality repeatedly, we can establish the following inequalities:

\[
\int_{\Sigma} \left\langle \Delta A^0, \nabla (2) H \right\rangle \gamma^4 \, d\mu \leq \eta \int_{\Sigma} |\Delta A^0|^2 \, \gamma^4 \, d\mu + c \eta^{-1} \int_{\Sigma} |\nabla (2) H|^2 \, \gamma^4 \, d\mu,
\]

\[
c \int_{\Sigma} |\nabla A^0| \, |\nabla H| \, |A^0| \, |H| \, \gamma^4 \, d\mu \leq c \left( \int_{\Sigma} |\nabla A^0|^2 \, |A^0|^2 \, \gamma^4 \, d\mu + \int_{\Sigma} |\nabla H|^2 \, H^2 \gamma^4 \, d\mu \right),
\]

\[
c c_\gamma \int_{\Sigma} |\nabla A^0| \, |A^0| \, H^2 \gamma^3 \, d\mu \leq \eta \int_{\Sigma} |\nabla A^0|^2 \, H^2 \gamma^4 \, d\mu + c \eta^{-1} c_\gamma^2 \int_{\Sigma} |A^0|^2 \, H^2 \gamma^2 \, d\mu,
\]

and

\[
c c_\gamma \int_{\Sigma} |\nabla A^0| \, |A^0|^2 \, \gamma^3 \, d\mu \leq c \left( \int_{\Sigma} |\nabla A^0|^2 |A^0|^2 \, \gamma^4 \, d\mu + c c_\gamma^2 \int_{\Sigma} |A^0|^4 \, \gamma^2 \, d\mu \right).
\]
The first and third inequalities hold for any \( \eta > 0 \). Therefore substituting back into (5.13) yields

\[
(1 - \eta) \int_{\Sigma} |\Delta A^0|^2 \gamma^4 \, d\mu + \left( \frac{1}{2} - \eta \right) \int_{\Sigma} |\nabla A^0|^2 H^2 \gamma^4 \, d\mu \\
\leq c \eta^{-1} \left( \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu + \int_{\Sigma} |\nabla H|^2 \gamma^4 \, d\mu \right) + c \eta^{-1} \int_{\Sigma} |\nabla A^0|^2 |A^0|^2 \gamma^4 \, d\mu \\
+ c \eta^{-1} c^2_\gamma \left( \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu + \int_{\Sigma} |A^0|^4 \gamma^2 \, d\mu \right). \tag{5.14}
\]

Next, using Theorem A.2 twice: once with \( u_1 = |\nabla A^0| |A^0| \gamma^2 \) and once more with \( u_2 = |A^0|^2 \gamma \), gives

\[
\int_{\Sigma} |\nabla A^0|^2 |A^0|^2 \gamma^2 \, d\mu \\
\leq c \left( \int_{\Sigma} |\nabla (2) A^0| |A^0| \gamma^2 \, d\mu + \int_{\Sigma} |\nabla A^0|^2 \gamma^2 \, d\mu + c_\gamma \int_{\Sigma} |\nabla A^0| |A^0| \gamma \, d\mu \right. \\
+ \left. \int_{\Sigma} |\nabla A^0| |A^0| |H| \gamma^2 \, d\mu \right)^2 \]

\[
\leq c \int_{\gamma > 0} |A^0|^2 \, d\mu \left( \int_{\Sigma} |\nabla (2) A^0|^2 \gamma^4 \, d\mu + \int_{\Sigma} |\nabla A^0|^2 H^2 \gamma^4 \, d\mu + c^2_\gamma \int_{\Sigma} |\nabla A^0|^2 \gamma^2 \, d\mu \right) \]

\[
+ c \left( \int_{\Sigma} |\nabla A^0|^2 \gamma^2 \, d\mu \right)^2,
\]

and

\[
c^2_\gamma \int_{\Sigma} |A^0|^4 \gamma^2 \, d\mu \\
\leq c c^2_\gamma \left( \int_{\Sigma} |\nabla A^0| |A^0| \gamma \, d\mu + c_\gamma \int_{\gamma > 0} |A^0|^2 \, d\mu + \int_{\Sigma} |A^0|^2 |H| \gamma \, d\mu \right)^2 \]

\[
\leq c c^2_\gamma \int_{\gamma > 0} |A^0|^2 \, d\mu \left( \int_{\Sigma} |\nabla A^0|^2 \gamma^2 \, d\mu + \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu + c^2_\gamma \int_{\gamma > 0} |A^0|^2 \, d\mu \right).
\]

Substituting the previous two inequalities into (5.14) and using the inequalities from Lemma 5.9 and Corollary 5.10 then gives

\[
\left( \frac{1}{2} - \eta - c c^{-1} \int_{\gamma > 0} |A^0|^2 \, d\mu \right) \left( \int_{\Sigma} |\Delta A^0|^2 \gamma^4 \, d\mu + \int_{\Sigma} |\nabla A^0|^2 H^2 \gamma^4 \, d\mu \right)
\]
\[ \leq c \eta^{-1} \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + c \eta^{-1} \int_{[\gamma > 0]} |A^\alpha|^2 \, d\mu \int_{\Sigma} |\nabla_{(2)} A^\alpha|^2 \gamma^4 \, d\mu \]
\[ + c \eta^{-1} c \gamma^4 \int_{[\gamma > 0]} |A^\alpha|^2 \, d\mu. \]  
(5.15)

Note that since all constants \( c \) are universal, for any \( \eta < \frac{1}{2} \) we may pick an \( \varepsilon_0 > 0 \) such that
\[ \eta + c \eta^{-1} \varepsilon_0 < \frac{1}{2}, \]
making the coefficient of the integral on the left hand side of (5.15) positive. The result therefore follows.

**Lemma 5.12.** Let \( f : \Sigma \to \mathbb{R}^3 \) be an immersion. Then the following identity holds:
\[ \Delta \nabla_i A^o_{jk} = \nabla_i \Delta A^o_{jk} + \frac{5}{2} R \nabla_i A^o_{jk} + R (g_{jk} \nabla_i H - g_{ik} \nabla_j H - g_{ij} \nabla_k H) + \nabla R * A^o, \]  
(5.16)
where \( R = 2K \) is the scalar curvature. Furthermore, there exists an \( \varepsilon_0 \) such that the following holds. If \( f \) satisfies (5.8), then there exists a universal constant \( c > 0 \) such that
\[ \int_{\Sigma} |\nabla_{(2)} A^\alpha|^2 \gamma^4 \, d\mu + \int_{\Sigma} |\nabla A^\alpha|^2 H^2 \gamma^4 \, d\mu + \int_{\Sigma} |\nabla H|^2 |A^\alpha|^2 \gamma^4 \, d\mu \]
\[ \leq c \int_{\Sigma} |\Delta A^\alpha|^2 \gamma^4 \, d\mu + c \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + c \gamma^4 \int_{[\gamma > 0]} |A^\alpha|^2 \, d\mu. \]  
(5.17)

**Proof.** For the first identity, we use the formula for the interchange of covariant derivative twice:
\[ \Delta \nabla_i A^o_{jk} = \nabla_i \Delta A^o_{jk} + R^{\alpha}_{\lambda} \nabla_{i} A^\alpha_{jk} + R^{\lambda}_{ijkl} \nabla^l A^\lambda_{jk} + \nabla Rm * A^o \]
\[ = g^{st} (\nabla_{ist} A^o_{jk} + R^{\alpha}_{\lambda st} \nabla_{l} A^\alpha_{jk} + R^{\lambda}_{ijkl} \nabla^l A^\lambda_{jk}) \]
\[ + R^{\lambda}_{ijkl} \nabla^l A^\lambda_{jk} + R^{\alpha}_{\lambda st} \nabla^l A^\alpha_{jk} + \nabla R * A^o \]
\[ = \nabla_i \Delta A^o_{jk} + g^{st} R^{\alpha}_{\lambda st} \nabla_{l} A^\alpha_{jk} + 2 \left( R^{\lambda}_{ijkl} \nabla^l A^\lambda_{jk} + R^{\alpha}_{\lambda st} \nabla^l A^\alpha_{jk} \right) + \nabla R * A^o. \]  
(5.18)
Next, using the identity

\[ R^i_{ijk} = \frac{1}{2} R (g_{ik} \delta^j_j - g_{jk} \delta^i_i), \]

we have

\[ g^{st} R^\lambda_{sit} \nabla_\lambda A^o_{jk} = \frac{1}{2} R g^{st} (g_{et} \delta^\lambda_t - g_{et} \delta^t_t) \nabla_\lambda A^o_{jk} = \frac{1}{2} R \nabla_i A^o_{jk}. \quad (5.19) \]

Note that the penultimate term in (5.18) can be written as

\[
2 \left( R^\lambda_{ij} \nabla^t A^o_{\lambda k} + R^\lambda_{ik} \nabla^t A^o_{j \lambda} \right) \\
= R \left( (g_{ij} \delta^\lambda_i - g_{ij} \delta^\lambda_j) \nabla^t A^o_{\lambda k} + (g_{ik} \delta^\lambda_i - g_{ik} \delta^\lambda_j) \nabla^t A^o_{j \lambda} \right) \\
= R \left( \nabla_j A^o_{ik} - \frac{1}{2} g_{ik} \nabla_k H + \nabla_k A^o_{ij} - \frac{1}{2} g_{ik} \nabla_j H \right). \quad (5.20)
\]

Here we have used the identity \( \nabla^i A^o_{ij} = \frac{1}{2} \nabla_j H \) in the last step. Next, using the fact that the tensor \( \nabla A \) is completely symmetric, we have

\[ \nabla_j A^o_{ik} = \nabla_i A^o_{jk} - \frac{1}{2} g_{ik} \nabla_j H = \nabla_i A^o_{jk} + \frac{1}{2} (g_{jk} \nabla_i H - g_{ik} \nabla_j H). \]

Similarly,

\[ \nabla_k A^o_{ij} = \nabla_i A^o_{jk} + \frac{1}{2} (g_{jk} \nabla_i H - g_{ij} \nabla_k H). \]

Substituting the previous two identities into (5.20), combining with (5.19) and then substituting back into (5.18) then proves (5.16).

To prove (5.17), we first combine the identity (5.16) and integration by parts:

\[
\int_{\Sigma} |\nabla (A^o)^2| \gamma^4 d\mu \\
= - \int_{\Sigma} \langle \nabla A^o, \Delta \nabla A^o \rangle \gamma^4 d\mu + \int_{\Sigma} \nabla (A^o)^2 \ast \nabla A^o \ast \nabla \gamma^3 d\mu \\
= - \int_{\Sigma} \langle \nabla A^o, \nabla \Delta A^o \rangle \gamma^4 d\mu - \frac{5}{2} \int_{\Sigma} |\nabla A^o|^2 R \gamma^4 d\mu \\
- \int_{\Sigma} R (g_{jk} \nabla_i H - g_{ik} \nabla_j H - g_{ij} \nabla_k H) \nabla^t (A^o)^t_{jk} \gamma^4 d\mu + \int_{\Sigma} \nabla A^o \ast \nabla R \ast A^o \gamma^4 d\mu \\
+ \int_{\Sigma} \nabla (A^o)^2 \ast \nabla A^o \ast \nabla \gamma^3 d\mu.
\]
\[
\begin{align*}
&= \int_{\Sigma} |\Delta A^o|^2 \gamma^4 \, d\mu + \frac{5}{2} \int_{\Sigma} |\nabla A^o|^2 R \gamma^4 \, d\mu + \int_{\Sigma} |\nabla H|^2 R \gamma^4 \, d\mu \\
&\quad + \int_{\Sigma} \nabla (2)A \ast \nabla A^o \ast \nabla \gamma \gamma^3 \, d\mu + \int_{\Sigma} \nabla A^o \ast \nabla R \ast A^o \gamma^4 \, d\mu,
\end{align*}
\]
(5.21)

where we have again used the identity \( \nabla^i A^o_{ij} = \frac{1}{2} \nabla_j H \) in the last step, and also the fact that \( A^o \) is traceless implies

\[
g_{jk} \nabla^i (A^o)^{jk} = \nabla^i \left( g_{jk} (A^o)^{jk} \right) = 0.
\]

Next, note that the identity

\[
R = H^2 - |A|^2 = \frac{1}{2} H^2 - |A|^2 = 2K
\]

implies

\[
\nabla R = H \nabla H + A^o \ast \nabla A^o,
\]

The identity (5.21) can be estimated in the following way. First note that

\[
\begin{align*}
&= \int_{\Sigma} |\nabla_{(2)} A^o|^2 \gamma^4 \, d\mu + \frac{5}{4} \int_{\Sigma} |\nabla A^o|^2 H^2 \gamma^4 \, d\mu + \int_{\Sigma} |\nabla H|^2 |A^o|^2 \gamma^4 \, d\mu \\
&\quad + \int_{\Sigma} \nabla A^o \ast \nabla R \ast A^o \gamma^4 \, d\mu + \int_{\Sigma} \nabla_{(2)} A^o \ast \nabla A^o \ast \nabla \gamma \gamma^3 \, d\mu \\
&\leq \int_{\Sigma} |\Delta A^o|^2 \gamma^4 \, d\mu + \frac{5}{2} \int_{\Sigma} |\nabla A^o|^2 |A^o|^2 \gamma^4 \, d\mu + \frac{1}{2} \int_{\Sigma} |\nabla H|^2 H^2 \gamma^4 \, d\mu \\
&\quad + c \int_{\Sigma} |\nabla A^o| |A^o| (|\nabla H| |H| + |\nabla A^o| |A^o|) \gamma^4 \, d\mu + c \gamma \int_{\Sigma} |\nabla_{(2)} A^o| |A^o| \gamma^3 \, d\mu \\
&\leq \int_{\Sigma} |\Delta A^o|^2 \gamma^4 \, d\mu + \left( \frac{5}{2} + c \right) \int_{\Sigma} |\nabla A^o|^2 |A^o|^2 \gamma^4 \, d\mu + \left( \frac{1}{2} + c \right) \int_{\Sigma} |\nabla H|^2 H^2 \gamma^4 \, d\mu \\
&\quad + \eta \int_{\Sigma} |\nabla_{(2)} A^o|^2 \gamma^4 \, d\mu + c \eta^{-1} \int_{\Sigma} |\nabla A^o|^2 \gamma^2 \, d\mu,
\end{align*}
\]
(5.22)

for any \( \eta > 0 \). Here we have used Young’s inequality twice to get to the last line: once to obtain
\[ \int_\Sigma |\nabla A^0| |A^0| (|\nabla H| H + |\nabla A^0| |A^0|) \gamma^4 \, d\mu \leq c \left( \int_\Sigma |\nabla A^0|^2 |A^0|^2 \gamma^4 \, d\mu + \int_\Sigma |\nabla H|^2 H^2 \gamma^4 \, d\mu \right), \]

and once more to obtain
\[
c_\gamma \int_\Sigma |\nabla (2) A^0| |\nabla A^0|^\gamma^3 \, d\mu \leq \eta \int_\Sigma |\nabla (2) A^0|^2 \gamma^4 \, d\mu + c \eta^{-1} c_\gamma^2 \int_\Sigma |\nabla A^0|^2 \gamma^2 \, d\mu.
\]

Therefore (5.22) becomes
\[
(1 - \eta) \int_\Sigma |\nabla (2) A^0|^2 \gamma^4 \, d\mu + \frac{5}{4} \int_\Sigma |\nabla A^0|^2 H^2 \gamma^4 \, d\mu + \int_\Sigma |\nabla H|^2 |A^0|^2 \gamma^4 \, d\mu \leq c \int_\Sigma |\nabla (2) A^0|^2 \gamma^4 \, d\mu + \left( \frac{1}{2} + c \right) \int_\Sigma |\nabla A^0|^2 |A^0|^2 \gamma^4 \, d\mu + c \eta^{-1} c_\gamma^2 \int_\Sigma |\nabla A^0|^2 \gamma^2 \, d\mu.
\]

Using Lemma 5.9 and Corollary 5.10 to estimate the three extraneous terms on the right hand side completes the proof. \(\square\)

**Corollary 5.13.** There exists an \(\varepsilon_0\) such that the following holds. If \(f : \Sigma \to \mathbb{R}^3\) satisfies (5.8) then there exists a universal constant \(c > 0\) such that
\[
\int_\Sigma |\nabla (2) A^0|^2 \gamma^4 \, d\mu + \int_\Sigma |\nabla A^0|^2 H^2 \gamma^4 \, d\mu + \int_\Sigma |\nabla H|^2 |A^0|^2 \gamma^4 \, d\mu \leq c \int_\Sigma |\nabla (2) A^0|^2 \gamma^4 \, d\mu + c c_\gamma^4 \int_{[\gamma > 0]} |A^0|^2 \, d\mu.
\]

**Proof.** Combining the results of Lemma 5.12 and Lemma 5.11 gives
\[
\int_\Sigma |\nabla (2) A^0|^2 \gamma^4 \, d\mu + \int_\Sigma |\nabla A^0|^2 H^2 \gamma^4 \, d\mu + \int_\Sigma |\nabla H|^2 |A^0|^2 \gamma^4 \, d\mu \leq c \int_\Sigma |\nabla (2) A^0|^2 \gamma^4 \, d\mu + c \int_{[\gamma > 0]} |A^0|^2 \, d\mu,
\]

Therefore if \(\varepsilon_0 > 0\) is small enough then we can absorb the first term on the right into
the left hand side of the inequality, which proves the result.

\[ \square \]

**Lemma 5.14.** There exists an \( \varepsilon_0 \) such that the following holds. If \( f : \Sigma^2 \to \mathbb{R}^3 \) satisfies (5.8) then there exists a universal constant \( c > 0 \) such that

\[
\int_{\Sigma} |\nabla A^o|^2 H^2 \gamma^4 \, d\mu + \int_{\Sigma} H^4 |A^o|^2 \gamma^4 \, d\mu \leq c \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + c \varepsilon_0^4 \int_{[\gamma > 0]} |A^o|^2 \, d\mu.
\]

**Proof.** Using an application of integration by parts, along with identity (5.5), gives

\[
\int_{\Sigma} |\nabla A^o|^2 H^2 \gamma^4 \, d\mu = - \int_{\Sigma} \left\langle A^o, \nabla (2) H + \frac{1}{2} H^2 A^o - |A^o|^2 A^o \right\rangle H^2 \gamma^4 \, d\mu + \int_{\Sigma} H \nabla A^o * A^o * \nabla H \gamma^4 \, d\mu + \int_{\Sigma} H^2 \nabla A^o * A^o * \nabla \gamma \gamma^3 \, d\mu,
\]

which implies the following inequality for any \( \eta > 0 \):

\[
\int_{\Sigma} |\nabla A^o|^2 H^2 \gamma^4 \, d\mu + \frac{1}{2} \int_{\Sigma} H^4 |A^o|^2 \gamma^4 \, d\mu \\
\leq \left( \int_{\Sigma} H^4 |A^o|^2 \gamma^4 \, d\mu \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \right)^{1/2} + \int_{\Sigma} |A^o|^4 H^2 \gamma^4 \, d\mu \\
+ c \int_{\Sigma} |\nabla A^o| |\nabla H| |A^o| |H| \gamma^4 \, d\mu + c \eta^{-1} \int_{\Sigma} |\nabla A^o| |A^o| H^2 \gamma^3 \, d\mu \\
\leq \left( \eta \int_{\Sigma} H^4 |A^o|^2 \gamma^4 \, d\mu + c \eta^{-1} \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \right) + \int_{\Sigma} |A^o|^4 H^2 \gamma^4 \, d\mu \\
+ c \left( \int_{\Sigma} |\nabla A^o|^2 H^2 \gamma^4 \, d\mu \int_{\Sigma} |\nabla H|^2 |A^o|^2 \gamma^4 \, d\mu \right)^{1/2} \\
+ \left( \eta \int_{\Sigma} |\nabla A^o|^2 H^2 \gamma^4 \, d\mu + c \eta^{-1} \varepsilon_0^2 \int_{\Sigma} |A^o|^2 H^2 \gamma^2 \, d\mu \right). \tag{5.23}
\]

Here we have used Young’s inequality inequality twice. Next, Corollary 5.13 gives

\[
\left( \int_{\Sigma} |\nabla A^o|^2 H^2 \gamma^4 \, d\mu \int_{\Sigma} |\nabla H|^2 |A^o|^2 \gamma^4 \, d\mu \right)^{1/2} \\
\leq c \left( \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + \varepsilon_0^4 \int_{[\gamma > 0]} |A^o|^2 \, d\mu \right).
\]
Similarly, by Corollary 5.9 we have

\[ \int_{\Sigma} |\nabla (2) H|^2 \gamma^4 \, d\mu \leq c \left( \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + c_4 \int_{\gamma > 0} |A^0|^2 \, d\mu \right), \]

and by Lemma 5.10 and the Cauchy-Schwarz inequality we have

\[ c_2^2 \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu \]

\[ \leq c c_2^2 \left( \int_{\gamma > 0} |A^0|^2 \, d\mu \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu \right)^{\frac{1}{2}} + c c_4 \int_{\gamma > 0} |A^0|^2 \, d\mu \]

\[ \leq c \left( \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + c_4 \int_{\gamma > 0} |A^0|^2 \, d\mu \right). \]

Therefore absorbing the small \( \eta \) quantities into the left hand side of inequality (5.23) gives

\[ (1 - \eta) \int_{\Sigma} |\nabla A^0|^2 H^2 \gamma^4 \, d\mu + \left( \frac{1}{2} - \eta \right) \int_{\Sigma} H^4 |A^0|^2 \gamma^4 \, d\mu \]

\[ \leq c \eta^{-1} \left( \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + c_4 \int_{\gamma > 0} |A^0|^2 \, d\mu \right) + \int_{\Sigma} |A^0|^4 H^2 \gamma^4 \, d\mu. \] (5.24)

To deal with the last term on the right hand side of (5.24) we use Theorem A.2 with \( u = |A^0|^2 |H| \gamma^2 \), as well as the inequalities from Corollaries 5.10 and 5.13:

\[ \int_{\Sigma} |A^0|^4 H^2 \gamma^4 \, d\mu \]

\[ \leq c \left( \int_{\Sigma} |\nabla A^0| |A^0| |H| \gamma^2 \, d\mu + \int_{\Sigma} |A^0|^2 |\nabla H| \gamma^2 \, d\mu \right) \]

\[ + c_4 \int_{\Sigma} |A^0|^2 |H| \gamma \, d\mu + \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu \]

\[ \leq c \int_{\gamma > 0} |A^0|^2 \, d\mu \left( \int_{\Sigma} |\nabla A^0|^2 H^2 \gamma^4 \, d\mu + \int_{\Sigma} H^4 |A^0|^2 \gamma^4 \, d\mu \right) \]

\[ + \int_{\Sigma} |\nabla H|^2 |A^0|^2 \gamma^4 \, d\mu + c_4^2 \int_{\Sigma} |A^0|^2 H^2 \gamma^2 \, d\mu \]

\[ \leq \int_{\gamma > 0} |A^0|^2 \, d\mu \left( \int_{\Sigma} |\nabla A^0|^2 H^2 \gamma^4 \, d\mu + \int_{\Sigma} H^4 |A^0|^2 \gamma^4 \, d\mu \right) \]
Here we have used Corollaries 5.10 and 5.13 in the penultimate step, and the Cauchy-Schwarz inequality in the last step. Substituting (5.25) into (5.24) then gives

\[
\left(1 - \eta - c \| A^0 \|^2_{L^2([\gamma > 0])}\right) \int_{\Sigma} |\nabla A^0|^2 H^2 \gamma^4 \, d\mu \\
+ \left(\frac{1}{2} - \eta - c \| A^0 \|^2_{L^2([\gamma > 0])}\right) \int_{\Sigma} H^4 |A^0|^2 \gamma^4 \, d\mu \\
\leq c \eta^{-1} \left(1 + \| A^0 \|^2_{L^2([\gamma > 0])}\right) \left( \int_{\Sigma} |\Delta H|^2 \gamma^4 \, d\mu + c_\gamma \| A^0 \|^2_{L^2([\gamma > 0])}\right).
\] (5.25)

Therefore since all constants $c$ are universal, for any $\eta > 0$ we may pick an $\varepsilon_0 > 0$ small enough such that both coefficients on the the left hand side of (5.25) are positive. The results then follows. 

The second result in the following theorem can be viewed as a higher-order analogue of $\varepsilon-$regularity, and should be compared to the work of Wheeler [104] and Kuwert-Schätzle [57] in their analysis of the Surface diffusion flow and Willmore flow respectively.

**Theorem 5.15** (Local $\varepsilon-$regularity). There exists an $\varepsilon_0 = \varepsilon_0$ such that the following holds. If $f : \Sigma^2 \to \mathbb{R}^3$ is an immersion satisfying

\[
\int_{f^{-1}(B_{2\rho}(x))} |A^0|^2 \, d\mu \leq \varepsilon_0, 
\] (5.26)
then for any $l \in \mathbb{N}$, $l \geq 2$, there exists a universal constant $c > 0$ such that

$$\|A^o\|_{\infty, f^{-1}(B_\rho(x))}^{2l} \leq c \|A^o\|_{2, f^{-1}(B_{2\rho}(x))}^{2l-8} \left( \|\Delta^{l/2} H\|_{2, f^{-1}(B_{2\rho}(x))}^2 + \rho^{-2l} \|A^o\|_{2, f^{-1}(B_{2\rho}(x))}^2 \right).$$

(5.27)

In particular, if the immersion satisfies $\Delta^p H(f) \equiv 0$ then there is a universal constant $c > 0$ such that

$$\|A^o\|_{\infty, f^{-1}(B_\rho(x))} \leq c \rho^{-\frac{2}{3}}.$$

In the statement of Theorem 5.15 we have used the notation of Proposition A.13 in which (with abuse of notation) we have $\Delta^\frac{1}{2} = \nabla$ and $\Delta$ acts from right to left so that $\Delta^{\frac{m+1}{2}} = \nabla \Delta^m$ for $m \in \mathbb{N}_0$.

**Proof.** Firstly using the multiplicative Sobolev inequality of Theorem A.6 with $\varphi = |A^o| \gamma^\frac{3}{2}$ gives

$$\|A^o\|_{\infty, |\gamma|=1}^{6} \leq c \|A^o\|_{2, |\gamma|>0}^2 \left( \int_\Sigma |\nabla A^o|^4 \gamma^6 \, d\mu + c_\gamma^4 \int_\Sigma |A^o|^4 \gamma^2 \, d\mu + \int_\Sigma |A^o|^4 H^4 \gamma^6 \, d\mu \right).$$

(5.28)

Next we use Theorem A.2 with $u = |\nabla A^o|^2 \gamma^3$ to estimate the first term on the right hand side:

$$\int_\Sigma |\nabla A^o|^4 \gamma^6 \, d\mu$$

$$\leq c \left( \int_\Sigma |\nabla (A^o)| \gamma^3 \, d\mu + c_\gamma \int_\Sigma |\nabla A^o|^2 \gamma^2 \, d\mu + \int_\Sigma |\nabla A^o|^2 |H| \gamma^3 \, d\mu \right)^2$$

$$\leq c \int_\Sigma |\nabla A^o|^2 \gamma^2 \, d\mu \left( \int_\Sigma |\nabla (A^o)|^2 \gamma^4 \, d\mu + \int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 \, d\mu \right)$$

$$+ c_\gamma^2 \left( \int_\Sigma |\nabla A^o|^2 \gamma^2 \, d\mu \right)^2$$

$$\leq c \left( \int_{|\gamma|>0} |A^o|^2 \, d\mu \right)^\frac{1}{2} \left( \int_{|\gamma|>0} |\Delta H|^2 \gamma^4 \, d\mu \right)^\frac{3}{2} + c_\gamma^6 \left( \int_{|\gamma|>0} |A^o|^2 \, d\mu \right)^\frac{3}{2}.$$  

(5.29)

Here we have used Corollary 5.13 and Corollary 5.10 in the last step.
We estimate the second term in (5.28) in a similar way. Using Theorem A.2 with $u = |A^o|^2 \gamma$ gives

$$
c_4 \int_\Sigma |A^o|^4 \gamma^2 \, d\mu \\
\leq c_\gamma c_4 \left( \int_\Sigma |\nabla A^o| |A^o| \gamma \, d\mu + c_\gamma \int_{\{\gamma > 0\}} |A^o|^2 \, d\mu + \int_\Sigma |A^o|^2 |H| \gamma^2 \, d\mu \right)^2 \\
\leq c_\gamma c_4 \int_{\{\gamma > 0\}} |A^o|^2 \, d\mu \left( \int_\Sigma |\nabla A^o|^2 \gamma^2 \, d\mu + \int_\Sigma |A^o|^2 H^2 \gamma^2 \, d\mu \right) \\
+ c_\gamma c_6 \left( \int_{\{\gamma > 0\}} |A^o|^2 \, d\mu \right)^2 \\
\leq c \left( \int_{\{\gamma > 0\}} |A^o|^2 \, d\mu \right)^{\frac{1}{2}} \left( c_\gamma c_4 \int_{\{\gamma > 0\}} |A^o|^2 \, d\mu \left( \int_\Sigma |\Delta H|^2 \gamma^4 \, d\mu \right)^{\frac{1}{2}} \\
+ c_\gamma c_6 \left( \int_{\{\gamma > 0\}} |A^o|^2 \, d\mu \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.
$$

Here we have used Corollary 5.10 in the penultimate step, and Young’s inequality with conjugate exponents $p = 3, p^* = 3/2$ in the final step.

To estimate the last term on the right hand side of (5.28) we use Theorem A.2 with $u = |A^o|^2 H^2 \gamma^3$, followed by the results of Lemma 5.14 and Corollaries 5.10 and 5.13:

$$
\int_\Sigma |A^o|^4 H^4 \gamma^2 \, d\mu \\
\leq c \left( \int_\Sigma |\nabla A^o| |A^o| H^2 \gamma^3 \, d\mu + \int_\Sigma |A^o|^2 |\nabla H| |H| \, d\mu + c_\gamma \int_\Sigma |A^o|^2 H^2 \gamma^2 \, d\mu \right. \\
+ \int_\Sigma |A^o|^2 |H|^3 \gamma^3 \, d\mu \right)^2 \\
\leq c \int_\Sigma |A^o|^2 H^2 \gamma^2 \, d\mu \left( \int_\Sigma |\nabla A^o|^2 H^2 \gamma^4 \, d\mu + \int_\Sigma |A^o|^2 H^4 \gamma^4 \, d\mu \right) \\
+ \int_\Sigma |\nabla H|^2 |A^o|^2 \gamma^4 \, d\mu + c_\gamma^2 \int_\Sigma |A^o|^2 H^2 \gamma^2 \, d\mu \right) \\
\leq c \left( \left( \int_{\{\gamma > 0\}} |A^o|^2 \, d\mu \int_\Sigma |\Delta H|^2 \gamma^4 \, d\mu \right)^{\frac{1}{2}} + c_\gamma^2 \int_{\{\gamma > 0\}} |A^o|^2 \, d\mu \right)^{\frac{1}{2}}.
$$
Here to get to the last step we have again used Young’s inequality twice with conjugate exponents \( p = 3, p^* = 3/2 \). Substituting (5.29), (5.30) and (5.31) in (5.28) then gives

\[
\|A^o\|_{\infty, [\gamma=1]} \leq c \left( \int_{[\gamma>0]} |A^o|^2 d\mu \right)^{\frac{3}{2}} \left( \int_{\Sigma} |\Delta H|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} + c_\gamma^6 \left( \int_{[\gamma>0]} |A^o|^2 d\mu \right)^{\frac{3}{2}}. (5.32)
\]

Next, for any \( l \geq 2 \) the inequality of Proposition A.13 gives

\[
\left( \int_{\Sigma} |\Delta H|^2 \gamma^{4l} d\mu \right)^{\frac{1}{2}} \leq c \left( \int_{[\gamma>0]} |A^o|^2 d\mu \right)^{\frac{3(l-2)}{2l}} \left( \int_{\Sigma} |\Delta^{\frac{l}{2}} H|^2 \gamma^{2l} d\mu \right)^{\frac{1}{2}} + c c_\gamma^6 \left( \int_{[\gamma>0]} |A^o|^2 d\mu \right)^{\frac{3}{2}}.
\]

Therefore (5.32) becomes

\[
\|A^o\|_{\infty, [\gamma=1]}^6 \leq c \|A^o\|_{2, [\gamma>0]}^{\frac{3l-4l}{2l}} \left( \|\Delta^{1/2} H\|_{2, [\gamma>0]}^\frac{3}{2} + c_\gamma^6 \|A^o\|_{2, [\gamma>0]}^\frac{6}{2} \right). (5.33)
\]

Raising both sides to the power \( l/3 \) gives

\[
\|A^o\|_{\infty, [\gamma=1]}^{2l} \leq c \|A^o\|_{2, [\gamma>0]}^{\frac{3l^2-6l+4l}{2l}} \left( \|\Delta^{1/2} H\|_{2, [\gamma>0]}^\frac{3}{2} + c_\gamma^6 \|A^o\|_{2, [\gamma>0]}^\frac{6}{2} \right). (5.33)
\]

To prove the statement of the theorem, we must choose a suitable cutoff function. Fix a \( x \in \mathbb{R}^3 \) and choose \( \gamma = \tilde{\gamma} \circ f \) where \( \tilde{\gamma} \) satisfies

\[
\chi_{B_\rho(x)} \leq \tilde{\gamma} \leq \chi_{B_{2\rho}(x)}.
\]

Here \( \chi_\Omega \) stands for the characteristic function over the set \( \Omega \subset \mathbb{R}^3 \).

With this choice of cutoff function it follows that \( f^{-1}(B_\rho(x)) \subset [\gamma = 1] \) and \( [\gamma > 0] \subset f^{-1}(B_{2\rho}(x)) \). Moreover, \( c_\gamma \leq c \rho^{-1} \) for some universal constant \( c > 0 \).
Since the inequality (5.33) holds under the assumption that \( \|A^o\|_{2,\gamma>0}^2 \leq \varepsilon_0 \) it follows that it holds under the assumption (5.26). This proves (5.27). The inequality (5.27) follows immediately under the assumption that \( \Delta^pH(f) \equiv 0 \).

Before proving the main theorem for this section, Theorem 5.4, we present a result of Li-Yau [65] which will assist us in the compact case to establish \( f \) as an embedding. Later on in Chapter 6 it will also allow us to establish \( f(\Sigma,t), t \in [0,T) \) as a one-parameter of embeddings.

**Theorem 5.16** (Li-Yau [65], Theorem 6). *If an immersion \( f : \Sigma \to \mathbb{R}^m, m \geq 3 \) has the property that\(^1\)

\[
\frac{1}{4} \int_{\Sigma} H^2 \, d\mu < 8\pi
\]

then \( f \) must be an embedding.

We are now in a position to prove the main theorem for this section, Theorem 5.4. In the non-compact case we will see that we utilise the \( \varepsilon \)-regularity result from the previous theorem.

**Proof of Theorem 5.4.** Assume that \( \Delta^pH \equiv 0 \). We prove the compact and non-compact cases separately, since the compact case is comparatively simple.

For the compact case, we first assume that \( p \geq 2 \). In this case, integrating by parts twice and using the assumption \( \Delta^pH \equiv 0 \) immediately gives

\[
\int_{\Sigma} |\Delta^{p-1}H|^2 \, d\mu = \int_{\Sigma} \Delta^{p-2}H \Delta^pH \, d\mu = 0,
\]

which implies that \( \Delta^{p-1}H \equiv 0 \). Continuing in this fashion \( p-2 \) more times leads to the equation \( \Delta H \equiv 0 \). Therefore \( \Delta^pH \equiv 0 \implies \Delta H \equiv 0 \). One more application of

\(^1\) The 1/4 here comes from the fact that some authors (such as Li and Yau) prefer to use \( H = \frac{1}{n} (\kappa_1 + \cdots + \kappa_n) \) as the definition of the mean curvature.
integration by parts gives

\[ \int_{\Sigma} |\nabla H|^2 \, d\mu = -\int_{\Sigma} H \Delta H \, d\mu = 0, \]

which implies that \( H \) is constant. Note that if \( p = 1 \) we could have simply applied that last step immediately. Therefore \( f \) is a closed constant mean curvature (CMC) surface. Furthermore, by the Gauss Bonnet theorem, it follows that if

\[ \int_{\Sigma} |A_o|^2 \, d\mu < 8\pi, \]

then

\[ \frac{1}{4} \int_{\Sigma} H^2 \, d\mu = \frac{1}{2} \int_{\Sigma} |A_o|^2 \, d\mu + \int_{\Sigma} K \, d\mu < 4\pi + 4\pi (1 - g) \leq 8\pi, \]

Where \( g \geq 0 \) is the genus of \( f(\Sigma) \). Therefore by Theorem 5.16, \( f \) is an embedding. Since by a classical theorem of Alexandrov [2] the only closed embedded CMC surfaces are spheres, the result then follows.

If \( f \) is instead non-compact then we recall that the \( \varepsilon \)–regularity result from Theorem 5.15 gives

\[ \|A_o\|_{\infty, f^{-1}(B_\rho(x))} \leq c\rho^{-1} \varepsilon_0^{2/3}. \]

Taking \( \rho \searrow \infty \) then implies that

\[ \|A_o\|_{\infty, \Sigma} \leq c \liminf_{\rho/\infty} \rho^{-1} \varepsilon_0^{2/3} = 0, \]

which proves that \( f \) is umbilic. A classical result from Codazzi (see Theorem A.1) then implies \( f(\Sigma) \) is a plane. This completes the proof.

\[ \square \]

**Remark 5.17.** We note that the method given here to prove Theorem 5.4 in the non-compact case is not the only one. Indeed, assume \( \Delta^p H \equiv 0 \). Then by combining
Corollary 5.10 and the interpolation inequalities of Proposition A.13, we have

\[ \int_{f^{-1}(B_\rho(0))} |\nabla A^o|^2 + |A^o|^2 H^2 d\mu \leq c \rho^{-2} \int_{f^{-1}(B_{2\rho}(0))} |A^o|^2 d\mu. \]

Therefore taking \( \rho \nearrow \infty \) yields

\[ |\nabla A^o|^2 + |A^o|^2 H^2 \equiv 0, \]

from which we ascertain, using Kato’s inequality \( |\nabla |A^o|| \leq |\nabla A^o| \), that \( |A^o| \) is constant.

Writing \( |A^o| \equiv c_0 \), we claim that \( c_0 \) must be equal to zero. For if not, then we would have

\[ \int_{\Sigma} |A^o|^2 d\mu = c_0^2 |\Sigma| = \infty, \]

(since \( f \) is non-compact) which contradicts the small-energy assumption (5.2). Therefore \( |A^o|^2 \equiv 0 \) and again using the classical result from Codazzi (Theorem A.1) implies that \( f(\Sigma) \) must be a plane.

The author has chosen to use the alternative method to prove Theorem 5.4 because it makes use of the \( \varepsilon \)-regularity result from Theorem 5.15 and also because he feels it is neater.
Chapter 6

Preserved sphericity

In this chapter we aim to show that an immersion evolving under the geometric poly-harmonic heat flow (GPHF), if initially sufficiently ‘close’ to a sphere in an averaged $L^2$ sense, will continue to become ‘more spherical’. The proof rests upon us being able to measure the averaged distance from our immersion by integrating the trace-free curvature over the surface to give the so-called ‘umbilic energy’ of the immersion $f$ (see (U1) from Chapter 5). We then study the evolution in time of this integral and show that, if initially small, this energy will decrease monotonically for the duration of the flow (this energy is said to be a Lyapunov functional). We will then apply the Li-Yau inequality from Theorem 5.16 to conclude that under the assumption of small initial umbilic energy, our family of immersions is in fact an embedding for the duration of the flow. Before stating the theorem we will need some supporting lemmata, all of which rest upon this small-umbilic-energy condition.

**Theorem 6.1** (Preserved sphericity). There exists an $\varepsilon_0$ depending only on $p$ such that the following holds. Let $f : \Sigma^2 \times [0, T) \to \mathbb{R}^3$ satisfy (GPHF). If

$$\int_{\Sigma} |A_0|^2 \, d\mu \bigg|_{t=0} \leq \varepsilon_0,$$

then...
then for $\tau < T$ we have the estimate
\[
\frac{d}{dt} \int_{\Sigma} |A^o|^2 \, d\mu \leq -\frac{1}{2} \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu.
\] (6.2)

Here we are using the notation of Proposition A.13 in which for $p = 2k$, we set $\Delta^{\frac{p+1}{2}} = \Delta^{k+\frac{1}{2}} = \nabla \Delta^k$.

Moreover, $\|A^o\|^2_2$ decays exponentially in time: there exists a $\delta > 0$ such that for $t \in [0, T)$ the following estimate holds:
\[
\int_{\Sigma} |A^o|^2 \, d\mu \leq \varepsilon_0 \exp(-\delta t).
\] (6.3)

Proof. First, note that by continuity and (6.1) there exists a $\delta > 0$ such that for $\tau \in I_\delta := [0, \delta)$, $\|A^o\|^2_2 |\tau| \leq 2 \varepsilon_0$.

Next, using the Gauss Bonnet Theorem and the evolution equations from Lemma 1.2, one has
\[
\frac{d}{dt} \int_{\Sigma} |A^o|^2 \, d\mu = \frac{1}{2} \frac{d}{dt} \int_{\Sigma} H^2 \, d\mu = (-1)^p \int_{\Sigma} H \Delta^{p+1} H \, d\mu + (-1)^p \int_{\Sigma} H \Delta^p H \ |A^o|^2 \, d\mu
\]
where we have used the identity $|A^o|^2 = |A|^2 - \frac{1}{2} H^2$. Integrating by parts $p + 1$ times on the first integral and $p - 1$ times on the second then yields
\[
\frac{d}{dt} \int_{\Sigma} |A^o|^2 \, d\mu + \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu = -\int_{\Sigma} \left( \Delta^{\frac{p+1}{2}} H \right) \left( \Delta^{\frac{p+1}{2}} \left( H \ |A^o|^2 \right) \right) \, d\mu \leq c \left( \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu \int_{\Sigma} \left| \Delta^{\frac{p+1}{2}} \left( H \ |A^o|^2 \right) \right|^2 \, d\mu \right)^{\frac{1}{2}}.
\] (6.4)

We have used Hölder’s inequality in the last step. The next step is to establish an inequality of the form
\[
\int_{\Sigma} \left| \Delta^{\frac{p+1}{2}} \left( H \ |A^o|^2 \right) \right|^2 \, d\mu \leq c \|A^o\|^2_2 \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu
\]
for some $\alpha > 0$ so that we can use (6.1) to prove the desired monotonicity result. Next note that by the identity (1.18) we have

$$\Delta^{p-1} (H |A^o|^2) \leq \Delta^{p-1} |A^o|^2 H + |P_3^{p-1} (A^o)|,$$

and therefore by the multiplicative Sobolev inequality of Theorem A.6 one has

$$\int_\Sigma \left| \Delta^{p-1} (H |A^o|^2) \right|^2 d\mu \leq c \int_\Sigma \left| \Delta^{p-1} |A^o|^2 \right|^2 H^2 d\mu + \int_\Sigma \left| P_6^{2(p-1)} (A^o) \right| d\mu$$

$$\leq c \int_\Sigma \left| \Delta^{p-1} |A^o|^2 \right|^2 H^2 d\mu + c \|A^o\|_4^4 \int_\Sigma \left| \nabla (p-1) A^o \right|^2 d\mu.$$

(6.5)

Our next job will be to estimate the first term on the right hand side of (6.5). We treat the cases $p$ is even and $p$ is odd separately.

If $p$ is even with $p = 2(k + 1), k \geq 0$, then using (5.11) with $\varphi = \Delta^k |A^o|^2$, followed by the multiplicative Sobolev inequality of Theorem A.6 gives

$$\int_\Sigma \left| \Delta^{k+1} |A^o|^2 \right|^2 d\mu$$

$$= \int_\Sigma |\nabla \varphi|^2 H^2 d\mu$$

$$\leq c \int_\Sigma |\Delta \varphi|^2 d\mu + c \int_\Sigma |\nabla A^o|^2 d\mu \int_\Sigma |\nabla \varphi|^2 d\mu$$

$$= c \int_\Sigma \left| \Delta^{k+1} |A^o|^2 \right|^2 d\mu + c \int_\Sigma |\nabla A^o|^2 d\mu \int_\Sigma |\nabla \Delta^k |A^o|^2 |^2 d\mu$$

$$\leq \int_\Sigma \left| P_4^{(k+1)} (A^o) \right| d\mu + \int_\Sigma |\nabla A^o|^2 d\mu \int_\Sigma \left| P_4^{2(k+1)} (A^o) \right| d\mu$$

$$\leq c \|A^o\|_\infty^2 \left( \int_\Sigma \left| \nabla (p) A^o \right|^2 d\mu + \int_\Sigma |\nabla A^o|^2 d\mu \int_\Sigma \left| \nabla (p-1) A^o \right|^2 d\mu \right).$$

Using the interpolation inequalities from Proposition A.13 and Lemma A.10 as well as Young’s inequality with conjugate exponents $\alpha = p/(p - 1), \alpha^* = p$, the last term on the right can be estimated as follows:

$$\int_\Sigma |\nabla A^o|^2 d\mu \int_\Sigma \left| \nabla (p-1) A^o \right|^2 d\mu$$
\[ \leq c \left( \| A^o \|_2^{\frac{2p}{p+1}} \left( \int_{\Sigma} | \Delta^{\frac{p+1}{2}} H^2 | d\mu \right)^{\frac{1}{p+1}} \right) \left( \| A^o \|_2^{\frac{2}{p+1}} \left( \int_{\Sigma} | \nabla_{(x)} A^o |^2 d\mu \right)^{\frac{p}{p+1}} \right) \]

\[ \leq c \int_{\Sigma} | \nabla_{(x)} A^o |^2 d\mu + c \| A^o \|_2^{\frac{4p+2}{p+1}} \left( \int_{\Sigma} | \Delta^{\frac{p+1}{2}} H^2 | d\mu \right)^{\frac{p}{p+1}}. \quad (6.6) \]

Therefore

\[ \int_{\Sigma} | \Delta^{\frac{p}{2}} | A^o |^2 |^2 H^2 d\mu \]

\[ \leq c \| A^o \|_\infty^2 \left( \int_{\Sigma} | \nabla_{(x)} A^o |^2 d\mu + \| A^o \|_2^{\frac{4p+2}{p+1}} \left( \int_{\Sigma} | \Delta^{\frac{p+1}{2}} H^2 | d\mu \right)^{\frac{p}{p+1}} \right). \quad (6.7) \]

We leave this inequality for the time being.

In the case \( p \) is odd with \( p = 2k + 1, k \geq 0 \), we use another identity similar to (5.11). This identity claims that for \( \varphi \in C^3 (\Sigma) \) under the small-energy assumption (6.1) one has

\[ \int_{\Sigma} | \nabla (2) \varphi |^2 H^2 d\mu + \int_{\Sigma} | \nabla \varphi |^2 H^4 d\mu \]

\[ \leq c \int_{\Sigma} | \nabla \varphi |^2 d\mu + c \int_{\Sigma} | \nabla H |^2 d\mu \int_{\Sigma} | \Delta \varphi |^2 d\mu + c \int_{\Sigma} | \Delta H |^2 d\mu \int_{\Sigma} | \nabla \varphi |^2 d\mu. \quad (6.8) \]

The proof of (6.8) is included at the beginning of Appendix A.

Applying (6.8) with \( \varphi = \Delta^{k-1} | A^o |^2 \) and the multiplicative Sobolev inequality of Theorem A.6 then gives

\[ \int_{\Sigma} | \Delta^{\frac{p-1}{2}} | A^o |^2 |^2 H^2 d\mu \]

\[ \leq c \int_{\Sigma} | \nabla (2) \varphi |^2 H^2 d\mu \]

\[ \leq c \int_{\Sigma} | \nabla \Delta \varphi |^2 d\mu + c \int_{\Sigma} | \nabla H |^2 d\mu \int_{\Sigma} | \Delta \varphi |^2 d\mu + c \int_{\Sigma} | \Delta H |^2 d\mu \int_{\Sigma} | \nabla \varphi |^2 d\mu \]

\[ = c \int_{\Sigma} | \nabla \Delta^k | A^o |^2 |^2 d\mu + c \int_{\Sigma} | \nabla H |^2 d\mu \int_{\Sigma} | \Delta^k | A^o |^2 |^2 d\mu \]

\[ + c \int_{\Sigma} | \Delta H |^2 d\mu \int_{\Sigma} | \nabla \Delta^{k-1} | A^o |^2 | d\mu \]

\[ \leq \int_{\Sigma} | P_{A^o}^{2p} (A^o) | d\mu + \int_{\Sigma} | \nabla H |^2 d\mu \int_{\Sigma} | P_{A^o}^{2(p-1)} (A^o) | d\mu \]
\[ + \int_{\Sigma} |\nabla H|^2 \, d\mu \int_{\Sigma} \left| P_{4}^{2(p-2)}(A^o) \right| \, d\mu \]
\[ \leq c(p) \|A^o\|_{\infty}^2 \left( \int_{\Sigma} |\nabla (p) A^o|^2 \, d\mu + \int_{\Sigma} |\nabla H|^2 \, d\mu \int_{\Sigma} |\nabla (p-1) A^o|^2 \, d\mu \right) \]
\[ + \int_{\Sigma} |\Delta H|^2 \, d\mu \int_{\Sigma} |\nabla (p-2) A^o|^2 \, d\mu \]

Estimating the last two terms in a similar fashion to (6.6), we once again arrive at

\[ \int_{\Sigma} \left| \Delta^{\frac{p-1}{2}} A^o \right|^2 \, H^2 \, d\mu \]
\[ \leq c(p) \|A^o\|_{\infty}^2 \left( \int_{\Sigma} |\nabla (p) A^o|^2 \, d\mu + \|A^o\|_{2}^{4p+2} \left( \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu \right)^{\frac{2}{p+1}} \right). \] (6.9)

Comparing with (6.7), we find that this form of inequality holds regardless of whether \( p \) is even or odd.

To prove the monotonicity result (6.2) it will therefore be enough to establish an inequality of the form

\[ \int_{\Sigma} |\nabla (p) A^o|^2 \, d\mu \leq c \|A^o\|_{2}^{\alpha} \left( \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu \right)^{\frac{\alpha}{p+1}} \]

for some \( \alpha \geq 0 \).

