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Abstract

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Circular dispersive shock waves in colloidal media

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Abstract. A dispersive shock wave (DSW), with a circular geometry, is studied in a colloidal medium. The colloidal particle interaction is based on the repulsive theoretical hard sphere model, where a series in the particle density, or packing fraction is used for the compressibility. Experimental results show that the particle interactions are temperature dependent and can be either repulsive or attractive, so the second term in the compressibility series is modified to allow for temperature dependent effects, using a power law relationship. The governing equation is a focusing nonlinear Schrödinger-type equation with an implicit nonlinearity. The initial jump in electric field amplitude is resolved via a DSW, which forms before the onset of modulational instability. A semi-analytical solution for the amplitude of the solitary waves in a DSW of large radius, is derived based on a combination of conservation laws and geometrical considerations. The effect of temperature and background packing fraction on the evolution of the DSW and the amplitude of the solitary waves is discussed and the semi-analytical solutions are found to be very accurate, when compared with numerical solutions.

Keywords. solitary waves, dispersive shock waves, nonlocal optical media, conservation laws

1 Introduction

The model equation for optical waves in colloidal media is a nonlinear Schrödinger (NLS)-type equation with implicit nonlinearity, where the nanoparticle interactions, or compressibility, are described in series form. DSW, which resolve initial discontinuities, are subject to modulational instability (MI) in focusing media and do not persist at long length scales. However DSW's can occur in focusing media over experimental length scales, in both a focusing photorefractive medium[1] and a nematic liquid crystal medium[2]. DSW's also occur in thermal defocussing media, such as a solution of carbon in water[3], where the initial jump in amplitude was generated by a boundary between two media of different refractive indexes. It is of significant theoretical interest to examine DSW in colloidal media as experimentally observable colloidal DSW's are also likely to exist.

DSW in a focusing nematic liquid crystal medium were considered[4]. As the focusing nematic liquid crystal medium is a nonlocal media the DSW persists

for an experimentally relevant propagation distance, as the onset of MI is delayed. Semi-analytical solutions for both one dimensional line and two dimensional circular DSW were found. DSW also occur in defocusing nematic liquid crystal media[5, 6]. Due to the large nonlocality the defocusing nematic DSW is qualitatively quite different to the defocusing NLS DSW. It was found that a Korteweg-de Vries (KdV) bore occurs consisting of bright waves, rather than the usual dark waves of the defocusing NLS DSW.

Circular DSW have also been examined in a range of non optical applications, such as water waves. Models such as the Euler water wave equations, Kadomtsev-Petviashvili equation and the two dimensional Benjamin-Ono (BO) equation have been considered[7, 8]. Simplified cylindrical KdV and BO equations were derived and circular DSW were studied via both numerical solutions and modulation theory.

Colloidal DSW were also examined[9]. The particle interaction model used was the Carnahan Starling (CS) hard sphere model. Three different types of solitary wave amplitude versus jump height diagrams were found, depending on the background packing fraction. An extension considered a temperature dependent series form for the colloidal particle interactions[10]. Results for the line (1+1)-D DSW, using both a hard disk (HD), valid in one-dimension, and hard sphere (HS), valid in two-dimensions, theoretical models, were examined.

This paper considers the evolution of a DSW in a colloidal medium, with a circular geometry. We use the semi-analytical colloidal solitary wave solutions and uniform soliton theory[12], together with geometrical arguments, to obtain semi-analytical expressions for the amplitude of the solitary waves generated by the circular (2+1)-D DSW. The semi-analytical results allow the effects of different background packing fractions and temperatures to be examined. The semi-analytical solutions are shown to be accurate, in comparison with the numerical solutions.

2 Governing equations

The NLS-type equation that governs the nonlinear propagation of the beam through a colloidal suspension[13] is

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\nabla^2 u + (\eta - \eta_0)u = 0, \quad |u|^2 = g(\eta) - g_0, \quad (1)$$

$$g(\eta) = \ln(\eta) + 2B_2\eta + \frac{3}{2}B_3\eta^2 + \dots, \quad g(\eta_0) = g_0,$$

where u is the electric field envelope, η is the packing fraction of the colloid particles, and η_0 is the background packing fraction. The propagation variable z plays a time like role. The colloidal particles drift towards the region of higher light intensity, due to the physical phenomena of electrostriction. Moreover, the colloidal particles have a higher refractive index than the background liquid medium. So when an optical beam passes through the medium the optical gradient force acts against particle diffusion, increasing the concentration of

colloidal particles and hence the refractive index, in regions of higher light intensity, allowing self-focusing to occur.