To establish this we first claim that there exists a constant \( c = c(p) > 0 \) such that

\[ \int_{\Sigma} |\nabla (p) A^o|^2 \, d\mu \leq c \int_{\Sigma} |\Delta^{\frac{p}{2}} H|^2 \, d\mu + c \int_{\Sigma} |\nabla (p-1) A^o|^2 \, H^2 \, d\mu. \]

This can be shown as follows. Firstly, by multiple applications of the interchange of covariant formula and the fact that in dimension \( n = 2 \), \( R_{i}^{k} = \frac{1}{2} R (g_{ik} \delta_{j}^{l} - g_{jk} \delta_{i}^{l}) \), we have

\[ \Delta \nabla (k) T = \nabla (k) \Delta T + \sum_{i=0}^{k} \nabla (i) \left( R \ast \nabla (k-i) T \right). \] (6.10)

Using integration by parts, identity (6.10) with \( T = A^o, k = p - 1 \), as well as the
identity (5.5), it follows that

\[
\int_{\Sigma} \left| \nabla (p) A^\circ \right|^2 d\mu \\
= - \int_{\Sigma} \left\langle \nabla (p-1) A^\circ, \Delta \nabla (p-1) A^\circ \right\rangle d\mu \\
= - \int_{\Sigma} \left\langle \nabla (k-1) A^\circ, \nabla (k-1) \Delta A^\circ \right\rangle d\mu + \sum_{i=0}^{p-1} \int_{\Sigma} \nabla (p-1) A^\circ \ast \nabla (i) \left( R \ast \nabla (p-1-i) A^\circ \right) d\mu \\
= - \int_{\Sigma} \left\langle \nabla (k-1) A^\circ, \nabla (p+1) H \right\rangle d\mu + \sum_{i=0}^{p-1} \int_{\Sigma} \nabla (p-1) A^\circ \ast \nabla (i) \left( R \ast \nabla (p-1-i) A^\circ \right) d\mu \\
= \int_{\Sigma} \left\langle \Delta \nabla (p-2) A^\circ, \nabla (p) H \right\rangle d\mu + \sum_{i=0}^{p-1} \int_{\Sigma} \nabla (p-1) A^\circ \ast \nabla (i) \left( R \ast \nabla (p-1-i) A^\circ \right) d\mu,
\]

which implies that

\[
\int_{\Sigma} \left| \nabla (p) A^\circ \right|^2 d\mu \leq c(p) \int_{\Sigma} \left| \nabla (p) H \right|^2 d\mu + \sum_{i=0}^{p-1} \int_{\Sigma} \nabla (p-1) A^\circ \ast \nabla (i) \left( R \ast \nabla (p-1-i) A^\circ \right) d\mu
\]

for some universal constant \( c > 0 \). By following a process similar to that above, we have

\[
\int_{\Sigma} \left| \nabla (p) H \right|^2 d\mu = \int_{\Sigma} \left| \Delta^{2} H \right|^2 d\mu + \sum_{i=0}^{p-1} \int_{\Sigma} \nabla (p-1) H \ast \nabla (i) \left( R \ast \nabla (p-1-i) H \right) d\mu \\
+ \sum_{i=0}^{p-2} \int_{\Sigma} \nabla (p) H \ast \nabla (i) \left( R \ast \nabla (p-2-i) H \right) d\mu.
\]

Combining the last two identities gives

\[
\int_{\Sigma} \left| \nabla (p) A^\circ \right|^2 d\mu \\
\leq c(p) \int_{\Sigma} \left| \Delta^{2} H \right|^2 d\mu + \sum_{i=0}^{p-1} \int_{\Sigma} \nabla (p-1) A^\circ \ast \nabla (i) \left( R \ast \nabla (p-1-i) A^\circ \right) d\mu \\
+ \sum_{i=0}^{p-1} \int_{\Sigma} \nabla (p-1) H \ast \nabla (i) \left( R \ast \nabla (p-1-i) H \right) d\mu \\
+ \sum_{i=0}^{p-2} \int_{\Sigma} \nabla (p) H \ast \nabla (i) \left( R \ast \nabla (p-2-i) H \right) d\mu.
\]
We need to estimate the extraneous terms on the right hand side of (6.11). For the first summation we use the identity $R = \frac{1}{2}H^2 - |A^o|^2$ to establish the following estimate:

\[
\sum_{i=0}^{p-1} \int_{\Sigma} \nabla_{(p-1)A^o} \cdot \nabla_i \left( R * \nabla_{(p-1)A^o} \right) d\mu \\
= \sum_{i=0}^{p-1} \int_{\Sigma} \nabla_{(p-1)A^o} \cdot \nabla_i \left( (H^2 + A^o * A^o) * \nabla_{(p-1)A^o} \right) d\mu \\
\leq \sum_{i=0}^{p-2} \int_{\Sigma} \nabla_{(p-1)A^o} \cdot \nabla_i \left( H^2 \nabla_{(p-1)A^o} \right) d\mu + \int_{\Sigma} \left| P_4^{2(p-1)} (A^o) \right| d\mu \\
= \int_{\Sigma} \left( \nabla_{(p-1)A^o} \right)^2 H^2 d\mu + \sum_{i=1}^{p-2} \int_{\Sigma} H \nabla_{(p-1)A^o} \cdot \nabla_i A^o + \sum_{i=1}^{p-2} \int_{\Sigma} H \nabla_{(p-1)A^o} \cdot \nabla_i H \nabla_{(p-1)A^o} d\mu \\
+ \int_{\Sigma} \left| P_4^{2(p-1)} (A^o) \right| d\mu \\
\leq c \int_{\Sigma} \left| \nabla_{(p-1)A^o} \right|^2 H^2 d\mu + \sum_{i=1}^{p-2} \int_{\Sigma} |\nabla_{(p-1)A^o}| |\nabla_{(p-1-i)A^o}| |\nabla_i A^o| |H| d\mu \\
+ \int_{\Sigma} \left| P_4^{2(p-1)} (A^o) \right| d\mu \\
\leq c \int_{\Sigma} \left| \nabla_{(p-1)A^o} \right|^2 H^2 d\mu + \int_{\Sigma} \left| P_4^{2(p-1)} (A^o) \right| d\mu \\
+ \sum_{i=1}^{p-2} \left( \int_{\Sigma} \left| \nabla_{(p-1)A^o} \right|^2 H^2 d\mu \int_{\Sigma} |\nabla_{(p-1-i)A^o}|^2 |\nabla_i A^o|^2 d\mu \right)^{\frac{1}{2}} \\
\leq c \int_{\Sigma} \left| \nabla_{(p-1)A^o} \right|^2 H^2 d\mu + \int_{\Sigma} \left| P_4^{2(p-1)} (A^o) \right| d\mu,
\]

where the constants on the right hand side depend only on $p$. Here we have used the Cauchy-Schwarz inequality in the last step along with the fact that

\[
\int_{\Sigma} |\nabla_{(p-1-i)A^o}|^2 |\nabla_i A^o|^2 d\mu \leq \int_{\Sigma} \left| P_4^{2(p-1)} (A^o) \right| d\mu.
\]

We have also used (1.18) to get to the third step. The other two summations in the right hand side of (6.11) are estimated in the same way. Therefore

\[
\int_{\Sigma} |\nabla_{(p)A^o}|^2 d\mu \leq c(p) \int_{\Sigma} |\Delta^2 H|^2 d\mu + c(p) \int_{\Sigma} |\nabla_{(p-1)A^o}|^2 H^2 d\mu + \int_{\Sigma} \left| P_4^{2(p-1)} (A^o) \right| d\mu.
\]

(6.12)
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The last term can be estimated by using the multiplicative Sobolev inequality of Theorem A.6, the interpolation inequalities of Proposition A.13 and Lemma A.10, as well as our \( L^\infty \) estimate for the trace-free curvature from Theorem 5.15:

\[
\int_\Sigma |P^2_{4(p-1)}(A^o)| \, d\mu \\
\leq c \|A^o\|_{\infty}^2 \int_\Sigma |\nabla(p-1)A^o|^2 \, d\mu \\
\leq c \left( \|A^o\|_2^{\frac{2(4p-3)}{3p}} \left( \int_\Sigma |\Delta^\frac{2}{3} H|^2 \, d\mu \right)^{\frac{1}{3p}} \right) \left( \|A^o\|_2^\frac{2}{3p} \left( \int_\Sigma |\nabla(p)A^o|^2 \, d\mu \right)^{\frac{p-1}{p}} \right) \\
\leq c \|A^o\|_2^2 \left( \int_\Sigma |\Delta^\frac{2}{3} H|^2 \, d\mu \right)^{\frac{1}{3p}} \left( \int_\Sigma |\nabla(p)A^o|^2 \, d\mu \right)^{\frac{p-1}{p}} \\
\leq c(p) \|A^o\|_{2}^2 \left( \int_\Sigma |\Delta^\frac{2}{3} H|^2 \, d\mu + \int_\Sigma |\nabla(p)A^o|^2 \, d\mu \right). 
\] (6.13)

Here in the last step we have used Young’s inequality with conjugate exponents \( \alpha = p/(p-1) \), \( \alpha^* = p \). For the penultimate term in (6.12), we combine Corollary A.16, Lemma A.17 and Lemma A.18 with \( m = p \) to give

\[
\int_\Sigma |\nabla(p-1)A^o|^2 H^2 \, d\mu \leq c \int_\Sigma |\nabla A^o|^2 H^{2(p-1)} \, d\mu + \int_\Sigma |A^o|^2 H^{2p} \, d\mu + c \int_\Sigma |\Delta^\frac{2}{3} H|^2 \, d\mu \\
\leq c \int_\Sigma |\Delta H|^2 H^{2(p-2)} \, d\mu + c \int_\Sigma |\Delta^\frac{2}{3} H|^2 \, d\mu \\
\leq c \int_\Sigma |\Delta^\frac{2}{3} H|^2 \, d\mu. 
\] (6.14)

Substituting (6.13) and (6.14) in (6.12) then implies

\[
(1 - c \|A^o\|_2^2) \int_\Sigma |\nabla(p)A^o|^2 \, d\mu \leq c \left( 1 + \|A^o\|_2^2 \right) \int_\Sigma |\Delta^\frac{2}{3} H|^2 \, d\mu. 
\]

Therefore if \( \epsilon_0 > 0 \) is small enough we obtain

\[
\int_\Sigma |\nabla(p)A^o|^2 \, d\mu \leq c \int_\Sigma |\Delta^\frac{2}{3} H|^2 \, d\mu \leq c \|A^o\|_{2, |\gamma| > 0}^{\frac{2}{p+1}} \left( \int_\Sigma |\Delta^\frac{p+1}{3} H|^2 \, d\mu \right)^{\frac{p+1}{p+1}},
\]

where we have used Proposition A.13 in the last step. Note that this last inequality
and Lemma A.10 together imply that

$$\int_{\Sigma} |\nabla(p-1)A^0|^2 d\mu \leq c \|A^0\|_{\frac{2}{p}}^2 \left( \int_{\Sigma} |\nabla(p)A^0|^2 d\mu \right)^{\frac{p-1}{p}}$$

$$\leq c \|A^0\|_{\frac{4}{p+1}}^4 \left( \int_{\Sigma} |A|^{\frac{p+1}{p}} H^2 |d\mu \right)^{\frac{p}{p+1}}.$$  

Substituting this into (6.9), and then back into (6.5) and using the interpolation inequalities of Proposition A.13 gives

$$\int_{\Sigma} |\Delta^{\frac{p-1}{2}} (H |A^0|^2)|^2 d\mu$$

$$\leq c \int_{\Sigma} |\Delta^{\frac{p-1}{2}} |A^0|^2|^2 H^2 d\mu + c \|A^0\|_{\infty}^4 \int_{\Sigma} |\nabla(p-1)A^0|^2 d\mu$$

$$\leq c \|A^0\|_{\infty}^2 \left( \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 d\mu + \|A^0\|_{\frac{4}{p+1}}^4 \left( \int_{\Sigma} |A|^{\frac{p+1}{p}} H^2 |d\mu \right)^{\frac{p}{p+1}} \right)$$

$$+ c \|A^0\|_{\infty}^4 \int_{\Sigma} |\nabla(p-1)A^0|^2 d\mu$$

$$\leq c \|A^0\|_{\infty}^2 \|A^0\|_{\frac{2}{p+1}}^2 \left( \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 d\mu \right)^{\frac{p}{p+1}}$$

$$+ c \|A^0\|_{\infty}^4 \|A^0\|_{\frac{4}{p+1}}^4 \left( \int_{\Sigma} |A|^{\frac{p+1}{p}} H^2 |d\mu \right)^{\frac{p}{p+1}}$$

$$\leq c \|A^0\|_{\frac{2(4p+1)}{3(4p+1)+2}}^2 + \frac{2}{p+1} \int_{\Sigma} |A|^{\frac{p+1}{2}} H^2 |d\mu + c \|A^0\|_{\frac{4}{p+1}}^4 \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 d\mu$$

$$\leq c \|A^0\|_{\frac{2}{p+1}}^2 \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 d\mu,$$

where \( c = c(p) \). Here we have used the \( L^\infty \) estimate for the trace-free curvature from Theorem 5.15 to get to the penultimate line. Finally, substituting this back into (6.4) yields

$$\frac{d}{dt} \int_{\Sigma} |A^0|^2 d\mu + \left( 1 - c \|A^0\|_{\frac{2}{p}}^2 \right) \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 d\mu \leq 0$$

for some constant \( c > 0 \) depending only on \( p \). Therefore if we choose \( \varepsilon_0 \) small enough such that \( c(2\varepsilon_0)^{2/3} \leq 1 \), then for times \( \tau \in I_{\delta} \) the inequality (6.3) holds. Since at time \( \tau \) the total umbilic energy has decreased from its initial value, we may repeat this process on the time interval \( I_{2\delta} := [\delta, 2\delta) \), and so forth, until we reach time \( T \). This proves (6.2).
To prove (6.3) we first use Hölder’s inequality, noting that by Lemma 1.2 the enclosed area $|\Sigma_t|$ is decreasing along the flow. This gives

$$\|A^o\|_4^2 \leq |\Sigma_t|^\frac{1}{2} \|A^o\|_4^2 \leq |\Sigma_0|^\frac{1}{2} \|A^o\|_4^2. \quad (6.15)$$

Next Theorem A.2 with $u = |A^o|^2$, along with the inequality of Corollary 5.10 and the interpolation inequality from Proposition A.13 to estimate the last term on the right hand side of (6.15) gives

$$\int_{\Sigma} |A^o|^4 \, d\mu \leq c \left( \int_{\Sigma} |\nabla A^o|^2 |A^o|^2 \, d\mu + \int_{\Sigma} |A^o|^2 |H|^2 \, d\mu \right)^2$$

$$\leq c \|A^o\|_2^2 \left( \int_{\Sigma} |\nabla A^o|^2 \, d\mu + \int_{\Sigma} |A^o|^2 H^2 \, d\mu \right)$$

$$\leq c \|A^o\|_2^2 \left( \int_{\Sigma} |\Delta H|^2 \, d\mu \right)^{\frac{1}{2}}$$

$$\leq c \|A^o\|_2^{2(p+1)} \left( \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu \right)^{\frac{1}{2(p+1)}}.$$

Combining this with (6.15) gives

$$\|A^o\|_2^2 \leq c \|A^o\|_2^{2(p+1)} |\Sigma_0|^{\frac{1}{2}} \left( \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu \right)^{\frac{1}{2(p+1)}}.$$

Therefore there exists a constant $\tilde{c} > 0$ depending only on $p$ such that

$$\frac{\tilde{c}}{|\Sigma_0|^{p+1}} \|A^o\|_2^2 \leq \frac{1}{2} \int_{\Sigma} |\Delta^{\frac{p+1}{2}} H|^2 \, d\mu.$$

Substituting this in (6.2), we find that

$$\frac{d}{dt} \|A^o\|_2^2 + \frac{\tilde{c}}{|\Sigma_0|^{p+1}} \|A^o\|_2^2 \leq 0.$$

Hence $\|A^o\|_2^2$ satisfies (6.2) with $\delta = \frac{\tilde{c}}{|\Sigma_0|^{p+1}}$. This finishes the proof.
Chapter 7

Construction of the blowup and long time existence

In Chapter 5 we established a family of gap lemmata for stationary solutions to the geometric polyharmonic heat flows (GPHF). The results therein were time-independent. In this chapter we return to looking at one-parameter families of solutions to the flow. This chapter culminates with Proposition 7.8, in which prove long time existence of the flow (that is, $T = \infty$), under the assumption of small initial trace-free curvature (6.1). To do this, we prove a barrage of lemmata and theorems which allow us to control the geometry of any blowup along the flow. By the lifespan theorem (Theorem 4.1) we already know that the only way a geometric polyharmonic heat flow can cease to exist is if its curvature concentrates. We will show (in the proof of Proposition 7.8) that assuming assuming a finite time singularity along with small initial concentration of trace-free curvature (which takes the form of (6.1)) leads to a contradiction, and so is impossible.

We start off by stating and proving Theorem 7.1. Here we assume a bound on the concentration of curvature in the ball $f^{-1}(B_{2\rho}(0))$ over some time interval. We use this assumption to establish a uniform bound on the $L^\infty$-norm of any derivative of curvature on the smaller ball $f^{-1}(B_{\rho}(0))$. 
**Theorem 7.1 (Interior Estimates).** Suppose $f : \Sigma^2 \times [0, T^*) \to \mathbb{R}^3$ satisfies (GPHF).

If
\[
\sup_{0 < t \leq T^*} \int_{f^{-1}(B_{2\rho}(x))} |A|^2 \, d\mu \leq \varepsilon_0 \quad \text{for} \quad T^* \leq c \rho^{2(p+1)}.
\] (7.1)

Then for any $m \in \mathbb{N}_0$ we have at time $t \in [0, T^*)$ the estimates
\[
\|\nabla (m)A\|_{2, f^{-1}(B_{\rho}(x))}^2 \leq c_m \varepsilon_0 t^{-m/(p+1)}
\] (7.2)
and
\[
\|\nabla (m)A\|_{\infty, f^{-1}(B_{\rho}(x))}^2 \leq c_k \varepsilon_0 t^{-(m+1)/(p+1)},
\] (7.3)
where $c_m = c_m \left( m, \rho, T^*, \|\nabla (m)A\|_{2, f^{-1}(B_{\rho}(x))} \big|_{t=0} \right)$.

**Proof.** The proof is similar to that of Theorem 3.5 from [57]. We let $x \in \mathbb{R}^3$ be chosen to satisfy the assumption (7.1). We then choose our cutoff function $\gamma = \tilde{\gamma} \circ f$ such that
\[
\chi_{B_{\rho}(x)} \leq \tilde{\gamma} \leq \chi_{B_{2\rho}(x)}.
\]
With this choice, $\|\nabla \gamma\|_{\infty} \leq c_\gamma \leq c \rho^{-1}$ for some absolute constant $c > 0$.

We now define a family of cutoff functions in time $\beta_j$ by
\[
\beta_j(t) = \begin{cases} 
0, & t \leq (j - 1) \frac{T^*}{m} \\
\frac{m}{T^*} (t - (j - 1) \frac{T^*}{m}), & (j - 1) \frac{T^*}{m} < t < j \frac{T^*}{m} \\
1, & t \geq j \frac{T^*}{m}
\end{cases}
\] (7.4)
where $j \in \{0, 1, \ldots, m\}$ and $m \in \mathbb{N} \cup \{0\}$. Note that $\beta_0 \equiv 1$ on $[0, T^*)$, $\beta_m(T^*) = 1$ and
\[
0 \leq \beta_j'(t) \leq \frac{m}{T^*} \beta_{j-1}.
\] (7.4)

Set $\Omega_j(t) := \beta_j \|\nabla (j)A\|_{2, \chi_{2\gamma}}^2$. We claim that for $0 \leq j \leq m$, $t \in (0, T^*)$,
\[
\Omega_j \leq c(m) \varepsilon_0(T^*)^{-\frac{j}{p+1}}.
\] (7.5)
The proof is by induction. Inequality (7.5) is trivially true for $j = 0$ by the assumption (7.1). Assume that it is true for $j = k$. Then, by the estimate of Proposition 3.3, as well as (7.4), we have

$$
\Omega'_{k+1}(t) = \beta_{k+1}(t) \left\| \nabla (k+1)A \right\|_{2,\gamma+2}^2 + \beta_{k+1}(t) \frac{d}{dt} \left\| \nabla (k+1)A \right\|_{2,\gamma+2}^2
\leq \left( \frac{k + 1}{T^*} \right) \beta_k(t) \left\| \nabla (k+1)A \right\|_{2,\gamma+2}^2
+ \beta_{k+1}(t) \left( c \varepsilon_0 \rho^{-2(k+p+2)} - \left\| \nabla (k+p+2)A \right\|_{2,\gamma+2}^2 \right).
$$

(7.6)

Next, using Lemma A.10 we have

$$
\left\| \nabla (k+1)A \right\|_{2,\gamma+2}^2 \leq c \left( \left\| \nabla (k)A \right\|_{2,\gamma+2}^{2(p+1)+2} \right) \left\| \nabla (k+p+2)A \right\|_{2,\gamma+2}^2 + \varepsilon_0 \rho^{-2(k+1)}.
$$

Substituting this into (7.6) and using Young’s inequality gives

$$
\Omega'_{k+1}(t) + \beta_{k+1}(t) \left\| \nabla (k+p+2)A \right\|_{2,\gamma+2}^2
\leq c(m) \left( T^* \right)^{-1} \beta_k(t) \left( \left\| \nabla (k)A \right\|_{2,\gamma+2}^{2(p+1)+2} \right) \left\| \nabla (k+p+2)A \right\|_{2,\gamma+2}^2 + \varepsilon_0 \rho^{-2(k+1)}
+ c \beta_{k+1}(t) \varepsilon_0 \rho^{-2(k+p+2)}
\leq c(m) \beta_k(t) \left( \eta^{-1} \left( T^* \right)^{-\frac{p+2}{p+1}} \left\| \nabla (k)A \right\|_{2,\gamma+2}^2 + \eta \left\| \nabla (k+p+2)A \right\|_{2,\gamma+2}^2 \right)
+ c \beta_{k+1}(t) \varepsilon_0 \rho^{-2(k+p+2)}
= c(m) \eta^{-1} \left( T^* \right)^{-\frac{p+2}{p+1}} \Omega_k(t) + c(m) \eta \beta_k(t) \left\| \nabla (k+p+2)A \right\|_{2,\gamma+2}^2
+ c \beta_{k+1}(t) \varepsilon_0 \rho^{-2(k+p+2)}.
$$

(7.7)

for any $\eta > 0$. Using the identities $\beta_k \leq \beta_{k+1}$ and $\beta_{k+1} \leq 1$, and choosing $\eta > 0$ small enough such that $c(m)\eta \leq 1$, we can absorb the second term on the right hand side of (7.7) into the left hand side, yielding

$$
\Omega'_{k+1}(t) \leq c(m) \left( T^* \right)^{-\frac{p+2}{p+1}} \Omega_k(t) + c(m) \varepsilon_0 \rho^{-2(k+p+2)}
\leq c(m) \left( T^* \right)^{-\frac{p+2}{p+1}} \left( T^* \right)^{-\frac{k}{p+1}} + c \varepsilon_0 \left( T^* \right)^{-\frac{k+p+2}{p+1}}
\leq c(m) \varepsilon_0 \left( T^* \right)^{-\frac{k+p+2}{p+1}}.
$$
Here we have used the inductive assumption as well as the assumption \( T^* \leq c \rho^{-2(p+1)} \).

Therefore using the fundamental theorem of calculus and the fact that \( \beta_{k+1}(0) = 0 \), we have

\[
\Omega_{k+1}(t) \leq c(m) \varepsilon_0 (T^*)^{-\frac{k+p+2}{p+1}} \int_0^t d\tau \\
= c(m) \varepsilon_0 (T^*)^{-\frac{k+p+2}{p+1}} t \\
\leq c(m) \varepsilon_0 (T^*)^{-\frac{k+1}{p+1}},
\]

which completes the inductive step and hence proves (7.5). Therefore at \( t = T^* \), we have

\[
||\nabla (m+2) A||_{2,f^{-1}(B_\rho(x))}^2 \leq ||\nabla (m) A||_{2,\gamma^2}^2 \leq c(m) \varepsilon_0 (T^*)^{-\frac{m}{p+1}}.
\]

Renaming \( T^* \) to \( t \) proves (7.2).

To prove (7.3), we use (7.2) as well as the localised \( L^\infty \) estimate from Proposition 3.4, which implies

\[
\left\| \nabla (m) A \right\|_{2, f^{-1}(B_\rho(x))}^{2(m+2)} \leq c \varepsilon_0 \left( \left\| \nabla (m+2) A \right\|_{2, \gamma^2}^{2(m+1)} + \rho^{-2(m+1)(m+2)} \varepsilon_0^{m+1} \right) \\
\leq c \varepsilon_0 \left( \left( \varepsilon_0 t^{-\frac{m+2}{p+1}} \right)^{m+1} + \rho^{-2(m+1)(m+2)} \varepsilon_0^{m+1} \right) \\
\leq c \varepsilon_0^{m+2} t^{-\frac{(m+1)(m+2)}{p+1}}.
\]

Taking the \( (m+2) \)-th root of both sides then finishes the proof.

In order to properly understand what is happening to our family of immersions \( f(\Sigma, t) \) in the limit \( t \nearrow T \), we will need to perform a blow-up analysis. This is a mathematical technique which involves dilating the flow in both the time and spatial directions, in order to ‘zoom in’ around the source of the blow-up. Recall that from (4.4) that if we define \( \rho \) by

\[
\rho(t) = \sup \left\{ r > 0 : \int_{f^{-1}(B_r(x))} |A|^2 \, d\mu \leq \varepsilon_0 \text{ for all } x \in \mathbb{R}^3 \right\},
\]
then \( \rho(t) \leq \frac{2(p+1)}{c(T-t)} \) (where \( c \) is the constant from Theorem 4.1), so that if our geometric polyharmonic heat flow encounters a finite time singularity (that is, if \( T < \infty \)) we necessarily have \( \rho \downarrow 0 \) as \( t \uparrow T \).

To study such a flow near the singularity, we have to rescale it to keep the curvatures bounded. We consider the same rescaling \( \tilde{f} \) as in Chapter 4:

\[
\tilde{f}(x,t) = \rho^{-1} f(x, \rho^{2(p+1)} t).
\]

The \( \rho^{-1} \) part acts to zoom in around the singularity, and in the events of a finite time singularity ‘scales up’ the spatial component of the flow around the curvature singularity (because \( \rho \to 0 \) in that case). The \( \rho^{2(p+1)} \) part needs to be incorporated in order to ensure that \( \tilde{f} \) flows by the geometric polyharmonic flow equation (GPHF) (see the calculations contained within Claim 4.5), but it also ‘scales down’ the time component (as \( \rho \) gets small). This time rescaling is necessary because the spatial rescaling implies that our flow \( \tilde{f} \) takes less time than \( f \) to cover large distances. In essence our rescaling acts to ‘zoom-in’ around the spatial component of the curvature singularity, whilst stretching out the time component. By choosing a suitable series of radii \( \{r_j\} \downarrow 0 \) and corresponding times \( \{t_j\} \) we are able to acquire greater understanding of the geometric properties of the singularity without having to worry about the maximal radius \( \rho \) shrinking to a point. This is, in essence, what the study of geometric blowups is all about.

We proceed by presenting a compactness theorem of Kuwert-Schätzle. As noted by the pair, the theorem is a localised version of the earlier result of Langer (see the main theorem in [61]). It will help us later when constructing our blowup.

**Theorem 7.2** (Kuwert-Schätzle [57], Theorem 4.2). Let \( f_j : \Sigma_j \to \mathbb{R}^m, \ m \geq 3 \) be a sequence of proper immersions, where each \( \Sigma_j \) is a surface without boundary. For \( R > 0 \) define

\[
\Sigma_j(R) := \{p \in \Sigma_j : |f_j(p)| < R\} = \Sigma_j \cap f_j^{-1}(B_R).
\]
Assume the bounds

\begin{align*}
\begin{cases}
\mu_j (\Sigma_j (R)) \leq c(R) \text{ for any } R > 0, \text{ and} \\
\| \nabla_{(k)} A \|_\infty \leq c(k) \text{ for any } k \in \mathbb{N}_0
\end{cases}
\end{align*}

hold. Then there is a proper immersion \( \tilde{f} : \tilde{\Sigma} \to \mathbb{R}^m \) (where \( \tilde{\Sigma} \) is also a 2-manifold without boundary) such that after passing to a subsequence one has

\[ f_j \circ \phi_j = \tilde{f} + u_j \quad \text{on } \tilde{\Sigma}(j) := \tilde{\Sigma} \cap \tilde{f}^{-1}(B_j), \]

satisfying the following properties:

\begin{align*}
\begin{cases}
\phi_j : \tilde{\Sigma}(j) \to U_j \subset \Sigma_j \text{ is a diffeomorphism,} \\
\Sigma_j (R) \subset U_j \text{ if } j \geq j(R), \\
u_j \in C^\infty (\tilde{\Sigma}(j), \mathbb{R}^3) \text{ is normal along } \tilde{f}(\tilde{\Sigma}), \text{ and} \\
\| \nabla_{(k)} u_j \|_{\infty, \Sigma_j} \to 0 \text{ as } j \to \infty \text{ for any } k \in \mathbb{N}_0.
\end{cases}
\end{align*}

The theorem basically says that on any ball \( B_R \), for sufficiently large \( j \) our sequence of immersions can be written as a normal graph \( \tilde{f} + u_j \) over the limit immersion \( \tilde{f} \) with small norm (after reparameterisation by our functions \( \phi_j \)).

Now let \( f : \Sigma^2 \times [0, T) \to \mathbb{R}^3 \) satisfy (GPHF), as usual, \( \Sigma^2 \) is a surface without boundary. We set

\[ \kappa(p, \tau) := \sup_{x \in \mathbb{R}^3} \int_{f^{-1}(B_r(x))} |A|^2 \ d\mu \bigg|_{t=\tau}. \]

Now we pick an arbitrarily decreasing sequence \( \{ r_j \} \searrow 0 \) of radii and assume that our curvature concentrates in finite time such that for each \( j \),

\[ t_j := \inf \{ t \geq 0 : \kappa(r_j, t) > \epsilon_1 \} < T. \]

Here \( \epsilon_1 := \frac{\epsilon_0}{c^*} \), where \( \epsilon_0, c^* \) are the same constants from the Lifespan Theorem. Note
that by construction, \( \{t_j\} \) must be a monotonically increasing sequence. By the definition of \( t_j \), we have

\[
\int_{f^{-1}(B_{r_j}(x))} |A|^2 \, d\mu \bigg|_{t=t_j} \leq \varepsilon_1 \text{ for any } x \in \mathbb{R}^3.
\]

However, for each \( j \) it is possible to find a point \( x_j \in \mathbb{R}^3 \) such that

\[
\int_{f^{-1}(B_{r_j}(x_j))} |A|^2 \, d\mu \bigg|_{t=t_j} \geq \varepsilon_1.
\]

(7.10)

To see this, we can take a sequence \( \{v\} \searrow \infty, v > 1 \) and consider the times \( \tau_j = t_j + v^{-1} \) and radii \( \lambda_j = r_j + v^{-2} \). We have

\[
\tau_j \downarrow t_j \text{ and } \lambda_j \nearrow r_j \text{ as } j \to \infty.
\]

Then by continuity, the definition of \( t_j \), and the fact that \( \tau_j > t_j \), for each \( v \) we can choose a point \( x_{j+v^{-1}} \in \mathbb{R}^3 \) such that

\[
\int_{f^{-1}(B_{\lambda_j}(x_{j+v^{-1}}))} |A|^2 \, d\mu \bigg|_{t=\tau_j} \geq \varepsilon_1,
\]

so that taking \( v \to \infty \) yields (7.10).

Now we consider a sequence of rescaled immersions

\[
f_j : \Sigma^2 \times \left[ -r_j^{-2(p+1)}t_j, r_j^{-2(p+1)}(T - t_j) \right) \to \mathbb{R}^3
\]

given by

\[
f_j (p, t) = \frac{1}{r_j} \left( f \left( p, t_j + r_j^{2(p+1)}t \right) - x_j \right).
\]

Here \( r_j, x_j, t_j \) are as previously defined.

From the definition of \( \kappa \), we define \( \kappa_j \) to be \( \kappa \) with respect to the immersion \( f_j \).
That is to say,
\[ \kappa_j (r, \tau) = \sup_{x \in \mathbb{R}^3} \int_{f_j^{-1}(B_r(x))} |A|^2 \, d\mu \bigg|_{t=\tau}. \]

Then it is easy to calculate that for a fixed \( x \in \mathbb{R}^3 \):
\[ y \in f_j^{-1}(B_1(0)) \bigg|_{t=0} \Leftrightarrow y \in f^{-1}(B_{r_j}(x_j)) \bigg|_{t=t_j}. \]

Thus by (7.10) we conclude that
\[ \int_{f_j^{-1}(B_1(0))} |A|^2 \, d\mu \bigg|_{t=0} \geq \epsilon_1. \]

Similarly, we find that for a fixed \( x \in \mathbb{R}^3 \):
\[ y \in f_j^{-1}(B_1(x)) \bigg|_{t=t_j} \Leftrightarrow y \in f^{-1}(B_{r_j}(r_jx + x_j)) \bigg|_{t=t_j+r_j^{2(p+1)}\tau}. \]

Then, using the definition of \( t_j \), we find that for \( \tau \leq 0 \):
\[ \kappa_j (1, \tau) = \sup_{x \in \mathbb{R}^3} \int_{f_j^{-1}(B_1(x))} |A|^2 \, d\mu \bigg|_{t=\tau} \leq \sup_{x \in \mathbb{R}^3} \int_{f^{-1}(B_{r_j}(x))} |A|^2 \, d\mu \bigg|_{t=t_j+r_j^{2(p+1)}\tau} \leq \epsilon_1. \]

Recall that the Lifespan Theorem tells us for our unscaled flow \( f \), the flow will cease to exists at time \( T \) when the curvature concentrates, with \( T \) satisfying \( T \geq c^{-1} \) (and also recall that without any loss of generality, for the unscaled case we assume that \( \rho = 1 \)).

In our rescaled scenario, this is equivalent to \( T \) satisfying \( T \geq t_j + r_j^{2(p+1)}c^{-1} \) for each \( j \). Equivalently,
\[ r_j^{-2(p+1)}(T - t_j) \geq c^{-1} \] for each \( j \).

Additionally, statement (4.3) of the Lifespan Theorem tells us that
\[ \sup_{x \in \mathbb{R}^3} \int_{f^{-1}(B_1(x))} |A|^2 \, d\mu \leq \epsilon_1 \] for \( 0 < \tau \leq c^{-1} \).
Or equivalently, by (7.11),

\[ \kappa_j (1, \tau) \leq \varepsilon_0 \text{ for } 0 < \tau \leq c^{-1}. \]

We then apply the interior estimates from Theorem 7.1 on the cylinder \( B_1 (x) \times (t - 1, t] \) to conclude that

\[ \| \nabla (k) A \|_{\infty, f_j} \leq c (k) \text{ for } -r_j^{-2(p+1)} t_j + 1 \leq t \leq c^{-1}, \tag{7.12} \]

so that the second condition of (7.8) from the hypothesis of Theorem 7.2 holds. For us to establish that the first condition holds as well, we just have to establish that \( \mu_j (\Sigma_j (R)) \leq c (R) \) for any \( R > 0 \). Now, by definition of \( \Sigma_j (R) \), we have

\[ \mu_j (\Sigma_j (R)) \leq | f_j^{-1} (B_R (0)) | \text{ for any } R > 0. \]

So it will be enough to prove that

\[ | f_j^{-1} (B_R (0)) | \leq c (R) \text{ for any } R > 0. \]

Next we state a lemma due to Simon [89] which will allow us to show that the first condition of (7.8) holds so that we may apply Lemma 7.2.

**Lemma 7.3** (Simon Monotonicity Formula [89], Equation (1.3)). Suppose \( f : \Sigma \to \mathbb{R}^3 \) is an immersed surface. Then for \( 0 < \sigma \leq \rho < \infty \) we have

\[ \frac{| \Sigma_\sigma |}{\sigma^2} \leq c \left( \frac{| \Sigma_\rho |}{\rho^2} + \int_{\Sigma_\rho} H^2 \, d\mu \right) \]

for some universal constant \( c > 0 \). Here \( c \) depends only on the dimension of the ambient space.
As noted in [57], Lemma 4.1, if $\Sigma$ is compact with no boundary then
\[
\frac{|\Sigma_{\sigma}|}{\sigma^2} \leq c \left( \int_{\Sigma} |A_{\sigma}|^2 \, d\mu + 4\pi \chi_{\Sigma} \right).
\]
Taking into account the monotonicity of the umbilic energy under the flow from Theorem 6.1, we obtain
\[
\frac{|\Sigma_{\sigma}|}{\sigma^2} \leq c (\varepsilon_0 + 4\pi \chi_{\Sigma})
\]
Thus
\[
\mu_j (\Sigma_j (R)) \leq c (R) \text{ for any } R > 0,
\]
and our sequence of immersions $f_j = f_j (\cdot, 0) : \Sigma^2 \to \mathbb{R}^3$ satisfy all of the conditions of Theorem 7.2. We conclude that there exists a limiting immersion $\tilde{f}_0 : \tilde{\Sigma} \to \mathbb{R}^3$. Letting $\phi_j : \tilde{\Sigma} (j) \to U_j \subset \Sigma$ be a series of functions as in (7.9), the reparameterisations
\[
f_j (\phi_j, \cdot) : \tilde{\Sigma} (j) \times [0, c^{-1}] \to \mathbb{R}^3
\]
are geometric polyharmonic heat flows. Also, by (7.9), our reparameterised flows \( \{f_j (\phi_j, \cdot)\} \) have initial data
\[
f_j (\phi_j, 0) = \tilde{f}_0 + u_j : \tilde{\Sigma} (j) \to \mathbb{R}^3,
\]
and satisfy the pointwise curvature bounds (7.12). The flows $f_j$ have initial data converging locally to the immersion $\tilde{f}_0 : \Sigma \to \mathbb{R}^3$. By converting the covariant derivatives of curvature into partial derivatives of the immersion functions $f_j$ as in Claim 4.9, we conclude from (7.12) that $f_j (\phi_j, \cdot) \to \tilde{f}$ locally in $C^k$ for any $k \in \mathbb{N}_0$, where $\tilde{f} : \tilde{\Sigma} \times [0, c^{-1}] \to \mathbb{R}^3$ is a geometric polyharmonic heat flow with initial data $f_0$.

We now prove three key properties of the blowup for geometric polyharmonic heat flows which hold under the assumption that
\[
\int_{\Sigma} |A_{\sigma}|^2 \, d\mu \bigg|_{t=0} \leq \varepsilon_0,
\]
(7.13)
for some sufficiently small constant $\varepsilon_0 > 0$. We often refer to (7.13) as the ‘small initial energy condition’.

These properties will be necessary to prove long time existence for the geometric polyharmonic heat flows and are listed as follows:

1. (Theorem 7.4) The blowup is stationary.

2. (Lemma 7.5) The blowup is not a union of planes.

3. (Lemma 7.6) If the blowup (denoted by $\tilde{\Sigma}$) contains a compact component $C$, then $\tilde{\Sigma} = C$.

We proceed by proving Property 1.

**Theorem 7.4.** Let $f : \Sigma^2 \times [0, T) \rightarrow \mathbb{R}^3$ be a geometric polyharmonic heat flow satisfying the small initial energy condition (7.13). Then the blowup $\tilde{f}$ constructed above is stationary. That is, $\Delta^p H(\tilde{f}) \equiv 0$ on $f(\tilde{\Sigma})$.

**Proof.** By Theorem 6.1, the Gauss-Bonnet Theorem, and the fact that each of the functions $\phi_j$ is a diffeomorphism, we have for $t \in [0, c^{-1}]$:

$$
\frac{d}{dt} \int_{\Sigma} |A_j|^2 \, d\mu_j \bigg|_{t=\tau} = \frac{1}{2} \frac{d}{dt} \int_{\Sigma} |H_j|^2 \, d\mu_j \bigg|_{t=\tau} = -\frac{1}{2} \int_{U_j} |\Delta^{p+1} H_j|^2 \, d\mu_j \bigg|_{t=\tau}.
$$

Here we have used the notation of Proposition A.13. Therefore

$$
\int_{U_j} |\Delta^{p+1} H_j|^2 \, d\mu_j \bigg|_{t=\tau} \leq -\frac{d}{dt} \int_{\Sigma} \left( |A_j^o|^2 + \frac{1}{2} |H_j|^2 \right) \, d\mu_j \bigg|_{t=\tau} = -\frac{d}{dt} \int_{\Sigma} |A_j|^2 \, d\mu_j \bigg|_{t=\tau}.
$$

(7.14)

By scale invariance it follows that

$$
\int_{\Sigma} |A_j|^2 \, d\mu_j \bigg|_{t=0} = \int_{\Sigma} |A|^2 \, d\mu \bigg|_{t=c^{-1}}, \quad \text{and} \quad \int_{\Sigma} |A_j|^2 \, d\mu_j \bigg|_{t=c^{-1}} = \int_{\Sigma} |A|^2 \, d\mu \bigg|_{t=t_j + r^2(r+1)c^{-1}}.
$$

Integrating identity (7.14) over the time interval $[0, c^{-1}]$ and using the fundamental
THEOREM OF CALCULUS THEN YIELDS
\[
\int_0^{c^{-1}} \int_{\tilde{\Sigma}(j)} |\Delta^{\frac{p+1}{2}} H(f_j(\phi_j, \tau))|^2 d\mu_{f_j(\phi_j, \cdot)} d\tau = \int_0^{c^{-1}} \int_{U_j} |\Delta^{\frac{p+1}{2}} H_j|^2 d\mu_j |_{t=\tau} d\tau \\
\leq \int_\Sigma |A_j|^2 d\mu_j |_{t=0} - \int_\Sigma |A_j|^2 d\mu_j |_{t=c^{-1}} \\
= \int_\Sigma |A|^2 d\mu |_{t=t_j} - \int_\Sigma |A|^2 d\mu |_{t=t_j+r_j^{2(p+1)c^{-1}}},
\]
so that taking \( j \to \infty \) implies
\[
\lim_{j \to \infty} \int_0^{c^{-1}} \int_{\tilde{\Sigma}(j)} |\Delta^{\frac{p+1}{2}} H(f_j(\phi_j, \tau))|^2 d\mu_{f_j(\phi_j, \cdot)} d\tau \leq 0,
\]
because \( \{r_j\} \searrow 0 \) and \( c > 0 \) is fixed. Hence as \( f_j(\phi_j, \cdot) \to \tilde{f} \) smoothly, we conclude that \( |\Delta^{\frac{p+1}{2}} H(\tilde{f})| \equiv 0 \). This tells us immediately that \( \Delta^p H(\tilde{f}) \equiv 0 \), which is what we wanted to prove.

Next we prove Property 2.

Lemma 7.5. Let \( f: \Sigma^2 \times [0, T) \to \mathbb{R}^3 \) be a geometric polyharmonic heat flow satisfying the small initial energy condition (7.13). Then the blowup \( \tilde{f} \) constructed above is not a union of planes.

Proof. By identity (7.10) and construction of \( \tilde{f} \), we have the inequality
\[
\int_{f^{-1}(B_1(0))} |A|^2 d\mu \geq \varepsilon_1 > 0,
\]
which follows from considering limiting values of the sequence \( f_j \) of immersions. This tells us that \( \tilde{f}(\tilde{\Sigma}) \) has a component that is not planar (otherwise it would have constant zero curvature).

Finally we prove Property 3.

Lemma 7.6. Let \( \tilde{f}: \tilde{\Sigma} \to \mathbb{R}^3 \) be the blowup constructed above. If \( \tilde{\Sigma} \) contains a compact component \( C \), then \( \tilde{\Sigma} = C \) and \( \Sigma \) is diffeomorphic to \( C \).
Proof. Let $C$ be the aforementioned compact component of $\tilde{\Sigma}$. Then, since $C$ is compact, for $j$ large enough, $\tilde{\Sigma}(j)$ contains $C$. Therefore since each $\phi_j$ is a diffeomorphism, for sufficiently large $j$, $\phi_j(C)$ is both closed and open (clopen) in $\Sigma$. Since $\Sigma$ is connected, its only clopen subsets are $\Sigma$ and $\emptyset$. We conclude that $\Sigma = \phi_j(C)$ because $C$ is not empty. Hence $\Sigma$ is diffeomorphic to $C$. Because each of the $\phi_j : C \to \Sigma$ functions are bijective and $\Sigma$ is closed, we may take a limit in $j$ in

$$C = \phi_j^{-1}(\Sigma)$$

and conclude that

$$C = \lim_{j \to \infty} \phi_j^{-1}(\Sigma) = \lim_{j \to \infty} \tilde{\Sigma}(j) = \tilde{\Sigma}. $$

This concludes our proof. \qed

We combine the three preceding results in the following theorem.

**Theorem 7.7.** Let $f : \Sigma^2 \times [0,T) \to \mathbb{R}^3$ be a geometric polyharmonic heat flow satisfying the small initial energy condition (7.13). Let $\tilde{f}$ be the blowup constructed above. Then none of the components of $\tilde{f}(\tilde{\Sigma})$ are compact, and the blowup has a component which is non-umbilic and stationary.

**Proof.** We claim that the enclosed surface area $\mu(\Sigma)$ is uniformly bounded away from zero. To prove this, note that by Theorem 6.1 each $f_t$ is an embedding for $t \in [0,T)$. Also for $t \in [0,T)$, Lemma 1.2 tells us that the enclosed volume, $\text{Vol}(\Sigma_t)$, does not change. We combine this with the isoperimetric inequality for $\mathbb{R}^3$ (see, for example, [80]) to conclude that for $t \in [0,T)$,

$$\mu(\Sigma_t) \geq \sqrt{36\pi \text{Vol}(\Sigma_0)} > 0. \quad (7.15)$$

Next assume (for the sake of contradiction) that $\tilde{f}(\tilde{\Sigma})$ has a compact component, say $D$. The properness of $\tilde{f}$ implies that $\tilde{f}^{-1}(D) \subseteq \tilde{\Sigma}$ is also compact. Lemma 7.5 then tells us that we must have $\tilde{f}^{-1}(D) = \tilde{\Sigma}$, so that $D = \tilde{f}(\tilde{\Sigma})$. That is to say, $\tilde{f}(\tilde{\Sigma})$
consists of a single compact component, which implies that the area $\tilde{f}(\tilde{\Sigma})$ is bounded. That is,

$$\lim_{j \to \infty} \mu_j(\Sigma) < \infty.$$ 

We next use the definition of the sequence of immersions $\{f_j\}$ to compute the area of the blowup. Firstly, a quick computation gives

$$\partial_k f_j \bigg|_{t=0} = \frac{1}{r_j} \partial_k f \bigg|_{t=t_j},$$

so that

$$g \bigg|_{t=t_j} = r_j^2 g_j \bigg|_{t=0}$$

where $g_j$ denotes the metric induced by the immersion $f_j$. The area of the blowup can then be calculated from the formula

$$\mu(\Sigma) \bigg|_{t=t_j} = \mu \bigg|_{t=t_j} = \sqrt{\det g} dp \bigg|_{t=t_j} = r_j^2 \sqrt{\det g_j} dp \bigg|_{t=0} = r_j^2 d\mu_j \bigg|_{t=0} = r_j^2 \mu_j(\Sigma) \bigg|_{t=0}.$$ 

Thus as $\{r_j\} \searrow 0$ by construction, this area must tend to zero:

$$\mu(\Sigma) \bigg|_{t=T} = \lim_{j \to \infty} \mu(\Sigma) \bigg|_{t=t_j} = \lim_{j \to \infty} r_j^2 \mu_j(\Sigma) \bigg|_{t=0} = 0.$$ 

This of course contradicts (7.15). Thus the assumption of $\tilde{f}(\tilde{\Sigma})$ having a compact component must have been false, and we conclude that $\tilde{f}(\tilde{\Sigma})$ has no compact components. Lemma 7.5 then tells us that there must be a component of $\tilde{f}(\tilde{\Sigma})$ with nonzero curvature. Hence the component identified above is non-compact and non-umbilic, which is
what we wished to show.

We are now in a position to prove long time existence of the flow under the small initial umbilic energy condition. We will see in the next chapter that this turns out to be the first part of the proof of Theorem 8.1, the main theorem for Part I of this thesis.

**Proposition 7.8** (Global existence of the flow). Suppose $f : \Sigma^2 \times [0, T) \to \mathbb{R}^3$ is a geometric polyharmonic heat flow. Then there exists an $\varepsilon_0 > 0$ depending only on $p$ such that if

$$
\int_{\Sigma} |A_o|^2 \, d\mu \bigg|_{t=0} \leq \varepsilon_0
$$

then $T = \infty$.

**Proof.** Suppose not i.e. suppose that for any $\varepsilon_0 > 0$ there exists a geometric polyharmonic flow $f : \Sigma^2 \times [0, T) \to \mathbb{R}^3$ satisfying (7.16) but with $T < \infty$. Since $T < \infty$, by Remark 4.4 we have that if we define $\rho(t)$ by

$$
\rho(t) = \sup \left\{ r > 0 : \int_{f^{-1}(B_r(x))} |A|^2 \, d\mu \bigg|_t \leq \varepsilon_0 \text{ for all } x \in \mathbb{R}^3 \right\},
$$

then

$$
\lim_{t \uparrow T} \int_{f^{-1}(B_{\rho(t)}(x(t)))} |A|^2 \, d\mu \geq \varepsilon_0,
$$

where $x(t)$ is taken to be the centre of the ball where the integral is maximised at time $t$. However, if $\varepsilon_0$ is chosen sufficiently small, then by Theorem 6.1 we know that the small initial trace-free assumption guarantees that $A^o$ remains small in $L^2$ on the time interval $[0, T)$. We can then construct a blowup $\tilde{f}$ as above, and use the scale-invariance of $\int_{\Sigma} |A^o|^2 \, d\mu$ to guarantee that the trace-free curvature remains small in $L^2$ right up to time $T$. We showed in Theorem 7.4 that the blowup $\tilde{f}$ is stationary, i.e., $\Delta^p H(\tilde{f}) \equiv 0$. Hence our blowup satisfies the hypothesis of Theorem 5.4, and we may conclude that $\tilde{f}(\Sigma)$ is either a plane or a sphere. But Theorem 7.7 tells us that $\tilde{f}(\Sigma)$ is non-compact with a non-umbilic component, and so we have reached a contradiction.
We conclude that $T = \infty$. \hfill \Box
Chapter 8

Smooth exponential convergence to spheres

We are almost ready to prove the main theorem for Part I of this thesis: global existence and exponential convergence to round spheres. Recall that by Proposition 7.8 from the previous chapter we already know that given an initial immersion $f_0$ with small umbilic energy, the geometric polyharmonic heat flow exists for all time, so we are already halfway there. We first present the main theorem here and then provide some supporting lemmas before proving the theorem at the end of the chapter.

**Theorem 8.1.** Suppose $f : \Sigma \times [0, T) \rightarrow \mathbb{R}^3$ is a geometric polyharmonic heat flow with smooth initial data. Then there exists an absolute constant $\varepsilon_0 > 0$ depending on $p$ such that if

$$\left| \widetilde{W}(f) \right|_{t=0} = \int_{\Sigma} |A_o|^2 \, d\mu \bigg|_{t=0} \leq \varepsilon_0$$

then the flow exists for all time as a one-parameter family of embeddings, and for some $x \in \mathbb{R}^3$, the family $\Sigma_t = f(\Sigma, t)$ converges exponentially fast to $S^2_{\sqrt{3 \frac{\text{Vol}(\Sigma_0)}{4\pi}}}$ in the $C^\infty$ topology. Here $\text{Vol}(\Sigma_0)$ is the signed enclosed volume of the initial immersion.

Before proving the theorem, we will include some supporting statements.

**Lemma 8.2.** Let $f : \Sigma^2 \times [0, T) \rightarrow \mathbb{R}^3$ be a geometric polyharmonic heat flow satisfying
the initial small umbilic energy condition (8.1). Then for any sequence of times \( t_j \uparrow \infty \) we may choose \( x_j \in \mathbb{R}^3 \) and \( \phi_j \in \text{Diff}(\Sigma, \mathbb{R}^3) \) (the space of diffeomorphisms from \( \Sigma \) to \( \mathbb{R}^3 \)) such that after passing to a subsequence, the immersions \( f(\phi_j, t) - x_j \) converge smoothly to an embedded round sphere.

**Proof.** We infer from Proposition 7.8 that under the initially small trace-free curvature assumption we have \( T = \infty \). Thus we pick an arbitrary time sequence \( \{t_j\} \) with \( t_j \uparrow \infty \). For an arbitrary \( p \in \Sigma \) we let \( x_j = f(p, t_j) \). By Theorem 7.1, we conclude that for every \( t_j \) we have

\[
\left\| \nabla (k) A \right\|_\infty \bigg|_{t=t_j} \leq c(k).
\]

By Lemma 1.2,

\[
d\mu(\Sigma) = - \int_\Sigma |\Delta^{\frac{4}{k+1}} H|^2 d\mu \leq 0,
\]

and so as \( \mu(\Sigma_0) \) is bounded we conclude that

\[
\mu(\Sigma) \bigg|_{t=t_j} \leq \mu(\Sigma_0) < \infty.
\]

In particular, if we consider the sequence of immersions \( f_j : \Sigma \rightarrow \mathbb{R}^3 \) given by \( f_j(p, t) = f(p, t_j) - x_j \) then Theorem 7.2 guarantees the existence of a proper immersion \( \tilde{f} : \tilde{\Sigma} \rightarrow \mathbb{R}^3 \) (where, of course, \( \tilde{\Sigma} \) is a surface without boundary) and a sequence \( \phi_j \in \text{Diff}(\Sigma, \mathbb{R}^3) \) such that

\[
f_j(\phi_j, t) = f(\phi_j, t_j) - x_j \rightarrow \tilde{f} \text{ as } j \uparrow \infty.
\]

(8.2)

Here the convergence is locally smooth. We now wish to construct an extension to this flow that has initial data equal to that of the blowup \( \tilde{f} \) whilst being a geometric polyharmonic heat flow. We define a new sequence of geometric polyharmonic heat flows \( h_j : \tilde{\Sigma}(j) \times [-t_j, \infty) \rightarrow \mathbb{R}^3 \) defined by

\[
h_j(p, t) = f(\phi_j(p), t_j + t) - x_j.
\]
Then each $h_j$ also satisfies the interior estimates and the bounded area hypothesis of Theorem 7.1. By (8.2) we conclude that

$$h_j (p, 0) = f (\phi_j (p), t_j) - x_j \to \tilde{f} \text{ as } j \nearrow \infty.$$ 

That is to say, at initial time the sequence $h_j$ converges locally in $C^\infty$ to the ‘blowup’ $\tilde{f}$ (remembering that we have now established that the ‘blowup’ is not actually a blowup but rather a limit immersion). Following the same line of argument as in the proof of Theorem 7.4 we conclude that

$$\int_{t_j}^{t_{j+1}} \int_{\Sigma} |\Delta \frac{t_{j+1}-t_j}{2} H|^2 d\mu d\tau \leq \int_{\Sigma} |A^o|^2 d\mu \bigg|_{t=t_j} - \int_{\Sigma} |A^o|^2 d\mu \bigg|_{t=t_{j+1}} \searrow 0 \text{ as } j \nearrow \infty,$$

and hence that $\tilde{f}(\tilde{\Sigma})$ is stationary (i.e., $\Delta^p H(\tilde{f}) \equiv 0$) because $t_j \nearrow \infty$ was arbitrary. We infer from the Gap Lemma (Theorem 5.4) that $\tilde{f}(\tilde{\Sigma})$ must be a union of planes and spheres. However by Lemma 7.6 we infer that $\tilde{\Sigma}$ (and hence $\tilde{f}(\Sigma)$, because $\tilde{f}$ is continuous) only has one connected component. So the only possibility is that $\tilde{f}(\tilde{\Sigma})$ is an embedded plane or sphere. Since here $\tilde{f}$ is compact, we conclude that it is an embedded sphere. This finishes the proof.

Lemma 8.2 allows us to conclude that our geometric polyharmonic heat flow with small initial umbilic energy does, in fact, converge to a round, embedded sphere (at least subsequentially modulo translation). All that remains is to prove the rate of convergence is exponentially fast. The term ‘rate of convergence’ here refers to the rate at which the pointwise norms of the trace-free curvature and covariant derivatives of curvature tend to zero. This definition is justified because the round sphere is the only complete compact surface in $\mathbb{R}^3$ with zero umbilic energy $\tilde{W}(f)$.

**Proposition 8.3 (Exponential Convergence).** Suppose $f : \Sigma^2 \times [0, T) \to \mathbb{R}^3$ is a geometric polyharmonic heat flow satisfying the initial small energy condition (8.1). Then there exist constants $c_0, c_k, \xi_0, \xi_k > 0$ such that for sufficiently large times the
following estimates hold:

\[ \| A^o \|_\infty \leq c_0 e^{-\xi_0 t} \quad \text{and} \quad \| \nabla (k) A \|_\infty \leq c_k e^{-\xi_k t} \quad (k \in \mathbb{N}). \]

Proof. Recall that by Theorem 6.1, there exist constants \( \tilde{c}_0, \tilde{\xi}_0 > 0 \) such that for \( t \in [0, T) \) the following estimate holds:

\[ \int_{\Sigma} |A^o|^2 \, d\mu \leq \tilde{c}_0 e^{-\tilde{\xi}_0 t}. \quad (8.3) \]

Next, by Proposition 3.3 as well as the interpolation inequality from Lemma A.10, we have

\[
\frac{d}{dt} \int_{\Sigma} |\nabla (k) A|^2 \, d\mu + 2 \int_{\Sigma} |\nabla (k+p+1) A|^2 \, d\mu \\
= \sum_{i=0}^{2p+k} \int_{\Sigma} \nabla (i) \left( \nabla (2p+k-i) A \ast R \right) \ast \nabla (k) A \, d\mu \\
= \sum_{i=0}^{2p+k} \int_{\Sigma} \nabla (i) \left( \nabla (2p+k-i) A \ast (H^2 + A^o \ast A^o) \right) \ast \nabla (k) A \, d\mu \\
\leq \int_{\Sigma} \left| \mathcal{P}_{2(k+p)} (A^o) \right| \, d\mu + c \int_{\Sigma} |\nabla (k+p) A|^2 H^2 \, d\mu \\
\leq c \left( \| A^o \|_\infty^2 + \| H \|_\infty^2 \right) \int_{\Sigma} |\nabla (k+p) A|^2 \, d\mu \\
\leq c \left( \| A^o \|_\infty^2 + \| H \|_\infty^2 \right) \| A^o \|_2^{\frac{2p+k}{2p+1}} \left( \int_{\Sigma} |\nabla (k+p+1) A|^2 \, d\mu \right)^{\frac{k+p}{k+p+1}} \\
\leq \eta \int_{\Sigma} |\nabla (k+p+1) A|^2 \, d\mu + c \left( \eta^{-1} \right) \| A \|^2_{\infty} \| A^o \|^2_{2, [\gamma > 0]} \quad (8.4)
\]

Here we have used the interpolation inequality of Lemma A.10 as well as the interpolation inequality of Proposition A.13. Next, by Theorem 8.2 we know that (along a subsequence of times) we have

\[ \| A \|_\infty^2 \rightarrow \frac{8\pi}{|\Sigma_\infty|} = \sqrt{\frac{128\pi^2}{9 \text{Vol} (\Sigma_0)^2}} \text{ as } t \rightarrow \infty, \]

where \( \Sigma_\infty \) stands for the limit sphere and \( |\Sigma_\infty| \) its enclosed area. In particular, there
exist finite \( \tilde{c}, \tilde{t} \) such that for times \( t \geq \tilde{t} \), \( \|A\|_\infty^2 \leq \tilde{c} \). By (8.4), we conclude that for \( t \geq \tilde{t} \) the following inequality holds:

\[
\frac{d}{dt} \|\nabla^{(k)} A\|_2^2 + \|\nabla^{(k+p+1)} A\|_2^2 \leq C \left( k, p, \varepsilon_0 \right) e^{-\tilde{\xi}t}.
\]

Therefore by interpolation, for every \( k \in \mathbb{N} \), there exist constants \( \tilde{C}_k, C_k \) such that for \( t \geq \tilde{t} \) we have

\[
\int_{\tilde{t}}^{\infty} \|\nabla^{(k)} A\|_2^2 d\tau \leq \tilde{C}_k \quad \text{and} \quad \|\nabla^{(k)} A\|_2^2 \leq C_k.
\]

(8.5)

Since we already know that \( \|A^o\|_2^2 \) is decaying exponentially fast in time (from (8.3)), interpolating\(^1\) then allows us to obtain exponential decay for every derivative of \( A \) for times larger than \( \tilde{t} \). To be more precise, for every \( k \in \mathbb{N} \) there exists universal constants \( \tilde{c}_k, \tilde{c}_k \) such that for times larger than \( \tilde{t} \),

\[
\|\nabla^{(k)} A\|_2^2 \leq \tilde{c}_k e^{-\tilde{\xi}_k t}.
\]

(8.6)

Next, we claim that under the assumption (8.1), for every \( k \in \mathbb{N}_0 \) there exists a universal constant \( c > 0 \) such that

\[
\|\nabla^{(k)} A^o\|_\infty^2 \leq c \|A^o\|_2^{\frac{2}{k+2}} \|\nabla^{(k+2)} A^o\|_2^{\frac{k+1}{2}}.
\]

(8.7)

To prove (8.6) we first employ the multiplicative Sobolev inequality from Theorem A.6 with \( u = |\nabla^{(k)} A^o|\):

\[
\|\nabla^{(k)} A^o\|_\infty^6 \leq c \|\nabla^{(k)} A^o\|_2^2 \left( \|\nabla^{(k+1)} A^o\|_4^4 + \|\nabla^{(k)} A^o\| H_{\frac{2}{3}}^4 \right).
\]

(8.8)

\(^{1}\)To see how this works for \( j = 1 \), note that using (8.5) and the interpolation inequality from Lemma A.10 gives

\[
\int_{\Sigma} |\nabla^2 A|^2 d\mu \leq 3 \int_{\Sigma} |\nabla A^o|^2 d\mu \leq c \left( \int_{\Sigma} |A^o|^2 d\mu \int_{\Sigma} |\nabla (2) A|^2 d\mu \right)^{\frac{1}{2}} \leq c \sqrt{\varepsilon_0 C_2} e^{-\frac{1}{2} \xi_1 t}.
\]

Here the last step follows because \( \|\nabla^{(2)} A\|_2^2 \) is uniformly bounded, while \( \|A^o\|_2^2 \) decays exponentially. Choosing \( \tilde{c}_1 = c \sqrt{\varepsilon_0 C_2}, \tilde{\xi}_1 = \xi_0/2 \) then gives the result for \( j = 1 \). Continuing this process recursively then gives the result for \( j \) equal to any natural number.
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Then using Theorem A.2 with \( u = |\nabla^{(k+1)} A^o|^2 \) as well as (A.49) gives

\[
\int_\Sigma |\nabla^{(k+1)} A^o|^4 \, d\mu \\
\leq c \left( \int_\Sigma |\nabla^{(k+2)} A^o| \, d\mu + \int_\Sigma |\nabla^{(k+1)} A^o|^2 \, d\mu \right) \\
\leq c \int_\Sigma |\nabla^{(k+1)} A^o|^2 \, d\mu \left( \int_\Sigma |\nabla^{(k+2)} A^o|^2 \, d\mu + \int_\Sigma |\nabla^{(k+1)} A^o|^2 \, d\mu \right) \\
\leq c \int_\Sigma |\nabla^{(k+1)} A^o|^2 \, d\mu \int_\Sigma |\nabla^{(k+2)} A^o|^2 \, d\mu.
\tag{8.9}
\]

Similarly, using Theorem A.2 with \( u = |\nabla^{(k)} A^o|^2 H^2 \) and (A.49),

\[
\int_\Sigma |\nabla^{(k)} A^o|^4 H^2 \, d\mu \\
\leq c \left( \int_\Sigma |\nabla^{(k+1)} A^o| \, d\mu + \int_\Sigma |\nabla^{(k)} A^o|^2 \, d\mu \right) \int_\Sigma |\nabla^{(k)} A^o|^2 \, d\mu \\
\leq c \int_\Sigma |\nabla^{(k+1)} A^o|^2 \, d\mu \left( \int_\Sigma |\nabla^{(k+2)} A^o|^2 \, d\mu + \int_\Sigma |\nabla^{(k)} A^o|^2 \, d\mu \right) \int_\Sigma |\nabla^{(k)} A^o|^2 \, d\mu \\
\leq c \int_\Sigma |\nabla^{(k+1)} A^o|^2 \, d\mu \int_\Sigma |\nabla^{(k+2)} A^o|^2 \, d\mu. \tag{8.10}
\]

Here to get to the last line we have used the inequality

\[
\int_\Sigma |\nabla^{(k)} A^o|^2 |\nabla H|^2 \, d\mu \leq c \int_\Sigma |\nabla^{(k)} A^o|^2 |\nabla A^o|^2 \, d\mu \\
\leq \int_\Sigma P_4^{2(k+1)} (A^o) \, d\mu \\
\leq c \| A^o \|_\infty \int_\Sigma |\nabla^{(k+1)} A^o|^2 \, d\mu \\
\leq c \left( \| A^o \|_{L_2}^{\frac{5}{2}} \left( \int_\Sigma |\nabla^{(2)} A^o|^2 \, d\mu \right)^{\frac{1}{2}} \right) \int_\Sigma |\nabla^{(k+1)} A^o|^2 \, d\mu \\
\leq c \| A^o \|_{L_2}^{\frac{13k+6}{12k+5}} \left( \int_\Sigma |\nabla^{(k+2)} A^o|^2 \, d\mu \right),
\]
which follows from Lemma A.12, (5.32), and the interpolation inequalities from Lemma A.10. Substituting (8.9) and (8.10) into (8.8) and employing the interpolation inequalities from Lemma A.10 then gives (8.7). This inequality, together with (8.6) and (1.17), then proves exponential decay in $L^\infty$ for all derivatives of curvature, as well as $A^\circ$. □

We are finally ready to prove the main theorem for Part I, Theorem 8.1.

Proof of Theorem 8.1. Firstly by Proposition 7.8 we know that under the small energy assumption (8.1) we have $T = \infty$. Next, we conclude from Lemma 1.2 and Proposition 8.3 that

$$|\partial_t g| \leq 2 |\Delta^p H| |A| \leq C e^{-\xi t},$$

for some constants $C, \xi > 0$. Hence by Lemma 4.6 we conclude that the time-dependent metrics $g_{ij}(t)$ converge uniformly to a positive definite metric $g_{ij}(\infty)$ as $t \to \infty$. Moreover, from Proposition 8.3 we have $\|\nabla_{(k)} A\|_\infty \leq c_k e^{-\xi kt} (k \geq 1), |A|$ bounded, and therefore the metrics converge in the $C^\infty$ topology and $g_{ij}(\infty)$ is smooth. Finally, again by Proposition 8.3, $\|A^\circ\|_\infty \leq c_0 e^{-\xi_0 t}$ and so $g_{ij}(\infty)$ is the metric of a sphere (see, for example Theorem A.1). Since the volume of the evolving surfaces remains stationary in time, we deduce the radius of this sphere is $\frac{S^2}{\sqrt{3\text{Vol}(\Sigma_0)/4\pi}}$.

Unfortunately, this procedure does not identify the centre of the sphere, and in principle there is nothing to say that the centre of this sphere is fixed (solutions could “drift off”- or even something more exotic- while still becoming round). Since the surface is converging uniformly to a sphere (which has constant positive Gauss curvature), and the mean and Gauss curvature functions are sufficiently smooth, we know that for sufficiently large times $t$, $f(t, \cdot)$ is convex and can be written as a radial graph over some interior point $p(t)$. Moreover, by Proposition 8.3 we have that

$$|f(t_2, x) - f(t_1, x)| \leq \int_{t_1}^{t_2} \|\nabla_{(2p)} A(s, x)\|_\infty ds \leq C(e^{-\xi t_1} - e^{-\xi t_2}) \quad \text{for } t_1 \leq t_2$$
for some constants $C, \xi > 0$. Taking $t_2 \nearrow \infty$ we obtain

$$|f_\infty(x) - f(t, x)| \leq Ce^{-\xi t},$$

where $f_\infty$ is the embedding of the limit sphere. This shows that the solution does not “drift off” as $t$ gets large, but rather that for large enough times, say $t \geq t^*$, $f(t, \cdot)$ can in fact be written as a radial graph over a single point $p^* \equiv p(t)$ that is contained in the interior set $\bigcap_{t \in [t^*, \infty)} \text{Int}(f(\cdot, t))$. Note that the point $p^*$ is not necessarily equal to the centre of the limit sphere. We linearise the geometric polyharmonic heat flow equation around this radial graph and analyse its eigenvalues.

We follow the same steps as Simonett in his study of the surface diffusion flow for immersed hypersurfaces [29], using some of the calculations that McCoy derives in his study of the mixed volume mean curvature flow [76]. For a fixed $t$ sufficiently large, we write our surface as a radial graph over the unit $2$-sphere. That is,

$$f(z, t) = \rho(z, t) z,$$

where $z \in S^2$ and $\rho$ is the “height” of our radial graph. For the remainder of the section, $\nabla$ will represent the covariant derivative on $S^2$. We compute the components of the induced metric $g$ for our immersion to be

$$g_{ij} = (\partial_i f, \partial_j f)$$

$$= (\rho \nabla_i z + z \nabla_i \rho, \rho \nabla_j z + z \nabla_j \rho)$$

$$= \rho^2 \sigma_{ij} + \nabla_i \rho \nabla_j \rho,$$

where $\sigma$ is the metric on $S^2$. We have used the identity $z \perp \nabla_i z$ and the fact that $|z|^2$ is constant. From [76] we have

$$A_{ij} = -\Phi(\rho)^{-\frac{3}{2}} (\rho \nabla_i \rho - 2 \nabla_i \rho \nabla_j \rho - \sigma_{ij} \rho^2),$$
and

\[ g^{ij} = \rho^{-2} \left( \sigma^{ij} - \Phi (\rho)^{-1} \nabla^i \rho \nabla^j \rho \right), \]

and hence

\[ H (\rho) = -\rho^{-1} \Phi (\rho)^{-\frac{1}{2}} \Delta_s \rho + \rho^{-1} \Phi (\rho)^{-\frac{3}{2}} \nabla^i \rho \nabla^j \rho + 2 \Phi (\rho)^{-\frac{1}{2}} + \Phi (\rho)^{-\frac{3}{2}} |\nabla \rho|^2. \quad (8.11) \]

Here

\[ \Phi (\rho) := \rho^2 + |\nabla \rho|^2 \]

and \( \Delta_s \) denotes the Laplace-Beltrami on \( S^2 \). Next we set \( \rho_\varepsilon = \rho_\infty + \varepsilon \eta \), where

\[ \rho_\infty = \sqrt[3]{3 \text{Vol}(\Sigma_0)} / 4\pi. \]

We note that with this substitution, we have

\[ \nabla_i \rho_\varepsilon = \varepsilon \nabla_i \eta, \quad \text{and} \quad \nabla_i \Phi (\rho_\varepsilon) = 2 \varepsilon (\rho_\varepsilon \nabla_i \eta + \varepsilon \nabla_i |\nabla \eta|^2). \]

A calculation using (8.11) gives

\[
\Delta H (\rho_\varepsilon) = g^{pq} \nabla_p \nabla_q H (\rho_\varepsilon) \\
= -\rho_\varepsilon^{-2} \left( \rho_\varepsilon^{-1} \Phi (\rho_\varepsilon)^{-\frac{1}{2}} \Delta_s^2 \rho_\varepsilon + \Phi (\rho_\varepsilon)^{-\frac{3}{2}} \Delta_s \Phi (\rho_\varepsilon) \right) + \varepsilon^2 \mathcal{L}_\Delta H (\rho_\varepsilon, \eta, \varepsilon),
\]

where \( \mathcal{L}_\Delta H \) is an operator satisfying \( \frac{d}{d\varepsilon} \varepsilon^2 \mathcal{L}_\Delta H \bigg|_{\varepsilon=0} = 0 \). Hence

\[
\frac{d}{d\varepsilon} \Delta H (\rho_\varepsilon) \bigg|_{\varepsilon=0} = -\frac{d}{d\varepsilon} \left( \rho_\varepsilon^{-2} \left( \rho_\varepsilon^{-1} \Phi (\rho_\varepsilon)^{-\frac{1}{2}} \Delta_s^2 \rho_\varepsilon + \Phi (\rho_\varepsilon)^{-\frac{3}{2}} \Delta_s \Phi (\rho_\varepsilon) \right) \right) \bigg|_{\varepsilon=0} \\
= -\frac{d}{d\varepsilon} \left( \varepsilon \rho_\varepsilon^{-3} \Phi (\rho_\varepsilon)^{-\frac{1}{2}} \Delta_s^2 \eta \right) \bigg|_{\varepsilon=0} - \frac{d}{d\varepsilon} \left( \varepsilon \rho_\varepsilon^{-2} \Phi (\rho_\varepsilon)^{-\frac{1}{2}} \left( 2 \rho_\varepsilon \Delta_s \eta + \varepsilon \Delta_s |\nabla \eta|^2 \right) \right) \bigg|_{\varepsilon=0} \\
= -\rho_\varepsilon^{-3} \Phi (\rho_\varepsilon)^{-\frac{1}{2}} \Delta_s^2 \eta \bigg|_{\varepsilon=0} - 2 \rho_\varepsilon^{-1} \Phi (\rho_\varepsilon)^{-\frac{1}{2}} \Delta_s \eta \bigg|_{\varepsilon=0} \\
= -\rho_\infty^{-4} (\Delta_s^2 \eta + 2 \Delta_s \eta). \]
Hence the linearisation of the surface surface diffusion flow around the stationary sphere solution with radius $\rho_\infty$ is
\[
\frac{\partial \eta}{\partial t} = -\rho_\infty^{-4} \left( \Delta_s^2 \eta + 2 \Delta_s \eta \right).
\]

Continuing in a similar fashion for the higher derivatives of mean curvature, we get
\[
\Delta^p H(\rho_\varepsilon) = -\rho_\varepsilon^{-2p} \left( \rho_\varepsilon^{-1} \Phi(\rho_\varepsilon)^{-\frac{1}{2}} \Delta^{p+1}_s \rho_\varepsilon + \Phi(\rho_\varepsilon)^{-\frac{3}{2}} \Delta^p_s \Phi(\rho_\varepsilon) \right) + \varepsilon^2 \mathcal{L}_{\Delta^p H}(\rho_\varepsilon, \eta, \varepsilon),
\]
where, like before, $\frac{d}{d\varepsilon} \varepsilon^2 \mathcal{L}_{\Delta^p H} \bigg|_{\varepsilon=0} = 0$. Therefore
\[
\frac{d}{d\varepsilon} \Delta^p H(\rho_\varepsilon) \bigg|_{\varepsilon=0} = -\rho_\varepsilon^{-2p+1} \Phi(\rho_\varepsilon)^{-\frac{1}{2}} \Delta^{p+1}_s \eta - 2 \rho_\varepsilon^{-2p} \Phi(\rho_\varepsilon)^{-\frac{3}{2}} \Delta^p_s \eta \bigg|_{\varepsilon=0}
\]
\[
= -\rho_\infty^{-2(p+1)} \left( \Delta^{p+1}_s \eta + 2 \Delta^p_s \eta \right).
\]

It follows that the linearisation of the geometric polyharmonic heat flow (GPHF) around our stationary sphere solution is
\[
\frac{\partial \eta}{\partial t} = (-1)^p \rho_\infty^{-2(p+1)} \left( \Delta^{p+1}_s \eta + 2 \Delta^p_s \eta \right) =: \mathcal{L} \eta.
\]

It is well-known (see, for example, [88]) that the eigenvalues $\lambda_l$ of the Laplacian $\Delta_s$ on $S^2$ are
\[
\lambda_l = -l(l+1), \quad l \in \mathbb{N}_0.
\]

From this, one can show that
\[
\sigma(\Delta^p_s) = \{ \lambda_l^p : l \in \mathbb{N}_0 \} \quad \text{and} \quad \sigma(\Delta^{p+1}_s) = \{ \lambda_l^{p+1} : l \in \mathbb{N}_0 \},
\]
where $\Delta^q_s$ denotes the $q^{th}$ repeated iteration of the spherical Laplacian. Hence the
eigenvalues $\mu_l$ ($l \in \mathbb{N}_0$) of $\mathcal{L}$ are given by

\[
\mu_l = (-1)^p \rho^{-2(p+1)} \left( (-1)^{p+1} l^{p+1} (l + 1)^{p+1} + 2(-1)^p l^p (l + 1)^p \right) \\
= -\rho^{-2(p+1)} l^p (l + 1)^p (l - 1)(l + 2).
\]

Therefore all of the eigenvalues of $\mathcal{L}$ are non-positive, and all but $\mu_0$ and $\mu_1$ are strictly negative. The first two eigenvalues $\lambda_0, \lambda_1$ (in the cases $l = 0, 1$) correspond to the two zero eigenvalues of the operator $\mathcal{L}$. That is

\[
\mu_0 = \mu_1 = 0.
\]

By Theorem 22.1 from [88], we have that the multiplicity of each eigenvalue $\lambda_l$ of the $\Delta_s$ on $\mathbb{S}^2$ is equal to the dimension of the space of homogeneous, harmonic polynomials of degree $l$ in $\mathbb{R}^3$. For the case $l = 0$ this is simply the dimension of the space of scalars, which is 1. For the case $l = 1$ we are interested in the dimension of the space of homogenous, harmonic polynomials of degree 1, which is 3 [10]. Hence our operator $\mathcal{L}$ has a zero eigenvalue of algebraic multiplicity 4.

Henceforth the proof follows along the lines of the proof of Theorem 2 in [29]. We want to construct two projections $\pi^s, \pi^c$ so that we can quotient out the zero eigenvalues of $\mathcal{L}$. First of all we define

\[
N := \text{span} \{ H_k : k = 0, 1, 2 \}
\]

where $\{ H_k : k = 0, 1, 2 \}$ is a basis for spherical harmonic polynomials of degree 1. It is possible to show that $\ker \mathcal{L} = N$ (see, for example [91]). Next, for $h \in C^\infty(\mathbb{S}^2)$ we define the operator

\[
Ph = \sum_{k=0}^{2} \langle h, H_k \rangle H_k,
\]
where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{C^\infty(S^2)}$ is the regular inner product on $C^\infty(S^2)$ defined by

$$\langle f, g \rangle := \langle f, g \rangle_{C^\infty(S^2)} = \int_{S^2} fg \, d\sigma.$$ 

$P$ acts as a continuous projection of $C^\infty(S^2)$ onto $N$. In [91] it is shown that $P$ and $\mathcal{L}$ commute, that is, $[P, \mathcal{L}] = 0$, and that $\text{ker} P = N_\perp$. Therefore we may split $C^\infty(S^2)$ into two orthogonal subspaces, with

$$C^\infty(S^2) \cong N \oplus N_\perp = \text{ker} \mathcal{L} \oplus \text{ker} P.$$ 

We define

$$\pi^c := P, \quad \pi^s := \text{id} - P.$$ 

Then it is easy to see that $\pi^c$ and $\pi^s$ are projections of $C^\infty(S^2)$ into $N = \text{ker} \mathcal{L}$ (the centre subspace of $\mathcal{L}$), and $N_\perp = \text{ker} P$ (the stable subspace of $\mathcal{L}$) respectively. One readily checks that

$$\sigma(\pi^c \mathcal{L}) = \{0\}.$$ 

Similarly, we find that

$$\sigma(\pi^s) \subset (-\infty, \mu_1] \subset (-\infty, 0).$$ 

Effectively, our projections are chosen to separate $N = \text{ker} \mathcal{L}$ (the center subspace of $\mathcal{L}$) and $N_\perp = \text{ker} P$ (the stable subspace of $\mathcal{L}$).

By using the same method as in the proof of Theorem 2 in [29], we can then guarantee local exponentially attraction of solutions to the geometric polyharmonic heat flow in the following sense: if $\mathcal{S}$ is a fixed Euclidean sphere and $\mathcal{X}$ denotes the set of all spheres which are sufficiently close to $\mathcal{S}$, then $\mathcal{X}$ attracts all embedded solutions which are close to $\mathcal{X}$ (in the sense of the “little Hölder spaces”: see, for example [5],[29]) at an exponential rate. In particular, if $\Sigma_0$ is sufficiently close to $\mathcal{S}$ in the sense of the little Hölder spaces, then $\Sigma_t$ exists globally and converges exponentially fast to some
sphere in $\mathcal{X}$ enclosing the same volume as $\Sigma_0$. The radius of this sphere is given by

$$\sqrt[3]{\frac{3 \text{Vol}(\Sigma_0)}{4\pi}}.$$ 

This completes Part I of the thesis.
Part II

On the anisotropic polyharmonic heat flow of closed plane curves
Chapter 9

Introduction

There has been a great deal of mathematical research pertaining to the study of planar curve flows, in essence because they are easier to deal with than curvature flows in higher-dimensional Riemannian manifolds (such as our geometric polyharmonic heat flow in Part I), especially those in which the ambient manifold may involve some topology or curvature. Nonetheless, the simplicity involved in the analysis of curve flows allows for some very robust and interesting mathematical results.

Any fitting introduction to planar curve flows would perhaps be insincere without a mention of the curve-shortening flow. This famous flow deforms a planar curve in such a way that its normal velocity is everywhere equal to its Euclidean curvature. Hence it is the special case of the mean curvature flow (MCF) where the evolving manifold is one-dimensional. If we take a curve immersed in $\mathbb{R}^2$, say $\gamma : I \to \mathbb{R}^2$ (where $I$ is an open interval, possibly equal to $\mathbb{R}$), then the Euclidean curvature $k$ at a point $p \in \gamma (I)$ is defined to be the reciprocal of the radius of the osculating circle to $\gamma$ at $p$. Furthermore, if $\gamma$ is parameterised by Euclidean arc length (which we denote by $s$), then the Euclidean curvature and $\gamma$ are related by the following formula:

$$\frac{\partial^2 \gamma}{\partial s^2} (s) = k (s) \nu (s).$$
Here \( \nu \) is a chosen Euclidean unit normal to \( \gamma \) at the point \( \gamma(s) \in \mathbb{R}^2 \). Hence the curve-shortening flow takes the form

\[
\frac{\partial \gamma}{\partial t}(s,t) = k(s,t) \nu(s,t) = \frac{\partial^2 \gamma}{\partial s^2}(s,t),
\]

which bears a striking resemblance to the heat equation \( (H) \) with \( n = 1 \). Indeed the curve-shortening flow is often regarded as the ‘natural’ heat equation for curves, and can be written as a pair of second-order parabolic quasilinear equations (see Section 11.1). Note that although \( (CSF) \) at first glance appears to be a linear system of differential equations, the non-linearity holds because of the parameter \( s(u) = \int_0^u |\gamma_u| \, du \).

It is rather unsurprising to learn that the flow exhibits instant smoothing capabilities (initial curves with bounded curvature become immediately smooth), a property which is often associated with parabolic flows. The curve-shortening flow is also the path of steepest descent for the length functional \( L \) in \( L^2 \):

\[
L(\gamma) := \int_{\gamma} ds = \int_{S^1} |\gamma_u| \, du.
\]

It also decreases the enclosed areas of closed curves monotonically in time with velocity

\[ A' = -2\omega\pi, \]

where \( \omega \) is the turning number of the initial curve. In addition, the curve-shortening flow \( (CSF) \) shares some nice properties with the heat equation and mean curvature flow, which make it a particularly attractive geometric flow from an analytical point of view. Namely the scalar maximum principle, which guarantees that the following properties hold:

(a) *The Comparison Principle:* Two initially disjoint flows remain disjoint;

(b) *Preserved Convexity:* Initially convex closed curves remain convex, and;
(c) *Preserved Simplicity:* Simple curves do not develop self-intersections over time.

The flow possesses a number of special self-similar solutions (which can be classified into six basic types; see [41]). Perhaps the easiest to understand is the homothetically shrinking family of circles. If take \( \gamma_0 = \gamma(S^1, 0) = S_{R_0}^1 \subset \mathbb{R}^2 \) (a circle with radius \( R_0 \)) to be the initial curve and evolve it under \((\text{CSF})\), then it is easy to see that the solution \( \gamma \) is a one-parameter family of shrinking circles with radius \( R(t) \), where \( R \) satisfies

\[
R(t) = \sqrt{R_0^2 - 2t}.
\]

Hence the flow reduces the initial embedded circle to a single point at time \( R_0^2/2 \).
Therefore if we take any closed curve that is initially contained in a circle of radius \( R_0 \), the Comparison Principle tells us that the curve will become extinct in finite time under the curve-shortening flow.

To demonstrate this more clearly, assume that \( \gamma : S^1 \times [0, T) \rightarrow \mathbb{R}^2 \) is a closed curve that solves \((\text{CSF})\) and initially satisfies \( |\gamma(s, 0)|^2 \leq R_0^2 \) for every \( s \in S^1 \). Then a quick calculation gives

\[
\frac{\partial}{\partial t} |\gamma|^2 = 2k(\gamma, \nu) = \frac{\partial^2}{\partial s^2} |\gamma|^2 - 2 |\gamma_s|^2 = \Delta_\gamma |\gamma|^2 - 2,
\]

where \( \Delta_\gamma = \partial_{ss} \) denotes the Laplace-Beltrami operator for the curve \( \gamma \). The scalar maximum principle (see Theorem 3.2 from [87], for example) then implies that \( |\gamma(s, t)|^2 \leq \phi(t) \), where \( \phi \) is the solution to the ordinary differential equation

\[
\frac{d\phi}{dt} = -2 \\
\phi(0) = R_0^2
\]

That is, \( |\gamma(s, t)| \leq \sqrt{R_0^2 - 2t} \), which implies that \( \gamma \) remains bounded by a circle of radius \( R(t) \) and therefore must shrink to a point at least as quickly. An analogous
argument works in higher dimensions for closed hypersurfaces evolving by the mean curvature flow (MCF).

Using the preceding argument (along with some deep analysis), Gage, Grayson and Hamilton were able to fully describe the behaviour of smooth initially embedded planar curves under the curve shortening flow with a series of papers beginning in 1983. Indeed, in 1983 Gage [32] proved that under the flow the Euclidean isoperimetric ratio $L^2/4\pi A$ (see Chapter 10) decreases monotonically. This implies that the flow tends to make embedded curves more circular, because the embedded circle minimises the isoperimetric ratio. Moreover, Gage proved that if a curve shortening flow did not encounter a premature curvature singularity (before shrinking to a point), then the isoperimetric ratio tends to unity and the curve flows to a ‘round point’. One year later, Gage and Hamilton showed that for initially convex curves such a premature curvature singularity cannot occur, although the related paper did not appear in the literature until 1986 [31]. Any initially convex curve must therefore shrink to a ‘round point’, or a round circle if the flow is scaled to preserve length. Quite amazingly the paper appeared after Huisken’s analogous work on the mean curvature flow [47], which came out in 1984.

A year later Matthew Grayson extended this result to closed curves that are initially embedded (that is, those without self-intersections) [39]. The was essentially proved by showing that embedded curves remain embedded and eventually becomes convex, before subsequently shrinking to a ‘round point’ by the result of Gage and Hamilton above. In this thesis we prove a lower bound for the ‘waiting time’ before the curve becomes convex (see Proposition 13.2), similar in concept to Grayson’s idea. Later work by Grayson includes extending the theory to the case in which the ambient manifold is a more general surface, showing that embedded curves either flow to ‘round points’ (as in the case of planar curves) or geodesics [40].

In 1993 Michael Gage considered the anisotropic analogue of the regular curve shortening flow, extending the Euclidean plane to the Minkowski plane [34]. In this setting
vectors are directionally dependent, being determined by the radius of a centrally-symmetric curve called the indicatrix (or Minkowski unit circle), and so some care is needed to define what is meant by ‘tangent’ and ‘normal’ vectors (see Section 10.2). Regardless, Gage managed to conclude that the analogous results for convex curves hold in this setting, with initially embedded convex curves flowing to a special convex curve called the isoperimetrix. The Minkowski plane is the setting for Part II of this thesis.

A few years later, Huisken developed an alternative method for proving Grayson’s Theorem, which involved studying the evolution of an extrinsic-intrinsic isometric ratio (involving chord lengths and intrinsic distances) under (CSF), and classifying of singularities that can occur [50]. Similarly, Hamilton also developed a self-contained method for proving Grayson’s Theorem by estimating the ratio of the isoperimetric profile of a curve shortening flow to that of a circle with the same area [43].

More recently, Andrews and Bryan have written a number of papers in the same vein of the aforementioned work of Huisken and Hamilton. In [8] a new isoperimetric estimate was proven for embedded curves evolving under the length-normalised curve shortening flow. In a related paper [7], a comparison theorem for the isoperimetric profile of simple closed curves under the normalised curve shortening flow is proven, along with both lower and upper curvature estimates. It should be noted that both of these papers contain a self-contained proof of Grayson’s theorem without the need of the monotonicity formula or an analysis of the classification of possible singularities, such as in Grayson’s and Huisken’s work. As one can see, the curve shortening flow has enjoyed a tremendous amount of interest in the mathematical community.

Although the curve shortening flow undeniably dominates the curve-flow landscape, a number of similar curve flows have been studied over the years. For example, in [26] Dziuk, Kuwert and Schätzle study the elastic flow of closed curves in $\mathbb{R}^n$. This fourth-
order flow is the steepest descent $L^2$-gradient flow of the elastic energy, defined by

$$E(\gamma) := \int_\gamma k^2 ds.$$ 

Here $k$ denotes the ordinary Euclidean curvature. It is also the one-dimensional counterpart to the Willmore flow. Wheeler later extended this in his analysis of the generalised Helfrich flow [107], of which the elastic flow is a special case. In [105] he studies the evolution of closed planar curves under the curve diffusion flow, which can be viewed as a fourth-order analogue of the curve shortening flow (see also [28] for a study of special soliton solutions). Unlike the curve shortening flow, however, this flow preserves signed enclosed area. The energy-based methods of Wheeler’s paper, based in turn on [26], along with the aforementioned anisotropic setting of Gage, provided much of the inspiration for Part II of this thesis.

We summarise the main contributions to Part II of this thesis as the following.

Chapter 9 Differential geometry of Euclidean plane curves. For the remainder of this chapter we introduce the standard notation and results pertaining to the study of regular curves in the Euclidean plane.

Chapter 10 The anisotropic setting - introducing the Minkowski plane. In this chapter we first introduce the basic concepts and properties of convex body geometry. We then go on to extend the notation and results of Chapter 9 to the Minkowski plane, which is equivalent to the Euclidean plane endowed with an anisotropic metric.

Chapter 11 The flow equation. In this chapter we introduce the parabolic geometric flow to be studies in Part II of this thesis. We also give a short outline of short time existence as well as calculating the evolution equations for basic geometric quantities associated with our geometric flow.

Chapter 12 The Minkowski normalised oscillation of curvature. In this chapter we introduce (and prove some fundamental properties of) a scale-invariant quantity $N_{osc}$ called
the Minkowski normalised oscillation of curvature, which can be viewed as a planar curve analogue of the total umbilic energy from Part I of the thesis. We prove that if \( \mathcal{K}_{osc} \) is initially small, and if the isoperimetric ratio is initially close to unity, then \( \mathcal{K}_{osc} \) does not more than double over the lifespan of the flow.

Chapter 13 Proof of the main theorem. In our final chapter in Part II of this thesis, we first characterise the finite-time singularities for the polyharmonic curve flows, showing that if the flow becomes extinct in finite time then we must encounter an \( L^2 \) curvature singularity as we approach the maximal time \( T \). This allows us to conclude long time existence for flows with initially small \( \mathcal{K}_{osc} \) and with an isoperimetric ratio initially close to unity. Under the same conditions we show that the polyharmonic curve flows converge exponentially fast to a homothetic rescaling of the isoperimetrix. We also establish an upper bound on the waiting time until our family of immersions becomes uniformly convex.

I will now give a brief introduction to the standard terminology pertaining to the study of regular curves in the Euclidean plane.

### 9.1 Basic differential geometry of Euclidean plane curves

We start by considering an immersed plane curve

\[
\gamma : \mathbb{S}^1 \to \mathbb{R}^2 \\
u \mapsto \gamma (u).
\]

We then define the Euclidean arc length function in the conventional way:

\[
s (u) = \int_0^u |\gamma_v| \, dv.
\]
We are only interested in parameterisations wherein the derivative of $\gamma$ is nowhere zero ($\gamma' \neq 0$). Such a curve is called regular. We note that for any regular curve the arc length function is necessarily strictly monotonic increasing (and hence injective). From the definition of arc length, it is intuitive to define the Euclidean arc length element by

$$ds = ds(u) = |\gamma_u| du.$$ 

We can reparameterise such a regular curve $\gamma$ with respect to arc length. Such a parameterisation is often called a unit speed curve because it satisfies

$$\gamma_s = u_s\gamma_u = \gamma_u/|\gamma_u|,$$

due to $|\gamma_s| = 1$.

If $\gamma : [a, b] \to \mathbb{R}^2$ is an immersed plane curve, the Euclidean length of $\gamma ([a, b])$ (which we will simply call the length of $\gamma$, when the context is clear), is

$$L_{ds}([a, b]) = L(\gamma) = L(\gamma ([a, b])) = \int_0^L ds = \int_a^b |\gamma_u| du,$$

which we will often just write as

$$\int_\gamma ds.$$

In general, if a function $f$ is defined along such a curve $\gamma$, we define the integral of $f$ along $\gamma$ by

$$\int_0^L f(s) \, ds = \int_a^b f(u) |\gamma_u| \, du,$$

or simply

$$\int_\gamma f \, ds.$$

Here $|\cdot|$ denotes the ordinary Euclidean norm, and $\gamma_u$ is taken to mean the derivative of $\gamma (v)$ with respect to $v$. We say a curve $\gamma : [a, b] \to \mathbb{R}^2$ is a simple, closed curve (or sometimes a Jordan) curve if it is a homeomorphic image of the circle. Given a unit speed curve $\gamma = \gamma (s)$, we define the Euclidean unit tangent to the curve by

$$\tau (s) = \gamma_s.$$
We have already seen that this vector is of Euclidean length 1. If we have a unit speed curve $\gamma$ with Euclidean unit tangent $\tau$ we may denote the angle between the positive $x$–axis and $\tau$ (taken in the positive direction) by $\theta$. We are interested in how this angle $\theta$ varies with respect to $s$. This is an important function which measures the rate at which the curve ‘bends’. We call this function $k$ the signed Euclidean curvature of the curve $\gamma$. Given the unit speed parameterisation $\gamma(s) = (x(s), y(s))$, the aforementioned angle $\theta$ satisfies

$$\theta(s) = \tan^{-1}(y_s/x_s),$$

where the right hand side is well-defined. An application of the chain rule gives

$$k(s) = \theta'(s) = (x_s y_{ss} - x_{ss} y_s) / |\gamma_s|^2 = (\gamma_{ss}, (-y_s, x_s)).$$

The Euclidean curvature of a general (not necessarily unit speed) parameterised curve $\gamma(u) = (x(u), y(u))$ can then be seen to be

$$k(u) = \frac{x_u y_{uu} - y_u x_{uu}}{(x_u^2 + y_u^2)^{3/2}}.$$

A simple closed curve is defined to be positively oriented if when traveling on it one has the curve interior to the left. Given a closed, positively oriented curve $\gamma(s)$ with Euclidean tangent $\tau(s)$, we may rotate $\tau$ by $\pi/2$ radians in the anticlockwise direction to give the inward facing Euclidean unit normal to the curve $n(s)$ (see Figure 9.1).
That is to say,
\[ n(s) = \text{rot}_\frac{\pi}{2}(\tau(s)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x'(s) \\ y'(s) \end{pmatrix} = \begin{pmatrix} -y'(s) \\ x'(s) \end{pmatrix}. \]

We often write this as \( n = (-y_s, x_s) \). We can then see from our earlier definition of curvature that for a unit speed curve the curvature can be written as
\[ k = (\gamma_{ss}, n). \]

Note that here we have used the opposite sign convention to when we defined the mean curvature in Section 1.1. The tangent and normal vectors are collectively called the Frenet-Serret frame for \( \gamma \), and form an orthonormal basis spanning \( \gamma \). By differentiating the identities \((\tau, \tau) = (n, n) = 1\) with respect to \( \theta \) one can see that \( n_\theta \) is purely tangential to \( \gamma \) (in the direction of \( \tau \)), and that \( \tau_\theta \) is purely normal to \( \gamma \) (in the direction of \( n \)). Combining this with the identity \((\tau, n) = 0\) gives us
\[ \begin{pmatrix} \tau \\ n \end{pmatrix}_\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ n \end{pmatrix}. \]

Combining this with the chain rule and the definition of \( k \) then gives the familiar Frenet-Serret formulae:
\[ \begin{pmatrix} \tau \\ n \end{pmatrix}_s = k \begin{pmatrix} \tau \\ n \end{pmatrix}_\theta = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} \tau \\ n \end{pmatrix}. \quad (\text{FS}) \]

Lastly, if \( \gamma : \mathbb{S}^2 \to \mathbb{R}^2 \) is an immersed closed plane curve of period \( P \), the signed Euclidean enclosed area of \( \gamma \) is given by
\[ A(\gamma) = -\frac{1}{2} \int_{0}^{P} (\gamma, n) |\gamma_u| \, du = -\frac{1}{2} \int_{\gamma} (\gamma, n) \, ds. \quad (9.1) \]

Equation (9.1) is simply the equation (1.26) but with the role of the inner and outer
unit normal vectors reversed.
Chapter 10

The anisotropic setting -
introducing the Minkowski plane

We now introduce the fundamental concepts of the Minkowski plane, which is a 2–dimensional vector space in which vector lengths are directionally-dependent. This vector space carries its own notions of geometric quantities such as lengths and curvature which are reminiscent of their Euclidean counterparts. To get a broader understanding of some of the finer details of Minkowski plane and related vector spaces, the author recommends reading the fantastic survey articles of Martini and Swanepoel [70, 71].

10.1 Introduction to convex bodies

We begin with a real vector space $X$ and a proper subset $K \subset X$ containing the zero vector $\vec{0}$. The subset $K$ is assumed to possess the following properties:

1. Convexity: meaning that any convex combination of vectors in $\mathcal{U}$ is also contained in $\mathcal{U}$, and;
2. Balanced: meaning that $\alpha K \subseteq K$ for every $|\alpha| \leq 1$.

Such a subset $K$ is said to be absolutely convex.

If $K$ is absolutely convex, then we can define a corresponding Minkowskian functional

$$p_K : X \to [0, \infty)$$

by

$$p_K (x) = \inf \{ r > 0 : x \in rK \}.$$ \hfill (10.1)

The properties 1 and 2 prescribed to $K$ above allow us to ascertain that $p_K$ is subadditive:

$$p_K (x + y) \leq p_K (x) + p_K (y) \quad \forall x, y \in X$$ \hfill (10.2)

and homogeneous:

$$p_K (\alpha x) = |\alpha| p_K (x) \quad \forall \alpha \in \mathbb{R}.$$ \hfill (10.3)

Note that the properties (10.2) and (10.3) imply that $p_K$ is a seminorm for the vector space $X$.

As a simple example, consider the $n$–dimensional Euclidean vector space $X = \mathbb{R}^n$ and the closed ball centred at the origin of fixed radius $\rho > 0$:

$$K_\rho := B_\rho (0) := \{ x \in \mathbb{R}^n : |x| \leq \rho \}.$$ 

The set $K_\rho$ is obviously absolutely convex by the definition above. Moreover, for any $r > 0$, one has

$$r K_\rho = B_{r \rho} (0) = \{ x \in \mathbb{R}^n : |x| \leq r \rho \},$$

and so the associated Minkowski functional is easily calculable:

$$p_{K_\rho} (x) = \inf \{ r > 0 : x \in r K_\rho \} = \inf \{ r > 0 : x \in B_{r \rho} (x) \} = \rho^{-1} |x|.$$
Here $|\cdot|$ is the ordinary norm in $\mathbb{R}^n$. Therefore the Minkowski functional in this case simply scales a vector by a factor of $\rho^{-1}$. It is isotropic (meaning that vectors of the same Euclidean length map to the same value under $p_{K,\rho}$, independent of their direction). It is worth noting that if $\rho = 1$ then $p_K$ simply gives the regular Euclidean vector length, $|\cdot|$.

Note that the ball centred at the origin is special in $\mathbb{R}^n$ in that it is invariant under all actions of $SO(n)$, the special orthogonal group. This means that it is invariant under rotations, and therefore will induce a Minkowskian functional which is isotropic.

For a generic absolutely convex body however this is certainly not the case, however, as you can quite clearly see by considering $K$ to be a non-spherical ellipsoid in $\mathbb{R}^3$ together with its interior. In this scenario a vector $x \in \mathbb{R}^3$ which is oriented in the direction of the longest semi-axis of $K$ attains a value of $p_K(x)$ that is smaller than or equal to the value of $p_K$ evaluated at any proper rotation of $x$:

$$p_K(x) \leq p_K(k \cdot x) \quad \forall k \in SO(3).$$

The Minkowskian functional in this case is anisotropic, meaning that it is not invariant under rotations (that is, it is directionally dependent).

This sets the scene for this part of the thesis. We now introduce the Minkowski plane $\mathcal{M}^2$, the anisotropic setting for our curve flow.

We consider a convex, centrally symmetric domain $\mathcal{U} \subset \mathbb{R}^2$ with symmetry centre $\bar{0}$. We assume that $\partial \mathcal{U}$ is smooth with strictly positive Euclidean curvature. $\partial \mathcal{U}$ can be expressed as $(r(\theta) \cos \theta, r(\theta) \sin \theta)$ with $\theta \in [0, 2\pi)$ and $r > 0$ and $r(\theta + \pi) = r(\theta)$.

For a vector $x \in \mathbb{R}^2$ with $x = |x| (\cos \theta, \sin \theta)$ (where $|\cdot|$ is the regular Euclidean norm) the Minkowski norm, $l(x)$, of $x$ is defined by

$$l(x) = \frac{|x|}{r(\theta)}.$$  \hspace{1cm} (10.4)

Here $\theta = \theta(x)$ is as defined earlier. One notes that because $\mathcal{U}$ is convex and centrally
symmetric then it is automatically absolutely convex, and then the Minkowski norm $l$ satisfies the definition (10.1) of the Minkowskian functional corresponding to the body $K = \mathcal{U}$. Aptly, we then define the *Minkowski plane*\(^1\) $(\mathcal{M}^2, d)$ as the vector space $\mathcal{M}^2 = \mathbb{R}^2$ equipped with the distance metric $d : \mathcal{M}^2 \times \mathcal{M}^2 \to [0, \infty)$ given by

$$d(x, y) = l(x - y).$$

Hence for any $x \in \partial \mathcal{U}$ we have $d(x, \vec{0}) = l(x) = 1$, and so we define $\partial \mathcal{U}$ to be the *Minkowski unit circle*, or *indicatrix* of $\mathcal{M}^2$. For example, if we are just working in $\mathbb{R}^2$ then our indicatrix $\partial \mathcal{U}$ is simply the Euclidean unit circle and the distance metric is the regular (isotropic) one defined by $d(x, y) = |x - y|$. Similarly, if $\partial \mathcal{U}$ is the Euclidean circle with radius $r$, then the corresponding distance metric is the one defined by $d(x, y) = |x - y|/r$. In this case the distance metric either isotropically enlarges or shrinks the length of vectors by a factor of $r$, depending upon whether $r < 1$ or $r > 1$, respectively.

The *polar dual* of $\mathcal{U}$, denoted $\mathcal{U}^*$ is given by

$$\mathcal{U}^* := \{ f \in (\mathcal{M}^2)^* : |f(x)| \leq 1 \ \forall x \in \mathcal{U} \} \subset (\mathcal{M}^2)^*.$$

It is a simple exercise to show that if $\mathcal{U}$ is closed convex and contains the origin then $\mathcal{U}^{**} = \mathcal{U}$. This set is also a closed convex set (in the sense that convex combinations of linear functionals in $(\partial \mathcal{U})^*$ are also contained in the set). The boundary of the polar dual is given by

$$\partial \mathcal{U}^* := \{ f \in \mathcal{U}^* : f(x) = 1 \ \text{for some} \ x \in \partial \mathcal{U} \}.$$

Recall that given a non-empty closed convex set $K \subset \mathbb{R}^2$, the *support function*

\(^1\)One should be careful to not confuse this definition of the Minkowski plane with the other, perhaps more familiar notion of the 2-dimensional Minkowski spacetime, which is a $1+1$-dimensional Lorentzian manifold which in local coordinates $(t, x)$ is endowed with the metric $ds^2 := -dt^2 + dx^2$.\]
\( h_K : \mathbb{R}^2 \to \mathbb{R} \) of \( K \) is given by

\[
h_K(x) := \sup \{ (x,k) : k \in K \},
\]

where \((\cdot,\cdot)\) is the ordinary inner product in \( \mathbb{R}^2 \). If \( K = \partial U \) is parameterised by the angle function \( \theta \) as before, then we define the polar radial support function \( h = h_U^\ast \) as the support function of the polar dual of \( U, U^* \). This function is also parameterised by \( \theta \) and is in fact given by the reciprocal of the radial function: \( h = r^{-1} \) [34].

### 10.2 Basic differential geometry of the Minkowski plane

In this section we reintroduce basic concepts such as curves and arc length in our new anisotropic setting.

Let \( \gamma : S^1 \to M^2 \) be a parameterised closed piecewise differentiable curve. The Minkowski norm (10.4) defined in the previous section helps us to define the length of \( \gamma \). We do this in the regular way; one first approximates the curve with a polygonal path of straight lines. The length of each of these lines can be calculated via (10.4). Taking the limit as the lengths of these lines gets arbitrarily small yields the Minkowski length of \( \gamma \):

\[
L(\gamma) = \int_{S^1} l(\gamma_u) \, du = \int_{S^1} d\sigma.
\]

Here the Minkowski arc length element is given by

\[
d\sigma(u) = l(\gamma_u) \, du = \frac{|\gamma_u|}{r(\gamma_u)} \, du = \frac{ds(u)}{r(\gamma_u)}.
\]  

(10.5)

Alternatively, we write

\[
d\sigma(s) = \frac{ds}{r(\tau(s))}
\]

where \( \tau = \gamma_s \) is the Euclidean unit length tangent vector.
Given our earlier parameterisation for $\partial U$ we define the Minkowski tangent and normal vector to a curve $\gamma = (x(\theta), y(\theta))$ by

$$T(\theta) = r(\theta) \tau(\theta) \quad \text{and} \quad N(\theta) = -h_\theta (\cos \theta, \sin \theta) + h(-\sin \theta, \cos \theta),$$

respectively, where $h$ is the polar radial support function, $h = r^{-1}$. It is worth noting again that the angle $\theta$ refers to the angle that the regular Euclidean tangent to $\gamma$ makes with the $x-$axis. Much like in the Euclidean case, the vectors $T$ and $N$ form a Minkowski frame for the curve $\gamma$, although it must be emphasised that generally they do not form an orthonormal basis for $\mathbb{R}^2$ restricted to $\gamma$.

**Claim 10.1.** For every $\theta$, the Minkowski tangent $T$ and normal vector $N$ span a parallelogram of unit (Euclidean) area.

**Proof.** We consider a triangle in the plane with two sides given by $T$ and $N$. Write

$$T(\theta) = r \tau \quad \text{and} \quad N(\theta) = -h_\theta \tau + h n$$

where $\tau = (\cos \theta, \sin \theta)$ and $n = (-\sin \theta, \cos \theta)$. We can write

$$N(\theta) = \sqrt{h^2 + h_\theta^2} \left( \cos (\theta + \varphi(\theta)), \sin (\theta + \varphi(\theta)) \right).$$

Here

$$\varphi(\theta) = \tan^{-1} \left( \frac{r}{r_\theta} \right),$$

where defined. Therefore the angle between $T$ and $N$ is given by $\varphi(\theta)$, and the Euclidean area of the triangle spanned by $T$ and $N$ s

$$A = \frac{1}{2} |T| \cdot |N| \sin \varphi(\theta) = \frac{r^2 \sqrt{h^2 + h_\theta^2}}{2 \sqrt{r^2 + r_\theta^2}} = \frac{1}{2}.$$ 

Since the area of the parallelogram in question is equal to twice the area of this triangle, this proves the claim. \qed
Accordingly, the enclosed area $\mathcal{A}$ of a closed curve $\gamma : S^1 \to \mathcal{M}^2$ is simply equal to

$$\mathcal{A} (\gamma) = -\frac{1}{2} \int_{\gamma} (\gamma, n) \, ds. \quad (10.6)$$

Note that measure in the integral is $ds$ instead of $d\sigma$ because the Minkowski area element is identical to its Euclidean counterpart.

The isoperimetrix (the reasoning for this particular nomenclature will be explained in the next section) $\mathcal{I}$ is then defined by the parameterisation

$$\mathcal{I} = \{ N (\partial U) (\theta) : \theta \in [0, 2\pi) \} = \{-h_\theta \tau + hn : \theta \in [0, 2\pi)\}. \quad (10.7)$$

Qualitatively we have traced out the Minkowski normal vector $N$ as we vary along the indicatrix $\partial U$. It can be shown (see, for example, [96]) that the isoperimetrix $\mathcal{I}$ is actually given by a $\pi/2$ counterclockwise rotation of the boundary of the polar dual $\partial U^*$.  

Claim 10.1 gives an easy way to construct the Minkowski normal to a curve, given an indicatrix and isoperimetrix $\partial U$ and $\mathcal{I}$, respectively: one simply takes the Minkowski tangent vector $T$ and rescales it so that its end lies on the isoperimetrix. It is then a matter of rotating this new vector counterclockwise until the area of the parallelogram spanned by it and $T$ is of unit area.

Let us now attempt to develop a Minkowski analogue of the Frenet-Serret equations. Given an indicatrix as defined in the previous section, along with a curve $\gamma$ with Euclidean tangent vector $\tau (\theta)$, we write Minkowski unit tangent vector $T$ in the same direction as $\tau (\theta)$ by

$$T (\theta) = r (\theta) \tau (\theta).$$

The Euclidean tangent has been multiplied by the radius of the indicatrix at the corresponding angle to insure $T$ is of Minkowski unit length. Taking the derivative of both
sides with respect to the angular parameter $\theta$ gives

$$T_\theta = r_\theta \tau + r_\theta n = r^2 (-h_\theta \tau + h n) = r^2 N.$$  

Similarly,

$$N_\theta = -h_{\theta\theta} \tau - h_\theta n + h_\theta \tau - h \tau = -(h + h_{\theta\theta}) = -h (h + h_{\theta\theta}) T.$$  

Combining this with the chain rule and the identity $\theta_s = k$, we arrive at the analogue of the Frenet-Serret equations in the Minkowski plane:

$$\begin{pmatrix} T \\ N \end{pmatrix}_\sigma = \begin{pmatrix} 0 & kh^{-3} \\ -k (h + h_{\theta\theta}) & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}. \quad (10.8)$$  

Next, let $\gamma$ be a closed plane curve. Let $\theta$ be the angle between the Minkowski tangent vector and the positive $x$-axis. That is to say, $T = T(\theta)$. It is relatively straightforward to check that the Euclidean curvature of the isoperimetrix $\partial \mathcal{U}$ at the point $N(\theta)$ is equal to

$$\hat{k} = \frac{1}{h + h_{\theta\theta}}. \quad (10.9)$$  

By defining $T^*, N^*$ to be the corresponding dual frames to $T$ and $N$ respectively (that is, $T^*, N^* \in (T\gamma)^*$ satisfy $T^*(T) = N^*(N) = 1, T^*(N) = N^*(T)$ where $(T\gamma)^*$ denotes the cotangent bundle of $\gamma$), one can show that the differential of the Minkowski length functional, $dl$ can be expressed in a particularly attractive way:

$$dl = T^*.$$  

(For a calculation of this, see [34]). It is then straightforward to show that for a one-parameter family of Minkowski immersions $\gamma : S^1 \times [0, T) \to \mathcal{M}^2$, the evolution equation

$$\partial_t^N \gamma = \dot{k}/\dot{k} = k (h + h_{\theta\theta}) =: \kappa. \quad (ACSF)$$
gives the steepest descent in $L^2 = H^0$ for the Minkowski length functional (see Section 11.3). Trying to keep in line with the Euclidean case (in which the curve shortening flow gives the steepest descent in $L^2$ for the Euclidean length functional), we therefore dub $\kappa = \kappa(\sigma,t)$ to be the Minkowski curvature associated to $\gamma(\sigma,t)$. This is not the only possible definition of the Minkowski curvature (see Section 10.4, for example), however the author sees it as the most suitable one given its aforementioned variational properties. Gage [34] (see also [84]) has studied the motion of a plane curve evolving with flow speed given by (ACSF), the so-called ‘anisotropic curve-shortening flow’, proving that flows that are renormalised to be area-preserving converge smoothly to a homothetic rescaling of $\partial U$ with enclosed area $\mathcal{A}(\gamma_0) := \mathcal{A}(\gamma(\cdot,0))$. This is clearly the Minkowski analogue of the regular Euclidean curve-shortening flow, which has been studied quite thoroughly in the mathematical community (see [1, 4, 33, 39], among many others).

Let us end this section by looking at a concrete example of an indicatrix and the isoperimetrix that it induces.

**Example 10.2.** Consider the square in $\mathbb{R}^2$ with corners at $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$ (see Figure 10.1). Denoting this square by $\partial U$, we can see that the figure is convex and centrally symmetric and therefore satisfies the criteria of being an indicatrix. Note that the radial function $r$ is not continuously differentiable at $\theta = \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$. This is will in turn affect the smoothness of the isoperimetrix $\tilde{I}$.

![Figure 10.1: A simple indicatrix given by the perimeter of a square with sides of length 2.](chart)

Let us try to find the isoperimetrix $\tilde{I}$ associated to $\partial U$. First we need to calculate the
radial polar function \( r \) that defines the geometry of our space. Elementary trigonometry tells us that

\[
r(\theta) = \begin{cases} 
  \sec \theta, & -\pi/4 \leq \theta < \pi/4 \\
  \csc \theta, & \pi/4 \leq \theta < 3\pi/4 \\
  -\sec \theta, & 3\pi/4 \leq \theta < 5\pi/4 \\
  -\csc \theta, & 5\pi/4 \leq \theta < 7\pi/4 
\end{cases}
\]

where we have identified \( \theta = -\pi/4 \) and \( \theta = 7\pi/4 \). Since \( \partial \mathcal{U} \) is not continuously differentiable at the corners of the square, we cannot calculate \( \tilde{\mathcal{I}} \) via the formula (10.7).

Luckily, there is an easier way. Recall that \( \tilde{\mathcal{I}} \) is given by a \( \pi/2 \) anticlockwise rotation of the boundary of the polar dual \( \partial \mathcal{U}^* \) which is defined by the polar function \( h = r^{-1} \). Since \( \partial \mathcal{U} \) is invariant under rotations of \( \pi/2 \), so is \( \tilde{\mathcal{I}} \). Therefore in our case the isoperimetrix \( \tilde{\mathcal{I}} \) is simply the boundary of the body whose radial function for each \( \theta \) is defined by the reciprocal of \( r(\theta) \). We calculate that

\[
\tilde{\mathcal{I}}(\theta) = \begin{cases} 
  \frac{1}{2} \{ (\cos 2\theta, \sin 2\theta) + (1, 0) \}, & -\pi/4 \leq \theta < \pi/4 \\
  \frac{1}{2} \{ (\cos (2\theta - \pi/2), \sin (2\theta - \pi/2)) + (0, 1) \}, & \pi/4 \leq \theta < 3\pi/4 \\
  -\frac{1}{2} \{ (\cos 2\theta, \sin 2\theta) + (1, 0) \}, & 3\pi/4 \leq \theta < 5\pi/4 \\
  -\frac{1}{2} \{ (\cos (2\theta - \pi/2), \sin (2\theta - \pi/2)) + (0, 1) \}, & 5\pi/4 \leq \theta < 7\pi/4 
\end{cases}
\]

We find that \( \tilde{\mathcal{I}} \) is given by the union of 4 semicircles of radius \( 1/2 \). Each point at which the semicircles intersect corresponds to a point where the indicatrix fails to be continuously differentiable. We present the indicatrix and isoperimetrix together in the following figure.