For the (2+1)-D circular colloidal DSW case, circular geometry is used

$$u = u(r, z), \quad \eta = \eta(r, z), \quad \text{where } r = \sqrt{x^2 + y^2}, \quad (2)$$

with $\nabla^2 u$ given by $u_{rr} + \frac{1}{r}u_r$. The boundary condition for the circular DSW is

$$u = \begin{cases} a_m e^{i\theta}, & 0 < r < r_0, \\ 0, & r > r_0, \end{cases} \quad \eta = \begin{cases} \eta_m, & 0 < r < r_0, \\ \eta_0, & r > r_0, \end{cases} \quad (3)$$

at $z = 0$ and $\theta = (1 - \exp -\alpha_1 r)kr$. a_m is the amplitude of the continuous wave inside the disk of radius r_0 . For $r > r_0$ there is no wave so a jump occurs at $r = r_0$. There is a jump in electric field amplitude of a_m and packing fraction of $\eta_m - \eta_0$, linked by the state equation, $a_m^2 = g(\eta_m) - g_0$.

The phase $\theta = (1 - \exp -\alpha_1 r)kr$ is used because we want the boundary condition to have a quadratic chirp near the origin, where $\theta \rightarrow \alpha_1 kr^2$ as $r \rightarrow 0$. By using this form, we are sure that the Laplacian ∇^2 for the boundary condition (3) is bounded in the limit as $r \rightarrow 0$. We choose $\alpha_1 = 0.2$ so that for $r \gtrsim 50$, the phase will be chirp free. Therefore, for any value at which r is large, we will get $\theta \rightarrow kr$, which represents the (1+1)-D continuous wave solution. However, unlike the (1+1)-D case, the continuous wave will only be the exact solution of the (2+1)-D governing equations for a steady state DSW at which the wavenumber $k = 0$. For any value of $k > 0$, the continuous wave in (3) is only a correct approximation for $r \gg 1$ and will evolve in z , especially near the origin $r = 0$.

The particle compressibility is given by,

$$Z = 1 + (4 - 100T^{-1})\eta + 10\eta^2 + 18\eta^3 + 28\eta^4 + 40\eta^5 + 54\eta^6 + \dots, \quad (4)$$

where T is the temperature. We include a power law form of temperature dependency for the second virial coefficient, which reflects experimental studies. As the non-dimensional temperature T varies, the second coefficient B_2 can change from positive to negative, which changes the particle interaction forces from repulsive to attractive. As the temperature becomes large, the model approaches the repulsive HS model[14] (see their Table 3). This choice of (4) allows us to explore the effects on the circular DSW as the temperature changes and the interaction forces vary between repulsive and attractive cases.

3 Semi-analytical solutions

In order to develop our semi-analytical approximation for the solitary wave amplitude in a (2+1)-D circular DSW we first consider the DSW for the (1+1)-D line bore. This has the boundary condition.

$$u = \begin{cases} a_m e^{ikx}, & x < 0, \\ 0, & x > 0, \end{cases} \quad \eta = \begin{cases} \eta_m, & x < 0, \\ \eta_0, & x > 0. \end{cases} \quad (5)$$

We assume that the bore generates a train of solitary waves of uniform amplitude. In order to obtain an approximation for the amplitude of the solitary waves, we use uniform soliton theory[15, 16]. The mass conservation equation for (1) is

$$i \frac{\partial}{\partial z} |u|^2 + \frac{1}{2} \frac{\partial}{\partial x} (u^* u_x - u u_x^*) = 0, \quad (6)$$

and the energy equation is given by

$$\begin{aligned} & i \frac{\partial}{\partial z} [|u_x|^2 - 2(\eta - \eta_0)|u|^2 + 2\eta \ln \eta + 2\eta_0 \ln \eta_0 - 2(\eta - \eta_0)(1 + g_0) \\ & + 6\eta + 2B_2\eta^2 + B_3\eta^3 - 6\eta_0 - 2B_2\eta_0^2 - B_3\eta_0^3 + \dots] \\ & + \frac{1}{2} \frac{\partial}{\partial x} [u_x^* u_{xx} - u_x u_{xx}^* - 2(\eta - \eta_0)(u^* u_x - u u_x^*)] = 0. \end{aligned} \quad (7)$$