To demonstrate how \( \partial \mathcal{U} \) and \( \tilde{\mathcal{I}} \) influence the geometry of curves in the plane, we consider an embedded circle of radius 2 in the following diagram.

One can see that the Minkowski tangent vectors (marked in blue) vary in length as we travel around the circle, getting largest in sections where the direction of the tangent
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MINKOWSKI PLANE

Figure 10.2: Here we combine the indicatrix from before, $\partial U$ (in blue) with its isoperimetrix $\tilde{I}$ (in red).

Figure 10.3: Here we demonstrate how the indicatrix and isoperimetrix shape the geometry of our plane. At several points along our large circle we have placed a Minkowski tangent and normal vectors (in blue and red respectively).

approaches $\{\pm \pi/2, \pm 3\pi/4\}$. This corresponds to the parts of the indicatrix in which the radius is the largest (the corners of the blue square). At these points the Minkowski normal changes orientation instantaneously because the failure of $\partial U$ to be continuously differentiable.

Note that in Figure 10.3 the Minkowski tangent and normal vectors always span a parallelogram of unit area, even though they are not necessarily perpendicular (see Claim 10.1).

Contrary to the previous example, for the remainder of this thesis we will assume that the indicatrix $\partial U$ (and hence the isoperimetrix $\tilde{I}$), is smooth. The reason for this is that the main result for Part II (see Theorem 13.1) establishes conditions under which the flow introduced in Chapter 11 converges smoothly and exponentially to a
homothetic rescaling of a smooth isoperimetric. It would be an interesting topic for further work to investigate the case in which the indicatrix is nonsmooth.

10.3 The anisotropic isoperimetric problem

In classical Euclidean geometry, there are varied notions of what is meant by the isoperimetric problem, the most well-known of which is the following: find a simple closed curve of largest possible area given prescribed length [16]. In Euclidean space with the usual definitions of length and area, the solution is of course given by the circle.

The aforementioned problem leads to a ratio that is satisfied by all simple closed curves in $\mathbb{R}^2$ known as the (Euclidean) isoperimetric ratio: 

$$I(\gamma) := \frac{L^2(\gamma)}{4\pi A(\gamma)}. \quad (10.10)$$

For simple closed curves, this ratio is greater than or equal to one with equality if and only if $\gamma$ is a circle. We have varied our notation for length and area here to differentiate from the Minkowski case, although both definitions overlap when the indicatrix is the usual Euclidean unit circle.

We want to try to find an anisotropic analogue to the Euclidean isoperimetric problem and isoperimetric ratio. Not only is this interesting in its own right, but it will be important in our analysis later on. In fact before any analysis takes place at all it is quite clear that some notion of the isoperimetric ratio will be needed, the reasoning for which we now outline. A curve solving the isoperimetric problem would be the Minkowskian analogue to the Euclidean circle which is the limit solution to many Euclidean geometric flows such as the curve shortening flow and curve diffusion flow, among others. Thus it would seem likely that a geometric flow in the Minkowski plane would also flow to a a closed curve solving the isoperimetric ratio. After all,
the purpose of geometric flows is largely to deform an object into something more geometrically desirable.

We present the isoperimetric problem (and its solution) in the following theorem.

**Theorem 10.3.** For a Minkowski plane $\mathcal{M}^2$ with associated indicatrix $\partial U$, a homothetic rescaling of $\tilde{I}$ gives the minimum Minkowski boundary length of all simple closed curves with a given enclosed area.

**Proof.** The result is quite standard (see, for example, [20]). However, for the sake of self-containment, we include an outline. We consider an immersed curve $\gamma$ and use the classical method of Lagrange multipliers to extremise the functional

$$L(\gamma) := \int_\gamma |\gamma_u| \, du$$

given a prescribed enclosed area

$$\mathcal{A}(\gamma) = -\frac{1}{2} \int_\gamma (x_u y_u - x y_u) \, du.$$ 

Here $|\gamma_u| = \sqrt{x_u^2 + y_u^2}$ and $\theta = \tan^{-1}(y_u / x_u)$ (where defined). The Lagrangian is therefore given by

$$L(x, y, x_u, y_u) = \frac{|\gamma_u|}{r(\theta)} - \frac{\lambda}{2} (x_u y_u - x y_u)$$

We therefore need to solve the two equations

$$\frac{\partial L}{\partial x} - \frac{d}{du} \left( \frac{\partial L}{\partial x_u} \right) = \frac{\partial L}{\partial y} - \frac{d}{du} \left( \frac{\partial L}{\partial y_u} \right) = 0. \quad (10.11)$$

We calculate

$$\frac{\partial L}{\partial x} = \frac{\lambda y_u}{2}, \quad \frac{\partial L}{\partial y} = -\frac{\lambda x_u}{2},$$

as well as

$$\frac{\partial L}{\partial x_u} = \left( \frac{r x_u + r \theta y_u}{r^2 |\gamma_u|} \right) - \frac{\lambda y_u}{2} \quad \text{and} \quad \frac{\partial L}{\partial y_u} = \left( \frac{r y_u - r \theta x_u}{r^2 |\gamma_u|} \right) + \frac{\lambda x_u}{2}.$$
Therefore the first of the equations in (10.11) is equivalent to

$$\lambda x_u = -\frac{d}{du} \left( \frac{ry_u - r_\theta x_u}{r^2 |\gamma_u|} \right),$$

which implies

$$\lambda x = -\left( \frac{ry_u - r_\theta x_u}{r^2 |\gamma_u|} \right) + A = -hy_s - h_\theta x_s + A$$

for some constant $A$. Similarly, the first of the equations in (10.11) is equivalent to

$$\lambda y_u = \frac{d}{du} \left( \frac{rx_u + r_\theta y_u}{r^2 |\gamma_u|} \right),$$

which implies

$$\lambda y = \left( \frac{rx_u + r_\theta y_u}{r^2 |\gamma_u|} \right) + B = hx_s - h_\theta y_s + B$$

for some constant $B$. Therefore our extremising curve $\gamma = (x, y)$ satisfies

$$\gamma = (x, y) = \frac{1}{\lambda} \left( -h_\theta \tau + hn + (A, B) \right),$$

where $\tau = (x_s, y_s)$, $n = (-y_s, x_s)$ are the Euclidean unit normal and tangent to the curve, respectively. Comparing to (10.7) we can see that $\gamma$ is a copy of the isoperimetrix that has been translated and rescaled homothetically.

It turns out (see, for example [97]) that for any simple closed curve immersed in the Minkowski plane $\mathcal{M}^2$, $\gamma : S^1 \to \mathcal{M}^2$ with positive enclosed area, the following inequality holds:

$$\mathcal{L}^2(\gamma) - 2\mathcal{A}(\gamma) \int_\gamma \kappa d\sigma \geq 0,$$

with equality if and only if $\gamma$ is a homothetic rescaling of the isoperimetrix $\tilde{\gamma}$. Therefore we define the ratio

$$\mathcal{I}(\gamma) := \frac{\mathcal{L}^2(\gamma)}{2\mathcal{A}(\gamma) \int_\gamma \kappa d\sigma}. \quad (10.12)$$

to be the anisotropic isoperimetric ratio associated to $\mathcal{M}^2$. For simple curves, ratio
is always greater than or equal to 1, with equality if and only if \( \gamma \) is a homothetic rescaling of the isoperimetrix \( \tilde{I} \) (hence its name). In this paper we will often refer to \( I \) simply as the ‘isoperimetric ratio’ for short, since it is in fact equal to its Euclidean counterpart in the case \( \mathcal{M} = \mathbb{R}^2 \).

The ratio given by (10.12) at first appears to only superficially resemble the classical isoperimetric ratio (10.10). However we note in the following proposition that the integral \( \int_{\gamma} \kappa d\sigma \) that appears in the denominator of equation (10.12) is in fact a topological invariant which depends solely on the structure inherited from the indicatrix \( \partial U \) and reduces to (10.10) in the Euclidean case. This is the generalisation of the Euclidean case in which the total curvature of a simple closed immersed plane curve \( \gamma : S^1 \to \mathbb{R}^2 \) is given by an integer multiple of \( 2\pi \):

\[
\int_{\gamma} k \, ds = 2\omega \pi.
\] (10.13)

Here \( \omega \) is the winding number of the unit tangent (or equivalently, the normal) vector around the origin. Equation (10.13) is a special case of the Gauss-Bonnet formula and is often referred to as the Gauss-Bonnet formula for simple closed curves.

**Proposition 10.4.** Let \( \gamma : S^1 \to \mathcal{M}^2 \) be a simple closed immersion in the Minkowski plane \( \mathcal{M} \) with associated indicatrix \( \partial U \) and isoperimetrix \( \tilde{I} \). Then

\[
\int_{\gamma} \kappa \, d\sigma = 2A(\tilde{I}),
\]

where \( A(\tilde{I}) \) is the enclosed area of the isoperimetrix \( \tilde{I} \). Therefore the (anisotropic) isoperimetric ratio (10.12) can be written as

\[
\mathcal{I}(\gamma) := \frac{L^2(\gamma)}{4A(\gamma) A(\tilde{I})}.
\] (I)

**Proof.** Using the identities \( d\sigma = h \, ds \) and \( d\theta = k \, ds \), we have

\[
\int_{\gamma} \kappa \, d\sigma = \int_{\gamma} k \, (h + h_{\theta\theta}) \, d\sigma = \int_{0}^{2\pi} h \, h_{\theta\theta} \, d\theta.
\] (10.14)
Next, using the notation $\tau = (\cos \theta, \sin \theta), n = (-\sin \theta, \cos \theta)$, the isoperimetrix $\mathcal{I}$ can be parameterised by

$$\tilde{\mathcal{I}}(\theta) = \{-h_\theta \tau + h n : \theta \in [0, 2\pi)\}.$$

Therefore a quick calculation gives

$$\tilde{\mathcal{I}}_\theta = -(h + h_{\theta \theta}) \tau,$$

and so the induced Euclidean arc length and normal to $\tilde{\mathcal{I}}$ are given by

$$d\tilde{s} = \sqrt{h + h_{\theta \theta}} \, d\theta, \quad \tilde{n} = -\sqrt{h + h_{\theta \theta}} n,$$

respectively.

This implies that the signed enclosed area of $\tilde{\mathcal{I}}$ is given by

$$\mathcal{A}(\tilde{\mathcal{I}}) = -\frac{1}{2} \int_{\mathcal{I}} (\tilde{\mathcal{I}}, \tilde{n}) \, d\tilde{s}$$

$$= \frac{1}{2} \int_0^{2\pi} (h + h_{\theta \theta}) (-h_\theta \tau + h n, n) \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} h (h + h_{\theta \theta}) \, d\theta.$$

Comparing to (10.14) we find that

$$\int_{\gamma} \kappa \, d\sigma = 2\mathcal{A}(\tilde{\mathcal{I}}),$$

which finishes the proof.

Recall that in the case we are simply in $\mathbb{R}^2$ with the regular Euclidean distance metric, then the indicatrix is given by the ordinary unit circle. In this case, $h = r^{-1} \equiv 1$ and so the formula (10.7) for the parameterisation of the isoperimetrix implies that it too is the unit circle, which has area $\pi$. Therefore substituting into (1) we obtain the classical isoperimetric ratio $L^2/4\pi A$, as expected.
Remark 10.5. It should be noted that there exist a number of different isoperimetric inequalities. For example, the Banchoff-Pohl inequality gives a lower bound on the length of a possibly non-simple curve. In the anisotropic setting (see, for example [94] for a proof of the inequality by anisotropic curve shortening flow), the inequality takes the form

\[ \mathcal{L}^2(\gamma) \geq 4\mathcal{A}(\tilde{I}) \int_{M \setminus \gamma} w^2(x) dA \]  

where \( w(x) \) denotes the winding number of \( x \) with respect to \( \gamma \). Equality holds in (10.15) if and only if \( \gamma \) is a (possibly multi-covered) homothetic rescaling of the isoperimetrix. The reader is also invited to read [13] for the classical version of this inequality, as well as [95] for the case in which the ambient manifold has negative curvature.

10.4 An alternative definition of Minkowski curvature

As mentioned above, the definition given for the anisotropic curvature \( \kappa \) is not the only possible definition of curvature in the Minkowski plane. Indeed, by comparing the Euclidean Frenet-Serret equations:

\[
\begin{pmatrix}
\tau \\
n
\end{pmatrix}_s = \begin{pmatrix}
0 & k \\
-k & 0
\end{pmatrix} \begin{pmatrix}
\tau \\
n
\end{pmatrix}
\]

with their Minkowski counterparts (10.8), the two obvious candidates for a curvature function present themselves. One is \( \kappa := k(h + h_{\theta\theta}) \), (which we have chosen for this thesis), the other \( \kappa^* := kh^{-3} \). In some ways (which we now highlight) \( \kappa^* \) would seem a more logical choice, although it does not exhibit the nice variational properties that \( \kappa \) does (see Section 11.3).

If we consider an immersed closed curve \( \gamma : S^1 \to M^2 \) then (10.5) implies that the induced metric on \( \gamma \) is given by \( g = d\sigma^2 = r^{-2} ds^2 \). Therefore the induced Laplace-
Beltrami operator is given by
\[ \Delta_\gamma = g^\sigma\sigma \partial_\sigma^2 = \partial^2_\sigma. \]
Applying \( \Delta_\gamma \) to our immersion function (see the Gauss-Weingarten relations C.6) then yields
\[ \Delta_\gamma \gamma = \partial_\sigma^2 \gamma = \partial_\sigma (r \partial_\sigma \gamma) = \partial_\sigma T = kh^{-3} N = \kappa^* N, \]
where we have used our earlier definition of \( \kappa^* \) as well as (10.8) in our calculation. Comparing to the identity
\[ \Delta_\gamma \gamma = \partial_\sigma^2 \gamma = k n \]
which holds for curves immersed in regular Euclidean space, one might hypothesise that using \( \kappa^* \) as the definition of curvature for our anisotropic setting might be fitting. Indeed the Minkowski-normal flow equation
\[ \partial_t \gamma = \kappa^* N = kh^{-3} N \quad (10.16) \]
does exhibit some interesting variational properties. A quick calculation proves that under (10.16) the Minkowski length of closed curves decreases monotonically:
\[ \frac{d}{dt} L(\gamma) = -\int_\gamma \kappa^* \cdot \kappa^* d\sigma = -\int_\gamma k^2 h^{-3} (h + h_{\theta\theta}) d\sigma \leq 0. \]
Here we have used the fact that by the isoperimetrix is assumed to be strictly convex and hence the reciprocal of its Euclidean curvature, \( \hat{k}^{-1} = h + h_{\theta\theta} \), is everywhere positive. Similarly, aided by relevant calculations found in Corollary 11.2 of the next chapter, we find that the enclosed area of \( \gamma \) (defined by (10.6)) is decreasing monotonically also:
\[ \frac{d}{dt} A(\gamma) = -\int_\gamma \kappa^* d\sigma = -\int_\gamma kh^{-3} d\sigma = -\int_\gamma kr^2 ds = -\int_\gamma r^2 (\theta) d\theta \leq 0. \]
Therefore without deeper analysis there is no obvious way to determine whether or not the isoperimetric ratio is increasing or decreasing. The author thinks that it would be a
worth while endeavour to study the behaviour of immersed curves under the alternate Minkowski curve shortening flow given by (10.16).
Chapter 11

The flow equation

In this chapter we introduce the geometric flow that is to be studied in Part II of the thesis.

For a fixed $p \in \mathbb{N}$, we consider a one parameter family of closed immersed curves $\gamma : S^1 \times [0, T) \to \mathcal{M}$ evolving with Minkowski normal velocity equal to $(-1)^p \kappa_{2p}$:

$$\partial_t \gamma (\sigma, t) = (-1)^p \kappa_{2p} \cdot N (\sigma, t).$$ (APH)

Here $\kappa_{2p} = \frac{\partial^{2p} \kappa}{\partial \sigma^2}$ refers to the $2p^{th}$ derivative of $\kappa$ with respect to $\sigma$, the Minkowski arc length parameter. We will only be considering initial curves $\gamma (\cdot, 0) = \gamma_0$ that are closed with winding number one. However, unless stated otherwise, results apply to initially non-embedded curves.

We will henceforth refer to a one-parameter family of closed curves evolving via (APH) as a $2 (p + 1)$-anisotropic polyharmonic curve flow, and it naturally generalises its lower-order Euclidean counterparts. In the case $p = 0$ and $p = 1$ we have the Minkowski curve shortening and curve diffusion flows respectively. It turns out that our energy-based methods will not work for the case $p = 0$. Therefore, we stress that we are only considering the cases $p \geq 1$ in this part of the thesis.
In particular, our problem is equivalent to solving the initial value problem

\[
\begin{align*}
\partial_t \gamma (\sigma, t) &= (-1)^p \kappa_{\sigma^{2p}} N (\sigma, t), \\
\gamma (\cdot, 0) &= \gamma_0 \in C^\infty (S^1).
\end{align*}
\] (11.1)

Here \( \gamma_0 \) is a prescribed, smooth closed curve. The flow can be seen to be a degenerate system of quasilinear parabolic differential equations of order \( 2(p + 1) \), from which we can derive short time existence using the earlier results from Section 2.3.

### 11.1 A note on short-time existence

Short-time existence results are quite standard for the flow being considered here, but shall be included for clarity and self-containment.

We rewrite the flow (APH) as a parabolic system on a section over \( \gamma \). We first denote by \( \Pi \in SU (2) \) the rotation matrix

\[
\Pi = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

Multiplying a vector in \( \mathbb{R}^2 \) by \( \Pi \) of course has the affect of rotating the vector \( \pi/2 \) radians in the anticlockwise direction.

With this notation, we observe that

\[
\kappa = (h + h_{\theta \theta}) k = |\gamma_u|^{-3} (h + h_{\theta \theta}) (\gamma_{uu}, (\Pi \cdot \gamma_u))
\]

where \( \Pi \cdot X, X \in \mathbb{R}^2 \) stands for ordinary matrix premultiplication of \( X \) by \( \Pi \). Using this along with the identity \( \partial_\sigma = (r |\gamma_u|^{-1}) \partial_u \) yields

\[
\kappa_{\sigma^{2p}} = (h + h_{\theta \theta}) \frac{r^{2p}}{|\gamma_u|^{2p+3}} (\gamma_{u2p+2}, (\Pi \cdot \gamma_u)) + \text{Lower order terms in } \gamma'.
\]
Moreover,
\[ N = -h_\theta \tau + h \nu = \frac{1}{|u|} (h (\Pi \cdot u) - h_\theta u_\theta). \]

Therefore in index notation the flow (APH) becomes
\[
\partial_t \gamma^\alpha = (-1)^p (h + h_\theta) \frac{r^{2p}}{|u|^{2(p+2)}} (h(\Pi \cdot u)^\alpha - h_\theta u_\theta \gamma^\alpha) \delta^\alpha_\beta (\Pi \cdot u)^\beta + ' \text{lower order terms in } \gamma' + \frac{1}{|u|^2 (p+2)} (h (\Pi \cdot u)^\alpha - h_\theta u_\theta \gamma^\alpha) \delta^\alpha_\beta (\Pi \cdot u)^\beta. \]

From (11.2) we can deduce that the flow is \textit{weakly parabolic} (see Section 2.1 for the definition). To see this we need to show that eigenvalues of the operator
\[
T_\beta^\alpha = A_\beta^{i_1j_1i_2j_2...i_{p+1}j_{p+1}} \cdot (\Pi \cdot u)^\alpha = 0
\]
are non-negative for any \( \xi \in \mathbb{R}^2 \). If we fix \( \xi \), then solving \( T \eta = \lambda \eta \) leads to the equation
\[
(h + h_\theta) \frac{r^{2p}}{|u|^{2(p+2)}} (h(\Pi \cdot u)^\alpha - h_\theta u_\theta \gamma^\alpha) (\Pi \cdot u), \eta) = \lambda \eta^\alpha. \quad (11.3)
\]

Now \( \eta \) can be written as \( \eta = \varphi u + \phi (\Pi \cdot u) \) everywhere. If \( \eta \) is not purely tangential (meaning that \( \phi \neq 0 \)), then taking an inner product of (11.3) with \( \Pi \cdot u \) and then dividing through by \( \phi |u|^2 \) gives
\[
\lambda = (h + h_\theta) \frac{r^{2p-1}}{|u|^{2(p+1)}} > 0.
\]

If instead \( \phi = 0 \) then \( \eta = \varphi u \) and solving the eigenvalue problem leads to
\[
\lambda u^\alpha = (h + h_\theta) \frac{r^{2p}}{|u|^{2(p+2)}} (h(\Pi \cdot u)^\alpha - h_\theta u_\theta \gamma^\alpha)((\Pi \cdot u), \gamma^\alpha) u^\beta = 0,
\]
where the last step follows from the fact that \((\Pi \cdot \gamma_u) \perp \gamma_u\). Therefore \(\lambda = 0\) and we conclude that \((\text{APH})\) is weakly parabolic. The principal symbol is degenerate in the tangential direction, much like the geometric polyharmonic heat flow (see Part I). In particular the Legendre-Hadamard condition does not hold for any positive constant because of this degeneracy: choosing \(\eta = \varphi \gamma_u\) ones notes that

\[
A_\beta^{i_1 j_1 \ldots i_{p+1} j_{p+1}} \xi_{i_1} \xi_{i_2} \cdots \xi_{i_{p+1}} \xi_{j_{p+1}} \eta_\alpha \eta_\beta = 0.
\]

We then proceed by considering a smooth family of reparameterisations exactly as in the proof of Theorem 2.1 in Section 2.3 to show that the flow is equivalent to a strongly parabolic system of differential equations, modulo the action of a one-parameter diffeomorphism group acting on the curve. Henceforth the proof is almost identical to the one contained in Section 2.3 and so we will not repeat it. This establishes local well-posedness of the initial value problem (11.1).

### 11.2 Associated evolution equations

In this section we calculate the preliminary geometric evolution equations associated with the \(2(p+1)\)-anisotropic polyharmonic curve flow \((\text{APH})\). Many of these are quite straightforward (but lengthy) to calculate.

**Lemma 11.1.** Suppose that \(\gamma : \mathbb{S}^1 \times [0, T) \to \mathcal{M}^2\) solves \((\text{APH})\), and that \(f : \mathbb{S}^1 \times [0, T) \to \mathbb{R}\) is a periodic function with the same period as \(\gamma\). Then

\[
\frac{d}{dt} \int_\gamma f \, d\sigma = \int_\gamma f_t \, d\sigma + \int_\gamma f \, d^t \sigma = \int_\gamma f_t + (-1)^{p+1} f \cdot \kappa_{\sigma} \kappa_{2\varphi} \, d\sigma.
\]

**Proof.** We first calculate the time derivatives of the Euclidean and Minkowski length elements, \(ds\) and \(d\sigma\) respectively. Firstly,

\[
\partial_t |\gamma_u|^2 = 2 \langle \gamma_u, \partial_u \gamma_t \rangle
\]
It follows immediately that

\[
\partial_t ds = \partial_t (|\gamma_u| \, du) = (-1)^{p+1} (h \cdot h_{\theta} \kappa_{\sigma_{2p+1}} + \kappa \cdot \kappa_{\sigma_{2p}}) \, ds. \tag{11.4}
\]

Next, by using the chain rule, it is relatively straightforward to calculate

\[
\partial_r (\theta) = \partial_r \left( \tan^{-1} \left( \frac{y_u}{x_u} \right) \right)
= \frac{\partial r (\tan^{-1} \left( \frac{y_u}{x_u} \right)) \, \partial (y_u/x_u)}{\partial (\tan^{-1} \left( \frac{y_u}{x_u} \right)) \, \partial (y_u/x_u) \, \partial t}
= \frac{r_{\theta}}{|\gamma_u|^2} \langle \left( -y_u, x_u \right), \partial_u \gamma_t \rangle
= r_{\theta} \langle n, \partial_u \gamma_t \rangle
= (-1)^{p} \, h_{\theta} \langle n, \partial_{\sigma} (\kappa_{\sigma_{2p}} N) \rangle
= (-1)^{p} \, h_{\theta} \langle n, \kappa_{\sigma_{2p+1}} N - \kappa \cdot \kappa_{\sigma_{2p}} T \rangle
= (-1)^{p} \, h_{\theta} \langle n, \kappa_{\sigma_{2p+1}} (hn - h_{\theta} \tau) - r \kappa \cdot \kappa_{\sigma_{2p}} \tau \rangle
= (-1)^{p} \, h^2 r_{\theta} \kappa_{\sigma_{2p+1}}
= (-1)^{p-1} \, h_{\theta} \kappa_{\sigma_{2p+1}}. \tag{11.5}
\]

Combining (11.4) and (11.5) allows us to calculate the evolution of the Minkowski length element:

\[
\partial_t ds = \partial_t (r^{-1} \, ds)
= -r^{-2} \left( (-1)^{p-1} \, h_{\theta} \kappa_{\sigma_{2p+1}} \right) \, ds + r^{-1} \left( (-1)^{p+1} \, (h \cdot h_{\theta} \kappa_{\sigma_{2p+1}} + \kappa \cdot \kappa_{\sigma_{2p}}) \right) \, ds
= (-1)^{p+1} \, r^{-1} \kappa \cdot \kappa_{\sigma_{2p}} \, ds
\]
\[ = (-1)^{p+1} \kappa \cdot \kappa_{\sigma^p} \, d\sigma. \]

It follows that if \( f : \mathbb{S}^1 \times [0,T) \to \mathbb{R} \) follows the hypothesis of the lemma, then

\[
\frac{d}{dt} \int_{\gamma} f \, d\sigma = \int_{\gamma} f_t \, d\sigma + \int_{\gamma} f \partial_t \, d\sigma = \int_{\gamma} f_t + (-1)^{p+1} f \cdot \kappa \cdot \kappa_{\sigma^p} \, d\sigma,
\]

which is the desired result.

We now use the preceding lemma to calculate the evolution equations for some geometric quantities, including Minkowski length, and enclosed area.

**Corollary 11.2** (Evolution equations for basic geometric quantities). Suppose that \( \gamma : \mathbb{S}^1 \times [0,T) \to \mathcal{M}^2 \) solves (APH). Then

\[
\frac{d}{dt} \mathcal{L}(\gamma) = -\int_{\gamma} \kappa_{\sigma^p}^2 \, d\sigma \leq 0 \quad \text{and} \quad \frac{d}{dt} \mathcal{A}(\gamma) = 0.
\]

Moreover,

\[
\frac{d}{dt} \int_{\gamma} \kappa \, d\sigma = 0 \quad \text{and} \quad \frac{d}{dt} \bar{\kappa} \geq 0,
\]

where \( \bar{f} \) refers to the average of a function \( f \) over \( \gamma \):

\[
\bar{f} := \frac{1}{\mathcal{L}} \int_{\gamma} f \, d\sigma.
\]

As a result, the isoperimetric ratio \( \mathcal{I} \) decreases monotonically along the flow, with

\[
\mathcal{I}(t) \leq \mathcal{I}(0) \exp \left( \frac{-2 \int_0^t \int_{\gamma} \kappa_{\sigma^p}^2 \, d\sigma \, d\tau}{\mathcal{L}(0)} \right).
\]

**Proof.** Applying Lemma 11.1 with \( f \equiv 1 \) and integrating by parts once gives the first claim of the corollary immediately:

\[
\frac{d}{dt} \mathcal{L} = (-1)^{p+1} \int_{\gamma} \kappa \cdot \kappa_{\sigma^p} \, d\sigma = -\int_{\gamma} \kappa_{\sigma^p}^2 \, d\sigma (\leq 0).
\]
For the second claim of the corollary, we employ equation (11.4):

\[
\frac{d}{dt} \int_\gamma (\gamma, n) \, ds = \int_\gamma \partial_t (\gamma, n) + (-1)^{p+1} (\gamma, n) (h \cdot h_{\theta K_{\sigma^{2p+1}} + \kappa \cdot \kappa_{\sigma^2}) \, ds. \tag{11.6}
\]

All that is left is to calculate \(n_t\). To do so, we will need to first calculate the commutator \([\partial_t, \partial_s]\). Some of the work from Lemma 11.1 will assist us along the way. We have

\[
\partial_{ts} = \partial_t \left( |\gamma_u|^{-1} \partial_u \right) = -|\gamma_u|^{-2} \partial_t |\gamma_u| \partial_u + \partial_{st}
\]

\[
= (-1)^p |\gamma_u|^2 \left( h \cdot h_{\theta K_{\sigma^{2p+1}} + \kappa \cdot \kappa_{\sigma^2}} \right) |\gamma_u| \partial_u + \partial_{st}
\]

\[
= \partial_{st} + (-1)^p \left( h \cdot h_{\theta K_{\sigma^{2p+1}} + \kappa \cdot \kappa_{\sigma^2}} \right) \partial_s. \tag{11.7}
\]

Therefore the time and Euclidean arc length derivatives do not commute.

Next, note that \(\tau = \gamma_s\), and so the identity (11.7) implies that

\[
\partial_t \tau = \partial_{st} \gamma + (-1)^p \left( h \cdot h_{\theta K_{\sigma^{2p+1}} + \kappa \cdot \kappa_{\sigma^2}} \right) \partial_s \gamma
\]

\[
= (-1)^p \partial_s \left( K_{\sigma^{2p}} N \right) + (-1)^p \left( h \cdot h_{\theta K_{\sigma^{2p+1}} + \kappa \cdot \kappa_{\sigma^2}} \right) \tau
\]

\[
= (-1)^p h \left( K_{\sigma^{2p+1}} N - \kappa \cdot \kappa_{\sigma^2} T \right) + (-1)^p \left( h \cdot h_{\theta K_{\sigma^{2p+1}} + \kappa \cdot \kappa_{\sigma^2}} \right) \tau
\]

\[
= (-1)^p h K_{\sigma^{2p+1}} (hn - h\theta \tau) + (-1)^{p+1} \kappa \cdot \kappa_{\sigma^2} \tau + (-1)^p \left( h \cdot h_{\theta K_{\sigma^{2p+1}} + \kappa \cdot \kappa_{\sigma^2}} \right) \tau
\]

\[
= (-1)^p h^2 K_{\sigma^{2p+1}} n. \tag{11.8}
\]

Next, \(|n|^2 = 1\) implies that \(\partial_t n \perp n\). Hence from (11.8) we have

\[
\partial_t n = (\partial_t n, \tau) \tau = - (n, \partial_t \tau) \tau = (-1)^{p+1} h^2 K_{\sigma^{2p+1}} \tau
\]

This implies

\[
\partial_t (\gamma, n) = (-1)^p (K_{\sigma^{2p}} N, n) + (-1)^{p+1} h^2 K_{\sigma^{2p+1}} (\gamma, \tau)
\]

\[
= (-1)^p h K_{\sigma^2} + (-1)^{p+1} h^2 K_{\sigma^{2p+1}} (\gamma, \tau).
\]
Substituting this back into (11.6) gives

\[
\frac{d}{dt} \int_{\gamma} (\gamma, n) \, ds = (-1)^p \int_{\gamma} \kappa_{2p} \, d\sigma + (-1)^{p+1} \int_{\gamma} h^2 \kappa_{2p+1} (\gamma, \tau) \, ds \\
+ (-1)^{p+1} \int_{\gamma} (\gamma, n) (h \cdot h_{\theta} \kappa_{2p+1} + \kappa \cdot \kappa_{2p}) \, ds. \quad (11.9)
\]

For the second last term in (11.9) we apply integration by parts once:

\[
(-1)^{p+1} \int_{\gamma} (\gamma, n) h \cdot h_{\theta} \kappa_{2p+1} \, ds \\
= (-1)^{p+1} \int_{\gamma} (\gamma, n) h_{\theta} \kappa_{2p} \, ds \\
= (-1)^p \int_{\gamma} \kappa_{2p} ((\gamma, n) h_{\theta} + (\gamma, n_s) h_{\theta} + (\gamma, n) h_{\theta s}) \, ds \\
= (-1)^{p+1} \int_{\gamma} k \cdot \kappa_{2p} h_{\theta} (\gamma, \tau) \, ds + (-1)^p \int_{\gamma} k \cdot h_{\theta} \kappa_{2p} (\gamma, n) \, ds. \quad (11.10)
\]

Here we have also used the identity \( \partial_s = k \partial_\theta \) and the fact that \( \partial_s \gamma = \tau \perp n \). Similarly, for the third last term in (11.9) we use integration by parts once more:

\[
(-1)^{p+1} \int_{\gamma} h^2 \kappa_{2p+1} (\gamma, \tau) \, ds \\
= (-1)^{p+1} \int_{\gamma} h \kappa_{2p} (\gamma, \tau) \, ds \\
= (-1)^p \int_{\gamma} \kappa_{2p} (h_s (\gamma, \tau) + h (\gamma, \tau_s) + h (\gamma, \tau_s)) \, ds \\
= (-1)^p \int_{\gamma} k \cdot \kappa_{2p} h_{\theta} (\gamma, \tau) \, ds + (-1)^p \int_{\gamma} \kappa_{2p} \, d\sigma \\
+ (-1)^p \int_{\gamma} k \cdot h \kappa_{2p} (\gamma, n) \, ds. \quad (11.11)
\]

Combining (11.10), (11.11) and substituting back into (11.9) gives

\[
\frac{d}{dt} \mathcal{A} = -\frac{1}{2} \frac{d}{dt} \int_{\gamma} (\gamma, n) \, ds \\
= (-1)^{p+1} \int_{\gamma} \kappa_{2p} \, d\sigma + \frac{1}{2} (-1)^{p+1} \int_{\gamma} k (h + h_{\theta \theta}) \kappa_{2p} (\gamma, n) \, ds \\
+ \frac{1}{2} (-1)^p \int_{\gamma} k \cdot \kappa_{2p} (\gamma, n) \, ds
\]
\[\begin{align*}
= (-1)^{p+1} \int_\gamma \kappa_{\sigma^2p} \, d\sigma \\
= 0.
\end{align*}\]

Here we have used \(\kappa = k(h + h_{\theta\theta})\), and the last step follows from the divergence theorem because \(\gamma(\cdot, t)\) is closed. This establishes the second claim of the corollary.

To begin calculating the evolution of the Minkowski curvature \(\kappa\), we first calculate the evolution of the Euclidean curvature \(k\):

\[
\partial_t k = \partial_t (\gamma_{ss}, n)
\]

\[
= (\partial_t \tau, n) + (\tau_s, \partial_t n)
\]

\[
= (\partial_t \tau, n)
\]

\[
= (\partial_t \tau + (-1)^p (h \cdot h_{\theta\kappa_{\sigma^2p+1}} + \kappa \cdot \kappa_{\sigma^2p}) \tau_s, n)
\]

\[
= (\partial_s \left((-1)^p h^2_{\kappa_{\sigma^2p+1}}\right), n) + (-1)^p k (h \cdot h_{\theta\kappa_{\sigma^2p+1}} + \kappa \cdot \kappa_{\sigma^2p})
\]

\[
= (-1)^p (2kh \cdot h_{\theta\kappa_{\sigma^2p+1}} + h^3_{\kappa_{\sigma^2p+2}}) + (-1)^p k (h \cdot h_{\theta\kappa_{\sigma^2p+1}} + \kappa \cdot \kappa_{\sigma^2p})
\]

\[
= (-1)^p (h^3_{\kappa_{\sigma^2p+2}} + 3kh \cdot h_{\theta\kappa_{\sigma^2p+1}} + \kappa \kappa \cdot \kappa_{\sigma^2p}) \cdot \tag{11.12}
\]

Here we have used \(\tau_s \perp \partial_t n\). Next we need to calculate the evolution of \(h, h_{\theta\theta}\). This turns out to be relatively straightforward. For any \(m \in \mathbb{Z} \cup \{0\}\) a simple calculation gives

\[
\partial_t h_{\theta^m} = \frac{1}{|\gamma_u|} \left(\tau, \partial_t (y_u, -x_u)\right) h_{\theta^{m+1}} = - (\tau, n_t) h_{\theta^{m+1}} = (-1)^p h^2 h_{\theta^{m+1}} \kappa_{\sigma^2p+1},
\]

and therefore

\[
\partial_t (h + h_{\theta\theta}) = (-1)^p h^2 (h + h_{\theta\theta})_\theta \kappa_{\sigma^2p+1} = (-1)^p k^{-1} h^3 (h + h_{\theta\theta})_\sigma \kappa_{\sigma^2p+1}. \tag{11.13}
\]

Combining (11.12) and (11.13) gives us

\[
\partial_t \kappa = \partial_t (k (h + h_{\theta\theta}))
\]
\begin{equation*}
= (-1)^p \left( h^3 \kappa_{\sigma 2p+2} + 3kh \cdot h_{\theta \kappa} \kappa_{\sigma 2p+1} + k \kappa \cdot \kappa_{\sigma 2p} \right) (h + h_{\theta \theta}) \\
+ (-1)^p h^3 (h + h_{\theta \theta}) \kappa_{\sigma 2p+1} \\
= (-1)^p \left( h^3 (h + h_{\theta \theta}) \kappa_{\sigma 2p+1} \right) + (-1)^p \kappa^2 \cdot \kappa_{\sigma 2p}.
\end{equation*}

Applying Lemma 11.1 then yields

\begin{equation*}
\frac{d}{dt} \int_\gamma \kappa \, d\sigma = \int_\gamma \partial_t \kappa \, d\sigma + (-1)^{p+1} \int_\gamma \kappa^2 \cdot \kappa_{\sigma 2p} \, d\sigma \\
= (-1)^p \int_\gamma \left( h^3 (h + h_{\theta \theta}) \kappa_{\sigma 2p+1} \right) \kappa_{\sigma 2p} \, d\sigma + (-1)^p \int_\gamma \kappa^2 \cdot \kappa_{\sigma 2p} \, d\sigma \\
+ (-1)^{p+1} \int_\gamma \kappa^2 \cdot \kappa_{\sigma 2p} \, d\sigma \\
= (-1)^p \int_\gamma \left( h^3 (h + h_{\theta \theta}) \kappa_{\sigma 2p+1} \right) \kappa_{\sigma 2p} \, d\sigma \\
= 0.
\end{equation*}

Here the last step follows from the divergence theorem because \( \gamma (\cdot, t) \) is closed. This gives the first claim of the lemma. The next claim follows immediately from combining the previous claim and the evolution equation for the length \( \mathcal{L} \):

\begin{equation*}
\frac{d}{dt} \tilde{\kappa} = -\mathcal{L}^{-2} \int_\gamma \kappa \, d\sigma \cdot \frac{d}{dt} \mathcal{L} = \int_\gamma \frac{\kappa \, d\sigma}{\mathcal{L}^2} \int_\gamma \kappa_{\sigma 2p} \, d\sigma \geq 0.
\end{equation*}

For the final claim regarding the isoperimetric ratio, simply combine the previous two results:

\begin{equation*}
\frac{d}{dt} \mathcal{I} (\gamma) = \frac{d}{dt} \left( \frac{\mathcal{L}^2 (\gamma)}{4 \mathcal{A}(\mathcal{I}) \cdot \mathcal{A} (\gamma)} \right) = -\frac{\mathcal{L} \int_\gamma \kappa_{\sigma 2p} \, d\sigma}{2 \mathcal{A}(\mathcal{I}) \cdot \mathcal{A} (\gamma)} \leq -\frac{2 \mathcal{I} \int_\gamma \kappa_{\sigma 2p} \, d\sigma}{\mathcal{L}(0)}.
\end{equation*}

Hence the claim follows. \( \square \)

Note that the result \( \frac{d}{dt} \mathcal{I} \leq 0 \) from the preceding lemma is critical. This is because for our main theorem (Theorem 13.1), we wish to prove convergence to a homothetic rescaling of the isoperimetrix \( \mathcal{I} \), which \textit{minimises} the ratio \( \mathcal{I} \) amongst all closed curves immersed in \( \mathcal{M}^2 \).
11.3 The gradient flow hierarchy

Consider a closed immersed curve in the Minkowski plane: \( \gamma : \mathbb{S}^1 \to \mathcal{M}^2 \), and its accompanying length functional \( \mathcal{L}(\gamma) = \int_{\gamma} d\sigma \). Then for Minkowski-normal variations \( \partial_t \gamma = \varphi N \) calculations from the previous proposition give

\[
\frac{d}{dt} \mathcal{L}(\gamma) = -\int_{\gamma} \kappa \cdot \varphi \, d\sigma.
\]

To determine the gradient flow for the energy \( \mathcal{L} \) in the Sobolev space \( H^p(\gamma) \) it is necessary to find the normal variation that satisfies the equation

\[
\partial_t \gamma = -\nabla_{H^p} \mathcal{L}(\gamma),
\]

where \( \nabla_{H^p} \mathcal{L}(\gamma) \) is the gradient associated to the functional \( \mathcal{L} \) in the natural inner product on the space \( H^p \). This is usually done by comparing (11.14) to the equation

\[
d\mathcal{L}_{\varphi}(\gamma) = (\nabla_{H^p} \mathcal{L}(\gamma), \varphi)_{H^p},
\]

which is often simple to compute using integration by parts.

In the case where \( p = 0 \), we have that \( H^0 = L^2 \), in which case the inner product is given by the familiar equation:

\[
(f, g)_{H^0} = (f, g)_{L^2} = \int_{\gamma} fg \, d\sigma.
\]

In this case by comparing (11.14) and (11.14) one can immediately see that

\[
\frac{d}{dt} \mathcal{L}(\gamma) = (\kappa, \varphi)_{H^0},
\]

which means that \( \nabla_{H^0} \mathcal{L}(\gamma) = \kappa N \) and establishes the anisotropic curve shortening flow (ACSF) as the gradient flow for the Minkowski length functional.
We next consider the dual spaces to the Sobolev spaces $H^p(\Sigma)$, which are denoted by $H^{-p}(\Sigma)$ and consist of the bounded linear functionals $L : H^p(\gamma) \to \mathbb{R}$. We assign to the dual space its dual pairing $(\cdot, \cdot)_{H^{-p}}$. It turns out that these spaces will be the natural setting for our family of curves (APH).

If $f \in L^2(\gamma)$ then recall (see, for example [30]) that we can define the $p^{th}$ distributional derivative of $f$ as the function satisfying

$$(\partial_p^p f, g)_{H^{-p}} = (-1)^p \int_\gamma f \partial_p^p g d\sigma$$

for every $g \in H^p(\Sigma)$.

We will use this definition to prove that the flows (APH) are gradient flows in the spaces $H^{-p}$, $p \in \mathbb{N}$. Indeed for sufficiently smooth flows of the form

$$\partial_t \gamma = \varphi N,$$

one can use (11.14) and (11.15) to conclude

$$\frac{d}{dt} \mathcal{L}(\gamma) = - \int_\gamma \kappa \varphi d\sigma$$

$$= (-1)^{p-1} ((-1)^p \int_\gamma \kappa \varphi d\sigma)$$

$$= (-1)^{p-1} ((-1)^p \int_\gamma \kappa \partial_p^p \psi d\sigma)$$

$$= (-1)^{p-1} (\partial_p^p \kappa, \psi)_{H^{-p}}$$

$$\geq - \|\partial_p^p \kappa\|_{H^{-p}}^1 \|\psi\|_{H^{-p}}^1.$$  \hspace{1cm} (11.16)

Here $\psi$ is the $p^{th}$ distributional anti-derivative of $\varphi$. Equality holds in the last line of (11.16) if and only if $\varphi$ is equal to (some positive multiple of) $(-1)^{p} \partial_p^p \kappa$, which is equivalent to saying that $\varphi = (-1)^p \partial_2^p \kappa$. Therefore the anisotropic polyharmonic curve flows form a natural hierarchy of gradient flows for the Sobolev spaces $H^{-p}, p \in \mathbb{N}_0$. The $p^{th}$ step in the hierarchy corresponds to a flow of order $(2p + 2)$, with the first two steps being the curve shortening flow and curve diffusion flow, respectively.
Chapter 12

The Minkowski normalised oscillation of curvature

In this chapter we introduce a scale-invariant quantity

\[ \mathcal{H}_{osc}(\gamma) = \mathcal{L}(\gamma) \int_{\gamma} (\kappa - \bar{\kappa})^2 d\sigma, \]

which we call the *Minkowski normalised oscillation of curvature*. This energy was introduced by Wheeler [105] in the Euclidean setting and is a natural one of our purposes: it is a Lyapunov functional for closed curves in the Minkowski plane under some natural assumptions. More specifically, if we are given a closed curve \( \gamma \) with turning number \( \omega \), then \( \mathcal{H}_{osc}(\gamma) = 0 \) if and only if \( \gamma \) is \( n \omega \)-covered homothetic rescaling of the isoperimetrix \( \tilde{I} \). To see this, note that \( \mathcal{H}_{osc}(\gamma) = 0 \) implies that \( \gamma \) has constant Minkowski curvature everywhere, say \( \kappa \equiv C \). That is to say, \( \hat{k} \equiv C (h + h_{\theta\theta})^{-1} \) where \( \hat{k} \) is the ordinary Euclidean curvature of \( \gamma \). But by (10.9), \( (h + h_{\theta\theta})^{-1} \) is the Euclidean curvature of the isoperimetrix, and so this implies \( \gamma \) must be a homothetic rescaling of \( \tilde{I} \). The prescribed turning number \( \omega \) then allows us to ascertain how many times it is covered.

Moreover, the normalised oscillation of curvature is scale-invariant (see Proposition
12.1). One may choose to view $\mathcal{K}_{osc}$ as a planar curve analogue of the total umbilic energy $\tilde{W}(f)$ from Part I of the thesis. We start off by proving some fundamental properties of the energy, before calculating its associated evolution equation. Many of the results henceforth will use the assumption of small energy, the same way we did for the total trace-free curvature in Part I. We will also use the assumption of small isoperimetric ratio (see, for example, Proposition 12.4), which is another natural ‘circularity’ assumption to make.

**Proposition 12.1.** The quantity $\mathcal{K}_{osc}$ is scale-invariant.

*Proof.* The proof is straightforward. We consider a closed planar curve $\gamma : S^1 \to \mathcal{M}$ and a homothetic rescaling $\tilde{\gamma} = \lambda \gamma$ with $\lambda > 0$. We will affix a ‘‘$\tilde{\cdot}$’’ symbol to all geometric quantities associated with $\tilde{\gamma}$. Then

$$\tilde{s}(u) = \int_0^u \sqrt{\langle \tilde{\gamma}_v, \tilde{\gamma}_v \rangle} \, dv = \lambda \int_0^u \sqrt{\langle \gamma_v, \gamma_v \rangle} \, dv = \lambda s(u),$$

and so $\partial \tilde{s} = \lambda^{-1} \partial s$. It is obvious that $\tilde{r} = r$, where $\tilde{r}, r$ denote the reciprocal of the radial support function of the indicatrix, because $\gamma_u$ and $\tilde{\gamma}_u$ are in the ‘same direction’. Hence

$$\tilde{\gamma}_s = \lambda^{-1} \gamma_s = \gamma_s.$$

Moreover, $\mathcal{L}(\tilde{\gamma}) = \lambda \mathcal{L}(\gamma)$, and

$$\tilde{\kappa}(u) = \frac{\tilde{x}' \tilde{y}'' - \tilde{y}' \tilde{x}''}{((\tilde{x}')^2 + (\tilde{y}')^2)^{3/2}} (\tilde{h} + \tilde{h}_{\theta\theta})$$

$$= \lambda^{-1} \frac{x' y'' - y' x''}{((x')^2 + (y')^2)^{3/2}} (h + h_{\theta\theta})$$

$$= \lambda^{-1} \kappa(u).$$

Therefore

$$\mathcal{K}_{osc}(\tilde{\gamma}) = \mathcal{L}(\tilde{\gamma}) \int_{\tilde{\gamma}} (\tilde{\kappa} - \tilde{\kappa})^2 \, d\tilde{\sigma}$$
\[
= \mathcal{L} \left( \tilde{\gamma} \right) \int_{\tilde{\gamma}} \tilde{\kappa}^2 \, d\tilde{\sigma} - \left( \int_{\tilde{\gamma}} \tilde{\kappa} \, d\tilde{\sigma} \right)^2 \\
= (\lambda \mathcal{L} \left( \gamma \right)) \int_{\gamma} \left( \lambda^{-1} \kappa \right)^2 \lambda \, d\sigma - \left( \int_{\gamma} \left( \lambda^{-1} \kappa \right) \, d\sigma \right)^2 \\
= \mathcal{L} \left( \gamma \right) \int_{\gamma} \kappa^2 \, d\sigma - \left( \int_{\gamma} \kappa \, d\sigma \right)^2 \\
= \mathcal{L} \left( \gamma \right) \int_{\gamma} (\kappa - \bar{\kappa})^2 \, d\sigma \\
= \mathcal{K}_{osc} \left( \gamma \right),
\]
showing that \( \mathcal{K}_{osc} \) is scale-invariant. \( \Box \)

One can deduce from our previous calculations that this quantity is a natural one, being that for a one parameter family of curves \( \gamma_t \) that solves \( \text{(APH)} \), \( \mathcal{K}_{osc} \left( t \right) \) is a bounded quantity in \( L^1 \left( [0, T) \right) \) (and in fact can be controlled a priori). Indeed, the fact that \( \int_{\gamma} (\kappa - \bar{\kappa}) \, d\sigma = 0 \) means that we can apply Lemma \( B.2 \), giving

\[
\mathcal{K}_{osc} = \mathcal{L} \int_{\gamma} (\kappa - \bar{\kappa})^2 \, d\sigma \leq \mathcal{L} \left( \frac{\mathcal{L}}{2\pi} \right)^2 \int_{\gamma} \kappa_{\sigma}^2 \, d\sigma
\]

Now the periodicity of \( \kappa \) implies that for every \( i \geq 1 \), \( \int_{\gamma} \kappa_{\sigma_i} \, d\sigma = 0 \), so we can apply Lemma \( B.2 \) to the right hand side of the above inequality \( p - 1 \) more times, yielding

\[
\mathcal{K}_{osc} \leq \mathcal{L} \left( \frac{\mathcal{L}}{2\pi} \right)^{2p} \int_{\gamma} \kappa_{\sigma_p}^2 \, d\sigma = -\frac{\mathcal{L}^{2p+1}}{(2\pi)^{2p}} \frac{d}{dt} \mathcal{L} = -\frac{1}{2 \left( p + 1 \right) (2\pi)^{2p}} \frac{d}{dt} \left( \mathcal{L}^{2(p+1)} \right).
\]

Here we have utilised the evolution of the length functional from Corollary \( 11.2 \). We conclude that for any \( t \in [0, T) \)

\[
\int_{0}^{t} \mathcal{K}_{osc} \left( \tau \right) \, d\tau \leq -\frac{1}{2 \left( p + 1 \right) (2\pi)^{2p}} \left( \mathcal{L}^{2(p+1)} \left( \gamma_t \right) - \mathcal{L}^{2p+1} \left( \gamma_0 \right) \right) \\
\quad \leq \frac{1}{2 \left( p + 1 \right) (2\pi)^{2p}} \mathcal{L}^{2(p+1)} \left( \gamma_0 \right), \tag{12.1}
\]

and ascertain from \( (12.1) \) that the normalised oscillation of curvature is a priori controlled in \( L^1 \) over the time of existence of the flow. Furthermore, by repeatedly using
Lemma B.3 in a similar fashion, one can easily obtain an $L^1$ bound for $\|\kappa - \bar{\kappa}\|_\infty^2$ over the interval $[0, T)$. Firstly

$$\|\kappa - \bar{\kappa}\|_\infty^2 \leq -\frac{1}{2p (2\pi)^{2p-1}} \frac{d}{dt} \mathcal{L}^{2p}.$$ 

Hence for any $t \in [0, T)$,

$$\int_0^t \|\kappa - \bar{\kappa}\|_\infty^2 \, d\tau \leq -\frac{1}{2p (2\pi)^{2p-1}} \left( \mathcal{L}^{2p}(\gamma_t) - \mathcal{L}^{2p}(\gamma_0) \right) \leq \frac{1}{2p (2\pi)^{2p-1}} \mathcal{L}^{2p}(\gamma_0).$$

(12.2)

It is clear from (12.1) and (12.2) as well as the discussion at the beginning of the chapter that if we can show our flow (APH) exists for all time ($T = \infty$) then it must converge (at least sequentially) to a homothetic rescaling of the isoperimetrix.

Next we formulate the evolution equation for the normalised oscillation of curvature.

Proposition 12.2. Suppose $\gamma : S^1 \times [0, T) \to \mathcal{M}^2$ is a closed solution to (APH), with turning number one. Define $Q := h^3 (h + h_{\theta\theta})$, where $h = r^{-1}$ is the radial support function of the indicatrix $\partial U$, as well as $Q_* := \min Q$. Then

$$\frac{d}{dt} \left( \mathcal{K}_{osc} + 8 \mathcal{J} (\tilde{I})^2 \ln \mathcal{L} \right) + 2 \mathcal{L} \int_{\gamma} Q_{\kappa_{p+1}}^2 \, d\sigma$$

$$= \mathcal{L} \int_{\gamma} \kappa_{\sigma^p} \left( (\kappa - \bar{\kappa}) + \bar{\kappa} (\kappa - \bar{\kappa})^2 \right)_{\sigma^p} \, d\sigma - 2 \mathcal{L} \sum_{l=0}^{p-1} \binom{p}{l} \int_{\gamma} Q_{\kappa_{\sigma^l} \kappa_{\sigma^{l+1}}} \, d\sigma. \quad (12.3)$$

Moreover, if $\mathcal{K}_{osc} \leq 1$ then there exists universal constants $c_i = c_i (p, \partial U) > 0$ such that

$$\frac{d}{dt} \left( \mathcal{K}_{osc} + 8 \mathcal{J} (\tilde{I})^2 \ln \mathcal{L} \right) + \frac{\|\kappa_{\sigma^p}\|_2^2}{\mathcal{L}} \mathcal{K}_{osc}$$

$$+ \mathcal{L} \left( 2 Q_* - c_1 \mathcal{K}_{osc} - c_2 \sqrt{\mathcal{K}_{osc}} \right) \int_{\gamma} \kappa_{\sigma_{p+1}}^2 \, d\sigma \leq 0$$

Therefore, if there exists a positive time $T^*$ such that

$$\mathcal{K}_{osc}(t) \leq \min \left\{ \frac{c_2^2 + 4c_1 Q_* - c_2 \sqrt{c_2^2 + 8c_1 Q_*}}{2c_1^2}, 1 \right\} =: 2K^* \text{ for } t \in [0, T^*), \quad (12.4)$$
then during this time we have the estimate

\[ \mathcal{K}_{osc}(t) + \int_0^t \frac{\|\kappa_p\|_2^2}{\mathcal{L}(\tau)} \mathcal{K}_{osc}(\tau) \, d\tau \leq \mathcal{K}_{osc}(0) + 8\mathcal{A}(\tilde{\mathcal{I}})^2 \ln \left( \frac{\mathcal{L}(0)}{\mathcal{L}(t)} \right). \]  

(12.5)

**Remark 12.3.** Although the right hand side of (12.4) depends on \( p \) (because the constants \( c_1, c_2 \) depend on \( p \)), the coefficients on the right hand side of (12.5) are independent of \( p \).