We integrate the conservation laws from $x = -\infty$ to $x = \infty$. As the boundary condition at $x = -\infty$ is non-zero, as described by (5), the x derivative terms are non-zero at $x = -\infty$. Integrating the conservation equations then gives

$$\begin{aligned} \frac{d}{dz} <M> &= k a_m^2, \quad \frac{d}{dz} <H> = k a_m^2 [k^2 - 2(\eta_m - \eta_0)], \quad \text{where} \\ M &= |u|^2, \quad H = [|u_x|^2 - 2(\eta - \eta_0)|u|^2 + 2\eta \ln \eta + 2\eta_0 \ln \eta_0 \\ &- 2(\eta - \eta_0)(1 + g_0) + 6\eta + 2B_2\eta^2 + B_3\eta^3 - 6\eta_0 - 2B_2\eta_0^2 \\ &- B_3\eta_0^3 + \dots], \quad <.\> = \int_{-\infty}^{\infty} . \, dx. \end{aligned} \quad (8)$$

From equation (8), we see that the conservation laws depend on k , which is the wavenumber of the initial condition. We then take the ratio of the two equations in (8) in the limit of $k \rightarrow 0$, and integrate, giving

$$<H> + 2(\eta_m - \eta_0) <M> = 0. \quad (9)$$

Equation (9) is the condition that describes, for small wavenumber, the mass to energy ratio generated by the initial condition. We assume that solitary waves generated by the bore have this same mass to energy ratio. However, for NLS-type equations, the solitary waves amplitude is independent of wavenumber, so (9) applies for all wavenumbers[4].

We now use the semi-analytical expression for a single colloidal solitary wave[10]

$$u(x, z) = a \operatorname{sech} \frac{x}{w} e^{i\sigma z}, \quad \eta(x) = \eta_0 + \alpha \operatorname{sech}^2 \frac{x}{\beta}. \quad (10)$$

Substituting this into (8) gives

$$\begin{aligned} <M> &= 2a^2 w, \quad <H> = P, \quad \text{where} \\ P &= \frac{2}{3} \frac{a^2}{w} - 4\alpha a^2 \Omega_1 - 4\alpha\beta(1 + g_0) + 4\beta\Theta_1 + 8B_2\alpha\beta\eta_0 + \frac{8}{3}B_2\alpha^2\beta \\ &+ 6B_3\eta_0^2\alpha\beta + 4B_3\eta_0\alpha^2\beta + \frac{16}{15}B_3\alpha^3\beta + 2(\eta_m - \eta_0)(2a^2 w). \end{aligned} \quad (11)$$

Substituting (11) into (9) then gives the transcendental equation

$$\begin{aligned} & \frac{2}{3} \frac{a^2}{w} - 4\alpha a^2 \Omega_1 - 4\alpha\beta(1 + g_0) + 4\beta\Theta_1 + 8B_2\alpha\beta\eta_0 + \frac{8}{3}B_2\alpha^2\beta \\ & + 6B_3\eta_0^2\alpha\beta + 4B_3\eta_0\alpha^2\beta + \frac{16}{15}B_3\alpha^3\beta + 4a^2w(\eta_m - \eta_0) = 0. \end{aligned} \quad (12)$$

where

$$\begin{aligned} & 3\alpha w(\Omega_1 - w\Omega_{1w}) - 1 = 0, \quad \sigma = -\frac{1}{2w^2} + \frac{\alpha}{w}(2\Omega_1 - w\Omega_{1w}), \\ & 4a^2\alpha(\Omega_1 - \beta\Omega_{1\beta}) - 4\beta(\alpha\Theta_{1\alpha} - \Theta_1) - \frac{8}{3}B_2\alpha^2\beta - 4B_3\eta_0\alpha^2\beta \\ & - \frac{32}{15}B_3\alpha^3\beta = 0, \\ & \alpha a^2\Omega_{1\beta} - 4\Theta_1 + 4\alpha(1 + g_0) - 8B_2\alpha\eta_0 - \frac{8}{3}B_2\alpha^2 - 6B_3\eta_0^2\alpha \\ & - 4B_3\eta_0\alpha^2 - \frac{16}{15}B_3\alpha^3 = 0. \end{aligned} \quad (13)$$