**Proof.** The first part is a direct calculation:

\[
\frac{d}{dt} \mathcal{K}_{osc} = \frac{d}{dt} \left( \mathcal{L} \int_\gamma (\kappa - \bar{\kappa})^2 \, d\sigma \right) \\
= -\int_\gamma \kappa_{p,\sigma}^2 \, d\sigma \cdot \int_\gamma (\kappa - \bar{\kappa})^2 \, d\sigma + 2\mathcal{L} \int_\gamma (\kappa - \bar{\kappa}) \partial_t \kappa \, d\sigma \\
\quad + (-1)^{p+1} \int_\gamma (\kappa - \bar{\kappa})^2 \kappa \cdot \kappa_{p,\sigma} \, d\sigma \\
= -\frac{\|\kappa_p\|_2^2}{\mathcal{L}} \mathcal{K}_{osc} + 2 (-1)^p \mathcal{L} \int_\gamma (\kappa - \bar{\kappa}) \left( h^3 (h + h_{\theta \theta}) \kappa_{p+1} \right)_\sigma + \kappa^2 \kappa_{p,\sigma} \right) \, d\sigma \\
\quad + (-1)^{p+1} \mathcal{L} \int_\gamma (\kappa - \bar{\kappa})^2 \kappa \cdot \kappa_{p,\sigma} \, d\sigma.
\]

Hence

\[
\frac{d}{dt} \mathcal{K}_{osc} + \frac{\|\kappa_p\|_2^2}{\mathcal{L}} \mathcal{K}_{osc} + 2\mathcal{L} \int_\gamma (Q\kappa)_{\sigma p} \kappa_{p,\sigma} \, d\sigma \\
= (-1)^p \mathcal{L} \int_\gamma (2 (\kappa - \bar{\kappa}) \kappa^2 - (\kappa - \bar{\kappa})^2 \kappa) \kappa_{p,\sigma} \, d\sigma \\
= \mathcal{L} \int_\gamma \kappa_{p,\sigma} \left( (\kappa - \bar{\kappa})^3 + \bar{\kappa} (\kappa - \bar{\kappa})^2 \right)_{\sigma p} \, d\sigma + 2\mathcal{L} \|\kappa_p\|_2^2 \\
= \mathcal{L} \int_\gamma \kappa_{p,\sigma} \left( (\kappa - \bar{\kappa})^3 + \bar{\kappa} (\kappa - \bar{\kappa})^2 \right)_{\sigma p} \, d\sigma - 8\mathcal{A}(\tilde{\mathcal{I}}) \frac{d}{dt} \ln \mathcal{L}.
\]

Applying the general Leibniz rule to the last term on the left hand side and rearranging gives (12.3). Next we need a general rule that tells us how to deal with derivatives of \( \mathcal{D} \). For any \( j \in \mathbb{N} \cup \{0\} \) we have

\[
\mathcal{D}_{\sigma j} = \sum_{i=1}^j \tilde{c}_i P_i^{-i} (\kappa), \ j \in \mathbb{N},
\]

(12.6)
where \( \tilde{c}_1 \) is a constant that depends upon the radial support function \( h \) and its derivatives \( h_\theta, h_{\theta^2}, \ldots, h_{\theta^i+2} \). For example, an application of the chain rule gives

\[
\mathcal{Q}_\sigma = \left( \frac{\partial}{\partial \theta} (h^3 (h + h_{\theta\theta})) \right) \cdot \frac{h}{(h + h_{\theta\theta})} \kappa = \tilde{c}_1 \kappa = \sum_{i=1}^{1} \tilde{c}_1 P_{1-i}^1 (\kappa),
\]

where \( \tilde{c}_1 (h, h_\theta, h_{\theta^2}, h_{\theta^3}) = (h^3 (h + h_{\theta\theta}))_\theta \cdot \frac{h}{(h + h_{\theta\theta})} \). The proof of this identity comes from a simple inductive argument.

Next note that by algebraic manipulation we can write any \( P \)-style combination of terms involving \( \kappa \) as a \( P \)-style combination of terms involving \( (\kappa - \bar{\kappa}) \) in the following way:

\[
P_k^n (\kappa) = \sum_{l=0}^{k-1} \bar{\kappa}^l P_{k-l}^n (\kappa - \bar{\kappa}). \quad (12.7)
\]

For example, using the fact that \( \bar{\kappa} \) is constant shows that

\[
P_2^2 (\kappa) = c_1 \partial_\sigma^2 \kappa \ast \kappa + c_2 \partial_\sigma \kappa \ast \partial_\sigma \kappa
\]

\[
= c_1 \partial_2 (\kappa - \bar{\kappa}) \ast (\kappa - \bar{\kappa} + \bar{\kappa}) + c_2 \partial_\sigma (\kappa - \bar{\kappa}) \ast \partial_\sigma (\kappa - \bar{\kappa})
\]

\[
= c_1 \partial_2 (\kappa - \bar{\kappa}) \ast (\kappa - \bar{\kappa}) + c_2 \partial_\sigma (\kappa - \bar{\kappa}) \ast \partial_\sigma (\kappa - \bar{\kappa}) + c_1 \bar{\kappa} \partial_\sigma^2 (\kappa - \bar{\kappa})
\]

\[
= P_2^2 (\kappa - \bar{\kappa}) + \bar{\kappa} P_2^2 (\kappa - \bar{\kappa})
\]

\[
= \sum_{l=0}^{1} \bar{\kappa}^l P_{2-l}^2 (\kappa - \bar{\kappa}).
\]

Combining (12.7) with (12.6) we obtain

\[
\mathcal{Q}_\sigma = \sum_{i=1}^{j} \tilde{c}_i \bar{\kappa}^{i-1} P_{i-1}^j (\kappa - \bar{\kappa}), \quad j \in \mathbb{N}.
\]

In order to use Lemma B.7 correctly to estimate the remaining terms in (12.3) we will need to make sure that the highest derivative of \( (\kappa - \bar{\kappa}) \) is of the order \( p + 1 \). We can do this effectively by integrating by parts. If \( l = p - 1 \) we integrate by parts to give

\[
\int_{\gamma} \mathcal{Q}_\sigma \kappa_{\sigma \rho} \kappa_{\sigma \rho+1} d\sigma = - \int_{\gamma} \mathcal{Q}_\sigma \kappa_{\sigma \rho} \kappa_{\sigma \rho+1} d\sigma - \int_{\gamma} \kappa^2_{\sigma \rho} \mathcal{Q}_\sigma^2 d\sigma.
\]
Hence

\[-2\mathcal{L} \int_{\gamma} D_{\sigma} \kappa_{\sigma} \kappa_{\sigma+1} \, d\sigma = \mathcal{L} \int_{\gamma} \kappa_{\sigma}^2 D_{\sigma}^2 \, d\sigma \]

\[= \mathcal{L} \sum_{i=1}^{2} \sum_{m=0}^{i-1} \int_{\gamma} (\kappa - \bar{\kappa})_{\sigma}^2 \tilde{c}_i (\bar{\kappa})^m p_{i-m}^2 (\kappa - \bar{\kappa}) \, d\sigma \]

\[\leq c \sum_{i=1}^{2} \sum_{m=0}^{i-1} \mathcal{L}^{1-m} \int_{\gamma} |P_{i-m+2}^{2(p+1)-i,p} (\kappa - \bar{\kappa})| \, d\sigma. \quad (12.8)\]

Note that this expansion does not depend on \( l \). Similarly, if \( l < p - 1 \) we have

\[-2\mathcal{L} \left( \frac{p}{l} \right) \int_{\gamma} D_{\sigma}^{l-1} \kappa_{\sigma} \kappa_{\sigma+1} \, d\sigma \]

\[= 2\mathcal{L} \left( \frac{p}{l} \right) \int_{\gamma} (\kappa - \bar{\kappa})_p (D_{\sigma}^{l-1} (\kappa - \bar{\kappa})_{\sigma+l} + D_{\sigma}^{l-1} (\kappa - \bar{\kappa})_{\sigma+l+2}) \, d\sigma \]

\[\leq c(p) \sum_{i=1}^{p-l} \sum_{m=0}^{i-1} \mathcal{L}^{1-m} \int_{\gamma} |P_{i-m+2}^{2(p+1)-i,p} (\kappa - \bar{\kappa})| \, d\sigma. \quad (12.9)\]

Since (12.8) is of the same form as (12.9) with \( l = p - 2 \), we only need to look at (12.9).

For \( m \in \{0, 1, \ldots, i-1\} \), Lemma B.7 gives us

\[\mathcal{L}^{1-m} \int_{\gamma} \left| \frac{P_{i-m+2}^{2(p+1)-i,p}}{P_{i-m+2}^{2(p+1)-i,p}} (\kappa - \bar{\kappa}) \right| \, d\sigma \]

\[\leq c \mathcal{L}^{1-m} \cdot \mathcal{L}^{1-(i-m+2)-2(p+1)+i} (\mathcal{K}_{osc}) \frac{1-m}{2+i+m} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{i+1}^2 \, d\sigma \right)^{1-\frac{i+m}{4(p+1)}} \]

\[\leq c \mathcal{L}^{-(2p+1)} (\mathcal{K}_{osc}) \frac{1-m}{2} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{i+1}^2 \, d\sigma \right)^{\frac{i+m}{4(p+1)}} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{i+1}^2 \, d\sigma \right)^{1-\frac{i+m}{4(p+1)}} \]

\[\leq c \mathcal{L} (\mathcal{K}_{osc}) \frac{1-m}{2} \int_{\gamma} \kappa_{i+1}^2 \, d\sigma. \quad (12.10)\]

Next we estimate the first term on the right hand side of (12.3) in a similar manner.

This term is much simpler through, because it does not involve any \( \mathcal{D} \) terms. We have

\[\mathcal{L} \int_{\gamma} \kappa_{\sigma} \left( (\kappa - \bar{\kappa})^3 + \bar{\kappa} (\kappa - \bar{\kappa})^2 \right) \, d\sigma \]

\[\leq c \mathcal{L} \int_{\gamma} \left| P_{4}^{2p} (\kappa - \bar{\kappa}) \right| \, d\sigma + c \int_{\gamma} \left| P_{3}^{2p} (\kappa - \bar{\kappa}) \right| \, d\sigma \]
\[ \leq c \mathcal{L}^{-2(p+1)} (\mathcal{K}_{osc})^{1 + \frac{2}{2(p+1)}} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{p+1}^{2} d\sigma \right)^{1 - \frac{1}{2(p+1)}} \\
+ c \mathcal{L}^{-2(p+1)} (\mathcal{K}_{osc})^{\frac{3}{2} + \frac{3}{2(p+1)}} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{p+1}^{2} d\sigma \right)^{1 - \frac{3}{2(p+1)}} \\
\leq c \mathcal{L}^{-2(p+1)} \mathcal{K}_{osc} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{p+1}^{2} d\sigma \right)^{1 - \frac{1}{2(p+1)}} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{p+1}^{2} d\sigma \right)^{1 - \frac{1}{2(p+1)}} \\
+ c \mathcal{L}^{-2(p+1)} \sqrt{\mathcal{K}_{osc}} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{p+1}^{2} d\sigma \right)^{1 - \frac{1}{2(p+1)}} \left( \mathcal{L}^{2p+3} \int_{\gamma} \kappa_{p+1}^{2} d\sigma \right)^{1 - \frac{1}{2(p+1)}} \\
\leq c \left( \mathcal{K}_{osc} + \sqrt{\mathcal{K}_{osc}} \right) \mathcal{L} \int_{\gamma} \kappa_{p+1}^{2} d\sigma. \]

Combining this and (12.10) and substituting into (12.3) then gives

\[ \frac{d}{dt} (\mathcal{K}_{osc} + 8\mathcal{A}(\hat{I})^{2} \ln \mathcal{L}) + \frac{||\kappa_{p}||_{2}^{2}}{\mathcal{L}} \mathcal{K}_{osc} \\
+ \mathcal{L} \left( 2\mathcal{D}_{*} - c\mathcal{K}_{osc} - c \sum_{i=1}^{p-1} (\mathcal{K}_{osc})^{i} \right) \int_{\gamma} \kappa_{p+1}^{2} d\sigma \leq 0. \]

Assuming that the inequality (12.4) holds, the previous inequality can be reduced to (12.4) where \( c_{i} = c_{i} (p, \partial U) \) are universal constants. Integrating this last inequality then gives equation (12.5) under the small-energy assumption (12.4).

We now use the previous proposition to show that under an initially small concentration of curvature (quantified in terms of the Minkowski oscillation of curvature, \( \mathcal{K}_{osc} \)), as well as a simple assumption on the initial isoperimetric ratio, a \( 2 \, (p + 1) \)-anisotropic polyharmonic curve flow (APH) never more-than doubles its initial oscillation of curvature (this argument can in fact be refined to obtain something stronger if one wishes).

\textbf{Proposition 12.4.} Suppose that \( \gamma : \mathbb{S}^{1} \times [0, T) \to \mathcal{M}^{2} \) solves (APH). Additionally, suppose that \( \gamma_{0} \) is a simple closed curve satisfying

\[ \mathcal{K}_{osc} (\gamma_{0}) \leq \mathcal{K}^{*} \text{ and } \mathcal{I} (\gamma_{0}) \leq \exp \left( \mathcal{K}^{*} / 8\mathcal{A}(\hat{I})^{2} \right). \quad (12.11) \]
Then

\[ \mathcal{H}_{\text{osc}}(\gamma) \leq 2K^* \text{ for } t \in [0, T) . \]

**Proof.** Assume for the sake of contradiction that \( \mathcal{H}_{\text{osc}} \) does not remain bounded by \( 2K^* \). Then we can find a maximal \( T^* < T \) such that

\[ \mathcal{H}_{\text{osc}}(\gamma) \leq 2K^* \text{ for } t \in [0, T^*) . \]

Hence by (12.5) the following identity holds for \( t \in [0, T^*) \):

\[ \mathcal{H}_{\text{osc}}(\gamma_t) \leq \mathcal{H}_{\text{osc}}(\gamma_0) + 8\mathcal{A}(\tilde{I})^2 \ln \left( \mathcal{L}(\gamma_0) / \mathcal{L}(\gamma_t) \right) . \] \hspace{1cm} (12.12)

Next the anisotropic isoperimetric inequality gives

\[ \frac{\mathcal{L}(\gamma_0)}{\mathcal{L}(\gamma_t)} \leq \frac{\mathcal{L}(\gamma_0)}{\sqrt{4\mathcal{A}(\tilde{I}) \cdot \mathcal{A}(\gamma_0)}} = \sqrt{\mathcal{J}(\gamma_0)} , \]

and so (12.12) implies that

\[ \mathcal{H}_{\text{osc}}(\gamma_t) \leq \mathcal{H}_{\text{osc}}(\gamma_0) + 4\mathcal{A}(\tilde{I})^2 \ln \mathcal{J}(\gamma_0) \leq K^* + K^*/2 = 3K^*/2 \] \hspace{1cm} (12.13)

which is strictly less than \( 2K^* \). The penultimate step here follows from the assumptions (12.11). Taking \( t \nearrow T^* \) in (12.13) contradicts the definition of \( T^* \), and so we conclude that such a maximal \( T^* \) can not exist. The claim then follows.

\[ \square \]

The preceding proposition will be vital in proving the main theorem in Chapter 13, particularly when showing long time existence of the flow (Theorem 13.5).
Chapter 13

Proof of the main theorem

We are almost ready to prove the major result of this part: long time existence and exponential convergence to homothetic rescaling of the isoperimetrix. We first state the main theorem (Theorem 13.1) before proving it in increments over the course of the chapter.

Theorem 13.1. Suppose that \( \gamma_0 : S^1 \to \mathcal{M}^2 \) is a regular smooth immersed closed curve with \( \mathcal{A}(\gamma_0) > 0 \). Define \( \mathcal{Q} := h^3 (h + h_{\theta\theta}) \), where \( h = r^{-1} \) is the radial support function corresponding to the indicatrix \( \partial \mathcal{U} \). Then there exists a constant \( \varepsilon_0 > 0 \) such that if \( \mathcal{K}_{\text{osc}}(\gamma_0) < \varepsilon_0 \) and \( \mathcal{I}(\gamma_0) < \exp(\varepsilon_0/8\mathcal{A}(\tilde{I})^2) \), then the \((p + 1)\)-anisotropic polyharmonic curve flow \( \gamma : S^1 \times [0, T) \to \mathcal{M}^2 \) with initial data \( \gamma(\cdot, 0) = \gamma_0 \) exists for all time (meaning \( T = \infty \)) and converges exponentially fast to a homothetic rescaling of the isoperimetrix \( \tilde{I} \) with enclosed area \( \mathcal{A}(\gamma_0) \).

We also obtain the following result which establishes an absolute upper bound on the waiting time until our family of immersions \( \gamma(t) \) becomes uniformly convex. The result should be compared to the work of Wheeler in his analysis of the curve diffusion flow in the Euclidean plane [105].
Proposition 13.2 (Upper bound on waiting time until uniform convexity). Suppose that \( \gamma: S^1 \times [0, T) \to M^2 \) satisfies the criteria of Theorem 13.1. Then

\[
\mathcal{L} \{ t \in [0, \infty) : \kappa (\cdot, t) \neq 0 \} \leq \frac{8 \mathcal{A}(\mathcal{I})^{p-1} \cdot \mathcal{A}(\gamma_0)^{p+1}}{(p + 1) \pi^{2p}} \left\{ (\mathcal{I}(\gamma_0))^{p+1} - 1 \right\},
\]

where \( \kappa (\cdot, t) \neq 0 \) is taken to mean there exists at least one \( \sigma \) with \( \kappa (\sigma, t) \leq 0 \).

The proof of Proposition 13.2 is included at the end of the chapter.

Lemma 13.3. Suppose that \( \gamma: S^1 \times [0, T) \to M^2 \) solves (APH). Then for any \( m \geq 0 \) we have

\[
\frac{d}{dt} \int_{\gamma} \kappa_{\sigma m}^2 \, d\sigma \leq c(m, p) \left( \int_{\gamma} \kappa^2 \, d\sigma \right)^{2(m+p)+3},
\]

for some constant \( c > 0 \). Here \( \kappa_{\sigma m} \) refers to the \( m \)th repeated derivative of \( \kappa \) with respect to \( \sigma \).

Proof. We first work out the commutator operator \([\partial_t, \partial_\sigma]\). This is relatively straightforward:

\[
\partial_t \sigma = \partial_t \left( r \left| \gamma_u \right|^{-1} \partial_u \right) = \left( (-1)^{p+1} h \cdot h_\sigma \kappa_{2p+1} \left| \gamma_u \right|^{-1} + r \left( -1 \right)^p \left( h \cdot h_\sigma \kappa_{2p+1} + \kappa \cdot \kappa_{2p} \right) \left| \gamma_u \right|^{-1} \right) \partial_u + \partial_{\sigma t} = \left( -1 \right)^p \kappa \cdot \kappa_{2p} \partial_\sigma + \partial_{\sigma t}.
\]

Hence

\[
[\partial_t, \partial_\sigma] = (-1)^p \kappa \cdot \kappa_{2p} \partial_\sigma.
\]

Repeated applications of (13.2) then gives, for any \( m \in \mathbb{N}_0 \):

\[
\partial_{t \kappa_{\sigma m}} = \partial_{\sigma}^m \partial_t \kappa + (-1)^p \sum_{i=0}^{m-1} \partial_{\sigma}^i \left( \kappa \cdot \kappa_{2p} \cdot \kappa_{m-i} \right) = \left( -1 \right)^p \partial_{\sigma}^{m+1} \left( \partial \kappa_{2p+1} \right) + (-1)^p \sum_{i=0}^{m} \partial_{\sigma}^i \left( \kappa \cdot \kappa_{2p} \cdot \kappa_{m-i} \right),
\]
which implies

\[
\frac{d}{dt} \int_\gamma \kappa_{\sigma m}^2 \, d\sigma \\
= 2 (-1)^p \int_\gamma \kappa_{\sigma m} \partial_\sigma^{m+1} (\mathcal{D}_{\sigma} \kappa_{\sigma m+1}) \, d\sigma + 2 (-1)^p \sum_{i=0}^{m} \int_\gamma \kappa_{\sigma m} \partial_\sigma^{i} (\kappa \cdot \kappa_{\sigma z^p} \cdot \kappa_{\sigma m-i}) \, d\sigma \\
+ (-1)^{p+1} \int_\gamma \kappa_{\sigma m}^2 \cdot \kappa \cdot \kappa_{\sigma z^p} \, d\sigma \\
= 2 (-1)^p \int_\gamma \kappa_{\sigma m} \partial_\sigma^{m+1} (\mathcal{D}_{\sigma} \kappa_{\sigma m+1}) \, d\sigma + 2 (-1)^p \sum_{i=1}^{m} \int_\gamma \kappa_{\sigma m} \partial_\sigma^{i} (\kappa \cdot \kappa_{\sigma z^p} \cdot \kappa_{\sigma m-i}) \, d\sigma \\
+ (-1)^p \int_\gamma \kappa_{\sigma m}^2 \cdot \kappa \cdot \kappa_{\sigma z^p} \, d\sigma. 
\] (13.3)

We have to consider the cases \( p > m, p \leq m \) separately in order to use Lemma A.12 correctly. The idea is that in general we do not want any derivatives of \( \kappa \) that are higher than order \( m + p \).

If \( p > m \), we write \( p = m + \alpha \) for some \( \alpha \in \mathbb{N} \). The equation (13.3) then becomes

\[
\frac{d}{dt} \int_\gamma \kappa_{\sigma m}^2 \, d\sigma \\
= 2 (-1)^{p+m+1} \int_\gamma \mathcal{D}_{\sigma} \kappa_{\sigma 2m+1} \cdot \kappa_{\sigma m+p+1+\alpha} \, d\sigma \\
+ 2 \sum_{i=1}^{m} (-1)^{m+i} \int_\gamma \kappa \cdot \kappa_{\sigma m-i} \cdot \kappa_{\sigma m+i} \cdot \kappa_{\sigma m+p+1+\alpha} \, d\sigma + (-1)^p \int_\gamma \kappa \cdot \kappa_{\sigma m}^2 \cdot \kappa_{\sigma m+p+1+\alpha} \, d\sigma \\
= -2 \int_\gamma \kappa_{\sigma m+p+1} \partial_\sigma^{\alpha} (\mathcal{D}_{\sigma} \kappa_{\sigma 2m+1}) \, d\sigma \\
+ 2 \sum_{i=1}^{m} (-1)^{m+i+\alpha} \int_\gamma \kappa_{\sigma m+p} \partial_\sigma^{\alpha} (\kappa \cdot \kappa_{\sigma m-i} \cdot \kappa_{\sigma m+i}) \, d\sigma \\
+ (-1)^{p+\alpha} \int_\gamma \kappa_{\sigma m+p} \partial_\sigma^{\alpha} (\kappa \cdot \kappa_{\sigma m}^2) \, d\sigma \\
= -2 \int_\gamma \mathcal{D}_{\sigma} \kappa_{\sigma m+p+1} \, d\sigma - 2 \sum_{j=1}^{\alpha} \binom{\alpha}{j} \int_\gamma \mathcal{D}_{\sigma} \kappa_{\sigma m+p+1} \cdot \kappa_{\sigma 2m+1+\alpha-j} \, d\sigma \\
+ 2 \sum_{i=1}^{m} (-1)^{m+i+\alpha} \int_\gamma \kappa_{\sigma m+p} \partial_\sigma^{\alpha} (\kappa \cdot \kappa_{\sigma m-i} \cdot \kappa_{\sigma m+i}) \, d\sigma \\
+ (-1)^{p+\alpha} \int_\gamma \kappa_{\sigma m+p} \partial_\sigma^{\alpha} (\kappa \cdot \kappa_{\sigma m}^2) \, d\sigma \\
\leq -2 \int_\gamma \mathcal{D}_{\sigma} \kappa_{\sigma m+p+1} \, d\sigma - 2 \sum_{j=1}^{\alpha} \binom{\alpha}{j} \int_\gamma \mathcal{D}_{\sigma} \kappa_{\sigma m+p+1} \cdot \kappa_{\sigma m+p+1-j} \, d\sigma
\]
In a similar manner to (12.8), we treat the cases \( j = 1, j < 1 \) separately in the sum on the right hand side of (13.4). For \( j = 1 \) we have

\[
\int_\gamma D_\sigma \kappa_{\sigma m+p+1} \cdot \kappa_{\sigma m+p} \, d\sigma = -\int_\gamma D_\sigma \kappa_{\sigma m+p+1} \cdot \kappa_{\sigma m+p} \, d\sigma - \int_\gamma D_{\sigma^2} \kappa_{\sigma m+p}^2 \, d\sigma,
\]

where we have used integration by parts. Hence

\[
\int_\gamma D_\sigma \kappa_{\sigma m+p+1} \cdot \kappa_{\sigma m+p} \, d\sigma = -\frac{1}{2} \int_\gamma D_{\sigma^2} \kappa_{\sigma m+p}^2 \, d\sigma
\]
\[
= \sum_{i=1}^{2} \int_\gamma \bar{c}_i P_i^{2-i} (\kappa) \kappa_{\sigma m+p}^2 \, d\sigma
\]
\[
\leq c \sum_{i=1}^{2} \int_\gamma \left| P_i^{2(m+p+1)-i,m+p} (\kappa) \right| \, d\sigma.
\]  \hspace{1cm} (13.5)

For \( j > 1 \) we can integrate by parts once, yielding

\[
\int_\gamma D_{\sigma j} \kappa_{\sigma m+p+1} \cdot \kappa_{\sigma m+p+1-j} \, d\sigma = -\int_\gamma \kappa_{\sigma m+p} \left( D_{\sigma j+1} \kappa_{\sigma m+p+1-j} + D_{\sigma j} \kappa_{\sigma m+p+2-j} \right) \, d\sigma
\]
\[
= \sum_{i=1}^{j+1} \int_\gamma \bar{c}_i \kappa_{\sigma m+p} \kappa_{\sigma m+p+1-j} P_i^{j+1-i} (\kappa) \, d\sigma + \sum_{i=1}^{j} \int_\gamma \bar{c}_i \kappa_{\sigma m+p} \kappa_{\sigma m+p+2-j} P_i^{j-i} (\kappa) \, d\sigma
\]
\[
\leq c \sum_{i=1}^{j+1} \int_\gamma \left| P_i^{2(m+p+1)-i,m+p} (\kappa) \right| \, d\sigma.
\]  \hspace{1cm} (13.6)

Substituting (13.5) and (13.6) into (13.4) then yields

\[
\frac{d}{dt} \int_\gamma \kappa_{\sigma m+p}^2 \, d\sigma + 2 \kappa_3 \int_\gamma \kappa_{\sigma m+p+1}^2 \, d\sigma \leq c \sum_{j=1}^{\alpha} \sum_{i=1}^{j+1} \int_\gamma \left| P_i^{2(m+p+1)-i,m+p} (\kappa) \right| \, d\sigma
\]
\[
\leq \varepsilon \int_\gamma \kappa_{\sigma m+p+1}^2 \, d\sigma + c \left( \varepsilon^{-1}, m, p \right) \left( \int_\gamma \kappa_{\sigma m+p+1}^2 \, d\sigma \right)^{2(m+p)+3}.
\]  \hspace{1cm} (13.7)

Here we have used inequality (B.8) from Lemma B.7 in the last step.
Next, if \( p \leq m \) then we write \( m = p + \alpha, \alpha \in \mathbb{N}_0 \) and equation (13.3) becomes

\[
\frac{d}{dt} \int_{\gamma} \kappa_{\sigma m}^2 d\sigma = 2 (-1)^{m+p+1} \int_{\gamma} \mathcal{D}_\kappa \kappa_{\sigma p+1} \kappa_{\sigma m+1} d\sigma + 2 \sum_{i=0}^{p} (-1)^{p+i} \int_{\gamma} \kappa \cdot \kappa_{\sigma m-i} \kappa_{\sigma m+i+1} d\sigma + 2 \sum_{i=p+1}^{p+\alpha} (-1)^{p+i} \int_{\gamma} \kappa \cdot \kappa_{\sigma m-i} \kappa_{\sigma m+p+i} d\sigma + \int_{\gamma} P_4^{2(m+p),m+p} (\kappa) d\sigma
\]

\[
\leq 2 (-1)^{m+p+1} \int_{\gamma} \mathcal{D}_\kappa \kappa_{\sigma p+1} \kappa_{\sigma m+p+1} d\sigma + 2 \sum_{i=p+1}^{p+\alpha} (-1)^{p+i} \int_{\gamma} \kappa \cdot \kappa_{\sigma m-i} \kappa_{\sigma m+p+i} d\sigma + \int_{\gamma} P_4^{2(m+p),m+p} (\kappa) d\sigma
\]

\[
= -2 \int_{\gamma} \kappa_{\sigma m+p+1} \partial_\sigma^\alpha (\mathcal{D}_\kappa \kappa_{\sigma p+1}) d\sigma + 2 \sum_{i=p+1}^{p+\alpha} \int_{\gamma} \kappa_{\sigma m+p} \partial_\sigma^{i-p} (\kappa \cdot \kappa_{\sigma m-i} \kappa_{\sigma 2p}) d\sigma + \int_{\gamma} P_4^{2(m+p),m+p} (\kappa) d\sigma
\]

Here we have used the fact that for \( i \leq p \), \( \max \{m - i, m + i, 2p\} \leq m + p \) in the second and last steps. Hence

\[
\frac{d}{dt} \int_{\gamma} \kappa_{\sigma m}^2 d\sigma + 2 \mathcal{D} \int_{\gamma} \kappa_{\sigma m+p+1}^2 d\sigma 
\leq -2 \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) \int_{\gamma} \kappa_{\sigma m+p+1} \mathcal{D}_\sigma \kappa_{\sigma m+i} \kappa_{\sigma 2p+1-i} d\sigma + \int_{\gamma} P_4^{2(m+p),m+p} (\kappa) d\sigma. \tag{13.8}
\]

Now, we have already dealt with terms in the summation on the right back in (13.5) and (13.6). Hence (13.8) becomes

\[
\frac{d}{dt} \int_{\gamma} \kappa_{\sigma m}^2 d\sigma + 2 \mathcal{D} \int_{\gamma} \kappa_{\sigma m+p+1}^2 d\sigma \leq c \sum_{j=1}^{\alpha+1} \sum_{i=1}^{j+1} \int_{\gamma} P_4^{2(m+p+1)-i,m+p} (\kappa) d\sigma
\]

\[
\leq \varepsilon \int_{\gamma} \kappa_{\sigma m+p+1}^2 d\sigma + c (\varepsilon^{-1}, m, p) \left( \int_{\gamma} \kappa_{\sigma m+p+1} d\sigma \right)^{2(m+p)+3},
\]

for any \( \varepsilon > 0 \). Here we have used (B.8) from Lemma B.7 in the last step. Comparing
this to (13.7) we see that the evolution equation for $\int_\gamma \kappa^2_{\sigma_m} \, d\sigma$ can be estimated in the same way, regardless of the sign of $m - p$. We conclude that

$$\frac{d}{dt} \int_\gamma \kappa^2_{\sigma_m} \, d\sigma + (2\mathcal{D}_* - \varepsilon) \int_\gamma \kappa^2_{\sigma_{m+p+1}} \, d\sigma \leq c(\varepsilon, m, p) \left( \int_\gamma \kappa^2 \, d\sigma \right)^{2(m+p)+3}.$$ 

Choosing $\varepsilon > 0$ sufficiently small then gives (13.1). \hfill \Box

The preceding lemma allows us to characterise finite-time singularities for the polyharmonic curve flows (APH). As a part of our program for proving long time existence ($T = \infty$), we will now show that if the flow becomes extinct in finite time, then we must encounter an $L^2$ curvature concentration as we approach the maximal time $T$. This is a similar argument used by Dziuk, Kuwert and Schätzle in their study of elastic curves in $\mathbb{R}^n$ [26].

**Lemma 13.4.** Suppose $\gamma : S^1 \times [0, T) \to \mathcal{M}^2$ is a maximal solution to (APH). If $T < \infty$, then

$$\left. \int_\gamma \kappa^2 \, d\sigma \right|_t \geq c(T - t)^{-1/2(p+1)}$$

for some universal constant $c > 0$.

**Proof.** By Lemma 13.3, it will be enough to prove that $\limsup_{t \searrow T} \int_\gamma \kappa^2 \, d\sigma = \infty$. We assume for the sake of contradiction that $\int_\gamma \kappa^2 \, d\sigma \leq \varrho < \infty$ for all $t < T$. Our aim is to show that this assumption implies that our one-parameter family of solutions is smooth right up to the maximal time of existence $T$. Local existence results will then allow us to extend the flow smoothly past time $T$ to an interval $[0, T + \delta)$, contradicting the maximality of $T$ (much like in the proof of the lifespan theorem in Chapter 4). Now, by (13.1) we have $\frac{d}{dt} \int_\gamma \kappa^2_{\sigma_m} \, d\sigma \leq c \cdot \varrho^{2(m+p)+3}$ for any $m \in \mathbb{N}_0$. Hence

$$\int_\gamma \kappa^2_{\sigma_m} \, d\sigma \leq \int_\gamma \kappa^2_{\sigma_m} \, d\sigma \bigg|_{t=0} + c \cdot \varrho^{2(m+p)+3} T \leq c_m(\gamma_0, \varrho, T).$$
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Combining this with Lemma A.8, we have for \( m \geq 1 \):

\[
\|\kappa^m\|_\infty^2 \leq \frac{L}{2\pi} \int_\gamma\kappa_{m+1}^2 d\sigma \leq \frac{L(\gamma_0)}{2\pi} \cdot c_{m+1}(\gamma_0, \varrho, T) \leq d_m(\gamma_0, \varrho, T),
\] (13.9)

where \( d_m \) is a new universal constant. Since we intend to show that \( \|\partial^m_u\gamma\|_\infty < c_m \) for every \( m \in \mathbb{N} \), we start by showing that

\[
\|\partial^m_u\gamma\|_\infty < \tilde{c}_m(\gamma_0, \varrho, T) \text{ for every } m \in \mathbb{N},
\] (13.10)

and proceed from there. Assume that (13.10) is true for \( m = 1, 2, \ldots, k \). Then for \( m = k + 1 \), multiple applications of the commutator formula (13.2) gives

\[
\partial_t\partial^{k+1}_\sigma \gamma = \partial^{k+1}_\sigma \partial_t \gamma - \sum_{i=0}^{k} \partial^i\sigma(\kappa \cdot \kappa_{\sigma} \partial^{k+1-i}_\sigma \gamma)
\]

\[
= -\partial^{k+1}_\sigma(\kappa_{\sigma} N) - \sum_{i=0}^{k} \partial^i\sigma(\kappa \cdot \kappa_{\sigma} \partial^{k+1-i}_\sigma \gamma).
\]

This implies that

\[
|\partial_t\partial^{k+1}_\sigma \gamma| \leq c \left( 1 + |\partial^{k+1}_\sigma \gamma| \right).
\] (13.11)

The constant \( c \) here depends only on the \( d_m \) constants from (13.9), along with the \( \tilde{c}_m, m = 1, 2 \ldots k \) from (13.10) and \( \partial U \). We have also used the identity

\[
\|\partial^m_u N\|_\infty \leq c(h, h_\theta, \ldots, h_{g_{m+1}}, \kappa, \kappa_{\sigma}, \ldots, \kappa_{\sigma m+1}) \leq c(\gamma_0, \varrho, T, \partial U),
\]

which can be attained inductively by combining the Frenet equations (10.8) and identity (12.6). Using the Kato inequality on (13.11) gives

\[
|\partial_t|\partial^{k+1}_\sigma \gamma|| \leq c \left( 1 + |\partial^{k+1}_\sigma \gamma| \right).
\]

It follows directly from the previous inequality that for any \( t \in [0, T) \) we have the
following estimate:
\[
\left\| \frac{\partial^{k+1} \gamma}{\partial t} \right\|_{\ell} \leq \left( 1 + \left\| \frac{\partial^{k+1} \gamma}{\partial t} \right\|_{0} \right) e^{ct} - 1 < \left( 1 + \left\| \frac{\partial^{k+1} \gamma}{\partial t} \right\|_{0} \right) e^{cT} - 1 < \tilde{c}_{k+1}(\varrho, \gamma_0, T).
\]

This completes the inductive step and proves (13.10). To jump from (13.10) to proving that
\[
\left\| \frac{\partial^{m} \gamma}{\partial t} \right\|_{\infty} < c_{m}(\varrho, \gamma_0, T, \partial U) \text{ for every } m \in \mathbb{N},
\]
we first employ the identities \( \partial_s = \frac{\kappa}{\kappa + h\theta} \partial_{\theta} \) and \( \partial_s = h \partial_{\sigma} \), which combined with (13.9) and (13.10) gives
\[
\left\| \frac{\partial^{m} \gamma}{\partial t} \right\|_{\infty} \leq c(h, \ldots, h_{\theta m}, \kappa, \ldots, \kappa_{\sigma m-1}) \sum_{i=1}^{m} \left\| \frac{\partial^{m} \gamma}{\partial \sigma} \right\|_{\infty}
\]
\[
< \bar{c}_{m}(\varrho, \gamma_0, T, \partial U)
\]
for every \( m \in \mathbb{N} \). Here \( \bar{c}_m \) is a new universal constant. Applying the same reasoning to (13.9) above also gives
\[
\left\| \kappa_{\sigma m} \right\|_{\infty} \leq c(\gamma_0, \varrho, T, \partial U).
\]

Next by following the inductive argument used in the proof of Theorem 3.1 from [26], we have
\[
\left\| \frac{\partial^{m} u}{\partial t} \right\|_{\infty} \leq c(\varrho, \gamma_0, T, \partial U) \text{ for every } m \in \mathbb{N}.
\]
Unfortunately, applying the Kato inequality on this result does not give (13.12) because the inequality is the opposite to what we need. Fortunately, following along in the inductive argument of [26] gives
\[
\left\| \frac{\partial^{m} \gamma}{\partial t} \right\|_{\infty} \leq \left\| \frac{\partial^{m} \gamma}{\partial t} \right\|_{\infty} + \left\| P \right\|_{\infty}
\]
where \( P = P(|\gamma_0|, \partial_u |\gamma_0|, \ldots, \partial^{m-1}_u |\gamma_0|, \kappa, \kappa_s, \ldots, \kappa_{s m+1}, \partial U) \) is a polynomial. Applying (13.13) and (13.14) to this inequality then proves identity (13.12). Therefore \( \gamma(\cdot, t) \) is smooth right up until time \( T \), and hence can be extended to some larger time.
interval $[0, T + \delta)$. This contradicts the maximality of $T$. Hence the assumption that

$$\limsup_{t \to T} \int_\gamma \kappa^2 d\sigma < \infty$$

must have been incorrect, and we conclude that

$$\limsup_{t \to T} \int_\gamma \kappa^2 d\sigma = \infty.$$ 

Finally, the inequality in Lemma 13.3 with $m = 0$ gives

$$-\frac{1}{2(m + p + 1)} \frac{d}{dt} \left( \int_\gamma \kappa^2 ds \right)^{-2(m+p+1)} \leq c.$$ 

Hence integrating over $[t, T), t < T$ gives

$$-\frac{1}{2(m + p + 1)} \left( \int_\gamma \kappa^2 ds \right)^{-2(m+p+1)} \bigg|_t^{c} \leq c (T - t),$$ 

and rearranging this proves the lemma.

The preceding characterisation of finite-time singularities for the polyharmonic curve flows (APH) allows us to prove the first part of Theorem 13.1: long time existence ($T = \infty$).

**Theorem 13.5.** Suppose that $\gamma : S^1 \times [0, T) \to \mathcal{M}^2$ solves (APH). Additionally, suppose that $\gamma_0$ is a simple closed curve satisfying

$$\mathcal{K}_{osc}(\gamma_0) \leq K^* \text{ and } \mathcal{I}(\gamma_0) \leq \exp \left( \frac{K^*}{8 \mathcal{A}(\hat{I})^2} \right).$$

Then $T = \infty$.

**Proof.** Suppose for the sake of contradiction that $T < \infty$. Then by Lemma 13.4 we have

$$\int \kappa^2 d\sigma \nearrow \infty \text{ as } t \nearrow T.$$ 

This implies that $\mathcal{K}_{osc}$ must diverge as we approach time $T$ as well, since

$$\mathcal{K}_{osc}(\gamma) = \mathcal{L} \int_\gamma \kappa^2 d\sigma - 4\mathcal{A}(\hat{I})^2 \geq \sqrt{4\mathcal{A}(\gamma_0) \cdot \mathcal{A}(\hat{I})} \int \kappa^2 d\sigma - 4\mathcal{A}(\hat{I})^2.$$
Here the last step follows from the isoperimetric inequality. However, this directly contradicts Proposition 12.4. We conclude that the assumption that $T < \infty$ must have been false.

Recall we know that if $\gamma : S^1 \times [0, T) \to M^2$ is a $2(p + 1)$-anisotropic polyharmonic curve flow and satisfies the hypothesis of Theorem 13.5, then $T = \infty$. Hence identity (12.1) then implies that under these same assumptions we have

$$\mathcal{H}_{osc} \in L^1([0, \infty)), \text{ with } \int_0^\infty \mathcal{H}_{osc}(\tau) \, d\tau \leq \frac{1}{2(p + 1)(2\pi)^{2p}} L^{2(p+1)}(\gamma_0) < \infty.$$

So we can conclude that the quantity $\mathcal{H}_{osc}$ must be approaching zero along a subsequence for sufficiently large times. However, at the present time we have not ruled out the possibility that $\mathcal{H}_{osc}$ gets smaller and smaller in time as $t$ gets large, whilst vibrating with higher and higher frequency, remaining in $L^1([0, \infty))$ whilst never actually fully dissipating to zero in any smooth sense. To rule out this from happening, it is enough to bound its time derivative (this is done in Theorem 13.7). To do this we will need to first show that $\|\kappa_{\sigma^p}\|_2^2$ remains bounded. We will address this issue with the following proposition.

**Proposition 13.6.** Suppose $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ solves (APH) and that $\gamma_0$ has positive enclosed area. There exists a $\varepsilon_0 > 0$ (with $\varepsilon_0 \leq \mathcal{K}_\sigma$) such that if

$$\mathcal{H}_{osc}(\gamma_0) < \varepsilon_0 \quad \text{and} \quad \mathcal{I}(0) < \exp\left(\frac{\varepsilon_0}{8.\mathcal{A}(\tilde{\mathcal{I}})^2}\right),$$

then $\|k_{\sigma^p}\|_2^2$ remains bounded for all time. In particular

$$\int_{\gamma} \kappa_{\sigma^p}^2 \, d\sigma \leq c(\gamma_0),$$

and so $\|k_{\sigma^p}\|_2^2$ can be controlled a priori.
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Proof. Applying (13.3) with \( m = p \) gives

\[
\frac{d}{dt} \int_\gamma \kappa_{\sigma}^2 d\sigma + 2 \Omega \int_\gamma \kappa_{\sigma}^2 \sigma_{\sigma+1} d\sigma \\
\leq 2 (-1)^p \sum_{i=1}^p \int_\gamma \kappa_{\sigma} \partial_{\sigma} \left( \kappa \cdot \kappa_{\sigma-p-i} \cdot \kappa_{\sigma+2p} \right) d\sigma + (-1)^p \int_\gamma \kappa_{\sigma}^2 \cdot \kappa \cdot \kappa_{\sigma+2p} d\sigma \\
= 2 \sum_{i=1}^p (-1)^{p+i} \int_\gamma \kappa \cdot \kappa_{\sigma-p-i} \cdot \kappa_{\sigma+2p} \cdot \kappa_{\sigma+2p} d\sigma + (-1)^p \int_\gamma \kappa_{\sigma}^2 \cdot \kappa \cdot \kappa_{\sigma+2p} d\sigma \\
= 2 \sum_{i=1}^p (-1)^{p+i} \int_\gamma (\kappa - \bar{\kappa}) (\kappa - \bar{\kappa})_{\sigma-p-i} (\kappa - \bar{\kappa})_{\sigma+2p} d\sigma \\
\quad + 2 \int_\gamma \kappa^2 (\kappa - \bar{\kappa})_{\sigma+2p} d\sigma + (-1)^p \int_\gamma \kappa_{\sigma}^2 \cdot \kappa \cdot \kappa_{\sigma+2p} d\sigma \\
\leq 2 \int_\gamma \kappa^2 (\kappa - \bar{\kappa})_{\sigma+2p} d\sigma + \int_\gamma \left| P_{4p,2p}^p (\kappa - \bar{\kappa}) \right| d\sigma \\
\quad + \mathcal{L}^{-1} \int_\gamma \left| P_{3,4p,2p}^p (\kappa - \bar{\kappa}) \right| d\sigma. \tag{13.15}
\]

Next, from the last result of Lemma A.12 as well as Lemma A.7, we have

\[
\int_\gamma \left| P_{4p,2p}^p (\kappa - \bar{\kappa}) \right| d\sigma \\
\leq c \mathcal{L}^{-4(p+1)} \left( \mathcal{K}_{\text{osc}} \right)^{1+ \frac{1}{4(2p+1)}} \left( \mathcal{L}^{2(2p+1)} \int_\gamma \kappa_{\sigma+2p}^2 d\sigma \right)^{1- \frac{1}{4(2p+1)}} \\
\leq c \mathcal{L}^{-4(p+1)} \mathcal{K}_{\text{osc}} \left( \mathcal{L}^{2(2p+1)} \int_\gamma \kappa_{\sigma+2p}^2 d\sigma \right)^{\frac{1}{4(2p+1)}} \left( \mathcal{L}^{2(2p+1)} \int_\gamma \kappa_{\sigma+2p}^2 d\sigma \right)^{1- \frac{1}{4(2p+1)}} \\
= c \mathcal{K}_{\text{osc}} \int_\gamma \kappa_{\sigma+2p}^2 d\sigma,
\]

as well as

\[
\mathcal{L}^{-1} \int_\gamma \left| P_{3,4p,2p}^p (\kappa - \bar{\kappa}) \right| d\sigma \\
\leq \mathcal{L}^{-4(p+1)} \left( \mathcal{K}_{\text{osc}} \right)^{\frac{3}{4} + \frac{3}{4(2p+1)}} \left( \mathcal{L}^{2(2p+1)} \int_\gamma \kappa_{\sigma+2p}^2 d\sigma \right)^{1- \frac{3}{4(2p+1)}} \\
\leq c \mathcal{L}^{-4(p+1)} \sqrt{\mathcal{K}_{\text{osc}}} \left( \mathcal{L}^{2(2p+1)} \int_\gamma \kappa_{\sigma+2p}^2 d\sigma \right)^{\frac{3}{4(2p+1)}} \left( \mathcal{L}^{2(2p+1)} \int_\gamma \kappa_{\sigma+2p}^2 d\sigma \right)^{1- \frac{3}{4(2p+1)}} \\
= c \sqrt{\mathcal{K}_{\text{osc}}} \int_\gamma \kappa_{\sigma+2p}^2 d\sigma.
\]
Hence (13.15) becomes

\[
\frac{d}{dt} \int_\gamma \kappa_{\sigma p}^2 \, d\sigma + \left( 2 \mathcal{Q}_* - c \mathcal{K}_{osc} - c \sqrt{\mathcal{K}_{osc}} \right) \int_\gamma \kappa_{\sigma 2p+1}^2 \, d\sigma \leq 2 \int_\gamma \kappa^2 (\kappa - \kappa)_{\sigma 2p}^2 \, d\sigma \quad (13.16)
\]

The term on the right can be estimated easily by using our earlier \( P \)-style estimates, along with Lemma B.7:

\[
2 \int_\gamma \kappa^2 (\kappa - \bar{\kappa})_{\sigma 2p}^2 \, d\sigma
\]

\[
= 2 \int_\gamma \left( (\kappa - \bar{\kappa})^2 + \bar{\kappa}^2 + 2\bar{\kappa} (\kappa - \bar{\kappa}) \right) (\kappa - \bar{\kappa})_{\sigma 2p}^2 \, d\sigma
\]

\[
\leq 2 \bar{\kappa}^2 \int_\gamma \kappa_{\sigma 2p}^2 \, d\sigma + \int_\gamma \left| \mathcal{P}_{4p,2p}^{4p,2p} (\kappa - \bar{\kappa}) \right| \, d\sigma + \mathcal{L}^{-1} \int_\gamma \left| \mathcal{P}_{3}^{4p,2p} (\kappa - \bar{\kappa}) \right| \, d\sigma
\]

\[
\leq \frac{2 \mathcal{A}(\tilde{\mathcal{I}})^2}{\mathcal{L}^2} \int_\gamma \kappa_{\sigma 2p}^2 \, d\sigma + c \left( \mathcal{K}_{osc} + \sqrt{\mathcal{K}_{osc}} \right) \int_\gamma \kappa_{\sigma 2p+1}^2 \, d\sigma
\]

\[
\leq \frac{2 \mathcal{A}(\tilde{\mathcal{I}})^2}{\mathcal{L}^2} \left( \varepsilon \mathcal{L}^2 \int_\gamma \kappa_{\sigma 2p+1}^2 \, d\sigma + \frac{1}{4\varepsilon^2 p} \mathcal{L}^{-4p-1} \mathcal{K}_{osc} \right)
\]

\[
+ c \left( \mathcal{K}_{osc} + \sqrt{\mathcal{K}_{osc}} \right) \int_\gamma \kappa_{\sigma 2p+1}^2 \, d\sigma
\]

\[
= 2 \mathcal{A}(\tilde{\mathcal{I}})^2 \varepsilon \int_\gamma \kappa_{\sigma 2p+1}^2 \, d\sigma + c \left( \mathcal{K}_{osc} + \sqrt{\mathcal{K}_{osc}} \right) \int_\gamma \kappa_{\sigma 2p+1}^2 \, d\sigma + \frac{\mathcal{A}(\tilde{\mathcal{I}})^2}{2\varepsilon^2 p} \mathcal{L}^{-4p-3} \mathcal{K}_{osc}.
\]

This inequality holds for any \( \varepsilon > 0 \). Hence substituting into (13.16), we have

\[
\frac{d}{dt} \int_\gamma \kappa_{\sigma p}^2 \, d\sigma + \left( 2 \mathcal{Q}_* - c \mathcal{K}_{osc} - c \sqrt{\mathcal{K}_{osc}} - 2 \mathcal{A}(\tilde{\mathcal{I}})^2 \varepsilon \right) \int_\gamma \kappa_{\sigma 2p+1}^2 \, d\sigma
\]

\[
\leq \frac{\mathcal{A}(\tilde{\mathcal{I}})^2}{2\varepsilon^2 p} \mathcal{L}^{-4p-3} \mathcal{K}_{osc}.
\]

Choosing \( \varepsilon \) sufficiently small and integrating over \([0, t]\) for \( t \leq T \) gives

\[
\int_\gamma \kappa_{\sigma p}^2 \, d\sigma \bigg|_t \leq c \int_0^t \mathcal{L}^{-4p-3} \mathcal{K}_{osc} \, d\tau \leq c \left( 2 \mathcal{A}(\gamma_0) \mathcal{A}(\tilde{\mathcal{I}})^2 \right)^{\frac{4p+3}{4p+1}} \int_0^t \mathcal{K}_{osc} \, d\tau \leq c(\gamma_0, p).
\]

Here we have used the isoperimetric inequality in the penultimate step, and the fact that \( \mathcal{K}_{osc} \in L^1 ([0, \infty)) \) in the last step. This completes the proof.

The preceding proposition allows us to ascertain that our flow actually converges.
in the $C^\infty$ topology. We present these results in the following theorem.

**Theorem 13.7.** Suppose $\gamma : S^1 \times [0, T) \to \mathcal{M}^2$ solves (APH) and that $\gamma_0$ is simple with positive enclosed area. There exists an $\varepsilon_0 > 0$ (with $\varepsilon_0 \leq K^*$) such that if

$$K_{osc}(\gamma_0) < \varepsilon_0 \quad \text{and} \quad \mathcal{I}(\gamma_0) < \exp\left(\frac{\varepsilon_0}{8\mathcal{A}(\tilde{I})^2}\right),$$

then $\gamma(S^1)$ converges in the $C^\infty$ topology to a homothetic rescaling and translation of the isoperimetrix $\partial U$ with area equal to $\mathcal{A}(\gamma_0)$.

**Proof.** We begin by showing that $K_{osc} \searrow 0$ as $t \nearrow \infty$. We will then discuss the ramification of this result. Recall from a previous discussion that to show $K_{osc} \searrow 0$, it will be enough to show that $|K'_{osc}|$ is bounded for all time. First by Theorem 13.5 we know that for $\varepsilon_0 > 0$ sufficiently small, $T = \infty$. Moreover by Proposition 12.4, $K_{osc} \leq 2\varepsilon_0$ for all time and by Proposition 12.2 we have the estimate

$$\left|\frac{d}{dt}K_{osc}\right| \leq \left(\frac{2\mathcal{A}(\tilde{I})^2 - K_{osc}}{\mathcal{L}}\right)\|\kappa_{p}\|^2 \leq \frac{2\mathcal{A}(\tilde{I})^2}{\sqrt{2\mathcal{A}(\tilde{I}) \cdot \mathcal{A}(\gamma_0)}}\|\kappa_{p}\|^2 < c(\gamma_0, p, \partial U).$$

We have used the results from Proposition 13.6 in the last step, and the isoperimetric inequality in the penultimate step. This immediately tells us that $K_{osc} \searrow 0$ as $t \nearrow \infty$. We will denote the limiting immersion by $\gamma_\infty$. That is,

$$\gamma_\infty := \lim_{t \to \infty} \gamma_t(S^1) = \lim_{t \to \infty} \gamma(\cdot, t).$$

Our earlier equations imply that $K_{osc}(\gamma_\infty) \equiv 0$. Note that because the isoperimetric inequality forces $\mathcal{L}(\gamma_\infty) \geq \sqrt{2\mathcal{A}(\tilde{I}) \cdot A(\gamma_0)} > 0$, we can not have $\mathcal{L} \searrow 0$ and so we may conclude that

$$\int_{\gamma_\infty} (\kappa - \bar{\kappa})^2 d\sigma = 0.$$

From our discussion at the very beginning of the chapter, it follows that $\gamma_\infty$ is a homothetic rescaling of the isoperimetrix $\tilde{I}$. Since the enclosed area does not change
under the polyharmonic flow, this homothetic rescaling indeed has enclosed (Euclidean) area equal to $\mathcal{A}(\gamma_0)$. We can use this to find the scaling factor for $\gamma_\infty$. □

The previous theorem tells us that for any $l \in \mathbb{N}$ there is a sequence of times $\{t_i\}$ such that

$$\left\{ \int_\gamma \kappa_{\sigma_i}^2 \, d\sigma \right\} \searrow 0.$$  

This is only sequential convergence, and much like in the case of $\mathcal{H}_{osc}$, doesn’t rule out the possibility of sharp ‘spikes’ (oscillations) in time. Like in that case, to overcome this we will need to control $\left| \frac{d}{dt} \int_\gamma \kappa_{\sigma_i}^2 \, d\sigma \right|$ and will essentially do so by bounding the quantity by a multiple of $\mathcal{H}_{osc}(0)$ which can be controlled a priori. This allows us to obtain classical exponential convergence.

**Theorem 13.8** (Exponential Convergence). Suppose $\gamma : \mathbb{S}^1 \times [0, T) \to \mathcal{M}^2$ solves (APH) as well as the assumptions of Theorem 13.7. Then for each $m \in \mathbb{N} \setminus \{0\}$ there is a time $t_m$ sufficiently large such that for $t \geq t_m$ there are constants $c_m, c^*_m$ with

$$\int_\gamma \kappa_{\sigma_m}^2 \, d\sigma \leq c_m e^{-c^*_m t}, \quad (13.17)$$

and

$$\| \kappa_{\sigma_m} \|_\infty^2 \leq (\mathcal{L}(\gamma_0) c_{m+1}/2\pi) e^{-c^*_m t}. \quad (13.18)$$

**Proof.** Using our earlier calculations from the proof of Lemma 13.3, we have

$$\frac{d}{dt} \int_\gamma \kappa_{\sigma_m}^2 \, d\sigma + 2\mathcal{D}_* \int_\gamma \kappa_{\sigma_{m+1}}^2 \, d\sigma$$

$$\leq \sum_{j=1}^{p-m} \sum_{i=1}^{j-1} \sum_{l=0}^{\tilde{\kappa}-1} \tilde{c}_i \mathcal{P}_{i-l}^{j-i} (\kappa - \tilde{\kappa}) \cdot (\kappa - \tilde{\kappa})_{\sigma_{m+p}} (\kappa - \tilde{\kappa})_{\sigma_{m+p-l}} \, d\sigma$$

$$+ \int_\gamma \left| \mathcal{P}_4^{2(m+p),m+p} (\kappa - \tilde{\kappa}) \right| \, d\sigma + \mathcal{L}^{-1} \int_\gamma \left| \mathcal{P}_3^{2(m+p),m+p} (\kappa - \tilde{\kappa}) \right| \, d\sigma$$

$$+ 2 (-1)^{m-p} \tilde{\kappa}^2 \int_\gamma \kappa_{\sigma_{m+p}}^2 \, d\sigma$$

$$\leq \sum_{j=1}^{p-m} \sum_{i=1}^{j-1} \mathcal{L}^{-1} \int_\gamma \left| \mathcal{P}_{i-l+2}^{2(m+p)-j,m+p} (\kappa - \tilde{\kappa}) \right| \, d\sigma + \int_\gamma \left| \mathcal{P}_4^{2(m+p),m+p} (\kappa - \tilde{\kappa}) \right| \, d\sigma$$
\[ \leq c \left( \mathcal{H}_{osc} + \sqrt{\mathcal{H}_{osc}} \right) \int_{\gamma} \kappa_{\sigma_m+p+1}^2 \, d\sigma + 2 \left( -1 \right)^{|m-p|} \kappa_{\sigma_m+p}^2 \int_{\gamma} \kappa_{\sigma_m+p}^2 \, d\sigma. \]

Next we claim that for any smooth closed curve \( \gamma \) and any \( l \in \mathbb{N} \) there exists a universal bounded constant \( c_l \) such that
\[ \int_{\gamma} \kappa_{\sigma_l}^2 \, d\sigma \leq c_l \mathcal{L}^2(\gamma) \mathcal{H}_{osc}(\gamma) \int_{\gamma} \kappa_{\sigma_l+1}^2 \, d\sigma. \] (13.19)

To prove this, we assume for the sake of contradiction that (13.19) is not true. Then there exists a series of immersions \( \{ \gamma_j \} \) such that
\[ R_j := \frac{\| \kappa_{\sigma_l} \|_{2,\gamma_j}^2}{\mathcal{L}^2(\gamma_j) \mathcal{H}_{osc}(\gamma_j) \| \kappa_{\sigma_l+1} \|_{2,\gamma_j}^2} \rightarrow \infty \text{ as } j \rightarrow \infty. \] (13.20)

But by Lemma A.7, for each \( j \) we have
\[ R_j \leq \frac{\varphi^2(\gamma_j) \| \kappa_{\sigma_l+1} \|_{2,\gamma_j}^2}{\mathcal{L}^2(\gamma_j) \mathcal{H}_{osc}(\gamma_j) \| \kappa_{\sigma_l+1} \|_{2,\gamma_j}^2} = \frac{1}{4\pi^2 \mathcal{H}_{osc}(\gamma_j)}, \]
and so the only way for (13.20) to occur is if
\[ \mathcal{H}_{osc}(\gamma_j) \searrow 0 \text{ as } j \rightarrow \infty. \] (13.21)

Then, as each \( \gamma_j \) satisfies the criteria of Theorem B.8, we conclude there is a subsequence of immersions \( \{ \gamma_{j_k} \} \) and an immersion \( \gamma_{\infty} \) such that \( \gamma_{j_k} \rightarrow \gamma_{\infty} \) in the \( C^1 \)-topology. Moreover, by (13.21), we have \( \mathcal{H}_{osc}(\gamma_{\infty}) = 0 \). But this implies \( \gamma_{\infty} \) must be a homothetic rescaling of the isoperimetrix \( \tilde{I} \), in which case both sides on inequality (13.19) are zero. Hence the inequality holds trivially for the immersion \( \gamma_{\infty} \) with any \( c_l \) we wish, and so in fact we do not have \( R_j \not\rightarrow \infty \). This contradicts (13.20), and we conclude that (13.19) must be true. Hence we conclude that
\[ \frac{d}{dt} \int_{\gamma} \kappa_{\sigma_m}^2 \, d\sigma + \left( 2 \mathcal{D} - c \left( \mathcal{H}_{osc} + \sqrt{\mathcal{H}_{osc}} \right) \right) \int_{\gamma} \kappa_{\sigma_m+p+1}^2 \, d\sigma \leq 0. \]
for some universal constant \( c \). Then, since \( \mathcal{H}_{osc} \searrow 0 \), there exists a time \( t_m \) such that
for \( t \geq t_m \),
\[
\frac{d}{dt} \int_{\gamma} \kappa_{\gamma m}^2 \, d\sigma \leq -\mathcal{Q}\int_{\gamma} \kappa_{\gamma m+p+1}^2 \, d\sigma \leq -\mathcal{Q}_* \left( \frac{2\pi}{\mathcal{L}} \right)^{2(p+1)} \int_{\gamma} \kappa_{\gamma m}^2 \, d\sigma. \quad (13.22)
\]

Here we have used Lemma A.7 \((p + 1)\) times. Integrating (13.22) over \([t_m, t]\) gives
\[
\int_{\gamma} \kappa_{\gamma m}^2 \, d\sigma \bigg|_t \leq \left( \int_{\gamma} \kappa_{\gamma m}^2 \, d\sigma \bigg|_{t_m} \exp(\mathcal{Q}_* (2\pi/\mathcal{L})^{2(p+1)} t_m) \right) \exp(-\mathcal{Q}_* (2\pi/\mathcal{L})^{2(p+1)} t).
\]

This gives (13.17) with an appropriate choice of \( c_m \) and \( c_m^* \). Combining this with Lemma A.8 then gives (13.18).

\[ \square \]

Using the same methods as in the proof of Theorem 8.1 in Part I of the thesis, for sufficiently large times we can write the immersion \( \gamma(t, \cdot) \) as a radial graph around some single point, and linearise to show that that the limit immersion (which is a homothetic rescaling and translation of the isoperimetrix) is exponentially attractive. Combining the results of Theorem 13.7 and Theorem 13.8 then proves our main result, Theorem 13.1.

We finish off with the proof of Proposition 13.2. It is similar in structure to the proof of Proposition 1.5 from [105].

**Proof of Proposition 13.2.** Recall that by the main theorem, we know that \( T = \infty \).

We may assume without loss of generality that there exists a time \( t_0 \) such that
\[
\begin{cases}
\kappa (\cdot, t) \not\equiv 0 \text{ for } t \in [0, t_0), \text{ and} \\
\kappa (\cdot, t) > 0 \text{ for } t \in [t_0, \infty).
\end{cases}
\]

We may also assume without loss of generality that
\[
t_0 > \frac{8\mathcal{A}(\mathcal{I})^{p-1} (\mathcal{A}(\gamma_0))^{p+1}}{(p+1)\pi^{2p}} \left\{ (\mathcal{A}(\gamma_0))^{p+1} - 1 \right\}, \quad (13.23)
\]
otherwise the proposition is trivially true. Next, our equation for the evolution of the Minkowski length from Corollary 11.2, along with Lemma B.2 and Lemma B.4 gives

\[
\frac{d}{dt} \mathcal{L} = - \int_\gamma \kappa_\sigma^2 \, d\sigma \leq - \left( \frac{2\pi}{\mathcal{L}} \right)^{2(p-1)} \int_\gamma \kappa_\sigma^2 \, d\sigma \leq - \frac{\pi^2}{\mathcal{L}^2} \left( \frac{2\pi}{\mathcal{L}} \right)^{2(p-1)} \int_\gamma \kappa^2 \, d\sigma. \quad (13.24)
\]

Note that we have used Wirtinger’s inequality (Lemma B.4) in the last step, which is valid because for times \([0, t_0]\) there exists a point on the curve with zero curvature. Next, Hölder’s inequality implies that

\[
\mathcal{A}^2 \left( \tilde{I} \right)^2 = \left( \int_\gamma \kappa \, d\sigma \right)^2 \leq \mathcal{L} \int_\gamma \kappa^2 \, d\sigma.
\]

Hence (13.24) becomes

\[
\frac{d}{dt} \mathcal{L}^{2(p+1)} \leq -2(p+1)\pi^2(2\pi)^{2(p-1)}\mathcal{A}(\tilde{I})^2.
\]

After integrating in time over \([0, t_0]\), we find that

\[
\mathcal{L}^{2(p+1)} (t_0) \leq \mathcal{L}^{2(p+1)} (0) - 2(p+1)\pi^2(2\pi)^{2(p-1)}\mathcal{A}(\tilde{I})^2 t_0.
\]

However, the choice of \(t_0\) from (13.23) then implies that

\[
\mathcal{L}^{2(p+1)} (t_0) < \left( 4\mathcal{A}(\tilde{I}) \mathcal{A} (t_0) \right)^{p+1},
\]

which contradicts the isoperimetric inequality (I). Hence the proposition must be true.

This completes Part II of the thesis.
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Appendix A

Inequalities pertaining to Part I

We start with the proofs of two inequalities that are used in Part I: (5.11) and (6.8).

Proof. Firstly note that by using the same methods as we did to prove (5.4), the following identity holds:

\[ \Delta \nabla \varphi = \nabla \Delta \varphi + \frac{1}{2} R \nabla \varphi = \nabla \Delta \varphi + \frac{1}{4} \nabla \varphi \left( H^2 - 2 |A^o|^2 \right) . \] (A.1)

Therefore using integration by parts gives

\[
\int_\Sigma |(\nabla (2) \varphi)|^2 \, \gamma^4 \, d\mu = - \int_\Sigma \langle \nabla \varphi, \nabla \Delta \varphi \rangle \, \gamma^4 \, d\mu + \int_\Sigma \nabla \varphi \cdot \nabla (2) \varphi \cdot \nabla \gamma \, \gamma^3 \, d\mu \\
\leq - \int_\Sigma \langle \nabla \varphi, \nabla \Delta \varphi \rangle \, \gamma^4 \, d\mu - \frac{1}{4} \int_\Sigma |\nabla \varphi|^2 \left( H^2 - 2 |A^o|^2 \right) \, d\mu \\
+ \int_\Sigma \nabla \varphi \cdot \nabla (2) \varphi \cdot \nabla \gamma \, \gamma^3 \, d\mu \\
= \int_\Sigma |\Delta \varphi|^2 \, \gamma^4 \, d\mu - \frac{1}{4} \int_\Sigma |\nabla \varphi|^2 \left( H^2 - 2 |A^o|^2 \right) \, d\mu \\
+ \int_\Sigma \nabla \varphi \cdot \nabla (2) \varphi \cdot \nabla \gamma \, \gamma^3 \, d\mu \\
\leq \int_\Sigma |\Delta \varphi|^2 \, \gamma^4 \, d\mu - \frac{1}{4} \int_\Sigma |\nabla \varphi|^2 \left( H^2 - 2 |A^o|^2 \right) \, d\mu \\
+ c \gamma \int_\Sigma |(\nabla (2) \varphi)| \, |\nabla \varphi| \, \gamma^3 \, d\mu . \] (A.2)
Next using Hölder’s inequality and the Peter-Paul inequality, we have

\[ c_\gamma \int_\Sigma |\nabla_2 \varphi| |\nabla \varphi| \gamma^3 d\mu \leq c_\gamma \left( \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu \right)^{\frac{1}{2}} \left( \int_\Sigma |\nabla_2 \varphi|^2 \gamma^4 d\mu \right)^{\frac{1}{2}} \leq \eta \int_\Sigma |\nabla_2 \varphi|^2 \gamma^4 d\mu + c (\eta^{-1}) c_\gamma \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu \]

for any \( \eta > 0 \). Substituting this back into (A.2) and applying Theorem A.2 gives

\[
(1 - \eta) \int_\Sigma |\nabla_2 \varphi|^2 \gamma^4 d\mu + \frac{1}{4} \int_\Sigma |\nabla \varphi|^2 H^2 \gamma^4 d\mu \\
\leq \int_\Sigma |\Delta \varphi|^2 \gamma^4 d\mu + \frac{1}{2} \int_\Sigma |\nabla \varphi|^2 |A^o|^2 \gamma^4 d\mu + c (\eta^{-1}) c_\gamma \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu \\
\leq \int_\Sigma |\Delta \varphi|^2 \gamma^4 d\mu + c \left( \int_\Sigma |\nabla_2 \varphi| |A^o| \gamma^2 d\mu + \int_\Sigma |\nabla \varphi| |\nabla A^o| \gamma^2 d\mu \right) + c (\eta^{-1}) c_\gamma \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu \\
\leq \int_\Sigma |\Delta \varphi|^2 \gamma^4 d\mu \\
+ c \int_{\gamma > 0} |A^o|^2 d\mu \cdot \left( \int_\Sigma |\nabla_2 \varphi|^2 \gamma^4 d\mu + c_\gamma \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu + \int_\Sigma |\nabla \varphi|^2 H^2 \gamma^4 d\mu \right) \\
+ c \int_\Sigma |\nabla A^o|^2 \gamma^2 d\mu \cdot \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu + c (\eta^{-1}) c_\gamma \int_\Sigma |\nabla \varphi|^2 \gamma^2 d\mu
\]

Therefore if \( \varepsilon_0 > 0 \) is sufficiently small, and if we choose \( \eta > 0 \) small enough, the inequality (5.11) follows.

To prove (6.8) we proceed similar to before. Using integration by parts and identity (A.1) gives

\[
\int_\Sigma |\nabla_2 \varphi|^2 H^2 d\mu = - \int_\Sigma \langle \nabla \varphi, \Delta \nabla \varphi \rangle H^2 d\mu + \int_\Sigma \nabla_2 \varphi \ast \nabla \varphi \ast \nabla H H d\mu \\
= - \int_\Sigma \langle \nabla \varphi, \nabla \Delta \varphi \rangle H^2 d\mu - \frac{1}{4} \int_\Sigma |\nabla \varphi|^2 H^4 d\mu + \frac{1}{2} \int_\Sigma |\nabla \varphi|^2 |A^o|^2 H^2 d\mu \\
+ \int_\Sigma \nabla_2 \varphi \ast \nabla \varphi \ast \nabla H H d\mu \\
\leq \left( \int_\Sigma |\nabla \varphi|^2 H^4 d\mu \right)^{\frac{1}{2}} \left( \int_\Sigma |\nabla \Delta \varphi|^2 d\mu \right)^{\frac{1}{2}} - \frac{1}{4} \int_\Sigma |\nabla \varphi|^2 H^4 d\mu \\
+ \frac{1}{2} \int_\Sigma |\nabla \varphi|^2 |A^o|^2 H^2 d\mu + c \left( \int_\Sigma |\nabla_2 \varphi|^2 H^2 d\mu \right)^{\frac{1}{2}} \left( \int_\Sigma |\nabla \varphi|^2 |\nabla H|^2 d\mu \right)^{\frac{1}{2}}
\]
APPENDIX A. INEQUALITIES PERTAINING TO PART I

\[ \leq \eta \cdot \left( \int_{\Sigma} |\nabla (2)\varphi|^2 H^2 \, d\mu + \int_{\Sigma} |\nabla \varphi|^2 H^4 \, d\mu \right) \]

\[ + c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla \varphi|^2 \, d\mu + c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla \varphi|^2 |\nabla H|^2 \, d\mu \]

\[ - \frac{1}{4} \int_{\Sigma} |\nabla \varphi|^2 H^4 \, d\mu + \frac{1}{2} \int_{\Sigma} |\nabla \varphi|^2 |A^o|^2 H^2 \, d\mu \]

for any \( \eta > 0 \). Here we have used the Peter-Paul inequality twice in the last step.

Therefore applying rearranging and applying Theorem A.2 with \( u = |\nabla \varphi| |A^o| |H| \), we have

\[ (1 - \eta) \int_{\Sigma} |\nabla (2)\varphi|^2 H^2 \, d\mu + \left( \frac{1}{4} - \eta \right) \int_{\Sigma} |\nabla \varphi|^2 H^4 \, d\mu \]

\[ \leq c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla \Delta \varphi|^2 \, d\mu + c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla \varphi|^2 |\nabla H|^2 \, d\mu \]

\[ + c \left( \int_{\Sigma} |\nabla (2)\varphi| |A^o| |H| \, d\mu + \int_{\Sigma} |\nabla \varphi| |\nabla A^o| |H| \, d\mu + \int_{\Sigma} |\nabla \varphi| |A^o| |\nabla H| \, d\mu \right) \]

\[ + \int_{\Sigma} |\nabla A^o|^2 |A^o|^2 H^2 \, d\mu \]

\[ \leq c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla \varphi|^2 \, d\mu \leq c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla \varphi|^2 |\nabla H|^2 \, d\mu \]

\[ + c \int_{\Sigma} |A^o|^2 \, d\mu \cdot \left( \int_{\Sigma} |\nabla (2)\varphi|^2 H^2 \, d\mu + \int_{\Sigma} |\nabla \varphi|^2 H^4 \, d\mu + \int_{\Sigma} |\nabla \varphi|^2 |\nabla H|^2 \, d\mu \right) \]

\[ + c \int_{\Sigma} |\nabla A^o|^2 \, d\mu \cdot \int_{\Sigma} |\nabla \varphi|^2 H^2 \, d\mu \]

Therefore

\[ (1 - \eta - c \|A^o\|^2)^2 \int_{\Sigma} |\nabla (2)\varphi|^2 H^2 \, d\mu + \left( \frac{1}{4} - \eta - c \|A^o\|^2 \right) \int_{\Sigma} |\nabla \varphi|^2 H^4 \, d\mu \]

\[ \leq c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla \Delta \varphi|^2 \, d\mu + c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla \varphi|^2 |\nabla H|^2 \, d\mu \]

\[ + c \int_{\Sigma} |\nabla A^o|^2 \, d\mu \cdot \int_{\Sigma} |\nabla \varphi|^2 H^2 \, d\mu \]

(A.3)

for any \( \eta > 0 \). Next, using Theorem A.2 with \( u = |\nabla \varphi| |\nabla H| \) gives

\[ \int_{\Sigma} |\nabla \varphi|^2 |\nabla H|^2 \, d\mu \]
\[ \leq c \left( \int_\Sigma |\nabla (2) \varphi| |\nabla H| \, d\mu + \int_\Sigma |\nabla \varphi| |\nabla (2) H| \, d\mu + \int_\Sigma |\nabla \varphi| |\nabla H| |H| \, d\mu \right)^2 \]
\[ \leq c \int_\Sigma |\nabla A^o|^2 \, d\mu \cdot \left( \int_\Sigma |\nabla (2) \varphi|^2 \, d\mu + \int_\Sigma |\nabla \varphi|^2 H^2 \, d\mu \right) \]
\[ + c \int_\Sigma |\nabla (2) H|^2 \, d\mu \cdot \int_\Sigma |\nabla \varphi|^2 \, d\mu. \]

Therefore, substituting this into (A.3) and using (5.11) yields

\[ (1 - \eta - c \|A^o\|_2^2) \int_\Sigma |\nabla (2) \varphi|^2 H^2 \, d\mu + \left( \frac{1}{4} - \eta - c \|A^o\|_2^2 \right) \int_\Sigma |\nabla \varphi|^2 H^4 \, d\mu \]
\[ \leq c (\eta^{-1}) \int_\Sigma |\nabla \Delta \varphi|^2 \, d\mu + c (\eta^{-1}) \int_\Sigma |\nabla A^o|^2 \, d\mu \cdot \left( \int_\Sigma |\nabla (2) \varphi|^2 \, d\mu + \int_\Sigma |\nabla \varphi|^2 H^2 \, d\mu \right) \]
\[ + c (\eta^{-1}) \int_\Sigma |\nabla (2) H|^2 \, d\mu \cdot \int_\Sigma |\nabla \varphi|^2 \, d\mu \]
\[ \leq c (\eta^{-1}) \int_\Sigma |\nabla \Delta \varphi|^2 \, d\mu \]
\[ + c (\eta^{-1}) \int_\Sigma |\nabla A^o|^2 \, d\mu \cdot \left( \int_\Sigma |\Delta \varphi|^2 \, d\mu + \int_\Sigma |\nabla A^o|^2 \, d\mu \cdot \int_\Sigma |\nabla \varphi|^2 \, d\mu \right) \]
\[ + c (\eta^{-1}) \left( \int_\Sigma |\Delta H|^2 \, d\mu + \left( \int_\Sigma |\nabla A^o|^2 \, d\mu \right)^2 \right) \cdot \int_\Sigma |\nabla \varphi|^2 \, d\mu \]
\[ \leq c (\eta^{-1}) \int_\Sigma |\nabla \Delta \varphi|^2 \, d\mu + c (\eta^{-1}) \left( \int_\Sigma |\Delta H|^2 \, d\mu + \left( \int_\Sigma |\nabla A^o|^2 \, d\mu \right)^2 \right) \cdot \int_\Sigma |\nabla \varphi|^2 \, d\mu \]
\[ + c (\eta^{-1}) \int_\Sigma |\nabla A^o|^2 \, d\mu \cdot \int_\Sigma |\Delta \varphi|^2 \, d\mu \]
\[ \leq c (\eta^{-1}) \int_\Sigma |\nabla \Delta \varphi|^2 \, d\mu + c (\eta^{-1}) \left( 1 + \|A^o\|_2^2 \right) \cdot \int_\Sigma |\Delta H|^2 \, d\mu \cdot \int_\Sigma |\nabla \varphi|^2 \, d\mu \]
\[ + c (\eta^{-1}) \int_\Sigma |\nabla A^o|^2 \, d\mu \cdot \int_\Sigma |\Delta \varphi|^2 \, d\mu. \]

Here we have used the results of Corollary 5.10 in the last step. It follows that if \( \varepsilon_0 > 0 \) is small enough and if \( \eta > 0 \) is chosen sufficiently small then the inequality (6.8) follows. \( \square \)

The following theorem is classical result originally due to Codazzi. Recall than an immersion is said to be umbilic at a point if the second fundamental form is zero there. If the second fundamental form is identically zero, then the immersion is said to be an
umbilic immersion.

**Theorem A.1.** Let $f : \Sigma^2 \to \mathbb{R}^3$ be a connected regular, umbilic immersion. Then $f$ is a plane or sphere.

**Proof.** We begin by showing that $f$ has constant curvature $K \geq 0$.

Since $f$ is umbilic, the principal curvatures $\kappa_1$ and $\kappa_2$ are equal everywhere. Let $\varphi : f (\Sigma) \to \mathbb{R}$ be a function such that $\varphi (p)$ is the common value of the principal curvature at $p \in f (\Sigma)$. Therefore $H (p) = 2 \varphi (p)$ and for a tensor $X \in Tf (\Sigma)$ we have

$$S (X) \big|_p = -g^{ik} X^k A_{ik} f_j \big|_p = -g^{ij} X^k \left( \frac{1}{2} H g_{ik} \right) f_j \big|_p = -\varphi X^j f_j \big|_p.$$  

Here $f_i$ denotes the partial derivative of $f$ with respect to the chosen coordinate system $\{x^i\}_{i=1,2}$ for $\Sigma$. Letting $X = f_1, f_2$ we obtain the following equations which hold at every point $p \in f (\Sigma)$:

$$S (f_1) = -\varphi f_1 \text{ and } S (f_2) = -\varphi f_2.$$  

Taking the derivative of the first identity with respect to $x^2$ and the second identity with respect to $x^1$ then yields

$$\partial_2 S (f_1) = - (\varphi_2 f_1 + \varphi f_{12}) \text{ and } \partial_2 S (f_1) = - (\varphi_1 f_2 + \varphi f_{12}). \quad (A.4)$$  

Next, since $\mathbb{R}^3$ is flat, $\partial_i \nu = \partial_i \nu$ where $\nu$ is a chosen unit normal to $f (\Sigma)$ in $\mathbb{R}^3$. Thus equating the two equations of (A.4) gives

$$\varphi_1 f_2 - \varphi_2 f_1 \equiv 0.$$  

The linear independence of $f_1$ and $f_2$ therefore shows that $\varphi_1 = \varphi_2 = 0$, and hence $\varphi$ is constant. Since the Gaussian curvature $K$ is given by $K = \varphi^2$ we conclude it is non-negative and constant.
Next we show that $f$ either maps $\Sigma$ into a plane (if $K = 0$) or sphere (if $K > 0$). We treat the cases separately:

- If $K = 0$ then $\partial_1 \nu = \partial_2 \nu = 0$ which implies $\nu$ is constant and therefore $f(\Sigma)$ is contained in the plane which is perpendicular to $\nu$.

- If instead $K > 0$ then choose an arbitrary point $x \in f(\Sigma)$, along with a unit normal vector $\nu(x)$ to $f(\Sigma)$ at $x$. We will show that $f(\Sigma)$ is contained in a sphere centred at the point $c := x + \varphi^{-1}\nu(x)$. Let $y$ be an arbitrary point in $f(\Sigma)$ and 
  \[ \gamma: (a, b) \to f(\Sigma) \]
  be a curve with $a < 0 < 1 < b$ and $\gamma(0) = x, \gamma(1) = y$. We extend $\nu(x)$ to a unit normal vector field $\nu \circ \gamma$ along $\gamma$. Then, by defining $\tilde{\gamma}: (a, b) \to \mathbb{R}^3$ by
  \[ \tilde{\gamma}(x) = \gamma(t) + \varphi^{-1}\nu(\gamma(t)). \]

By the definition of the shape operator, it follows that

\[ \tilde{\gamma}'(t) = \gamma'(t) + \varphi^{-1}(\nu \circ \gamma)'(t) \]
\[ = \gamma'(t) - \varphi^{-1}S(\gamma'(t)) \]
\[ = \gamma'(t) - \varphi^{-1}(\varphi \gamma'(t)) \]
\[ = 0, \]

and $\tilde{\gamma}$ is a constant which is equal to $\tilde{\gamma}(1) =: c$. We conclude that

\[ |\gamma(t) - c| = \varphi^{-1}|\nu| = |\varphi|^{-1}, \]

which implies $\gamma$ maps to a sphere of radius $|\varphi|^{-1}$ centred at $c$. This proves the result.
Theorem A.2 (Michael-Simon Sobolev Inequality, [77] Theorem 2.1). Let \( f : \Sigma \rightarrow \mathbb{R}^m \) be an immersed closed hypersurface, and \( u \in C^1(\Sigma) \) be non-negative. Then

\[
\int_{\Sigma} u^{\frac{n}{n-(n-1)}} d\mu \leq c_{\text{MSS}} \left( \int_{\Sigma} |\nabla u| + |\vec{H}| u d\mu \right)^{\frac{n}{n-(n-1)}}.
\]

Here \( c_{\text{MSS}} = c_{\text{MSS}}(n) \) is explicitly given by \( c_{\text{MSS}} = \left( \frac{4^m (m+1)/(m-1)}{\omega_n^{1/(m-1)}} \right) \), where \( \omega_n \) is the Euclidean volume of the \( n \)-dimensional unit ball, and \( \vec{H} \) denotes the mean curvature vector.

Lemma A.3 (Gronwall’s Inequality). Let \( \alpha, \beta, u \) be real valued functions on the interval \( I \) which is either of the form \([a,b], [a,\infty)\) (with \( a < b \)) or \([a,\infty)\). Assume \( \beta, u \) are continuous and \( \alpha^- \) is integrable on every closed and bounded subinterval of \( I \).

If \( \beta \) is non-negative and if \( u \) satisfies

\[
u(t) \leq \alpha(t) + \int_a^t \beta(s) u(s) ds, \quad \forall t \in I,
\]

then

\[
u(t) \leq \alpha(t) + \int_a^t \left( \alpha(s) \beta(s) e^{\int_a^s \beta(r) dr} ds \right), \quad \forall t \in I.
\]

If, in addition, the function \( \alpha \) is non-decreasing, then

\[
u(t) \leq \alpha(t) e^{\int_a^t \beta(s) ds}, \quad \forall t \in I.
\]

Proof. For the first assertion, define

\[
v(s) = e^{-\int_a^s \beta(r) dr} \int_a^s \beta(r) u(r) dr.
\]

Then, by the chain rule and the fundamental theorem of calculus,

\[
v'(s) = \left( u(s) - \int_a^s (\beta(r) u(r) dr) \right) \beta(s) e^{-\int_a^s \beta(r) dr}
\]

\[
\leq \alpha(s) \beta(s) e^{-\int_a^s \beta(r) dr}, \quad \text{by (A.5)}.
\]
We can then integrate both sides and apply the fundamental theorem of calculus, which tells us that

\[ v (t) \leq \int_a^t \alpha (s) \beta (r) e^{-\int_a^s \beta (r) dr} ds, \quad \forall t \in I. \quad (A.6) \]

Here we have used \( v (a) = 0 \). Now, by the definition of \( v \), we have

\[ \int_a^t \beta (s) u (s) ds = e^{\int_a^t \beta (r) dr} v (t) \]
\[ \leq e^{\int_a^t \beta (r) dr} \int_a^t \alpha (s) \beta (r) e^{-\int_a^s \beta (r) dr} ds, \quad \text{by (A.6)} \]
\[ = \int_a^t \alpha (s) \beta (r) e^{-\int_a^s \beta (r) dr} ds. \]

Substituting this into our initial inequality for \( u \) then gives us

\[ u (t) \leq \alpha (t) + \int_a^t \beta (s) u (s) ds \leq \alpha (t) + \int_a^t \alpha (s) \beta (r) e^{-\int_a^s \beta (r) dr} ds, \]

which is the first assertion of the lemma.

For the second assertion, we assume that \( \alpha \) is non-decreasing and note that this means \( \alpha (s) \leq \alpha (t) \) for \( s \in [a, t] \). Hence our inequality from the first assertion then becomes

\[ u (t) \leq \alpha (t) \left( 1 + \int_a^t \beta (s) e^{\int_a^s \beta (r) dr} ds \right). \quad (A.7) \]

We then let \( F (s) = e^{\int_a^s \beta (r) dr}, f (s) = -\beta (s) e^{\int_a^s \beta (r) dr}, \) which means \( F' = -f \). It follows from the fundamental theorem of calculus that

\[ \int_a^t \beta (s) e^{\int_a^s \beta (r) dr} ds = -\int_a^t f (s) ds = F (a) - F (t) = e^{\int_a^t \beta (r) dr} - 1. \]

Substituting this into (A.7) then yields

\[ u (t) \leq \alpha (t) \left( 1 + \int_a^t \beta (s) e^{\int_a^s \beta (r) dr} ds \right) \]
\[ = \alpha (t) \left( 1 + \left( e^{\int_a^t \beta (r) dr} - 1 \right) \right) \]
\[
\alpha(t) e^\int_a^t \beta(r) dr,
\]
which is the statement of the second assertion. 

**Remark A.4.** In the case \( u \) is non-negative and \( \alpha \equiv 0 \), it follows that \( u \equiv 0 \).

The next lemma appears in Hamilton’s seminal paper.

**Lemma A.5** (Hamilton [42], Lemma 12.5). Assume that \( f(k) \) is a real-valued function of the integer \( k \) for \( 0 \leq k \leq n \). If \( f(k) \) satisfies

\[
f(k) \leq C f(k - 1)^{\frac{1}{2}} f(k + 1)^{\frac{1}{2}},
\]

then

\[
f(k) \leq C^{k(n-k)} f(0)^{1 - \frac{k}{n}} f(n)^{\frac{k}{n}}.
\]

**Theorem A.6** (Kuwert and Schätzle [58], Theorem 5.6). Let \( f : \Sigma^n \to \mathbb{R}^{n+1} \) be a smooth immersion. For \( u \in C^1_c(\Sigma), n < p \leq \infty, 0 \leq m \leq \infty \) and \( 0 < \alpha \leq 1 \) where \( \frac{1}{\alpha} = \frac{1}{n} - \frac{1}{p} \) the following inequality holds:

\[
\|u\|_{\infty} \leq c \|u\|_m^{1-\alpha} \left( \|\nabla u\|_p + \|Hu\|_p \right)^{\alpha}.
\]

Here \( c = c(n,m,p) \).

**Lemma A.7.** Let \( 1 \leq p, q, r \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \). Also, assume \( \alpha, \beta \geq 0 \) with \( \alpha + \beta = 1 \), and \( m \in \mathbb{N} \). Then, for \( s \geq \max \{ \alpha q, \beta p \} \) and \( -\frac{1}{p} \leq t \leq \frac{1}{q} \) we have

\[
c_{\gamma}^m \left( \int_{\Sigma} |\nabla T|^{2r} \gamma^s \, d\mu \right)^{\frac{1}{2}} \\
\leq c_1 \cdot c_{\gamma}^m \left( \int_{\Sigma} |T|^q \gamma^{s(1-tq)} \, d\mu \right)^{\frac{1}{q}} \left( \int_{\Sigma} |\nabla(2)T|^{p} \gamma^{s(1+tp)} \, d\mu \right)^{\frac{1}{p}} \\
+ c_2 \cdot c_{\gamma}^{m+1} \left( \int_{\Sigma} |T|^q \gamma^{s-\alpha q} \, d\mu \right)^{\frac{1}{q}} \left( \int_{\Sigma} |\nabla T|^{p} \gamma^{s-\beta p} \, d\mu \right)^{\frac{1}{p}}
\]

for some universal constants \( c_1 = c_1(r), c_2 = c_2(s) > 0 \).
\textbf{Proof.} The result follows from an application of integration by parts and the Hölder inequality:

\begin{align*}
& c^m_\gamma \int_\Sigma |\nabla T|^{2r} \gamma^s \, d\mu \\
& \leq (2r - 2) c^m_\gamma \int_\Sigma |\nabla (2T)| |\nabla T|^{2r-2} |T| \gamma^s \, d\mu + s \cdot c^m_\gamma \int_\Sigma |\nabla T|^{2r-1} |T| |\nabla \gamma| \gamma^{s-1} \, d\mu \\
& \leq (2r - 2) c^m_\gamma \int_\Sigma |\nabla (2T)| |\nabla T|^{2r-2} |T| \gamma^s \, d\mu + s \cdot c^{m+1}_\gamma \int_\Sigma |\nabla T|^{2r-1} |T| \gamma^{s-1} \, d\mu \\
& \leq \left( \int_\Sigma |\nabla T|^{2r} \gamma^s \, d\mu \right)^{\frac{r-1}{r}} c^m_\gamma \left( 2r - 2 \right) \left( \int_\Sigma |T|^q \gamma^{s(1-tq)} \, d\mu \right)^{\frac{1}{q}} \left( \int_\Sigma |\nabla (2T)| \gamma^{s(1+tp)} \, d\mu \right)^{\frac{1}{p}} \\
& \quad + s \cdot c_\gamma \left( \int_\Sigma |T|^q \gamma^{s-\alpha q} \, d\mu \right)^{\frac{1}{q}} \left( \int_\Sigma |\nabla T|^p \gamma^{s-\beta p} \, d\mu \right)^{\frac{1}{p}}.
\end{align*}

Now, assuming that \( \nabla T \neq 0 \) (In fact, if \( \nabla T \equiv 0 \) then the lemma is trivially true), we can divide through, yielding

\begin{align*}
& c^m_\gamma \left( \int_\Sigma |\nabla T|^{2r} \gamma^s \, d\mu \right)^{1 - \frac{r-1}{r}} \\
& \leq (2r - 2) c^m_\gamma \left( \int_\Sigma |T|^q \gamma^{s(1-tq)} \, d\mu \right)^{\frac{1}{q}} \left( \int_\Sigma |\nabla (2T)| \gamma^{s(1+tp)} \, d\mu \right)^{\frac{1}{p}} \\
& \quad + s \cdot c^{m+1}_\gamma \left( \int_\Sigma |T|^q \gamma^{s-\alpha q} \, d\mu \right)^{\frac{1}{q}} \left( \int_\Sigma |\nabla T|^p \gamma^{s-\beta p} \, d\mu \right)^{\frac{1}{p}}.
\end{align*}

Noting that \( 1 - \frac{r-1}{r} = \frac{1}{r} \), we obtain the desired result. \qed

\textbf{Lemma A.8.} For \( s \geq 2 \) and \( m \geq 1 \) we have

\begin{align*}
& c^m_\gamma \left( \int_\Sigma |\nabla T|^p \gamma^s \, d\mu \right)^{\frac{1}{p}} \\
& \leq \delta \cdot c^{m-1}_\gamma \left( \int_\Sigma |\nabla (2T)|^p \gamma^{s+p} \, d\mu \right)^{\frac{1}{p}} + c_\delta \cdot c^{m+1}_\gamma \left( \int_\Sigma |T|^p \gamma^{s-p} \, d\mu \right)^{\frac{1}{p}},
\end{align*}

where \( \delta, c_\delta = c_\delta (\delta, p, s) > 0 \) and \( \delta > 0 \) can be made as small as desired.

\textbf{Proof.} We let \( p = q = 2r, \alpha = 1, \beta = 0, t = \frac{1}{s} \) and making the substitutions \( m \mapsto 2m \)
in the previous Lemma, and then use Young’s Inequality:

\[
\begin{align*}
&c_\gamma^{2m} \left( \int_\Sigma |\nabla T|^p \gamma^s d\mu \right)^{\frac{2}{p}} \\
&\leq c_1 \cdot c_\gamma^{2m} \left( \int_\Sigma |T|^p \gamma^{s-p} d\mu \right)^{\frac{1}{p}} \left( \int_\Sigma |(2) T|^p \gamma^{s+p} d\mu \right)^{\frac{1}{p}} \\
&+ c_2 \cdot c_\gamma^{2m+1} \left( \int_\Sigma |T|^p \gamma^{s-p} d\mu \right)^{\frac{1}{p}} \left( \int_\Sigma |\nabla T|^p \gamma^s d\mu \right)^{\frac{1}{p}} \\
&\leq \frac{1}{2} c_\gamma^{2m} \left( \int_\Sigma |\nabla T|^p \gamma^s d\mu \right)^{\frac{2}{p}} + \frac{\delta^2}{2} c_\gamma^{2m-2} \left( \int_\Sigma |(2) T|^p \gamma^{s+p} d\mu \right)^{\frac{2}{p}} \\
&+ c_3 \cdot c_\gamma^{2m+2} \left( \int_\Sigma |T|^p \gamma^{s-p} d\mu \right)^{\frac{2}{p}},
\end{align*}
\]

where \( c_3 = c_3(\delta, p, s) > 0 \) and \( \delta > 0 \) can be made as small as desired. Absorbing and then multiplying out by 2 then yields

\[
\begin{align*}
&c_\gamma^m \left( \int_\Sigma |\nabla T|^p \gamma^s d\mu \right)^{\frac{1}{p}} \\
&\leq \delta \cdot c_\gamma^{m-1} \left( \int_\Sigma |(2) T|^p \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + c_3 \cdot c_\gamma^{m+1} \left( \int_\Sigma |T|^p \gamma^{s-p} d\mu \right)^{\frac{1}{p}},
\end{align*}
\]

meaning that

\[
\begin{align*}
&c_\gamma^m \left( \int_\Sigma |\nabla T|^p \gamma^s d\mu \right)^{\frac{1}{p}} \\
&\leq \delta \cdot c_\gamma^{m-1} \left( \int_\Sigma |(2) T|^p \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + \sqrt{c_3} \cdot c_\gamma^{m+1} \left( \int_\Sigma |T|^p \gamma^{s-p} d\mu \right)^{\frac{1}{p}},
\end{align*}
\]

which is the statement of the Lemma. Note that the last line follows from the inequality

\[(A^2 + B^2)^{\frac{1}{2}} \leq A + B \text{ for } A, B \geq 0. \]

\[\square\]

**Lemma A.9.** For \( k \in \mathbb{N}, s \geq kp \) and \( m \geq 1 \) we have

\[
\begin{align*}
&c_\gamma^m \left( \int_\Sigma |(k) T|^p \gamma^s d\mu \right)^{\frac{1}{p}} \\
&\leq \delta \cdot c_\gamma^{m-1} \left( \int_\Sigma |(k+1) T|^p \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + c_\delta \cdot c_\gamma^{m+k} \left( \int_\Sigma |T|^p \gamma^{s-kp} d\mu \right)^{\frac{1}{p}},
\end{align*}
\]
where $\delta, c_\delta = c_\delta (\delta, p, s) > 0$ and $\delta$ can be made as small as desired.

**Proof.** The case $k = 1$ was covered in the previous Lemma. Assume the case is true for a general $k$, that is assume

$$
\begin{align*}
c^n_\gamma \left( \int_\Sigma |\nabla (k) T|^p \gamma^s d\mu \right)^{\frac{1}{p}} \\
&\leq \lambda \cdot c^{n-1}_\gamma \left( \int_\Sigma |\nabla (k+1) T|^p \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + c \cdot c^{n+k}_\gamma \left( \int_\Sigma |T|^p \gamma^{s-kp} d\mu \right)^{\frac{1}{p}},
\end{align*}
$$

(A.8)

for all $n \geq 1$ and all $s \in \mathbb{R}$ with $s \geq kp$.

We will then use this to go on and show

$$
\begin{align*}
c^m_\gamma \left( \int_\Sigma |\nabla (k+1) T|^p \gamma^s d\mu \right)^{\frac{1}{p}} \\
&\leq \delta \cdot c^{m-1}_\gamma \left( \int_\Sigma |\nabla (k+2) T|^p \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + c \cdot c^{m+k}_\gamma \left( \int_\Sigma |T|^p \gamma^{s-(k+1)p} d\mu \right)^{\frac{1}{p}},
\end{align*}
$$

(A.9)

for all $m \geq 1$ and all $s \in \mathbb{R}$ with $s \geq (k+1)p$. Let $m \geq 1$. Then by our previous Lemma, we have

$$
\begin{align*}
c^m_\gamma \left( \int_\Sigma |\nabla (k+1) T|^p \gamma^s d\mu \right)^{\frac{1}{p}} \\
&\leq \delta \cdot c^{m-1}_\gamma \left( \int_\Sigma |\nabla (k+2) T|^p \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + c^* \cdot c^{m+1}_\gamma \left( \int_\Sigma |\nabla (k) T|^p \gamma^{s-p} d\mu \right)^{\frac{1}{p}}.
\end{align*}
$$

Substituting assumption (A.8) with $n = m + 1$, $\lambda = \frac{1}{2c^*}$ (where $c^*$ is the same constant in our above inequality) and $s \mapsto s - p$ we then have

$$
\begin{align*}
c^m_\gamma \left( \int_\Sigma |\nabla (k+1) T|^p \gamma^s d\mu \right)^{\frac{1}{p}} \\
&\leq \delta \cdot c^{m-1}_\gamma \left( \int_\Sigma |\nabla (k+2) T|^p \gamma^{s+p} d\mu \right)^{\frac{1}{p}} + c^* \left( \frac{1}{2c^*} \cdot c^m_\gamma \left( \int_\Sigma |\nabla (k+1) T|^p \gamma^s d\mu \right)^{\frac{1}{p}} + c \cdot c^{(m+1)+k}_\gamma \left( \int_\Sigma |T|^p \gamma^{(s-p)-kp} d\mu \right)^{\frac{1}{p}} \right).
\end{align*}
$$
Absorbing and multiplying out by 2 then yields (A.9). Hence our inductive step is complete for the case $k + 1$. Remembering that the previous lemma provided the base case for the induction process, we have now proven the Lemma.

Lemma A.10. Suppose $T$ is a tensor field, and let $k \in \mathbb{N}$ and $s \geq 2k$. Then there exists a constant $c = c(k, s)$ such that

\[
\int_{\Sigma} |\nabla (k) T|^2 \gamma^s d\mu \leq c \left( \int_{\Sigma} |T|^2 \gamma^{s-2k} d\mu \right)^{\frac{k}{k+m}} \left( \int_{\Sigma} |\nabla (k+1) T|^2 \gamma^{s+2m} d\mu \right)^{\frac{k}{k+m}} + c \cdot c_{\gamma}^{(m+1)+k} \left( \int_{\Sigma} |T| \gamma^{(s-p)-kp} d\mu \right)^{\frac{k}{p}}.
\]  

(A.10)

Moreover, for any $k, m \in \mathbb{N}$ the following estimate holds:

\[
\int_{\Sigma} |\nabla (k) T|^2 \gamma^s d\mu \leq c \left( \int_{\Sigma} |T|^2 \gamma^{s-2k} d\mu \right)^{\frac{m}{k+m}} \left( \int_{\Sigma} |\nabla (k+m) T|^2 \gamma^{s+2m} d\mu \right)^{\frac{k}{k+m}} + c \cdot c_{\gamma}^{2k} \int_{\Sigma} |T|^2 \gamma^{s-2k} d\mu.
\]  

(A.11)

Proof. Firstly we prove the intermediate inequality

\[
\int_{\Sigma} |\nabla (k) T|^2 \gamma^s d\mu \leq c \left( \int_{\Sigma} |\nabla (k-1) T|^2 \gamma^{s-2} d\mu \right)^{\frac{1}{2}} \left( \int_{\Sigma} |\nabla (k+1) T|^2 \gamma^{s+2} d\mu \right)^{\frac{1}{2}} + c \cdot c_{\gamma}^{2k} \int_{\Sigma} |T|^2 \gamma^{s-2k} d\mu.
\]  

(A.12)

which holds for any tensor $T$ and $k, s \in \mathbb{N}$ with $s \geq 2k$. To prove (A.12) we use
integration by parts and the results of Lemma A.9:

\[
\int_{\Sigma} |\nabla_{(k)} T|^2 \gamma^s \, d\mu \\
= \int_{\Sigma} \nabla_{(k-1)} T * \nabla_{(k+1)} T * \gamma^s \, d\mu + \int_{\Sigma} \nabla_{(k)} T * \nabla_{(k-1)} T * \nabla \gamma \gamma^{s-1} \, d\mu \\
\leq c \left( \int_{\Sigma} |\nabla_{(k-1)} T|^2 \gamma^{s-2} \, d\mu \right)^{\frac{1}{2}} \cdot \left( \int_{\Sigma} |\nabla_{(k+1)} T|^2 \gamma^{s+2} \, d\mu \right)^{\frac{1}{2}} \\
+ c c_{\gamma} \left( \int_{\Sigma} |\nabla_{(k-1)} T|^2 \gamma^{s-2} \, d\mu \right)^{\frac{1}{2}} \cdot \left( \int_{\Sigma} |\nabla_{(k)} T|^2 \gamma^s \, d\mu \right)^{\frac{1}{2}} \\
\leq c \left( \int_{\Sigma} |\nabla_{(k-1)} T|^2 \gamma^{s-2} \, d\mu \right)^{\frac{1}{2}} \cdot \left( \int_{\Sigma} |\nabla_{(k)} T|^2 \gamma^s \, d\mu \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \int_{\Sigma} |\nabla_{(k)} T|^2 \gamma^s \, d\mu + c c_{\gamma} \int_{\Sigma} |\nabla_{(k-1)} T|^2 \gamma^{s-2} \, d\mu \\
\leq c \left( \int_{\Sigma} |\nabla_{(k-1)} T|^2 \gamma^{s-2} \, d\mu \right)^{\frac{1}{2}} \cdot \left( \int_{\Sigma} |\nabla_{(k)} T|^2 \gamma^s \, d\mu \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \int_{\Sigma} |\nabla_{(k)} T|^2 \gamma^s \, d\mu + \eta \int_{\Sigma} |\nabla_{(k)} T|^2 \gamma^s \, d\mu \\
+ c (\eta^{-1}) c_{\gamma}^{2k} \int_{\Sigma} |T|^2 \gamma^{s-2k} \, d\mu.
\]

Here we have used the Cauchy-Schwarz inequality in the penultimate line and the results from Lemma A.9 in the last line. Choosing \( \eta > 0 \) sufficiently small and absorbing into the left hand side proves (A.12). We use this inequality to prove (A.10). First note that by (A.12) we already know that (A.10) holds for \( k = 1 \). Assume that it holds for a general \( m \in \mathbb{N} \):

\[
\int_{\Sigma} |\nabla_{(m)} T|^2 \gamma^s \, d\mu \\
\leq c \left( \int_{\Sigma} |T|^2 \gamma^{s-2m} \, d\mu \right)^{\frac{1}{m+1}} \left( \int_{\Sigma} |\nabla_{(m+1)} T|^2 \gamma^{s+2} \, d\mu \right)^{\frac{m}{m+1}} + c c_{\gamma}^{2m} \int_{\Sigma} |T|^2 \gamma^{s-2m} \, d\mu.
\]

Here we require that \( s \geq 2m \). We proceed inductively. By (A.12) with \( k = m + 1 \) it follows that

\[
\int_{\Sigma} |\nabla_{(m+1)} T|^2 \gamma^s \, d\mu
\]
\[ \begin{align*}
\leq & \ c \left( \int_{\Sigma} |\nabla_{(m)} T|^2 \gamma^{s-2} \, d\mu \right)^\frac{1}{2} \cdot \left( \int_{\Sigma} |\nabla_{(m+2)} T|^2 \gamma^{s+2} \, d\mu \right)^\frac{1}{2} \\
& + \ c \, c^2_{\gamma (m+1)} \int_{\Sigma} |T|^2 \gamma^{-2(m+1)} \, d\mu \\
\leq & \ c \left( \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu \right)^\frac{1}{m+1} \left( \int_{\Sigma} |\nabla_{(m+1)} T|^2 \gamma^{s} \, d\mu \right)^\frac{m}{m+1} \\
& + \ c \, c^2_{\gamma} \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu \frac{1}{2} \times \left( \int_{\Sigma} |\nabla_{(m+2)} T|^2 \gamma^{s+2} \, d\mu \right)^\frac{1}{2} \\
& + \ c \, c^2_{\gamma (m+1)} \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu.
\end{align*} \]

Next, Young’s inequality with conjugate exponents \( p = \frac{2(m+1)}{m} \) and \( p^* = \frac{2(m+1)}{m+2} \) implies that
\[ \left( \int_{\Sigma} |T|^2 \gamma^{-2(m+1)} \, d\mu \right)^\frac{1}{m+1} \left( \int_{\Sigma} |\nabla_{(m+1)} T|^2 \gamma^{s} \, d\mu \right)^\frac{m}{m+1} \left( \int_{\Sigma} |\nabla_{(m+2)} T|^2 \gamma^{s+2} \, d\mu \right)^\frac{1}{2} \]
\[ \leq \eta \int_{\Sigma} |\nabla_{(m+1)} T|^2 \gamma^{s} \, d\mu \\
+ \ c \left( \eta^{-1} \right) \left( \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu \right)^\frac{1}{m+2} \left( \int_{\Sigma} |\nabla_{(m+2)} T|^2 \gamma^{s+2} \, d\mu \right)^\frac{m+1}{m+2} \]
for any \( \eta > 0 \). Similarly, Young’s inequality again with conjugate exponents \( p = \frac{2(m+1)}{m} \) and \( p^* = \frac{2(m+1)}{m+2} \) gives
\[ \begin{align*}
& c^m_{\gamma} \left( \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu \right)^\frac{1}{2} \left( \int_{\Sigma} |\nabla_{(m+2)} T|^2 \gamma^{s+2} \, d\mu \right)^\frac{1}{2} \\
& \quad = \left( \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu \right)^\frac{1}{m+1} \left( \int_{\Sigma} |\nabla_{(m+2)} T|^2 \gamma^{s} \, d\mu \right)^\frac{m}{m+1} \\
& \quad \times \left( c^m_{\gamma} \left( \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu \right)^\frac{1}{m+2} \right) \left( \int_{\Sigma} |\nabla_{(m+2)} T|^2 \gamma^{s+2} \, d\mu \right)^\frac{m+1}{m+2} \\
& \quad \leq \ c \left( \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu \right)^\frac{1}{m+2} \left( \int_{\Sigma} |\nabla_{(m+2)} T|^2 \gamma^{s} \, d\mu \right)^\frac{m}{m+2} \\
& \quad + \ c \, c^2_{\gamma (m+1)} \int_{\Sigma} |T|^2 \gamma^{s-2(m+1)} \, d\mu.
\end{align*} \]
Substituting the previous two inequalities into (A.13) then gives

\[(1 - \eta) \int_\Sigma |\nabla_{(m+1)}T|^2 \gamma^s d\mu \]
\[\leq c (\eta^{-1}) \left( \int_\Sigma |T|^2 \gamma^{s-2(m+1)} d\mu \right)^{\frac{1}{m+2}} \left( \int_\Sigma |\nabla_{(m+2)}T|^2 \gamma^{s+2} d\mu \right)^{\frac{m+1}{m+2}}
\[+ c c_\gamma^{2(m+1)} \int_\Sigma |T|^2 \gamma^{s-2(m+1)} d\mu.\]

for any \(\eta > 0\). Choosing \(\eta\) sufficiently small then completes the inductive step and therefore proves (A.10).

To prove (A.11) we again use induction. First note that (A.11) holds for \(m = 1\) by (A.10). Assume that (A.11) holds for some \(m \in \mathbb{N}\):

\[\int_\Sigma |\nabla_{(k)}T|^2 \gamma^s d\mu \]
\[\leq c \left( \int_\Sigma |T|^2 \gamma^{s-2k} d\mu \right)^{\frac{k}{k+m}} \left( \int_\Sigma |\nabla_{(k+m)}T|^2 \gamma^{s+2m} d\mu \right)^{\frac{m}{k+m}}
\[+ c c_\gamma^2 \int_\Sigma |T|^2 \gamma^{s-2k} d\mu.\]

(A.14)

Next applying (A.10) with the tensor \(\nabla_{(k+m)}T\) gives:

\[\int_\Sigma |\nabla_{(k+m)}T|^2 \gamma^{s+2m} d\mu \]
\[\leq c \left( \int_\Sigma |T|^2 \gamma^{s-2k} d\mu \right)^{\frac{1}{k+m}} \left( \int_\Sigma |\nabla_{(k+m+1)}T|^2 \gamma^{s+2(m+1)} d\mu \right)^{\frac{k+m}{k+m+1}}
\[+ c c_\gamma^2 \int_\Sigma |T|^2 \gamma^{s-2k} d\mu.\]

Substituting this into (A.14) then yields

\[\int_\Sigma |\nabla_{(k)}T|^2 \gamma^s d\mu \]
\[\leq c \left( \int_\Sigma |T|^2 \gamma^{s-2k} d\mu \right)^{\frac{m}{k+m}}
\[\times \left( \int_\Sigma |T|^2 \gamma^{s-2k} d\mu \right)^{\frac{k}{k+m+1}} \left( \int_\Sigma |\nabla_{(k+m+1)}T|^2 \gamma^{s+2(m+1)} d\mu \right)^{\frac{k+m}{k+m+1}}\]
\[ + c c_2^{2(k+m)} \int_\Sigma |T|^2 \gamma^{s-2k} \, d\mu \right)^{b \over k+m} + c c_2^k \int_\Sigma |T|^2 \gamma^{s-2k} \, d\mu \]
\[ \leq c \left( \int_\Sigma |T|^2 \gamma^{s-2k} \, d\mu \right)^{m+1 \over k+m+1} \left( \int_\Sigma |\nabla_{(k+m+1)}T|^2 \gamma^{s+2(m+1)} \, d\mu \right)^{b \over k+m+1} + c c_2^k \int_\Sigma |T|^2 \gamma^{s-2k} \, d\mu. \]

This completes the inductive step and proves (A.11). \qed

**Lemma A.11.** For \( j \in \{1, 2, \ldots, k\} \) we have

\[ \left( \int_\Sigma |\nabla_{(j)}T|^{\frac{2k}{j}} \gamma^s \, d\mu \right)^{\frac{j}{\pi}} \leq c \left\| T \right\|_{\infty, |\gamma| > 0}^{1-\frac{j}{\pi}} \left( \left( \int_\Sigma |\nabla_{(k)}T|^2 \gamma^s \, d\mu \right)^{\frac{1}{2}} + c \gamma \left\| T \right\|_2 \right)^{\frac{j}{\pi}}, \]

where \( c > 0 \) is some universal constant.