So (12) and (13) give the amplitudes a and α , and the widths w and β , of the electric field and packing fraction solitary waves in the (1+1)-D line DSW. These solitary wave properties are given in terms of the initial jump in amplitude a_m .

For large radius $r \gg 1$ and $k = 0$ the circular (2+1)-D and line (1+1)-D DSW are approximately the same as $\nabla^2 u = u_{rr} + \frac{1}{r}u_r \approx u_{rr}$. For non-zero wavenumber the properties of the (1+1)-D DSW are unchanged, the waves just move to the right at speed $V = k$. For the circular (2+1)-D DSW however, the case of nonzero k changes the properties of the DSW due to the effects of geometric spreading. Hence we modify the predictions of uniform soliton theory by combining it with a simple geometric optics analysis. This analysis shows that the electric field amplitude decreases like $a \sim r^{-1/2}$ for large r .

Moreover, as the DSW evolves, the maximum amplitude of the DSW varies in a complicated manner with z since there is an interaction between the individual waves of the DSW. For an expanding circular DSW, the z -weighted averages of the electric field in a domain extending from $z = 0$ to $z = z_1$ is

$$\frac{a}{z_1} \int_0^{z_1} \frac{dz}{(1 + \frac{Vz}{r_0})^{\frac{1}{2}}} = \frac{2a}{Vz_1} [(r_0^2 + kr_0z_1)^{\frac{1}{2}} - r_0], \quad (14)$$

4 Results

The evolution of the circular (2+1)-D DSW, at different background packing fraction values and temperatures, is now considered, where the compressibility model (4) is used for the particle interactions. The numerical solutions of the colloidal equations (1) will be compared with the semi-analytical solutions (12) and (13). We use a hybrid Runge-Kutta finite difference scheme for the numerical solutions.

Figure 1: Electric field amplitude $|u|$ versus r for a circular (2+1)-D DSW. The numerical solution of (1) is shown for $k = 0$ (solid blue line) at $z = 151$, $k = 1$ (dashed red line) at $z = 210$ and $k = 1.5$ (dotted green line) at $z = 271$. The other parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$, $\alpha_1 = 0.2$ and $T \rightarrow \infty$.

Figure 1 shows $|u|$ versus r for a circular (2+1)-D DSW. Shown is the numerical solution of (1) for $k = 0$ at $z = 151$, $k = 1$ at $z = 210$ and $k = 1.5$ at $z = 271$. The parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$, $\alpha_1 = 0.2$ and $T \rightarrow \infty$. We plot the solution at the z value for which the initial solitary wave has reached its maximum amplitude. For $k = 0$ the DSW is stationary and the largest wave peak occurs at $z = 151$. For $k = 1.0$, the largest peak occurs at a longer length scale of $z = 210$. Finally, when $k = 1.5$, the largest peak occurs at $z = 271$. We note that $|u| \rightarrow a_m$ as $r \rightarrow 0$ for the stationary circular DSW, the continuous wave is an exact solution of the governing equations. In the limit of large temperature, $T \rightarrow \infty$, the particle interaction model is the classical repulse HS one.