**Proof.** For \( j \in \{1, 2, \ldots, k\} \) and some fixed \( m \in \mathbb{N} \) with \( m \geq k \), let

\[ a_j = c^{-j+\frac{k}{\gamma}} \left( \int_\Sigma |\nabla_{(j)}T|^{\frac{2k}{j}} \gamma^s \, d\mu \right)^{1 \over \pi} \quad \text{and} \quad b_j = c^{\frac{m+\frac{k}{\gamma}}{j}} \left( \int_\Sigma |T|^{\frac{2k}{j}} \, d\mu \right)^{1 \over \pi}. \]

It then follows that

\[ a_0 = c^m \left\| T \right\|_{\infty, |\gamma| > 0} \quad \text{and} \quad b_k = c^{m+\frac{k}{\gamma}} \left( \int_\Sigma |T|^{\frac{2k}{j}} \, d\mu \right)^{1 \over 2} \quad (k \geq 1), \]

and

\[ b_0 = c^m \left\| T \right\|_{\infty, |\gamma| > 0} \quad \text{and} \quad b_k = c^{m+1} \left( \int_\Sigma |T|^{\frac{2k}{j}} \, d\mu \right)^{1 \over 2} \quad (k \geq 1). \]

It then follows from Lemma A.7 that

\[ a_j^2 \leq c \cdot c_2^{(m-j+\frac{k}{\gamma})} \left( \int_\Sigma |\nabla_{(j+1)}T|^{\frac{2k}{j+1}} \gamma^s \, d\mu \right)^{1 \over 2 \pi} \left( \int_\Sigma |\nabla_{(j-1)}T|^{\frac{2k}{j-1}} \gamma^s \, d\mu \right)^{1 \over 2 \pi} \]
\[ + c \cdot c_2^{(m-j+\frac{k}{\gamma})+1} \left( \int_\Sigma |\nabla_{(j-1)}T|^{\frac{2k}{j-1}} \gamma^s \, d\mu \right)^{1 \over 2 \pi} \left( \int_\Sigma |\nabla_{(j)}T|^{\frac{2k}{j}} \gamma^s-\frac{2k}{j} \, d\mu \right)^{1 \over 2 \pi} \]
\[ \leq c \cdot a_{j-1} \left( a_{j+1} + c^{m-j+\frac{k}{\gamma}} \left( \int_\Sigma |\nabla_{(j)}T|^{\frac{2k}{j}} \gamma^s-\frac{2k}{j} \, d\mu \right)^{1 \over 2 \pi} \right). \quad \text{(A.15)} \]
Then, from Lemma A.9 we can write

\[
c^{m-j+\frac{j+1}{k}} \left( \int_\Sigma |\nabla_j T|^{\frac{2k}{j+1}} \gamma^s \frac{2k}{j+1} \, d\mu \right)^{\frac{j+1}{2k}} \\
\le c \cdot c^{m-(j+1)+j+1} \left( \int_\Sigma |\nabla_{(j+1)} T|^{\frac{2k}{j+1}} \gamma^s \, d\mu \right)^{\frac{j+1}{2k}} \\
+ c \cdot c^{m+\frac{j+1}{k}} \left( \int_\Sigma |T|^{\frac{2k}{j+1}} \gamma^s \frac{2k}{j+1} \, d\mu \right)^{\frac{j+1}{2k}} \\
\le c (a_{j+1} + b_{j+1}). \tag{A.16}
\]

Substituting (A.16) into (A.15) then yields

\[
a_j^2 \le c \cdot a_{j-1} (a_{j+1} + b_{j+1}). \tag{A.17}
\]

Hölder’s Inequality implies

\[
b_j^2 \le b_{j-1} \cdot b_{j+1}, \tag{A.18}
\]

and so we can combine (A.17) and (A.18), giving

\[
(a_j + b_j)^2 \le c (a_{j-1} + b_{j-1}) (a_{j+1} + b_{j+1})
\]

for some universal constant \( c > 0 \). This means that \( f(j) = f_j = a_j + b_j \) satisfies the convexity condition of Lemma A.5. Applying the lemma then gives

\[
a_j \le f_j \le c^{\frac{j(k-j)}{k}} \cdot f_0^{1-\frac{j}{k}} f_k^j,
\]

for some universal constant \( c > 0 \). Noting that

\[
f_0 = 2c^m \|T\|_{\infty, \gamma>0} \quad \text{and} \quad f_k = c^{m-k+1} \left( \int_\Sigma |\nabla_k T|^{\frac{2k}{j+1}} \gamma^s \, d\mu \right)^{\frac{j+1}{2}} + c^{m+1} \left( \int_\Sigma |T|^{\frac{2k}{j+1}} \gamma^s \, d\mu \right)^{\frac{j+1}{2}}.
\]

this then becomes

\[
c^{m-j+\frac{j+1}{k}} \left( \int_\Sigma |\nabla_j T|^{\frac{2k}{j+1}} \gamma^s \, d\mu \right)^{\frac{j+1}{2k}}
\]
\[ c \cdot c_\gamma^{m-\frac{j}{2}} \|T\|_{\infty,\gamma>0}^{-\frac{j}{2}} \left( c_\gamma^{m-\frac{k}{2}} \left( \int_\Sigma |\nabla(k)T|^2 \gamma^s \, d\mu \right)^{\frac{1}{2}} + c_\gamma^{m+1} \left( \int_\Sigma |T|^2 \, d\mu \right)^{\frac{1}{2}} \right)^{\frac{j}{2}} \]

\[ = c \cdot c_\gamma^{m-j+\frac{j}{2}} \|T\|_{\infty,\gamma>0}^{-\frac{j}{2}} \left( \left( \int_\Sigma |\nabla(k)T|^2 \gamma^s \, d\mu \right)^{\frac{1}{2}} + c_\gamma^{m+1} \left( \int_\Sigma |T|^2 \, d\mu \right)^{\frac{1}{2}} \right)^{\frac{j}{2}} \, . \]

Here we have replaced \( c^{\frac{j(k-j)}{2}} \) by \( c \) for simplicity. Dividing through by \( c_\gamma^{m-j+\frac{j}{2}} \) then gives the desired result. Note that this last step is valid because \( c_\gamma > 0 \) and \( m-j+\frac{j}{k} \geq 0. \)

**Lemma A.12.** For \( i_1, i_2, \ldots, i_r \leq k \) satisfying \( \sum_{j=1}^r i_j = 2k \) we have

\[ \int_\Sigma \nabla(i_1)T \ast \cdots \ast \nabla(i_r)T \cdot \gamma^s \, d\mu \leq c \|T\|_{\infty,\gamma>0}^{-\frac{r-2}{2}} \left( \int_\Sigma |\nabla(k)T|^2 \gamma^s \, d\mu + c_\gamma^{2k} \|T\|_{2,\gamma>0}^2 \right) \, , \]

where \( c > 0 \) is some universal constant.

**Proof.** For simplicity, we can reindex and assume that \( i_j > 0 \) for \( j = 1, 2, \ldots, l \) and \( i_j = 0 \) for \( j > l \). Then, by direct calculation, we have

\[ \int_\Sigma \nabla(i_1)T \ast \cdots \ast \nabla(i_r)T \cdot \gamma^s \, d\mu \]

\[ \leq \|T\|_{\infty,\gamma>0}^{-\frac{r-l}{2}} \prod_{j=1}^l \left( \int_\Sigma |\nabla(i_j)T|^2 \gamma^s \, d\mu \right)^{\frac{i_j}{2r}} \]

\[ \leq c \|T\|_{\infty,\gamma>0}^{-\frac{r-l}{2}} \prod_{j=1}^l \left( \|T\|_{\infty,\gamma>0}^{-\frac{1}{2}} \left( \left( \int_\Sigma |\nabla(k)T|^2 \gamma^s \, d\mu \right)^{\frac{1}{2}} + c_\gamma \|T\|_2^2 \right)^{\frac{1}{2}} \right) \]

\[ = c \|T\|_{\infty,\gamma>0}^{-\frac{r-l}{2}} \|T\|_{\infty,\gamma>0}^{-\frac{l-1}{2}} \sum_{j=1}^l \left( \left( \int_\Sigma |\nabla(k)T|^2 \gamma^s \, d\mu \right)^{\frac{1}{2}} + c_\gamma \|T\|_2^2 \right)^{\frac{1}{2}} \]

\[ = c \|T\|_{\infty,\gamma>0}^{-\frac{r-2}{2}} \left( \int_\Sigma |\nabla(k)T|^2 \gamma^s \, d\mu + c_\gamma^{2k} \|T\|_{2,\gamma>0}^2 \right) \, , \]

which is the statement of the lemma. Here the third line follows immediately from Lemma A.10.

**Proposition A.13.** Let \( m \in \mathbb{N}, n \in \mathbb{N}_0 \) and let \( f : \Sigma \to \mathbb{R}^3 \) be a manifold and \( \gamma \) a
cutoff function as in Section 1.2. Let

\[ S_l := \int_{\Sigma} |\Delta^{\frac{l}{2}} H|^{2} \gamma^{2l} d\mu, \quad l \in \mathbb{N}, \]

where (with abuse of notation) we take \( \Delta^{\frac{l}{2}} := \nabla \text{ and } \Delta \text{ acts from right to left so that} \)

\[ \Delta^{\frac{l+1}{2}} = \Delta^{\frac{1}{2}} \circ \Delta^{\frac{l}{2}} = \nabla \Delta^{\frac{l}{2}}. \]

Let \( l \geq 1 \). Then if \( \|A^o\|_{2, \{\gamma > 0\}}^2 \leq \varepsilon_0 \) for \( \varepsilon_0 > 0 \) sufficiently small there exists an absolute constant \( c > 0 \) such that

\[ S_l \leq c \left( \|A^o\|_{2, \{\gamma > 0\}}^\frac{2l}{l+m+1} S_{m+1}^{\frac{l}{m+1}} + c_\gamma^2 \|A^o\|_{2, \{\gamma > 0\}}^2 \right). \quad (A.19) \]

It follows that for \( m \in \mathbb{N} \)

\[ S_l \leq c \left( \|A^o\|_{2, \{\gamma > 0\}}^\frac{2m}{m+1} S_{m+1}^{\frac{l}{m+1}} + c_\gamma^2 \|A^o\|_{2, \{\gamma > 0\}}^2 \right). \quad (A.20) \]

**Proof.** The proof is done by combining induction and multiple applications of integration by parts.

First we prove (A.19). Inequality (5.12) proves the identity for \( l = 1 \). Next assume that the statement is true for \( l = m \in \mathbb{N} \):

\[ S_m \leq c \left( \|A^o\|_{2, \{\gamma > 0\}}^\frac{m+1}{m+1} S_{m+1}^{\frac{1}{m+1}} + c_\gamma^2 \|A^o\|_{2, \{\gamma > 0\}}^2 \right). \quad (A.21) \]

Then integration by part and Young’s inequality as well as the assumption (A.21) gives

\[ S_{m+1} = \int_{\Sigma} \Delta^\frac{m}{2} H \ast \Delta^\frac{m+2}{2} H \gamma^{2(m+1)} d\mu + \int_{\Sigma} \Delta^\frac{m}{2} H \ast \Delta^\frac{m+1}{2} H \ast \nabla \gamma \gamma^{2m+1} d\mu \]

\[ \leq c S_m^{\frac{1}{2}} S_{m+2}^{\frac{1}{2}} + \eta S_{m+1} + c \left( \eta^{-1} \right) c_\gamma^2 S_m \]

\[ \leq c \left( \|A^o\|_{2, \{\gamma > 0\}}^\frac{2m}{m+1} S_{m+1}^{\frac{m}{m+1}} + c_\gamma^2 \|A^o\|_{2, \{\gamma > 0\}}^2 \right)^{\frac{1}{2}} S_{m+2}^{\frac{1}{2}} \]

\[ + \eta S_{m+1} + c \left( \eta^{-1} \right) c_\gamma^2 \left( \|A^o\|_{2, \{\gamma > 0\}}^\frac{2m}{m+1} S_{m+1}^{\frac{m}{m+1}} + c_\gamma^2 \|A^o\|_{2, \{\gamma > 0\}}^2 \right). \]
Next using Young’s inequality with conjugate exponents $p = \frac{2(m+1)}{m}$, $p^* = \frac{2(m+1)}{m+2}$ gives

$$\|A^o\|_{2, [\gamma > 0]}^{\frac{2}{m+1}} S_{m+2}^{\frac{m}{m+1}} S_{m+2}^{\frac{m}{2}} \leq \eta S_{m+1} + c \left( \eta^{-1} \right) \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{m+1}} S_{m+2}^{\frac{m}{2}}.$$ 

Similarly

$$c^m \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} S_{m+2}^{\frac{m}{l+2}} \leq c \left( \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} S_{m+2}^{\frac{l+m}{l+2}} + c^2(m+1) \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} \right)$$

and

$$c \left( \eta^{-1} \right) c^\gamma \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} S_{m+2}^{\frac{m}{l+2}} \leq \eta S_{m+1} + c \left( \eta^{-1} \right) c^2(m+1) \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}}.$$ 

Therefore

$$(1 - 3\eta) S_{m+1} \leq c \left( \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} S_{m+2}^{\frac{l+m}{l+2}} + c^2(m+1) \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} \right),$$

and so choosing $\eta > 0$ small enough proves the inductive step. This proves (A.19).

To prove (A.20) we again use induction. The statement is true for $m = 1$ by (A.20).

Assume the statement is true for general $m \in \mathbb{N}$:

$$S_l \leq c \left( \|A^o\|_{2, [\gamma > 0]}^{\frac{2m}{l+m}} S_{l+m}^{\frac{l}{l+m}} + c_\gamma^2 \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} \right).$$

Then taking into account (A.20) this becomes

$$S_l \leq c \|A^o\|_{2, [\gamma > 0]}^{\frac{2m}{l+m}} \left( \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+m+1}} S_{l+m+1}^{\frac{l+m}{l+m+1}} + c_\gamma^2(l+m) \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} \right) \|A^o\|_{2, [\gamma > 0]}^{\frac{l}{l+m}} + c c_\gamma^2 \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}}$$

$$\leq c \left( \|A^o\|_{2, [\gamma > 0]}^{\frac{2m}{l+m+1}} S_{l+m+1}^{\frac{l+m}{l+m+1}} + c_\gamma^2 \|A^o\|_{2, [\gamma > 0]}^{\frac{2}{l+2}} \right).$$

This proves the inductive step and hence (A.20).

**Theorem A.14.** Let $f : \Sigma \to \mathbb{R}^3$ be a closed immersion satisfying $\|A^o\|_{2}^2 \leq \varepsilon_0$ for
some $\varepsilon_0 > 0$ sufficiently small. Then there exists a universal constant such that

$$\|\nabla H\|_\infty^6 \leq c \|A^o\|_2^2 \left( \int_{\Sigma} |\nabla \Delta H|^2 d\mu \right)^2.$$ 

**Proof.** The multiplicative Sobolev theorem from Theorem A.6 with $p = 4, m = 2$ immediately gives

$$\|\nabla H\|_\infty^6 \leq c \int_{\Sigma} |\nabla H|^2 d\mu \cdot \left( \int_{\Sigma} |\nabla (2) H|^4 d\mu + \int_{\Sigma} |\nabla H|^4 H^2 d\mu \right). \quad (A.22)$$

Using Theorem A.2 with $u = |\nabla (2) H|^2$ to estimate the first term in parentheses on the right gives

$$\int_{\Sigma} |\nabla (2) H|^4 d\mu \leq c \left( \int_{\Sigma} |\nabla (3) A| \ |\nabla (2) A| d\mu + \int_{\Sigma} |\nabla (2) A|^2 |H| d\mu \right)^2 \leq c \int_{\Sigma} |\nabla (2) H|^2 d\mu \cdot \left( \int_{\Sigma} |\nabla (3) H|^2 d\mu + \int_{\Sigma} |\nabla (2) H|^2 H^2 d\mu \right). \quad (A.23)$$

We leave this inequality for the time being, and go on to estimate the rest of the terms on the right hand side of (A.22). Using Theorem A.2 with $u = |\nabla H|^2 H^2$ gives

$$\int_{\Sigma} |\nabla H|^4 H^4 d\mu \leq c \left( \int_{\Sigma} |\nabla (2) A| |\nabla H| H^2 d\mu + \int_{\Sigma} |\nabla H|^3 |H| d\mu + \int_{\Sigma} |\nabla H|^2 |H|^3 d\mu \right)^2 \leq c \left( \int_{\Sigma} |\nabla (2) H|^2 d\mu + \int_{\Sigma} |\nabla H|^2 H^2 d\mu \right) \cdot \left( \int_{\Sigma} |\nabla H|^2 H^4 d\mu + \int_{\Sigma} |\nabla A^o|^4 d\mu \right). \quad (A.24)$$

Here we have used 1.18 in the last line. The last term on the right can be estimated using Lemma A.12 and the $L^\infty$ estimate for the trace-free curvature from Chapter 5, the identity from Corollary 5.13, as well as the interpolation inequality from Proposition A.13:

$$\int_{\Sigma} |\nabla A^o|^4 d\mu \leq \int_{\Sigma} |P^4_{4,1} (A^o)| d\mu.$$
\[ \leq c \| A^o \|_2^2 \int_\Sigma |\nabla_{(2)} A^o|^2 d\mu \]
\[ \leq c \| A^o \|_2^{\frac{5}{3}} \left( \int_\Sigma |\Delta H|^2 d\mu \right)^{\frac{3}{2}} \]
\[ \leq c \| A^o \|_2^8 \int_\Sigma |\nabla \Delta H|^2 d\mu. \quad (A.25) \]

Next we claim that the following equation holds for a universal constant \( c > 0 \):
\[ \int_\Sigma |\nabla_{(3)} H|^2 d\mu + \int_\Sigma |\nabla_{(2)} H|^2 H^2 d\mu + \int_\Sigma |\nabla H|^2 H^4 d\mu \leq c \int_\Sigma |\nabla \Delta H|^2 d\mu. \quad (A.26) \]

The proof of (A.26) is similar to that of many of the lemmata in Chapter 5. Firstly, using the identity (5.4) and integration by parts gives
\[ \int_\Sigma |\nabla_{(2)} H|^2 H^2 d\mu \]
\[ = - \int_\Sigma \langle \nabla H, \Delta \nabla H \rangle H^2 d\mu + \int_\Sigma H \nabla_{(2)} H \ast \nabla H \ast \nabla H d\mu \]
\[ \leq \left( \int_\Sigma |\nabla H|^2 H^4 d\mu \cdot \int_\Sigma |\nabla \Delta H|^2 d\mu \right)^{\frac{1}{2}} - \frac{1}{4} \int_\Sigma |\nabla H|^2 H^4 d\mu \]
\[ + \frac{1}{2} \int_\Sigma |\nabla H|^2 |A^o|^2 H^2 d\mu + c \left( \int_\Sigma |\nabla_{(2)} H|^2 H^2 d\mu \cdot \int_\Sigma |\nabla A^o|^4 d\mu \right)^{\frac{1}{2}} \]
Here we have used Hölder’s inequality as well as the identity 1.18 in the last line. Therefore, using Young’s inequality on the first and last terms and rearranging the previous inequality gives
\[ (1 - \eta) \int_\Sigma |\nabla_{(2)} H|^2 H^2 d\mu + \left( \frac{1}{4} - \eta \right) \int_\Sigma |\nabla H|^2 H^4 d\mu \]
\[ \leq c (\eta^{-1}) \int_\Sigma |\nabla \Delta H|^2 d\mu + \frac{1}{2} \int_\Sigma |\nabla H|^2 |A^o|^2 H^2 d\mu + c (\eta^{-1}) \int_\Sigma |\nabla A^o|^4 d\mu \]
\[ \leq \left( c (\eta^{-1}) + c \| A^o \|_2^\frac{5}{3} \right) \int_\Sigma |\nabla \Delta H|^2 d\mu + \frac{1}{2} \int_\Sigma |\nabla H|^2 |A^o|^2 H^2 d\mu \quad (A.27) \]
for any \( \eta > 0 \). Here we have also used (5.29) to estimate the term \( \int_\Sigma |\nabla A^o|^4 d\mu \) in the last step. Next using Theorem A.2 with \( u = |\nabla H| |A^o| |H| \) we can estimate the last on
the right hand side of (A.27):

\[
\int_{\Sigma} |\nabla H|^2 |A^\alpha|^2 H^2 \, d\mu \\
\leq c \left( \int_{\Sigma} |\nabla_{(2)} H| |A^\alpha| |H| \, d\mu + \int_{\Sigma} |\nabla H| |\nabla A^\alpha| |H| \, d\mu + \int_{\Sigma} |\nabla H|^2 |A^\alpha| \, d\mu \right) \\
+ \int_{\Sigma} |\nabla H| |A^\alpha| H^2 \, d\mu \right)^2 \\
\leq c \|A^\alpha\|_2^2 \left( \int_{\Sigma} |\nabla_{(2)} H|^2 H^2 \, d\mu + \int_{\Sigma} |\nabla H^2 H^4 \, d\mu \right) \\
+ c \int_{\Sigma} |\nabla H|^2 \, d\mu \cdot \left( \int_{\Sigma} |\nabla H|^2 |A^\alpha|^2 \, d\mu + \int_{\Sigma} |\nabla A^\alpha|^2 H^2 \, d\mu \right). \tag{A.28}
\]

The last term in (A.28) can be estimated using Corollary 5.10 and Corollary 5.13, as well the interpolation inequality from Proposition A.13:

\[
\int_{\Sigma} |\nabla H|^2 \, d\mu \cdot \left( \int_{\Sigma} |\nabla H|^2 |A^\alpha|^2 \, d\mu + \int_{\Sigma} |\nabla A^\alpha|^2 H^2 \, d\mu \right) \\
\leq c \|A^\alpha\|_2^2 \left( \int_{\Sigma} |\Delta H|^2 \, d\mu \right) \leq c \|A^\alpha\|_2^2 \int_{\Sigma} |\nabla \Delta H|^2 \, d\mu.
\]

Substituting this estimate in (A.28) and then back into (A.27) gives

\[
(1 - \eta - c \|A^\alpha\|_2^2) \int_{\Sigma} |\nabla_{(2)} H|^2 H^2 \, d\mu + \left( \frac{1}{4} - \eta - \|A^\alpha\|_2^2 \right) \int_{\Sigma} |\nabla H|^2 H^4 \, d\mu \\
\leq \left( c (\eta^{-1}) + c \|A^\alpha\|_2^2 + c \|A^\alpha\|_2 \right) \int_{\Sigma} |\nabla \Delta H|^2 \, d\mu,
\]

and so if \( \eta > 0 \) is chosen small enough, and if \( \varepsilon_0 \) is small enough, then we arrive at

\[
\int_{\Sigma} |\nabla_{(2)} H|^2 H^2 \, d\mu + \int_{\Sigma} |\nabla H|^2 H^4 \, d\mu \leq c \int_{\Sigma} |\nabla \Delta H|^2 \, d\mu, \tag{A.29}
\]

which gives us most of the inequality (A.26). To prove the rest, we need an interpolation inequality similar to that of (5.4) but for third derivatives of \( H \). We claim that

\[
\Delta \nabla_{ij} H = \nabla_{ij} \Delta H + 2 R \nabla_{ij} H - R g_{ij} \Delta H + \nabla R \star \nabla H. \tag{A.30}
\]
To prove (A.30) we use the interchange of covariant derivatives as well as (5.4):

\[
\Delta \nabla_{ij} H = \nabla^t \left( \nabla_{tj} H + R_{tij}^\lambda \nabla_{\lambda} H \right) \\
= g^{st} \left( \nabla_{istj} H + R_{sitj}^\lambda \nabla_{\lambda j} H + R_{sij}^\lambda \nabla_{t\lambda} H \right) + R_{tij}^\lambda \nabla^t \nabla_{\lambda} H + \nabla R * \nabla H \\
= \nabla_i \Delta \nabla_j H + g^{st} \left( R_{sitj}^\lambda \nabla_{\lambda j} H + R_{sij}^\lambda \nabla_{t\lambda} H \right) + R_{tij}^\lambda \nabla^t \nabla_{\lambda} H + \nabla R * \nabla H \\
= \nabla_i \left( \nabla_j \Delta H + \frac{1}{2} \nabla_j H R \right) + g^{st} \left( R_{sitj}^\lambda \nabla_{\lambda j} H + R_{sij}^\lambda \nabla_{t\lambda} H \right) + R_{tij}^\lambda \nabla^t \nabla_{\lambda} H \\
+ \nabla R * \nabla H \\
= \nabla_{ij} \Delta H + \frac{1}{2} \nabla_{ij} H R + g^{st} \left( R_{sitj}^\lambda \nabla_{\lambda j} H + R_{sij}^\lambda \nabla_{t\lambda} H \right) + R_{tij}^\lambda \nabla^t \nabla_{\lambda} H + \nabla R * \nabla H.
\]

Identity (5.4) was used in the fourth line. By substituting the identity

\[ R_{ijk}^l = \frac{1}{2} R \left( g_{ik} \delta^j_l - g_{jk} \delta^i_l \right), \]

we then obtain (A.30). Therefore, integrating by parts and using the identities (5.4) and (A.30) gives

\[
\int \Sigma |\nabla(3) H|^2 d\mu \\
= - \int \Sigma \left\langle \nabla(2) H, \Delta \nabla(2) H \right\rangle d\mu + \int \Sigma \nabla(2) H * \nabla R * \nabla H d\mu \\
= - \int \Sigma \left\langle \nabla(2) H, \nabla(2) \Delta H \right\rangle H^2 d\mu - 2 \int \Sigma |\nabla(2) H|^2 H^2 d\mu + \int \Sigma |\Delta H|^2 R d\mu \\
+ \int \Sigma \nabla(2) H * \nabla R * \nabla H d\mu \\
= \int \Sigma \left\langle \Delta \nabla H, \nabla \Delta H \right\rangle d\mu - 2 \int \Sigma |\nabla(2) H|^2 H^2 d\mu + \int \Sigma |\Delta H|^2 R d\mu \\
+ \int \Sigma \nabla(2) H * \nabla R * \nabla H d\mu \\
= \int \Sigma \left\langle \nabla \Delta H + \frac{1}{2} \nabla H R, \nabla \Delta H \right\rangle d\mu - 2 \int \Sigma |\nabla(2) H|^2 H^2 d\mu + \int \Sigma |\Delta H|^2 R d\mu \\
+ \int \Sigma \nabla(2) H * \nabla R * \nabla H d\mu.
\]

Here (A.30) was used in the first line, and (5.4) was used in the last line. Incorpor-
rating the identity \( R = \frac{1}{2} H^2 - |A^o|^2 \) as well as the inequality \(|\Delta H| = |\langle g, \nabla(2)H \rangle| \leq \sqrt{2} |\nabla(3)H| \) which follows from the Cauchy-Schwarz inequality, the previous identity can be rearranged to yield

\[
\int_{\Sigma} |\nabla(3)H|^2 \, d\mu + \int_{\Sigma} |\nabla(2)H|^2 H^2 \, d\mu + \frac{1}{2} \int_{\Sigma} |\Delta H|^2 |A^o|^2 \, d\mu
\]

\[
= \int_{\Sigma} |\nabla \Delta H|^2 \, d\mu + \frac{1}{4} \int_{\Sigma} |\Delta H|^2 H^2 \, d\mu + \int_{\Sigma} |\nabla(2)H|^2 |A^o|^2 \, d\mu
\]

\[
+ \int_{\Sigma} \nabla(2)H \ast \nabla R \ast \nabla H \, d\mu
\]

\[
\leq \int_{\Sigma} |\nabla \Delta H|^2 \, d\mu + \frac{1}{2} \int_{\Sigma} |\nabla(2)H|^2 H^2 \, d\mu + \int_{\Sigma} |\nabla(2)H|^2 |A^o|^2 \, d\mu
\]

\[
+ \int_{\Sigma} \nabla(2)H \ast \nabla R \ast \nabla H \, d\mu.
\]

Therefore

\[
\int_{\Sigma} |\nabla(3)H|^2 \, d\mu + \frac{1}{2} \int_{\Sigma} |\nabla(2)H|^2 H^2 \, d\mu + \frac{1}{2} \int_{\Sigma} |\Delta H|^2 |A^o|^2 \, d\mu
\]

\[
\leq \int_{\Sigma} |\nabla \Delta H|^2 \, d\mu + \int_{\Sigma} |\nabla(2)H|^2 |A^o|^2 \, d\mu + \int_{\Sigma} \nabla(2)H \ast \nabla R \ast \nabla H \, d\mu. \quad (A.31)
\]

Using Theorem A.2 with \( u = |\nabla(2)H| |A^o| \) to estimate the penultimate term on the right hand side gives

\[
\int_{\Sigma} |\nabla(2)H|^2 |A^o|^2 \, d\mu
\]

\[
\leq c \left( \int_{\Sigma} |\nabla(3)H| |A^o| \, d\mu + \int_{\Sigma} |\nabla(2)H| |\nabla A^o| \, d\mu + \int_{\Sigma} |\nabla(2)H| |A^o| |H| \, d\mu \right)^2
\]

\[
\leq c \|A^o\|^2_2 \left( \int_{\Sigma} |\nabla(3)H|^2 \, d\mu + \int_{\Sigma} |\nabla(2)H|^2 H^2 \, d\mu \right) + c \int_{\Sigma} |\nabla A^o|^2 \, d\mu \cdot \int_{\Sigma} |\nabla(2)H|^2 \, d\mu
\]

\[
\leq c \|A^o\|^2 \left( \int_{\Sigma} |\nabla(3)H|^2 \, d\mu + \int_{\Sigma} |\nabla(2)H|^2 H^2 \, d\mu \right) + c \|A^o\|^2 \left( \int_{\Sigma} |\Delta H|^2 \, d\mu \right)^{\frac{3}{2}}
\]

\[
\leq c \|A^o\|^2 \left( \int_{\Sigma} |\nabla(3)H|^2 \, d\mu + \int_{\Sigma} |\nabla(2)H|^2 H^2 \, d\mu \right) + c \|A^o\|^2 \int_{\Sigma} |\nabla \Delta H|^2 \, d\mu. \quad (A.32)
\]

Here we have used Corollary 5.10 and 5.13, to get to the fourth line, as well as the interpolation inequality from Proposition (A.13) to get to the last line. The last term
in (A.14) can be estimated using Young’s inequality and the identity $R = \frac{1}{2} H^2 - |A^o|^2$:

$$
\int_\Sigma \nabla (2) H \ast \nabla R \ast \nabla H \, d\mu = \int_\Sigma \nabla (2) H \ast (H \nabla H + A^o \ast \nabla A^o) \ast \nabla H \, d\mu \\
\leq \eta \int_\Sigma |\nabla (2) H|^2 H^2 \, d\mu + c(\eta^{-1}) \int_\Sigma |\nabla A^o|^4 \, d\mu + c \int_\Sigma |\nabla (2) H|^2 |A^o|^2 \, d\mu \\
\leq \eta \int_\Sigma |\nabla (2) H|^2 H^2 \, d\mu + c(\eta^{-1}) \parallel A^o \parallel^8 \int_\Sigma |\nabla \Delta H|^2 \, d\mu \\
+ c \parallel A^o \parallel^2 \left( \int_\Sigma |\nabla (3) H|^2 \, d\mu + \int_\Sigma |\nabla (2) H|^2 H^2 \, d\mu \right).
$$

(A.33)

Here we have used the identity (1.18), as well as (A.25) to estimate $\int_\Sigma |\nabla A^o|^4 \, d\mu$ and (A.32) to estimate $\int_\Sigma |\nabla (2) H|^2 |A^o|^2 \, d\mu$, to get to the last step. Finally, substituting (A.32) and (A.33) into (A.31) and rearranging gives

$$
\left(1 - c \parallel A^o \parallel^2\right) \int_\Sigma |\nabla (3) H|^2 \, d\mu + (1 - \eta - c \parallel A^o \parallel^2) \frac{1}{2} \int_\Sigma |\nabla (2) H|^2 H^2 \, d\mu \\
+ \frac{1}{2} \int_\Sigma |\Delta H|^2 |A^o|^2 \, d\mu \\
\leq \left(1 + c(\eta^{-1}) \parallel A^o \parallel^4 + c \parallel A^o \parallel^2\right) \int_\Sigma |\nabla H|^2 \, d\mu.
$$

Choosing $\eta > 0$ small enough to be absorbed, and if $\varepsilon_0 > 0$ is small enough, then multiplying through gives

$$
\int_\Sigma |\nabla (3) H|^2 \, d\mu + \int_\Sigma |\nabla (2) H|^2 H^2 \, d\mu + \int_\Sigma |\Delta H|^2 |A^o|^2 \, d\mu \leq c \int_\Sigma |\nabla \Delta H|^2 \, d\mu,
$$

and combining with (A.29) gives (A.26). We are able to then use (A.26) and (A.25) as well as the interpolation inequalities of Proposition A.13 and Corollary 5.13 to estimate (A.23) and (A.24):

$$
\int_\Sigma |\nabla (2) H|^4 \, d\mu + \int_\Sigma |\nabla H|^4 H^4 \, d\mu \\
\leq c \int_\Sigma |\nabla (2) H|^2 \, d\mu \cdot \left( \int_\Sigma |\nabla (3) H|^2 \, d\mu + \int_\Sigma |\nabla (2) H|^2 H^2 \, d\mu \right) \\
+ c \left( \int_\Sigma |\nabla (2) H|^2 \, d\mu + \int_\Sigma |\nabla H|^2 H^2 \, d\mu \right) \cdot \left( \int_\Sigma |\nabla H|^2 H^4 \, d\mu + \int_\Sigma |\nabla A^o|^4 \, d\mu \right) \\
\leq c \int_\Sigma |\Delta H|^2 \, d\mu \cdot \int_\Sigma |\nabla \Delta H|^2 \, d\mu.
$$
Substituting this estimate into (A.22) and using Proposition A.13 to obtain

\[ \int_{\Sigma} |\nabla H|^2 d\mu \leq c \|A^o\|_2^\frac{4}{3} \left( \int_{\Sigma} |\nabla \Delta H|^2 d\mu \right)^{\frac{5}{3}} \]

then means that (A.22) becomes

\[ \|\nabla H\|_\infty^6 \leq c \left( \|A^o\|_2^\frac{4}{3} \left( \int_{\Sigma} |\nabla \Delta H|^2 d\mu \right)^{\frac{5}{3}} \right) \cdot \left( \|A^o\|_2^\frac{4}{3} \left( \int_{\Sigma} |\nabla \Delta H|^2 d\mu \right)^{\frac{5}{3}} \right) \]

\[ = c \|A^o\|_2^2 \left( \int_{\Sigma} |\nabla \Delta H|^2 d\mu \right)^2. \]

This finishes the proof.

**Theorem A.15.** Suppose \( f : \Sigma \to \mathbb{R}^3 \) is a closed immersion satisfying \( \|A^o\|_2^2 \leq \varepsilon_0 \) for \( \varepsilon_0 > 0 \) sufficiently small. Fix \( m \in \mathbb{N} \) and denote \( Q_l, 0 \leq l \leq m \) by

\[ Q_l := \int_{\Sigma} |\nabla (m-l)A^o|^2 H^{2l} d\mu. \]

Then for \( 1 \leq l \leq m - 1 \) the recurrence relation

\[ Q_l \leq c Q_{l-1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}} + c \|A^o\|_2^2 (Q_0 + Q_{l+1} + S_m) \]

holds. Here we have used the notation from Proposition A.13.

**Proof.** Integration by parts and Hölder’s inequality, followed by the Young’s inequality immediately gives

\[ Q_l = \int_{\Sigma} \nabla (m-(l+1))A^o \ast \nabla (m-(l-1))A^o H^{2l} d\mu \]

\[ + \int_{\Sigma} \nabla (m-l)A^o \ast \nabla (m-(l+1))A^o \ast \nabla H H^{2l-1} d\mu \]

\[ \leq c Q_{l-1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}} + c \|\nabla H\|_\infty Q_l^{\frac{1}{2}} \left( \int_{\Sigma} |\nabla (m-(l+1))A^o|^2 H^{2(l-1)} d\mu \right)^{\frac{1}{2}} \]
\[ \leq c Q_{l-1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}} + \eta Q_l + c (\eta^{-1}) \| \nabla H \|_\infty^2 \int_\Sigma |(m-(l+1)) A^\alpha|^2 H^{2(l-1)} \, d\mu. \]

We then absorb and multiply out to give

\[ Q_l \leq c Q_{l-1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}} + c \| \nabla H \|_\infty^2 \int_\Sigma |(m-(l+1)) A^\alpha|^2 H^{2(l-1)} \, d\mu. \tag{A.34} \]

Next we apply Hölder’s inequality with conjugate exponents \( \alpha = (l+1)/(l-1) \), \( \alpha^* = (l+1)/2 \) to the last term on the right hand side:

\[ \int_\Sigma |(m-(l+1)) A^\alpha|^2 H^{2(l-1)} \, d\mu \leq Q_{l+1}^{\frac{l-1}{l+1}} \left( \int_\Sigma |(m-(l+1)) A^\alpha|^2 \, d\mu \right)^{\frac{2}{l+1}}. \tag{A.35} \]

The interpolation inequality of Lemma A.10 implies

\[ \int_\Sigma |(m-(l+1)) A^\alpha|^2 \, d\mu \leq c \| A^\alpha \|_2^{\frac{2(l+1)}{m}} \left( \int_\Sigma |(m) A^\alpha|^2 \, d\mu \right)^{\frac{m-(l+1)}{m}} \]

\[ = c \| A^\alpha \|_2^{\frac{2(l+1)}{m}} Q_0^{\frac{m-(l+1)}{m}}, \]

meaning that (A.35) becomes

\[ \int_\Sigma |(m-(l+1)) A^\alpha|^2 H^{2(l-1)} \, d\mu \leq c \| A^\alpha \|_2^{\frac{4}{l}} Q_0^{\frac{2(m-(l+1))}{m}} Q_{l+1}^{\frac{l-1}{l+1}}. \tag{A.36} \]

Next, incorporating our estimate for \( \| \nabla H \|_\infty^2 \) along with the interpolation inequalities from Proposition A.13, we have

\[ \| \nabla H \|_\infty^2 \leq c \| A^\alpha \|_2^2 \left( \int_\Sigma |\nabla \Delta H|^2 \, d\mu \right)^{\frac{2}{3}} \]

\[ \leq c \| A^\alpha \|_2^{\frac{4}{3} + \frac{4(m-3)}{3m}} \left( \int_\Sigma |\Delta^{\frac{2}{3}} H|^2 \, d\mu \right)^{\frac{2}{3}} \]

\[ = c \| A^\alpha \|_2^{\frac{4}{3} + \frac{4(m-3)}{3m}} S_m^{\frac{2}{3}}. \tag{A.37} \]

Here we have used the notation from Proposition A.13 in the last line. Substituting
(A.36) and (A.37) into (A.34) then gives

$$Q_l \leq c Q_{l-1}^{1/2} Q_{l+1}^{1/2} + c \| A^o \|_2^2 Q_0^{2(m-(l+1))} Q_{l+1}^{1/2} S_m^{1/2}. $$

Using the generalised Young’s inequality with conjugate exponents $\alpha = \frac{m(l+1)}{2(m-(l+1))}$, $\beta = \frac{l+1}{l-1}$ and $\delta = \frac{m}{2}$ on the last the three terms then finishes the proof.

Corollary A.16. Assume the notation and hypothesis of Proposition A.13 and Theorem A.15. Then for every $0 \leq l \leq m - 1$ there is a universal constant $c > 0$ such that

$$Q_0 + \cdots + Q_l \leq c Q_{l+1} + c S_m.$$

Proof. The proof is relatively simple, and uses induction. The statement is true for $l = 0$ by (6.12). Assume that the statement is true for $l = n$:

$$Q_0 + \cdots + Q_n \leq c Q_{n+1} + c S_m \text{ for some } n. \quad (A.38)$$

Next, using the inequality of Theorem A.15 with $l = n + 1$ followed by Young’s inequality gives

$$Q_{n+1} \leq c Q_n^{1/2} Q_{n+2}^{1/2} + c \| A^o \|_2^2 (Q_0 + Q_{n+1} + S_m)$$

$$\leq \eta Q_n + (c (\eta^{-1}) + c \| A^o \|_2^2) Q_{n+2} + c \| A^o \|_2^2 (Q_0 + S_m)$$

$$\leq \left( \eta + c \| A^o \|_2^2 \right) Q_{n+1} + (c (\eta^{-1}) + c \| A^o \|_2^2) Q_{n+2} + c \| A^o \|_2^2 S_m$$

for any $\eta > 0$. Here we have use the inductive assumption (A.38) to get to the last line. Therefore by choosing $\eta > 0$ small enough, and if $\varepsilon_0 > 0$ is sufficiently small, we may absorb the first term on the right to gives

$$Q_{n+1} \leq c Q_{n+2} + c S_m.$$
Thus by using the assumption (A.38) we have

\[ Q_0 + \cdots + Q_{n+1} = \{Q_0 + \cdots + Q_n\} + Q_{n+1} \leq c Q_{n+1} + c S_m \leq c Q_{n+2} + c S_m, \]

which finishes the inductive step and the proof of the corollary. \(\square\)

**Lemma A.17.** Fix \(m \in \mathbb{N}\). Assume the notation and hypothesis of Proposition A.13 and Theorem A.15. Then there exists a universal constant \(c > 0\) such that

\[ Q_{m-1} + Q_m \leq c \int_{\Sigma} |\Delta H|^2 H^{2(m-2)} \, d\mu + c S_m. \]

**Proof.** Firstly, using integration by parts, the identity 5.5 and Hölder’s inequality gives

\[
Q_{m-1} := \int_{\Sigma} |\nabla A^\alpha|^2 H^{2(m-1)} \, d\mu \\
= -\int_{\Sigma} \left( A^\alpha, \nabla (2) H + \frac{1}{2} H^2 A^\alpha - |A^\alpha|^2 A^\alpha \right) H^{2(m-1)} \, d\mu \\
+ \int_{\Sigma} \nabla A^\alpha \ast A^\alpha \ast \nabla H H^{2m-3} \, d\mu \\
\leq \left( Q_{m-1} \cdot \int_{\Sigma} |\nabla (2) H|^2 H^{2(m-2)} \, d\mu \right)^\frac{1}{2} \left( Q_{m-1} \cdot \int_{\Sigma} |A^\alpha|^4 H^{2(m-1)} \, d\mu \right)^\frac{1}{2} - \frac{1}{2} Q_m + \int_{\Sigma} |A^\alpha|^4 H^{2(m-1)} \, d\mu \\
+ c \|\nabla H\|_\infty \left( Q_{m-1} \cdot \int_{\Sigma} |A^\alpha|^2 H^{2(m-2)} \, d\mu \right)^\frac{1}{2} \\
\leq 2\eta Q_{m-1} + c \left( \eta^{-1} \right) \int_{\Sigma} |\nabla (2) H|^2 H^{2(m-2)} \, d\mu - \frac{1}{2} Q_m + \int_{\Sigma} |A^\alpha|^4 H^{2(m-1)} \, d\mu \\
+ c \left( \eta^{-1} \right) \|\nabla H\|_\infty^2 \int_{\Sigma} |A^\alpha|^2 H^{2(m-2)} \, d\mu
\]

for any \(\eta > 0\). Here we have used Young’s inequality twice in the last step to get to the last line. Letting \(\eta = \frac{1}{4}\) and absorbing into the left then gives

\[
Q_{m-1} + Q_m \leq c \int_{\Sigma} |\nabla (2) H|^2 H^{2(m-2)} \, d\mu + \int_{\Sigma} |A^\alpha|^4 H^{2(m-1)} \, d\mu + \|\nabla H\|_\infty^2 \int_{\Sigma} |A^\alpha|^2 H^{2(m-2)} \, d\mu.
\]

(A.39)
Next, using Theorem A.2 with \( u = |A^o|^2 |H|^{m-1} \) gives us the estimate

\[
\int_{\Sigma} |A^o|^4 H^{2(m-1)} \, d\mu \\
\leq c \left( \int_{\Sigma} |\nabla A^o| |A^o| |H|^{m-1} \, d\mu + \int_{\Sigma} |A^o|^2 |\nabla H| |H|^{m-2} \, d\mu + \int_{\Sigma} |A^o|^2 |H|^m \, d\mu \right) \\
\leq c \|A^o\|_2^2 \left( Q_{m-1} + Q_m + \|\nabla H\|_\infty^2 \int_{\Sigma} |A^o|^2 H^{2(m-2)} \, d\mu \right).
\]

Substituting back into (A.39) and absorbing yields

\[
(1 - c \|A^o\|_2^2) \cdot (Q_{m-1} + Q_m) \\
\leq c \int_{\Sigma} |\nabla H|^{2} H^{2(m-2)} \, d\mu + c \cdot (1 + \|A^o\|_2^2) \cdot \|\nabla H\|_\infty^2 \int_{\Sigma} |A^o|^2 H^{2(m-2)} \, d\mu.
\]

We now employ Hölder’s inequality with conjugate exponents \( \alpha = m / (m - 2) \), \( \alpha^* = m/2 \) to give

\[
\int_{\Sigma} |A^o|^2 H^{2(m-2)} \, d\mu \leq c \|A^o\|_2^\frac{4}{m} Q_m^{\frac{m-2}{m}}.
\]

Combining this with the estimate for \( \|\nabla H\|_\infty^2 \) from (A.37) and Young’s inequality with conjugate exponents \( \alpha = \frac{m}{m-2} \), \( \alpha^* = \frac{m}{2} \) gives

\[
(1 - c \|A^o\|_2^2) \cdot (Q_{m-1} + Q_m) \\
\leq c \int_{\Sigma} |\nabla H|^{2} H^{2(m-2)} \, d\mu + c \cdot (1 + \|A^o\|_2^2) \cdot \|A^o\|_2^\frac{m-2}{m} S_m^2 \\
\leq c \int_{\Sigma} |\nabla H|^{2} H^{2(m-2)} \, d\mu + c \|A^o\|_2^2 (Q_m + S_m).
\]

Therefore if \( \varepsilon_0 > 0 \) is sufficiently small it follows that

\[
Q_{m-1} + Q_m \leq c \int_{\Sigma} |\nabla H|^{2} H^{2(m-2)} \, d\mu + c \|A^o\|_2^2 S_m. \tag{A.40}
\]

We leave this inequality for the time being.

Next, we pay attention to the first term on the right hand side of (A.40). Employing
the identity of \ref{eq:5.4} and two applications of integration by parts yields

\[\int_\Sigma |\nabla(2)H|^2 H^{2(m-2)} \, d\mu = - \int_\Sigma \left\langle \nabla H, \nabla \Delta H + \frac{1}{4} \nabla H H^2 - \frac{1}{2} \nabla H |A^o|^2, \nabla H \right\rangle H^{2(m-2)} \, d\mu + \int_\Sigma \nabla(2) H \ast \nabla H \ast \nabla H H^{2(m-3)} \, d\mu = \int_\Sigma |\Delta H|^2 H^{2(m-2)} \, d\mu - \frac{1}{4} \int_\Sigma |\nabla H|^2 H^{2(m-1)} \, d\mu + \frac{1}{2} \int_\Sigma |\nabla H|^2 |A^o|^2 H^{2(m-2)} \, d\mu + \int_\Sigma \nabla(2) H \ast \nabla H \ast \nabla H H^{2m-5} \, d\mu. \tag{A.41}\]

The last term on the right can be estimated using our earlier estimate for \(\|\nabla H\|_\infty\) from \ref{eq:A.37}, Hölder's inequality, Young's inequality with conjugate exponents \(\alpha = 2, \beta = 2(m-1)/(m-3), \delta = m-1\), and the interpolation inequality of Proposition A.13:

\[\int_\Sigma \nabla(2) H \ast \nabla H \ast \nabla H H^{2m-5} \, d\mu \leq c \|\nabla H\|_\infty \int_\Sigma |\nabla(2) H| |\nabla H| |H|^{2m-5} \, d\mu \leq c \|\nabla H\|_\infty \left( \int_\Sigma |\nabla(2) H|^2 H^{2(m-2)} \, d\mu \right)^{1/2} \times \left( \int_\Sigma |\nabla H|^2 H^{2(m-1)} \, d\mu \right)^{1/(m-1)} \leq \eta \left( \int_\Sigma |\nabla(2) H|^2 H^{2(m-2)} \, d\mu + \int_\Sigma |\nabla H|^2 H^{2(m-1)} \, d\mu \right) + c (\eta^{-1}) \|\nabla H\|_\infty^2 \cdot \int_\Sigma |\nabla H|^2 \, d\mu \leq \eta \left( \int_\Sigma |\nabla(2) H|^2 H^{2(m-2)} \, d\mu + \int_\Sigma |\nabla H|^2 H^{2(m-1)} \, d\mu \right) + c (\eta^{-1}) \left( \|A^o\|_{L^2} \frac{(m-1)(m-2)}{m} S_m \right) \cdot \left( \|A^o\|_{L^2}^2 \frac{2(m-1)}{m} S_m \right) \leq \eta \left( \int_\Sigma |\nabla(2) H|^2 H^{2(m-2)} \, d\mu + \int_\Sigma |\nabla H|^2 H^{2(m-1)} \, d\mu \right) + c (\eta^{-1}) \|A^o\|_{L^2}^{m-1} S_m. \tag{A.42}\]

which holds for any \(\eta > 0\). We estimate the penultimate term in \eqref{eq:A.41} by using
Theorem A.2 with $u = |\nabla H| |A^o| |H|^{m-2}$:

$$
\int_{\Sigma} |\nabla H|^{2} |A^o|^{2} H^{2(m-2)} d\mu \\
\leq c \left( \int_{\Sigma} |\nabla (2)H| |A^o| |H|^{m-2} d\mu + \int_{\Sigma} |\nabla H| |\nabla A^o| |H|^{m-2} d\mu \right) \\
+ \int_{\Sigma} |\nabla H|^{2} |A^o| |H|^{m-3} d\mu + \int_{\Sigma} |\nabla H| |A^o| |H|^{m-1} d\mu \right)^{2} \\
\leq c \|A^o\|^{2}_{2} \left( \int_{\Sigma} |\nabla (2)H|^{2} H^{2(m-2)} d\mu + \int_{\Sigma} |\nabla H|^{2} H^{2(m-1)} d\mu \right) \\
+ c \left( \int_{\Sigma} |\nabla H|^{2} |A^o|^{2} d\mu + \int_{\Sigma} |\nabla A^o|^{2} H^{2} d\mu \right) \cdot \int_{\Sigma} |\nabla H|^{2} H^{2(m-3)} d\mu. \tag{A.43}
$$

Next, using the inequality of Corollary 5.13 followed by the interpolation inequality of Proposition A.13 we are able to obtain the estimate

$$
\int_{\Sigma} |\nabla H|^{2} |A^o|^{2} d\mu + \int_{\Sigma} |\nabla A^o|^{2} H^{2} d\mu \leq c \int_{\Sigma} |\Delta H|^{2} d\mu \leq c \|A^o\|^{2} \frac{(m-2)}{2} S_{m}^{2}.
$$

Similarly, using Hölder’s inequality with conjugate exponents $\alpha = (m-1)/2$, $\alpha^* = (m-1)/(m-3)$ and Proposition A.13 once more gives

$$
\int_{\Sigma} |\nabla H|^{2} H^{2(m-3)} d\mu \\
\leq c \left( \int_{\Sigma} |\nabla H|^{2} d\mu \right)^{\frac{m-2}{2}} \left( \int_{\Sigma} |\nabla H|^{2} H^{2(m-1)} d\mu \right)^{\frac{m-4}{m-1}} \\
\leq c \|A^o\|^{\frac{m-2}{2}} S_{m}^{\frac{2}{m-1}} \left( \int_{\Sigma} |\nabla H|^{2} H^{2(m-1)} d\mu \right)^{\frac{m-4}{m-1}}. 
$$

Therefore

$$
\left( \int_{\Sigma} |\nabla H|^{2} |A^o|^{2} d\mu + \int_{\Sigma} |\nabla A^o|^{2} H^{2} d\mu \right) \cdot \int_{\Sigma} |\nabla H|^{2} H^{2(m-3)} d\mu \\
\leq c \|A^o\|^{2} \frac{2}{S_{m}^{1}} \left( \int_{\Sigma} |\nabla H|^{2} H^{2(m-1)} d\mu \right)^{\frac{m-3}{m-1}} \\
\leq c \|A^o\|^{2} \left( S_{m} + \int_{\Sigma} |\nabla H|^{2} H^{2(m-1)} d\mu \right), \tag{A.44}
$$

where we have used Young’s inequality with conjugate exponents $\alpha = \frac{(m-1)}{2}$, $\alpha^* = \frac{(m-1)}{(m-3)}$.
in the last step. Substituting (A.43) and (A.44) into (A.42) and absorbing into the left hand side gives for any $\eta > 0$:

$$
(1 - \eta - c\|A^0\|_2^2) \int_\Sigma |\nabla (2) H|^2 H^{2(m-2)} d\mu \\
+ \left(\frac{1}{4} - \eta - c\|A^0\|_2^2\right) \int_\Sigma |\nabla H|^2 H^{2(m-1)} d\mu
\leq \int_\Sigma |\Delta H|^2 H^{2(m-2)} d\mu + c\left(\eta^{-1}\|A^0\|_2^2 S_m.\right)
$$

Therefore is $\eta > 0$ is chosen small enough, and if $\varepsilon_0 > 0$ is sufficiently small then

$$
\int_\Sigma |\nabla (2) H|^2 H^{2(m-2)} d\mu + \int_\Sigma |\nabla H|^2 H^{2(m-1)} d\mu \leq c \int_\Sigma |\Delta H|^2 H^{2(m-2)} d\mu + c\|A^0\|_2^2 S_m.
$$

Finally, combining this with (A.40) finishes the proof.

**Lemma A.18.** Fix $m \in \mathbb{N}$ and define

$$
R_l := \int_\Sigma |\Delta^{\frac{l}{2}} H|^2 H^{2(m-l)} d\mu, \quad 1 \leq l \leq m.
$$

If we assume the notation and hypothesis of Proposition A.13 and Theorem A.15, then the inductive inequality holds for some universal constant $c > 0$:

$$
R_l \leq c R_{l-1}^{\frac{1}{2}} R_{l+1}^{\frac{1}{2}} + c\|A^0\|_2^2 (R_{l-1} + S_m). \tag{A.45}
$$

Furthermore for any $1 \leq l \leq m - 1$ we have

$$
R_1 + \cdots + R_l \leq c R_{l+1} + c\|A^0\|_2^2 S_m. \tag{A.46}
$$

**Proof.** For the first interpolation inequality we proceed in a similar manner to the proof of Theorem A.15. Using integration by parts, our estimate for $\|\nabla H\|_\infty$ from (A.37) and Young’s inequality with conjugate exponents $\alpha = \frac{(m-(l-1))}{(m-(l+1))}$, $\alpha^* = \frac{(m-(l-1))}{2}$ gives

$$
R_l = \int_\Sigma \Delta^{\frac{l-1}{2}} H * \Delta^{\frac{l+1}{2}} H H^{2(m-l)} d\mu + \int_\Sigma \Delta^{\frac{l-1}{2}} H * \Delta^{\frac{l}{2}} H * \nabla H H^{2(m-l)-1} d\mu
$$
\[
\leq c R_{l-1}^{\frac{1}{2}} R_{l+1}^{\frac{1}{2}} + c \|\nabla H\|_\infty R_l^{\frac{1}{2}} \left( \int_\Sigma |\Delta^{\frac{l-1}{2}} H|^2 H^{2(m-(l+1))} \, d\mu \right)^{\frac{1}{2}}
\]
\[
\leq c R_{l-1}^{\frac{1}{2}} R_{l+1}^{\frac{1}{2}} + c \|\nabla H\|_\infty R_l^{\frac{1}{2}} R_{l-1}^{\frac{m-(l+1)}{2}} \left( \int_\Sigma |\Delta^{\frac{l-1}{2}} H|^2 \, d\mu \right)^{\frac{1}{m-(l+1)}}
\]
\[
\leq c R_{l-1}^{\frac{1}{2}} R_{l+1}^{\frac{1}{2}} + c \|A^o\|_2 R_l^{\frac{m-(l+1)}{2}} \left( \int_\Sigma |\Delta^{\frac{l-1}{2}} H|^2 \, d\mu \right)^{\frac{1}{m-(l+1)}}.
\]

Using Young’s then leaves us with the inequality

\[
R_l \leq c R_{l-1}^{\frac{1}{2}} R_{l+1}^{\frac{1}{2}} + \eta R_l + c \left( \eta^{-1} \right) \|A^o\|_2^2 R_{l-1}^{\frac{m-(l+1)}{2}} \left( \int_\Sigma |\Delta^{\frac{l-1}{2}} H|^2 \, d\mu \right)^{\frac{1}{m-(l+1)}}
\]

for any \( \eta > 0 \). Choosing \( \eta \) small enough, absorbing and multiplying through yields

\[
R_l \leq c R_{l-1}^{\frac{1}{2}} R_{l+1}^{\frac{1}{2}} + c \|A^o\|_2^2 R_{l-1}^{\frac{m-(l+1)}{2}} S_m^{\frac{1}{m-(l+1)}}.
\]

Finally, using Young’s inequality on the last term with conjugate exponents \( \alpha = \frac{m-(l-1)}{m-(l+1)} \), \( \alpha^* = \frac{(m-(l-1))}{2} \) proves (A.45).

The proof of (A.46) then follows in the same manner as the proof of Corollary A.16, with the base case \( l = 1 \) already being proven in Lemma A.17.

\[\square\]

**Corollary A.19.** Suppose \( f : \Sigma \to \mathbb{R}^3 \) is a closed immersion satisfying \( \|A^o\|_2^2 \leq \varepsilon_0 \) for \( \varepsilon_0 > 0 \) sufficiently small. Then for any \( j \in \mathbb{N} \) the following estimate holds for some universal constant \( c > 0 \):

\[
\left\| \nabla_{(j)} A^o \right\|_\infty^2 \leq c \|A^o\|_2^{\frac{2}{2+j}} \left( \int_\Sigma |\nabla_{(j+1)} A^o|^4 \, d\mu \right)^{\frac{j+1}{2}}.
\]

**Proof.** Firstly, the multiplicative Sobolev inequality Theorem A.6 implies that

\[
\left\| \nabla_{(j)} A^o \right\|_\infty^6 \leq c \int_\Sigma |\nabla_{(j)} A^o|^2 \, d\mu \cdot \left( \int_\Sigma |\nabla_{(j+1)} A^o|^4 \, d\mu + \int_\Sigma |\nabla_{(j)} A^o|^4 H^4 \, d\mu \right). \tag{A.47}
\]

Next the interpolation inequality of Lemma A.10 gives

\[
\int_\Sigma |\nabla_{(j)} A^o|^2 \, d\mu \leq c \|A^o\|_2^{\frac{j}{1+j}} \left( \int_\Sigma |\nabla_{(j+2)} A^o|^2 \, d\mu \right)^{\frac{j}{1+j}}.
\]
Similarly, we combine Theorem A.2 twice with \( u_1 = |\nabla_{j+1} A^o|^2 \) and \( u_2 = |\nabla_j A^o|^2 H^2 \), inequality (A.49), the multiplicative Sobolev inequality from Lemma A.12, as well as the \( L^\infty \) estimate for the trace-free curvature from Theorem 5.15 yields

\[
\int_\Sigma |\nabla_{j+1} A^o|^4 \, d\mu + \int_\Sigma |\nabla_j A^o|^4 \, H^4 \, d\mu \\
\leq c \left( \int_\Sigma |\nabla_{j+1} A^o|^2 \, d\mu + \int_\Sigma |\nabla_j A^o|^2 \, H^2 \, d\mu \right) \\
\times \left( \sum_{i=0}^2 \int_\Sigma |\nabla_{j+2-i} A^o|^2 H^{2i} \, d\mu + \int_\Sigma |\nabla_j A^o|^2 |\nabla H|^2 \, d\mu \right) \\
\leq c \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu \cdot \left( \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu + \int_\Sigma |\nabla_j A^o|^2 |\nabla H|^2 \, d\mu \right) \\
\leq c \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu \cdot \left( \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu + \| A^o \|^2 \int_\Sigma |\nabla_{j+1} A|^2 \, d\mu \right) \\
\leq c \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu \cdot \left( \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu + \| A^o \|^2 \int_\Sigma |\nabla_{j+1} A|^2 \, d\mu \right) \\
\leq c \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu \cdot \left( \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu + \| A^o \|^2 \int_\Sigma |\nabla_{j+1} A|^2 \, d\mu \right) \\
\leq c \| A^o \|_2^{\frac{2}{3}} \left( \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu \right)^{1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{4}}.
\]

We have also used the interpolative inequality from Lemma A.10 in the last step.

Therefore (A.47) implies

\[
\| \nabla_j A^o \|_\infty^6 \leq c \| A^o \|_2^{\frac{6}{7+2}} \left( \int_\Sigma |\nabla_{j+2} A^o|^2 \, d\mu \right)^{\frac{3(j+1)}{7+2}}.
\]

Taking the cube root of both sides then finishes the proof. \(\square\)

**Theorem A.20.** Suppose \( f : \Sigma \to \mathbb{R}^3 \) is a closed immersion satisfying \( \| A^o \|^2_2 \leq \varepsilon_0 \) for \( \varepsilon_0 > 0 \) sufficiently small. Fix \( m \in \mathbb{N} \) and denote \( Q_l, 1 \leq l \leq m - 1 \) by

\[
Q_l := \int_\Sigma |\nabla_{(m-l)} A^o|^2 \, H^{2l} \, d\mu.
\]
Then the recurrence inequality

$$Q_l \leq Q_{l-1}^{\frac{3}{2}} Q_{l+1}^{\frac{3}{2}} + c \| A^o \|_2^\frac{3}{2} \left( Q_{l+1} + \| (m) A^o \|_2^2 \right)$$

holds.

**Proof.** Integration by parts and Hölder’s inequality immediately gives

$$Q_l := \int_\Sigma |\nabla (m-l) A^o|^2 H^{2l} d\mu$$

$$= \int_\Sigma \nabla (m-(l+1)) A^o \ast \nabla (m-l) A^o \ast H^{2l} d\mu$$

$$+ \int_\Sigma \nabla (m-(l+1)) A^o \ast \nabla (m-l) A^o \ast \nabla H \ast H^{2l-1} d\mu$$

$$\leq c Q_{l-1}^{\frac{3}{2}} Q_{l+1}^{\frac{3}{2}} + c \| \nabla H \|_\infty Q_{l+1}^{\frac{1}{2}} \left( \int_\Sigma |\nabla (m-(l+1)) A^o|^2 H^{2(l-1)} d\mu \right)^{\frac{1}{2}}$$

$$\leq c Q_{l-1}^{\frac{3}{2}} Q_{l+1}^{\frac{3}{2}} + c \| \nabla H \|_\infty Q_{l+1}^{\frac{3}{2}} \left( \int_\Sigma |\nabla (m-(l+1)) A^o|^2 d\mu \right)^{\frac{1}{2}}.$$

Using Young’s inequality, absorbing and multiplying out then gives us

$$Q_l \leq c Q_{l-1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}} + c \| \nabla H \|_\infty \left( \int_\Sigma |\nabla (m-(l+1)) A^o|^2 d\mu \right)^{\frac{1}{2+\frac{1}{2}}} \int_\Sigma |\nabla (m-(l+1)) A^o|^2 d\mu$$

Incorporating our estimate from for \( \| \nabla H \|_\infty \) from (A.37), the interpolation inequalities from Lemma A.10 as well as Young’s inequality with conjugate exponents \( p = \frac{l+1}{l-1}, \) \( p^* = \frac{l+1}{2} \) gives

$$Q_l \leq c Q_{l-1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}} + c \| A^o \|_2^{\frac{3}{2}} \left( \int_\Sigma |\nabla (m) A^o|^2 d\mu \right)^{\frac{1}{2}} \left( \int_\Sigma |\nabla (m-(l+1)) A^o|^2 d\mu \right)^{\frac{1}{2+\frac{1}{2}}} Q_{l+1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}}$$

$$\leq c Q_{l-1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}} + c \| A^o \|_2^{\frac{3}{2} + \frac{4(m-3)}{3m}} \left( \int_\Sigma |\nabla (m) A^o|^2 d\mu \right)^{\frac{2}{m}} \left( \int_\Sigma |\nabla (m) A^o|^2 d\mu \right)^{\frac{2(l+1)}{m}} Q_{l+1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}}$$

$$\leq c Q_{l-1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}} + c \| A^o \|_2^{\frac{4}{m} + \frac{2}{m}} \left( \int_\Sigma |\nabla (m) A^o|^2 d\mu \right)^{\frac{2}{m}} Q_{l+1}^{\frac{1}{2}} Q_{l+1}^{\frac{1}{2}}.$$

\[\leq c Q_{l-1}^2 Q_{l+1}^2 + c \|A^0\|_2^2 \left( Q_{l+1} + \int_{\Sigma} |\nabla (m) A^0|^2 \, d\mu \right),\]

which finishes the proof. \(\square\)

**Theorem A.21.** Suppose \(f : \Sigma^2 \to \mathbb{R}^3\) is a closed immersion satisfying \(\|A^0\|_2^2 \leq \varepsilon_0\) for \(\varepsilon_0\) sufficiently small. Then

\[Q_m + Q_{m-1} \leq c Q_m^\frac{1}{2} Q_{m-2}^\frac{1}{2} + c \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} \, d\mu.\]

**Proof.** Integration by parts, Hölder’s inequality and identity (5.5), followed by an application of Theorem A.2 with \(u = |A^0|^2 |H|^{m-1}\) gives

\[Q_{m-1}\]

\[= \int_{\Sigma} |\nabla A^0|^2 |H|^{2(m-1)} \, d\mu\]

\[= - \int_{\Sigma} \left( A^0, \nabla (2) H + \frac{1}{2} H^2 A^0 - |A^0|^2 A^0 \right) H^{2(m-1)} \, d\mu + \int_{\Sigma} \nabla A^0 \ast \nabla H \ast A^0 \ast H^{2m-3} \, d\mu\]

\[\leq c Q_m^\frac{1}{2} Q_{m-2}^\frac{1}{2} - \frac{1}{2} Q_m + \int_{\Sigma} |A^0|^4 H^{2(m-1)} \, d\mu + \int_{\Sigma} |A^0|^2 |\nabla H| |H|^{m-2} \, d\mu\]

\[\leq c Q_m^\frac{1}{2} Q_{m-2}^\frac{1}{2} - \frac{1}{2} Q_m + c \left( \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} \, d\mu \right)^2 + c Q_{m-1}^{2(m-3)} \left( \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} \, d\mu \right) \frac{1}{2(m-1)}\]

\[\leq c Q_m^\frac{1}{2} Q_{m-2}^\frac{1}{2} - \frac{1}{2} Q_m + c \|A^0\|_2^2 \left( Q_{m-1} + Q_m + \int_{\Sigma} |\nabla H|^2 |A^0|^2 H^{2(m-2)} \, d\mu \right) + c Q_{m-1}^{2(m-3)} \left( \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} \, d\mu \right)^{\frac{1}{2(m-1)}}\]

\[\leq c Q_m^\frac{1}{2} Q_{m-2}^\frac{1}{2} - \frac{1}{2} Q_m + c \|A^0\|_2^2 \left( Q_{m-1} + Q_m + \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} \, d\mu \right) + c Q_{m-1}^{2(m-3)} \left( \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} \, d\mu \right) \frac{1}{2(m-1)}\]
\[ \leq c Q_m^2 Q_m^{l-2} - \frac{1}{2} Q_m + c \| A^0 \|_2^2 \left( Q_m - Q_m + \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} d\mu \right) \]

\[ + \eta Q_m - c (\eta^{-1}) \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} d\mu. \]

Therefore

\[ (1 - c \| A^0 \|_2^2 - \eta) Q_m - \left( \frac{1}{2} - c \| A^0 \|_2^2 \right) Q_m \]

\[ \leq c Q_m^{l/2} Q_m^{l-2} + c (\eta^{-1}) + c \| A^0 \|_2^2 \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} d\mu. \]

Choosing \( \eta > 0 \) small enough, and if \( \varepsilon_0 \) is small enough, finishes the proof. \( \Box \)

**Corollary A.22.** Suppose \( f : \Sigma^2 \to \mathbb{R}^3 \) is a closed immersion satisfying \( \| A^0 \|_2^2 \leq \varepsilon_0 \) for \( \varepsilon_0 \) sufficiently small. Then for \( 1 \leq l \leq \)

\[ Q_m + Q_{m-1} + \cdots + Q_{m-l} \leq c Q_{m-(l+1)} + c \| A^0 \|_2^2 Q_0 + c \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} d\mu. \quad (A.48) \]

Therefore it follows that

\[ \sum_{i=0}^{m-1} \int_{\Sigma} \left| \nabla_{(i)} A^0 \right|^2 H^{2(m-i)} d\mu \leq c \int_{\Sigma} \left| \nabla_{(m)} A^0 \right|^2 d\mu. \quad (A.49) \]

**Proof.** By Theorem A.21 and Young’s inequality,

\[ (1 - \eta) Q_m + Q_{m-1} \leq c (\eta^{-1}) Q_{m-2} + c \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} d\mu, \]

and so by choosing \( \eta > 0 \) small enough we find that the statement is true for \( l = 1 \). Next assume the statement is true for a general \( l \):

\[ Q_m + Q_{m-1} + \cdots + Q_{m-l} \leq c Q_{m-(l+1)} + c \| A^0 \|_2^2 Q_0 + c \int_{\Sigma} |\nabla A^0|^2 |A^0|^{2(m-1)} d\mu. \quad (A.50) \]

Then by Theorem A.2,

\[ Q_{m-(l+1)} \leq c Q_{m-(l+2)}^{1/2} Q_{m-l}^{1/2} + c \left[ Q_{m-l} + Q_0 \right]. \]
Substituting this into (A.50) and using Young’s inequality then gives

\[ Q_m + \cdots + (1 - c \| A^o \|^2 - \eta) Q_{m-t} + Q_{m-(t+1)} \leq c (\eta^{-1}) Q_{m-(t+2)} + c \| A^o \|^\frac{4}{3} Q_0 + c \int_\Sigma |\nabla A^o|^2 |A^o|^{2(m-1)} d\mu \]

for any \( \eta > 0 \). Therefore if \( \varepsilon_0, \eta > 0 \) are small, we find that the statement is true for \( l + 1 \). Thus the proof of (A.48) follows by mathematical induction.

To prove (A.49) we just have to estimate the term \( \int_\Sigma |\nabla A^o|^2 |A^o|^{2(m-1)} d\mu \). Using Theorem A.2 with \( u = |\nabla A^o| |A^o|^{m-1} \) and Young’s inequality gives

\[ \int_\Sigma |\nabla A^o|^2 |A^o|^{2(m-1)} d\mu \leq c \left( \int_\Sigma |(\nabla A^o)|^2 d\mu \right)^\frac{1}{2} \cdot \left( \int_\Sigma |A^o|^2 |A^o|^{2(m-1)} d\mu \right) \]

\[ \leq c \left( \int_\Sigma |(\nabla A^o)|^2 d\mu \right)^\frac{1}{2} \cdot \left( \int_\Sigma |A^o|^2 |A^o|^{2(m-1)} d\mu \right) + c \| A^o \|^2 \int_\Sigma |\nabla A^o|^2 |A^o|^{2(m-1)} d\mu + c \int_\Sigma |\nabla A^o|^2 |A^o|^{2(m-1)} d\mu \]

\[ \leq (\eta + c \| A^o \|^2) \int_\Sigma |\nabla A^o|^2 |A^o|^{2(m-1)} d\mu \]

\[ + c \| A^o \|^2 (\int_\Sigma |(\nabla A^o)|^2 d\mu)^\frac{1}{2} \cdot \int_\Sigma |A^o|^{2(m-1)} d\mu \]

\[ + c \| A^o \|^2 Q_{m-1} \leq (\eta + c \| A^o \|^2) \int_\Sigma |\nabla A^o|^2 |A^o|^{2(m-1)} d\mu \]

\[ + c \| A^o \|^2 (\int_\Sigma |(\nabla A^o)|^2 d\mu)^\frac{1}{2} \cdot \int_\Sigma |A^o|^{2(m-1)} d\mu \]

\[ + c \| A^o \|^2 Q_{m-1} \].

(A.51)
Meanwhile using Theorem A.2 with \( u = |A^o|^{m-1} \) gives the inductive inequality

\[
\int_\Omega |A^o|^{2(m-1)} \, d\mu \leq c \left( \int_\Omega |\nabla A^o|^2 \, d\mu + \int_\Omega |A^o|^{2m-2} \, d\mu + \int_\Omega |A^o|^{m-1} \|H\| \, d\mu \right)^2
\]

\[
\leq c \left( \int_\Omega |\nabla A^o|^2 \, d\mu + \int_\Omega |A^o|^2 H^2 \, d\mu \right) \cdot \int_\Omega |A^o|^{2(m-2)} \, d\mu,
\]

which implies

\[
\int_\Omega |A^o|^{2(m-1)} \, d\mu \leq c \|A^o\|_2^2 \left( \int_\Omega |\nabla A^o|^2 \, d\mu + \int_\Omega |A^o|^2 H^2 \, d\mu \right)^{m-2}
\]

\[
\leq c \|A^o\|_2^2 \left( \|A^o\|_{\frac{3}{2}}^\frac{4m-2}{3} \left( \int_\Omega |\nabla \Delta H|^2 \, d\mu \right) \right)^{m-2}
\]

\[
\leq c \|A^o\|_2^{4m-2} \left( \|A^o\|_{\frac{3}{2}}^{2(m-3)} \left( \int_\Omega |\nabla (\_m) A^o|^2 \, d\mu \right) \right)^{\frac{m-2}{3}}
\]

\[
= c \|A^o\|_2^{2(m-2m+2)} \left( \int_\Omega |\nabla (\_m) A^o|^2 \, d\mu \right)^{\frac{m-2}{m}}
\]

Substituting this into (A.51) gives

\[
(1 - \eta - c \|A^o\|_2^2) \int_\Omega |\nabla A^o|^2 |A^o|^{2(m-1)} \, d\mu
\]

\[
\leq c \|A^o\|_2^{2(m-1)} \int_\Omega |\nabla (\_m) A^o|^2 \, d\mu + c \left( \eta^{-1} \right) \|A^o\|_2^2 Q_{m-1}
\]

for any \( \eta > 0 \). Therefore if \( \eta, \varepsilon_0 > 0 \) are small enough the estimate

\[
\int_\Omega |\nabla A^o|^2 |A^o|^{2(m-1)} \, d\mu \leq c \|A^o\|_2^{2(m-1)} \int_\Omega |\nabla (\_m) A^o|^2 \, d\mu + c \|A^o\|_2^2 Q_{m-1}
\]

holds for some universal constant \( c > 0 \). Substituting into (A.48) with \( l = m - 1 \) gives

\[
\sum_{i=0}^{m-1} \int_\Omega |\nabla (\_i) A^o|^2 H^{2(m-i)} \, d\mu
\]

\[
\leq c \left( 1 + \|A^o\|_2^2 + \|A^o\|_{\frac{4}{3}}^\frac{4}{3} \right) \int_\Omega |\nabla (\_m) A^o|^2 \, d\mu + c \|A^o\|_2^2 \int_\Omega |\nabla A^o|^2 H^{2(m-1)} \, d\mu.
\]
Therefore if $\varepsilon_0 > 0$ is small enough, we can absorb and multiply out and the estimate (A.49) holds. \qed
Appendix B

Interpolation inequalities pertaining to Part II

In this chapter we frequently refer to a quantity $\mathcal{K}_{osc}$. This is the anisotropic normalised oscillation of curvature introduced in Chapter 12:

$$\mathcal{K}_{osc}(\gamma) := L(\gamma) \int_{\gamma} (\kappa - \bar{\kappa})^2 d\sigma.$$ 

**Lemma B.1.** Let $\gamma : S^1 \rightarrow M^2$ be a smooth closed curve. Then for any $m \in \mathbb{N}$ we have

$$\int_{\gamma} \kappa_{\sigma m}^2 d\sigma \leq \varepsilon L^2 \int_{\gamma} \kappa_{\sigma m+1}^2 d\sigma + \frac{1}{4\varepsilon^m} L^{-(2m+1)} \mathcal{K}_{osc},$$

where $\varepsilon > 0$ can be made as small as desired.

**Proof.** We will prove the lemma inductively. The case $m = 1$ can be checked quite easily, by applying integration by parts and the Cauchy-Schwarz inequality:

$$\int_{\gamma} \kappa_{\sigma}^2 d\sigma = \int_{\gamma} (\kappa - \bar{\kappa})^2 d\sigma = -\int_{\gamma} (\kappa - \bar{\kappa}) (\kappa - \bar{\kappa})_{\sigma} d\sigma$$

$$\leq \left( \int_{\gamma} (\kappa - \bar{\kappa})^2 d\sigma \right)^{\frac{1}{2}} \left( \int_{\gamma} \kappa_{\sigma}^2 d\sigma \right)^{\frac{1}{2}}$$
\[ \leq \varepsilon \mathcal{L}^2 \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma + \frac{1}{4\varepsilon} \mathcal{L}^{-2} \int_\gamma (\kappa - \bar{\kappa})^2 \, d\sigma \]
\[ = \varepsilon \mathcal{L}^2 \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma + \frac{1}{4\varepsilon} \mathcal{L}^{-3} \mathcal{K}_{osc}. \]

Next assume that the statement is true for \( j = m \):

\[ \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma \leq \varepsilon \mathcal{L}^2 \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma + \frac{1}{4\varepsilon} \mathcal{L}^{-2} \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma + \frac{1}{4\varepsilon} \mathcal{L}^{-3} \mathcal{K}_{osc} \] (B.1)

where \( \varepsilon > 0 \) can be made as small as desired.

Again performing integration by parts and the Cauchy-Schwarz inequality, we have for any \( \varepsilon > 0 \):

\[ \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma = \frac{\varepsilon}{2} \mathcal{L}^2 \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma + \frac{1}{2\varepsilon} \mathcal{L}^{-2} \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma. \] (B.2)

Substituting the inductive assumption (B.1) into (B.2) then gives

\[ \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma \leq \frac{\varepsilon}{2} \mathcal{L}^2 \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma + \frac{1}{2\varepsilon} \mathcal{L}^{-2} \left( \varepsilon \mathcal{L}^2 \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma + \frac{1}{4\varepsilon} \mathcal{L}^{-2} \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma \right), \]

meaning that

\[ \frac{1}{2} \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma \leq \frac{\varepsilon}{2} \mathcal{L}^2 \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma + \frac{1}{2} \frac{1}{4\varepsilon} \mathcal{L}^{-2} \int_\gamma \kappa_{\sigma^2}^2 \, d\sigma. \]

Multiplying out by 2 then gives us the inductive step, completing the lemma.

**Lemma B.2.** Let \( f : \mathbb{R} \to \mathbb{R} \) be an absolutely continuous and periodic function of period \( P \). Then, if \( \int_0^P f \, dx = 0 \) we have

\[ \int_0^P f^2 \, dx \leq \left( \frac{P}{2\pi} \right)^2 \int_0^P f_x^2 \, dx, \]
with equality if and only if

\[ f(x) = A \cos \left( \frac{2\pi}{P} x \right) + B \sin \left( \frac{2\pi}{P} x \right) \]

for some constants \( A, B \).

**Proof.** We will use the Fourier series expansion. Since \( f \) is periodic with period \( P \) we may write \( f \) as

\[ f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} a_k \cos \left( \frac{2k\pi x}{P} \right) + \sum_{k=1}^{\infty} b_k \sin \left( \frac{2k\pi x}{P} \right). \]

Here \( a_k \) and \( b_k \) are given by

\[ a_k = \frac{2}{P} \int_0^P f(x) \cos \left( \frac{2k\pi x}{P} \right) \, dx \quad \text{and} \quad b_k = \frac{2}{P} \int_0^P f(x) \sin \left( \frac{2k\pi x}{P} \right) \, dx. \]

Note the condition \( \int_0^P f \, dx = 0 \) implies that \( a_0 = 0 \). Therefore we have

\[ f(x) = \sum_{k=1}^{\infty} a_k \cos \left( \frac{2k\pi x}{P} \right) + \sum_{k=1}^{\infty} b_k \sin \left( \frac{2k\pi x}{P} \right), \]

and

\[ f'(x) = -\frac{2\pi}{P} \sum_{k=1}^{\infty} k a_k \sin \left( \frac{2k\pi x}{P} \right) + \frac{2\pi}{P} \sum_{k=1}^{\infty} k a_k \cos \left( \frac{2k\pi x}{P} \right). \]

Integrating \( f^2 \) then gives

\[ \int_0^P f^2 \, dx = \sum_{k,l=1}^{\infty} a_k a_l \int_0^P \cos \left( \frac{2k\pi x}{P} \right) \cos \left( \frac{2l\pi x}{P} \right) \, dx \]

\[ + \sum_{k,l=1}^{\infty} a_k b_l \int_0^P \cos \left( \frac{2k\pi x}{P} \right) \sin \left( \frac{2l\pi x}{P} \right) \, dx \]

\[ + \sum_{k,l=1}^{\infty} a_l b_k \int_0^P \cos \left( \frac{2l\pi x}{P} \right) \sin \left( \frac{2k\pi x}{P} \right) \, dx \]

\[ + \sum_{k,l=1}^{\infty} b_k b_l \int_0^P \sin \left( \frac{2k\pi x}{P} \right) \sin \left( \frac{2l\pi x}{P} \right) \, dx. \]
\[= \frac{P}{2} \sum_{k,l=1}^{\infty} \delta_k^l a_k a_l + 0 + \frac{P}{2} \sum_{k,l=1}^{\infty} \delta_k^l b_k b_l \]
\[= \frac{P}{2} \sum_{k=1}^{\infty} \left( a_k^2 + b_k^2 \right). \tag{B.3} \]

Equation (B.3) is known as **Parseval’s identity**. By a similar process, we can derive the equation
\[\int_0^P f^2(x) \, dx = \left( \frac{P}{2} \right) \left( \frac{2\pi}{P} \right)^2 \sum_{k=1}^{\infty} k^2 \left( a_k^2 + b_k^2 \right). \tag{B.4} \]

Comparing at (B.3) and (B.4), it is obvious that
\[\int_0^P f^2(x) \, dx \geq \left( \frac{2\pi}{P} \right)^2 \int_0^P f^2(x) \, dx, \]
with equality if and only if \( a_k = b_k = 0 \) for \( k \geq 2 \). This proves the result.

**Lemma B.3.** Let \( f : \mathbb{R} \to \mathbb{R} \) be an absolutely continuous and periodic function of period \( P \). Then, if \( \int_0^P f \, dx = 0 \) we have
\[\|f\|_{\infty}^2 \leq \frac{P}{2\pi} \int_0^P f^2(x) \, dx. \]

**Proof.** Since \( \int_0^P f \, dx = 0 \) and \( f \) is \( P \)-periodic we conclude that there exists distinct \( 0 \leq p < q < P \) such that
\[f(p) = f(q) = 0. \]

Next, the fundamental theorem of calculus tells us that for any \( x \in (0, P) \),
\[\frac{1}{2} (f(x))^2 = \int_p^x f f_x \, dx = \int_q^x f f_x \, dx. \]

Hence
\[(f(x))^2 = \int_p^x f f_x \, dx - \int_x^q f f_x \, dx \leq \int_p^q |f f_x| \, dx \]
\[ \leq \int_0^P |f f_x| \, dx \]
\[ \leq \left( \int_0^P f^2 \, dx \cdot \int_0^P f_x^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq \frac{P}{2\pi} \int_0^P f_x^2 \, dx, \]

where the last step follows from Lemma B.2. We have also utilised Hölder’s inequality with \( p = q = 2 \).

**Lemma B.4** (Wirtinger’s inequality [25]). Let \( f : \mathbb{R} \to \mathbb{R} \) be an absolutely continuous and periodic function of period \( P \). If there exists a point \( p \in [0, P] \) such that \( f(p) = 0 \), then

\[ \int_0^P f^2 \, dx \leq \left( \frac{P}{\pi} \right)^2 \int_0^P f_x^2 \, dx. \]

**Proof.** Because of periodicity, we may assume that \( f(0) = f(P) = 0 \). Then by considering the odd extension to \( f \), on the interval \([0, P]\) we may write

\[ f(x) = \sum_{k=1}^{\infty} b_k \sin \left( \frac{k\pi x}{P} \right), \]

where the coefficients \( b_k \) are given by

\[ b_k = \frac{2}{P} \int_0^P f(x) \sin \left( \frac{k\pi x}{P} \right) \, dx. \]

Similarly to the proof of Lemma (B.2), we find that

\[ \int_0^P f^2 \, dx = \frac{P}{2} \sum_{k=1}^{\infty} b_k^2, \quad \text{and} \quad \int_0^P f_x^2 \, dx = \left( \frac{P}{\pi} \right)^2 \sum_{k=1}^{\infty} k^2 b_k^2. \]

It follows immediately that

\[ \int_0^P f_x^2 \, dx \geq \left( \frac{\pi}{P} \right)^2 \int_0^P f^2 \, dx \]

with equality if and only if \( b_k = 0 \) for \( k \geq 2 \). This proves the result. \( \square \)
Lemma B.5 ([26], Lemma 2.4). Let $\gamma : \mathbb{S}^1 \to \mathcal{M}^2$ be a smooth closed curve. Let $\phi : \mathbb{S}^1 \to \mathbb{R}$ be a sufficiently smooth function. Then for any $l \geq 2, K \in \mathbb{N}$ and $0 \leq i < K$ we have

$$L_{i+1}^{\frac{i+1}{4}} \left( \int_{\gamma} (\phi)^2 \, d\sigma \right)^{\frac{1}{4}} \leq c(K) \mathcal{L}^{\frac{1-\alpha}{2}} \left( \int_{\gamma} \phi^2 \, d\sigma \right)^{\frac{1-\alpha}{2}} \|\phi\|_{K,2}^\alpha.$$ 

Here $\alpha = \frac{i+1}{K}$, and

$$\|\phi\|_{K,2} := \sum_{j=0}^{K} \mathcal{L}^{j+\frac{1}{2}} \left( \int_{\gamma} (\phi)^2 \, d\sigma \right)^{\frac{1}{2}}.$$ 

In particular, if $\phi = \kappa - \bar{\kappa}$, then

$$L_{i+1}^{\frac{i+1}{4}} \left( \int_{\gamma} (\kappa - \bar{\kappa})^2 \, d\sigma \right)^{\frac{1}{4}} \leq c(K) (\mathcal{K}_{osc})^{\frac{1-\alpha}{2}} \|\kappa - \bar{\kappa}\|_{K,2}^\alpha.$$ 

Proof. Although the ambient manifold is the Minkowski plane, the proof is identical to that of Lemma 2.4 from [26] and is of a standard interpolative nature. Note that we use $k - \bar{k}$ in the identity (as opposed to Dziuk, Kuwert and Schätzle who use $k$). \hfill \Box

Lemma B.6 (Proposition 2.5, [26]). Let $\gamma : \mathbb{S}^1 \to \mathcal{M}^2$ be a smooth closed curve. Let $\phi : \mathbb{S}^1 \to \mathbb{R}$ be a sufficiently smooth function. Then for any term $P_{\nu}^\mu(\phi)$ (where $P_{\nu}^\mu(\cdot)$ denotes the same $P$-style notation from (1.12)) with $\nu \geq 2$ which contains only derivatives of $\kappa$ of order at most $K - 1$, we have

$$\int_{\gamma} |P_{\nu}^\mu(\phi)| \, d\sigma \leq c(K, \mu, \nu) \mathcal{L}^{1-\mu-\nu} \left( \mathcal{L} \int_{\gamma} \phi^2 \, d\sigma \right)^{\frac{\nu-\mu}{2}} \|\phi\|_{K,2}^{\eta},$$

In particular, for $\phi = \kappa - \bar{\kappa}$ we have the estimate

$$\int_{\gamma} |P_{\nu}^\mu(\kappa - \bar{\kappa})| \, d\sigma \leq c(K, \mu, \nu) \mathcal{L}^{1-\mu-\nu} (\mathcal{K}_{osc})^{\frac{\nu-\mu}{2}} \|\kappa - \bar{\kappa}\|_{K,2}^{\eta} \quad \text{(B.5)}$$

where $\eta = \frac{\mu+\frac{\nu}{2}-1}{K}$. 

Proof. Using Hölder’s inequality and Lemma B.5 with \( K = \nu \), if \( \sum_{j=1}^{\nu} i_j = \mu \) we have

\[
\int_\gamma |\phi_{\sigma^{i_1}} \ast \cdots \ast \phi_{\sigma^{i_\nu}}| \, d\sigma \leq \prod_{j=1}^{\nu} \left( \int_\gamma |\phi_{\sigma^{i_j}}| \, d\sigma \right)^{\frac{1}{\nu}}
\]

\[
= \mathcal{L}^{1-\mu-\nu} \prod_{j=1}^{\nu} \mathcal{L}^{i_j+1-\frac{1}{\nu}} \left( \int_\gamma |\phi_{\sigma^{i_j}}| \, d\sigma \right)^{\frac{1}{\nu}}
\]

\[
\leq c(K, \mu, \nu) \mathcal{L}^{1-\mu-\nu} \prod_{j=1}^{\nu} \left( \mathcal{L} \int_\gamma \phi^2 \, d\sigma \right)^{\frac{1}{\nu}} \|\phi\|_{K,2}^{\alpha_j} \tag{B.6}
\]

where \( \alpha_j = \frac{i_j + \frac{1}{\nu} - \frac{1}{K}}{2} \). Now

\[
\sum_{j=1}^{\nu} \alpha_j = \frac{1}{K} \sum_{j=1}^{\nu} \left( i_j + \frac{1}{2} - \frac{1}{\nu} \right) = \frac{\mu + \frac{\nu}{2} - 1}{K} = \eta,
\]

and so substituting this into (B.6) gives the first inequality of the lemma. It is then a simple matter of substituting \( \phi = \kappa - \bar{\kappa} \) into this result to prove statement (B.5). \( \square \)

Lemma B.7 ([26]). Let \( \gamma : S^1 \to \mathcal{M}^2 \) be a smooth closed curve and \( \phi : S^1 \to \mathbb{R} \) a sufficiently smooth function. Then for any term \( P^\mu_\nu(\phi) \) with \( \nu \geq 2 \) which contains only derivatives of \( \kappa \) of order at most \( K - 1 \), we have for any \( \varepsilon > 0 \)

\[
\int_\gamma |P^\mu_\nu(\phi)| \, d\sigma \leq c(K, \mu, \nu) \mathcal{L}^{1-\mu-\nu} \left( \mathcal{L} \int_\gamma \phi^2 \, d\sigma \right)^{\frac{\nu - \eta}{2}} \left( \mathcal{L}^{2K+1} \int_\gamma \phi^2_\kappa \, d\sigma + \mathcal{L} \int_\gamma \phi^2 \, d\sigma \right)^{\frac{1}{2}} \tag{B.7}
\]

Moreover if \( \mu + \frac{1}{2} \nu < 2K + 1 \) then \( \eta < 2 \) and we have for any \( \varepsilon > 0 \)

\[
\int_\gamma |P^\mu_\nu(\phi)| \, d\sigma \leq \varepsilon \int_\gamma \phi^2_\kappa \, d\sigma + c \cdot \varepsilon^{-\frac{\nu}{2-\eta}} \left( \int_\gamma \phi^2 \, d\sigma \right)^{\frac{\nu - \eta}{2-\eta}} + c \left( \int_\gamma \phi^2 \, d\sigma \right)^{\mu+\nu-1} \tag{B.8}
\]

In particular, for \( \phi = \kappa - \bar{\kappa} \), we have the estimate

\[
\int_\gamma |P^\mu_\nu(\kappa - \bar{\kappa})| \, d\sigma \leq c(K, \mu, \nu) \mathcal{L}^{1-\mu-\nu} \left( \mathcal{H}_{osc}^{\frac{\nu - \eta}{2-\eta}} \right) \left( \mathcal{L}^{2K+1} \int_\gamma (\kappa - \bar{\kappa})^2 \, d\sigma \right)^{\frac{1}{2}}.
\]

Here, as before, \( \eta = \frac{\mu + \frac{\nu}{2} - 1}{K} \).
Proof. Combining the previous lemma with the following standard interpolation inequality from that follows from repeated applications of Lemma B.1 (and is also found in [11])

$$\|\phi\|_{K,2}^2 \leq c(K) \left( \mathcal{L}^{2K+1} \int_{\gamma} \phi_{\sigma K}^2 d\sigma + \mathcal{L} \int_{\gamma} \phi^2 d\sigma \right)$$

yields the identity (B.7) immediately. To prove (B.8) we simply combine (B.7) with the Cauchy-Schwarz identity. The final identity of the Lemma follow by letting $\phi = \kappa - \bar{\kappa}$ in (B.7) and combining this with the identity

$$\mathcal{H}_{osc} \leq \mathcal{L} \left( \frac{\mathcal{L}^2}{4\pi^2} \right)^K \int_{\gamma} (\kappa - \bar{\kappa})_{\sigma K}^2 d\sigma = c(K) \mathcal{L}^{2K+1} \int_{\gamma} (\kappa - \bar{\kappa}_{\sigma K}^2 d\sigma,$$

which is a direct consequence of applying Lemma B.2 $(p+1)$ times repeatedly. \qed

**Theorem B.8** (Breuning [19], Theorem 1.1). Let $q \in \mathbb{R}^n$, $m, p \in \mathbb{N}$ with $p > m$. Additionally, let $A, V \geq 0$ be some fixed constants. Let $\mathfrak{T}$ denote the set of all mappings $f : \Sigma \to \mathbb{R}^n$ with the following properties:

- $\Sigma$ is an $m$-dimensional, compact manifold (without boundary)
- $f$ is an immersion in $W^{2,p}(\Sigma, \mathbb{R}^n)$ satisfying

$$\|A(f)\|_p \leq A, \ Vol(\Sigma) \leq V, \ and \ q \in f(\Sigma).$$

Then for every sequence $f^i : \Sigma^i \to \mathbb{R}^n$ in $\mathfrak{T}$ there is a subsequence $f^j$, a mapping $f : \Sigma \to \mathbb{R}^n$ in $\mathfrak{T}$ and a sequence of diffeomorphisms $\phi^j : \Sigma \to \Sigma^j$ such that $f^j \circ \phi^j$ converges in the $C^1$-topology to $f$.

The preceding theorem is an important tool in proving exponential convergence of solutions to the polyharmonic curve flow in Theorem 13.8.
Appendix C

Some important formulae from Riemannian hypersurface theory

In this Appendix we present (and prove) some of the major formulae and ideas for hypersurfaces of Euclidean space. The list is by no means complete, but contains all the tools required for this thesis. For a more detailed overview of Riemannian submanifold theory, the author suggests the (extremely thorough) article by Chen [21]. Also, for an introduction to working with submanifolds in higher codimensions, the author recommends the classic textbook of do Carmo [24].

We assume that \( f : \Sigma^n \to \mathbb{R}^{n+1} \) is an immersion.

In \( \mathbb{R}^{n+1} \) we adopt a local orthonormal frame \( \{ \tau_0, \tau_1, \ldots, \tau_n \} \) such that restricted to \( \Sigma \) we have \( \tau_0 = \nu \) (a chosen unit normal vector field to \( \Sigma \)) and \( \tau_i = \partial_i f = \frac{\partial f}{\partial x_i} \) for \( i = 1, 2, \ldots, n \). The components of the induced metric \( g \) on \( M \) are then given by

\[
g_{ij} = (\partial_i f, \partial_j f).\]

The induced metric extends naturally to general tensor fields; if \( S, T \) are \((l, m)\) tensors
defined on $M$:

$$S = S_{q_1...q_l}^{p_1...p_m} \partial_{q_1} \otimes \cdots \otimes \partial_{q_l} \otimes dx^{p_1} \otimes \cdots \otimes dx^{p_m}, T = T_{j_1...j_l}^{i_1...i_m} \partial_{j_1} \otimes \cdots \otimes \partial_{j_l} \otimes dx^{i_1} \otimes dx^{i_m},$$

(with respect to a local coordinate frame $\{\partial_i\}$ and corresponding dual frame $\{dx^i\}$), then the inner product of $S$ and $T$ is given by

$$\langle S, T \rangle = g^{q_1j_1} \cdots g^{q_lj_l} g_{i_1...i_m} S_{p_1...p_m}^{q_1...q_l} T_{i_1...i_m}^{j_1...j_l}.$$ 

We denote $|S| = \sqrt{\langle S, S \rangle}$ to be the norm of a tensor $S$ with respect to the induced metric $g$. For most of the proofs in this chapter we adopt a local orthonormal frame $\{\tau_0, \tau_1, \ldots, \tau_n\}$ such that at a point we have

$$\langle \tau_i, \tau_j \rangle = \delta^j_i.$$ 

The covariant derivative $\nabla$ on $\Sigma$, is related to the usual (partial) derivative $D$ on $\mathbb{R}^{n+1}$ by the formula

$$\nabla_X Y = D_X Y - (D_X Y, \nu) \nu \text{ for } X, Y \in TM.$$ 

Alternatively (in our local orthonormal frame) one writes

$$\nabla_X Y = (D_X Y, \tau_i) \tau_i,$$

where the summation is taken from 1 to $n$ (and so does not include $\tau_0 = \nu$). As noted in Chapter 1, $\nabla$ is the unique torsion-free connection on $M$ compatible with the metric $g = f^*(\cdot, \cdot)$. If $T$ is a $p$-tangent tensor field such that (in our local orthonormal frame)

$T = T_{i_1...i_p} \tau_{i_1} \otimes \cdots \otimes \tau_{i_p}$, then the covariant derivative of $T$, $\nabla T$, is the $p + 1$-tensor field $\nabla T = \nabla_{\tau_{i_{p+1}}} T \otimes \tau_{i_{p+1}}$. The shape operator of $\Sigma$ at the point $x$ is the tensor field defined by

$$S(X) = (D_X \nu)^T \text{ for } X \in T\Sigma,$$
where \( T \) denotes the tangential component. The \((0, 2)\)–tensor \( A \) defined by

\[
A(X, Y) := (S(X, \nu), Y) = ((D_X \nu)^T, Y) = (D_X \nu, Y)
\]

is then called the second fundamental form. Here the last step follows from the fact that any component of \( D_X \nu \) that is not tangential to \( \Sigma \) is zero when taken as an inner product with \( Y \). We use \( \nabla_i, D_i \) as shorthand for \( \nabla_{\partial_i}, D_{\partial_i} \) respectively. It is quite easy to see that \( A \) is symmetric (that is, \( A_{ij} = A_{ji} \)) using that \( D \) is torsion-free. Note that in our familiar orthonormal frame \( A \) is written as \( A_{ij} \tau_i \otimes \tau_j \), with

\[
A_{ij} = (D_{\tau_i} \nu, \tau_j) = - (D_{\tau_i} \tau_j, \nu).
\]

It is sometimes convenient to write \( A_{ij} = D_{\mu_i} \nu \). We call the trace of \( A \) with respect to the induced metric \( g \) the mean curvature of \( M \):

\[
H = g^{ij} A_{ij} = g^{ij} (D_i \nu, \partial_j f) = - g^{ij} (\partial_{ij} f, \nu).
\]

The Christoffel symbols (sometimes called “connection coefficients”) \( \Gamma_{ij}^k \) of \( M \) are expressed by the formula

\[
\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}), \tag{C.1}
\]

and satisfy the equation

\[
\nabla_i \partial_j = \Gamma_{ij}^k \partial_k. \tag{C.2}
\]

Note that for our local orthonormal frame (at a single point), the covariant derivative (connection) \( \nabla_{\partial_i} \) is simply a partial derivative, and so (C.1) implies that locally we have \( \Gamma = 0 \). We define the Riemannian \((0, 4)\)–tensor by \( R = R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l \), where

\[
R_{ijkl} = (\nabla_{ij} \partial_k f - \nabla_{ji} \partial_k f, \partial_l f). \tag{C.3}
\]
Here $\nabla_{ij}$ is shorthand for the Hessian:

$$\nabla_{ij} = \partial_i \partial_j := \partial_i \partial_j - \partial_j \partial_i. \quad (C.4)$$

Here we have used the identity (C.2). Sometimes it is more convenient to write Riem in its $(1,3)$ form:

$$R^l_{ijk} := g^{ls} R_{ijkl}. \quad (C.5)$$

Riem possesses various symmetries and antisymmetries. For example,

$$R_{ijkl} = R_{klij} = -R_{jikl} = -R_{ijlk}$$

From (C.3) we can take the trace over the second and fourth components with respect to the metric $g$ to form the Ricci tensor $\text{Ric} := R_{ij} dx^i \otimes dx^j$:

$$R_{ik} := g^{il} R_{ijkl} = g^{il} \langle \nabla_{ij} \partial_k - \nabla_{ji} \partial_k, \partial_l \rangle.$$ 

Similarly, tracing once more gives the scalar curvature $R$:

$$R := g^{ik} R_{ik} = g^{ik} g^{jl} \langle \nabla_{ij} \partial_k - \nabla_{ji} \partial_k, \partial_l \rangle.$$ 

**Claim C.1** (Formula for Interchange of Covariant Derivatives). For a general vector $X$ and covector $Y$, the following equations hold:

$$\nabla_{ij} X^k = \nabla_{ji} X^k + R^k_{ij\lambda} X^\lambda,$$

and

$$\nabla_{ij} Y_k = \nabla_{ji} Y_k + R^\lambda_{ijk} Y_\lambda.$$
Proof. This is relatively straightforward. We prove both equations in one by working in our orthonormal frame \( \{ \tau_i \} \) in which the musical (canonical) isomorphism between the tangent and cotangent space is (locally) given by the identity map. Now because \( Rm \) is a tensor field, it is in particular multilinear. Hence

\[
R_{ij\lambda k}X_\lambda = Rm(\tau_i, \tau_j, \tau_\lambda, \tau_k)X_\lambda
\]
\[
= Rm(\tau_i, \tau_j, X, \tau_k)
\]
\[
= ((\nabla_{\tau_j, \tau_i} - \nabla_{\tau_i, \tau_j})X, \tau_k)
\]
\[
= \nabla_{ij}X_k - \nabla_{ji}X_k, \text{ by definition.}
\]

This completes the proof. \( \square \)

Claim C.2 (Codazzi Equations). For an immersion \( f : \Sigma \to \mathbb{R}^{n+1} \) the gradient of the second fundamental form is completely symmetric:

\[
\nabla_i A_{jk} = \nabla_j A_{ki} = \nabla_k A_{ij}. \tag{C.5}
\]

Proof. Using the identity \( A_{ij} = D_j \nu_i \) as well as Claim C.1 gives

\[
\nabla_k A_{ij} = (\nabla_k A, \partial_i \otimes \partial_j)
\]
\[
= (D_k A, \partial_i \otimes \partial_j)
\]
\[
= (D_k D \nu, \partial_i \otimes \partial_j)
\]
\[
= D_{kj} \nu_i
\]
\[
= D_{jk} \nu_i
\]
\[
= \nabla_j A_{ki},
\]

where the penultimate step follows from the fact the partial derivatives in \( \mathbb{R}^{n+1} \) commute. \( \square \)

Claim C.3 (Gauss Equations). Let \( f : \Sigma \to \mathbb{R}^{n+1} \) be an immersion. Then the Rie-
mannian curvature tensor of $\Sigma$ satisfies

$$R_{ijkl} = A_{ik} A_{jk} - A_{il} A_{jk}. \quad (C.6)$$

Furthermore the Ricci tensor and Scalar curvature satisfy

$$R_{ik} = H A_{ik} - A_i^l A_{lk}$$

and

$$R = H^2 - |A|^2$$

respectively.

Proof. First we note that for $Y, Z \in T\Sigma$ we have

$$D_Y Z = \nabla_Y Z + (D_Y Z, \nu) \nu.$$ 

Hence for $X \in TM$

$$D_X D_Y Z = D_X \nabla_Y Z + X (D_Y Z, \nu) \nu + (D_Y Z, \nu) D_X \nu.$$ 

By considering the tangential components and remembering that $\nabla_Y Z \in TM$ gives

$$(D_X D_Y Z)^T = \nabla_X \nabla_Y Z + (D_Y Z, \nu) (D_X \nu)^T = \nabla_X \nabla_Y Z - A(Y, Z) S(X).$$

Next we choose our local orthonormal frame as usual. Hence using the notation of $(C.4)$,

$$\begin{align*}
\left( \nabla_{\tau_j, \tau_i, \tau_k, \tau_l} \right) \\
= \left( \nabla_{\tau_j} \nabla_{\tau_k, \tau_l \otimes \tau_i} \right) \\
= \left( D_{\tau_j} \nabla_{\tau_k, \tau_l \otimes \tau_i} \right) \\
= \left( D_{\tau_j} (\nabla_{\tau_k} \tau_l \otimes \tau_i) \right) \tau_l \otimes \tau_i \\
= (D_{\tau_j} (\nabla_{\tau_k} \tau_l \otimes \tau_i)) \, \tau_l \otimes \tau_i
\end{align*}$$
where we have used \( g_{ij} A_{jl} \delta^i_k = 0 \), the definition of \( A_{ij} \), and the definition of \( \nabla \). Hence

\[
R_{ijkl} = \left( \left( \nabla_{r_i} - \nabla_{r_j} \right) \tau_k, \tau_l \right) \\
= \left( \left( D_{r_i} \tau_k, \tau_l \right) - \left( D_{r_j} \tau_k, \tau_l \right) \right) + A_{ik} A_{jl} - A_{il} A_{jk} \\
= A_{ik} A_{jl} - A_{il} A_{jk},
\]

where we have used the fact that partial derivatives in \( \mathbb{R}^{n+1} \) commute in the last step.

This proves (C.6). To prove the other Ricci tensor identity, we simply trace the identity (C.6) with the metric tensor:

\[
R_{ik} := g^{il} R_{ijkl} = g^{il} (A_{ik} A_{jl} - A_{il} A_{jk}) = HA_{ik} - A^i_l A_{lk},
\]

and the scalar curvature identity is then proved similarly:

\[
R := g^{ik} R_{ik} = g^{ik} (HA_{ik} - A^i_l A_{lk}) = H^2 - |A|^2.
\]

\( \square \)

**Definition C.4** (Sectional Curvature). *If \((M, g)\) is a Riemannian manifold and \(X_p, Y_p\) are two linearly independent tangent vectors at the point \(p \in M\), then

\[
K(X \wedge Y) = \frac{g(R(X, Y) Y, X)}{g(X, X) g(Y, Y) - g(X, Y)^2}
\]

(C.7)
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(taken to be evaluated at \( p \)) is defined to be the sectional curvature of the two-dimensional plane \( X_p \wedge Y_p \), which is a subspace of the tangent space \( T_p M \).

**Remark C.5** (Sectional Curvature in dimension 2). If \( \dim M = 2 \), then there are only two linearly independent directions in the tangent space, and in this case \( K(X_p \wedge Y_p) = K_p \) (the Gaussian curvature of \( M \) at the point \( p \)). Hence (C.7) can be rearranged to form

\[
Rm(X, Y, X, Y) = K(g(X, X)g(Y, Y) - g(X, Y)^2).
\]

In index notation, this is read as

\[
R_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk}). \tag{C.8}
\]

**Claim C.6** (Gauss-Weingarten Relations). If \( f: \Sigma^\alpha \rightarrow \mathbb{R}^{n+1} \) is an immersion, then

\[
\nabla_{ij} f^\alpha = -A_{ij} \nu^\alpha \tag{C.9}
\]

and

\[
\partial_i \nu^\alpha = A^j_i \partial_j f^\alpha. \tag{C.10}
\]

**Proof.** Let \( \varphi = f^\alpha \). A calculation then gives

\[
\nabla_{ij} \varphi = (D_{\tau_i} \nabla \varphi, \tau_j)
= (D_{\tau_i} (D\varphi - (D\varphi, \nu) \nu), \tau_j)
= D_{ij} \varphi - D_{\tau_i} (D\varphi, \nu) (\nu, \tau_j) - (D\varphi, \nu) (D_{\tau_i} \nu, \tau_j)
= D_{ij} \varphi - A_{ij} (D\varphi, \nu).
\]

Therefore

\[
\nabla_{ij} f^\alpha = D_{ij} f^\alpha - A_{ij} (Df^\alpha, \nu).
\]

But

\[
Df^\alpha = \delta^\alpha_\beta \partial_\beta f^\alpha \partial_\gamma = df^\alpha,
\]
and therefore

\[(Df^\alpha, \nu) = df^\alpha(\nu) = \nu^\alpha.\]

This implies

\[\nabla_{ij}f^\alpha = D_{ij}f^\alpha - A_{ij}\nu^\alpha.\]

Combining this with the identity

\[D_{ij}f^\alpha = \partial_i f^\beta \partial_j f^\gamma D_{\beta\gamma} f^\alpha = \partial_i f^\beta \partial_j f^\gamma \partial_{\beta\gamma} f^\alpha = 0\]

then proves (C.9). To prove (C.10), we first note that by metric compatibility and by the fact that \((\nu, \nu) \equiv 1\), one has \(D\nu \in TM\). In particular,

\[\partial_i \nu^\alpha = g^{lm}(\partial_i \nu, \partial_l f) \partial_m f^\alpha = g^{lm}A_{il} \partial_m f^\alpha = A^j_i \partial_j f^\alpha,\]

which finishes the proof. \(\square\)

**Corollary C.7** (Laplace’s Identity). Let \(f : \Sigma^n \to \mathbb{R}^{n+1}\) be an immersion. Then

\[\Delta f^\alpha = \vec{H}^\alpha := -H \nu^\alpha.\] (C.11)

**Proof.** The proof follows immediately from the previous claim by tracing the identity (C.9) with the metric tensor:

\[\Delta f^\alpha := g^{ij}\nabla_{ij}f^\alpha = -g^{ij}A_{ij}\nu^\alpha = -H \nu^\alpha =: \vec{H}^\alpha.\]

\(\square\)