For this model, when $k = 0$, the highest peak of the DSW occurs at $(a, \alpha) = (2.76, 0.31)$ at $r = 581$. These values are qualitatively similar to what have been obtained using the CS particle interaction formula[9] with only 2% difference in the a value. When we look at the case where $k \neq 0$, the circular DSW does not have the same profile as the stationary case as the circular DSW will propagate outwards, so the field intensity becomes low, and a central dark zone will be formed. This is qualitatively similar to what have been obtained for DSW in

other optical media (see Figures 2 and 6[17] and Figure 5[18]). When $k = 1$, the peak amplitude is $(a, \alpha) = (2.38, 0.23)$ at $r = 788$ and when $k = 1.5$, the peak amplitude is $(a, \alpha) = (1.91, 0.13)$ at $r = 982$. If we compare these results to the CS model[9], for $k = 1$, there is 4% difference in the value of a while for $k = 1.5$, the difference is 3%. The locations for the highest wave correspond to numerical values of $k = 0.996$ and $k = 1.479$, respectively. The numerical propagation constants for the circular DSW obtained here are very close to the theoretical $V = k$ values for (1+1)-D case.

By using geometric optics and numerical results for the stationary circular DSW we get a prediction for the amplitude of the largest wave in the spreading DSW of $a = (580.5/787.95)^{1/2}(2.76) = 2.37$ and $\alpha = (580.5/787.95)(0.3) = 0.23$ for $k = 1$. For $k = 1.5$ we get predictions of $a = 2.13$ and $\alpha = 0.18$. Comparison of these results to the actual numerical amplitudes of the expanding DSWs is excellent for $k = 1$ with almost 100% accuracy in a and α . However for $k = 1.5$, the prediction is still good but varies by about 10% and 33% for the amplitudes a and α .

The semi-analytical solutions (12) and (13) predict $a = 2.54$ and $\alpha = 0.252$ for the parameters of figure 1. For the stationary circular DSW, with $r_0 = 600$ and $z_1 = 1200$, the average maximum amplitudes are $a = 2.56$ and $\alpha = 0.273$, which represent variations of only 1% and 8% for a and α . However, for the expanding circular DSW, the predictions of uniform soliton theory must be combined with (14). For the case where $k = 1$, by using $z_1 = 1200$ and other related information from figure 1, the predictions are $a = 1.77$ and $\alpha = 0.13$ while the numerical averages are found to be $a = 1.64$ and $\alpha = 0.092$. From these solutions, we can see there exists 7% and 33% variation in the amplitudes a and α . When we look at the case where $k = 1.5$, the theoretical predictions are $a = 1.31$ and $\alpha = 0.06$ while the numerical averages are $a = 1.14$ and 0.04 , thus there are variation by 12% and 34% in the amplitudes a and α . Hence we can conclude that the theoretical predictions are relatively close to the numerical solutions and the theoretical approach proposed here works well.

Figure 2 shows $|u|$ versus r for a circular (2+1)-D DSW. Shown is the numerical solution of (1) at $z = 500$. The other parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$, $\alpha_1 = 0.2$ and $T \rightarrow \infty$. Shown are the cases for $k = 0$ and $k = 1$. When the circular DSW is stationary, the result obtained is qualitatively similar to the line DSW, see Figure 7(a)[10]. As for the line DSW case, the individual waves do not completely separate, so they continue to interact with each other and are not ordered by amplitude. At $k = 0$, the largest solitary wave has $a = 2.71$ and $\alpha = 0.29$ at $r = 521$ while the leading edge of the DSW is located at $r = 575$. For $k = 1$, the leading edge is now at $r = 1071$ where the DSW propagates outwards. The amplitude of the largest wave in the expanding DSW is $a = 1.14$ and $\alpha = 0.03$. The locations for the highest wave correspond to numerical values of $k = 0.998$ and $k = 1.483$, respectively, which are close to the theoretical estimates of $k = 1$ and $k = 1.5$.

Figure 3 shows $|u|$ versus r for (2+1)-D circular DSW. Shown is the numerical solutions of (1). The other parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$ and $\alpha_1 = 0.2$ and (a) $T = 50$, (b) $T = 100$, and (c) $T = 500$. We

Figure 2: Electric field amplitude $|u|$ versus r for a circular (2+1)-D DSW. The numerical solution of (1) is shown for $k = 0$ (solid blue line) and $k = 1$ (dashed red line) at $z = 500$. The other parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$, $\alpha_1 = 0.2$ and $T \rightarrow \infty$.

k	T	z	r	a	α
0	50	119	582	2.81	0.41
	100	142	581	2.78	0.33
	500	151	581	2.69	0.31
1	50	163	742	2.49	0.37
	100	200	778	2.42	0.26
	500	209	788	2.38	0.22
1.5	50	220	905	2.22	0.31
	100	257	982	1.98	0.16
	500	271	982	1.91	0.13

Table 1: Data for the first fully formed solitary wave for the (2+1)-D circular DSW. The parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$ and $\alpha_1 = 0.2$. Numerical solutions of (1) are shown.

k	T	a	α
0	50	2.73	0.43
	100	2.59	0.39
	500	2.53	0.25
1	50	2.48	0.32
	100	2.40	0.25
	500	2.30	0.23
1.5	50	2.25	0.31
	100	2.15	0.21
	500	2.13	0.18

Table 2: Semi-analytical solutions for the first fully formed solitary wave for the (2+1)-D circular DSW. The parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$ and $\alpha_1 = 0.2$.

consider z values for which the first solitary wave has fully formed; Table 1 provides all the numerical data for these solitary waves. It can be seen that the z values at which the first solitary wave has formed increases as the temperature increases, while the amplitude of the fully formed solitary wave decreases, as the temperature increases.

Table 2 shows the semi-analytical solutions for the largest wave in the (2+1)-D circular DSW. The $k = 0$ results are those of uniform soliton theory for the line (1+1)-D DSW case. These results are very close to the actual numerical amplitudes of the stationary and expanding DSWs. The predictions of the amplitude a are very good for all cases, as compared to the numerical results, with only a maximum of 11% error. However, the predictions for the α values are higher in comparison with the numerical results with a maximum of 43% error.

(a)

(b)

(c)

10

Figure 3: Electric field amplitude $|u|$ versus r for the circular (2+1)-D DSW. Shown are numerical solutions of (1) for (a) $T = 50$, (b) $T = 100$, and (c) $T = 500$ for $k = 0$ (solid blue line), $k = 1$ (dashed red line) and $k = 1.5$ (dotted green line). The other parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$ and $\alpha_1 = 0.2$.

(a)

(b)

(c)

Figure 4: Electric field amplitude $|u|$ versus r for the circular (2+1)-D DSW. Shown are numerical solutions of (1) for (a) $T = 50$, (b) $T = 100$, and (c) $T = 500$ for $k = 0$ (solid blue line) and $k = 1$ (dashed red line) at $z = 500$. The other parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$ and $\alpha_1 = 0.2$.

k	T	r	a	α
0	50	525	2.66	0.41
	100	563	3.00	2.75
	500	521	2.72	2.76
1	50	1033	2.11	0.28
	100	1028	1.20	0.036
	500	1025	1.15	0.035

Table 3: Numerical data for the circular (2+1)-D DSW for $|u|$ at $z = 500$. The parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$ and $\alpha_1 = 0.2$.

		Semi-analytical		Numerical	
k	T	a	α	a	α
0	50	2.44	0.24	2.55	0.26
	100	2.28	0.22	2.26	0.20
	500	2.05	0.16	2.03	0.15
1	50	1.57	0.16	1.63	0.18
	100	0.89	0.022	0.86	0.021
	500	0.85	0.017	0.82	0.016

Table 4: Semi-analytical and numerical solutions for the circular (2+1)-D DSW for $z = 500$. The parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$ and $\alpha_1 = 0.2$.

Figure 4 shows $|u|$ versus r for the circular (2+1)-D DSW. Shown is the numerical solutions of (1) at $z = 500$. The other parameters are $\eta_0 = 1 \times 10^{-2}$, $\eta_m = 2.43 \times 10^{-2}$, $r_0 = 600$ and $\alpha_1 = 0.2$ and (a) $T = 50$, (b) $T = 100$, and (c) $T = 500$. The highest amplitudes for the circular DSW are recorded in table 3 for the stationary DSW $k = 0$ and a expanding DSW with $k = 1$. For a fixed z value, the maximum amplitude can be significantly different to the long z average, hence the wide differences from the long term averages in table 4. Table 4 shows the z -weighted numerical average amplitudes, using (14) and the uniform soliton theory (12) and (13). We see that as temperature increases, the average maximum amplitude decreases. The difference between the theoretical and numerical solutions are quite small with a maximum of 4% and 11% errors in the a and α values.

5 Conclusions

This paper combines semi-analytical solutions for colloidal solitary waves, uniform soliton theory and geometric optics, to predict the amplitude of solitary waves that form in circular colloidal DSW. The theoretical approach is remarkably successful in predicting numerical amplitudes in the circular DSW, both for the initial solitary wave and the long- z average solitary wave amplitude. It is hoped that this theoretical study will encourage additional experimental investigations of colloidal dispersive shock waves and temperature dependent particle interaction effects. The model and semi-analytical solutions developed here could be easily used by experimental groups simply by selecting appropriate virial coefficients, which correspond to their experimental colloidal medium. Hence we believe that the colloidal model equations and solutions presented here provide an extremely useful testbed for exploring different colloidal media and particle interaction models.

References

- [1] W. Wan, D. V. Dylov, C. Barsi, and J. W. Fleischer, Diffraction from an edge in a self-focusing medium, *Opt. Lett.*, **35** (2010) 2819.
- [2] M. Peccianti, C. Conti, and G. Assanto, Optical modulational instability in a non-local medium, *Phy. Rev. E*, **68** (2003) 025602.
- [3] J. Wang, J. Li, D. Lu, Q. Guo and W. Hu, Observation of surface dispersive shock waves in a self-defocusing medium, *Phy. Rev. A*, **91** (2015) 063819.
- [4] G. Assanto, T. R. Marchant, and N. F. Smyth, Collisionless shock resolution in nematic liquid crystals, *Phy. Rev. A*, **78** (2008) 063808.
- [5] N. F. Smyth, Dispersive shock waves in nematic liquid crystals, *Physica D*, **333** (2016) 301-309.

- [6] G. A. El and N. F. Smyth, Radiating dispersive shock waves in non-local optical media, *Proc. Roy. Soc. A*, **472** (2016) 20150633.
- [7] M. J. Ablowitz, A. Demirci and Y.P. Ma, Dispersive shock waves in the Kadomtsev-Petviashvili and two-dimensional Benjamin-Ono equations, *Physica D*, **333** (2016) 84-98.
- [8] K.R. Khusnutdinova and X. Zhang, Nonlinear ring waves in a two-layer fluid, *Physica D*, **333** (2016) 208-211.
- [9] T.R. Marchant and N.F. Smyth, Semi analytical solutions for dispersive shock waves in colloidal media, *J. Phys. B: At. Mol. Opt.*, **45** (2012) 145401.
- [10] A. Azmi and T.R. Marchant, Dispersive shock waves in colloids with temperature dependent compressibility, *J. Non. Opt. Phys. Mater.*, **23**(4) (2014) 1450043.
- [11] J. Tian and Y. Gui, Equations of state for fluids: empirical temperature dependence of the second virial coefficient, *J. Phys. Chem. B*, **111** (2007) 10970.
- [12] A. Azmi and T.R. Marchant, Colloidal solitary waves with temperature dependent compressibility, *J. Opt.*, **16** (2014) 055203.
- [13] M. Matuszewski, W. Krolikowski, and Y.S. Kivshar, Spatial solitons and light-induced instabilities in colloidal media, *Opt. Express*, **16** (2008) 1371.
- [14] J. Tian, H. Jiang, Y. Gui, and A. Mulero, Equation of state for hard-sphere fluids offering accurate virial coefficients, *Phys. Chem.: Chem. Phys.*, **11** (2008) 11213.
- [15] R.H.J. Grimshaw and N.F. Smyth, Resonant flow of a stratified fluid over topography, *J. Fluid Mech.*, **169** (1986) 429.
- [16] T.R. Marchant and N.F. Smyth, Approximate techniques for dispersive shock wave in nonlinear media, *J. Nonlinear Opt. Phys.*, **21** (2012) 1250035.
- [17] W. Wan, S. Jia, and J. W. Fleischer. Dispersive superfluid-like shock waves in nonlinear optics. *Nature Phys.*, **3** (2007) 46–51.
- [18] N. Ghofaniha, C. Conti, G. Ruocco, and S. Trillo. Shocks in nonlocal media. *Phy. Rev. Lett.*, **99** (2007) 043903.