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Growth in an Economy with Automation and Human Capital Accumulation

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This paper presents a model of endogenous growth with directed technical change to examine the implications of automation on output growth and employability of unskilled workers. Automation occurs to replace unskilled labor with industrial robots in producing intermedi- ate varieties. This automation is partly offset by horizontal innovation that creates new varieties in which unskilled labor has employment advantage. Unskilled labor also benefits from dynamics of a labor augmenting technology that helps improve its productivity. As the key production factor, human capital takes part in every economic activity, ranging from working in research labs to educating its labor force or producing varieties. We find that over the long run, there exists a balanced growth path along which each of the production factors, robots and unskilled labor, is employed to produce a fixed range of varieties, alongside human capital. Output grows either at the exogenous rate of labor augmenting technology or at the rate of human capital accumulation in the fully endogenous model. At long- run equilibrium, automation will be strengthened if research devoted to its development becomes more efficient. However, it will be dis- couraged by an improvement in either the variety creation or labor augmenting technology or by a surge in the rate of time preference. Keywords: Automation, horizontal innovation, endogenous growth, human capital, robots, directed technological change.

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capital, human, automation, accumulation, growth, economy

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This paper presents a model of endogenous growth with directed technical change to examine the implications of automation on output growth and employability of unskilled workers. Automation occurs to replace unskilled labor with industrial robots in producing intermediate varieties. This automation is partly offset by horizontal innovation that creates new varieties in which unskilled labor has employment advantage. Unskilled labor also benefits from dynamics of a labor augmenting technology that helps improve its productivity. As the key production factor, human capital takes part in every economic activity, ranging from working in research labs to educating its labor force or producing varieties. We find that over the long run, there exists a balanced growth path along which each of the production factors, robots and unskilled labor, is employed to produce a fixed range of varieties, alongside human capital. Output grows either at the exogenous rate of labor augmenting technology or at the rate of human capital accumulation in the fully endogenous model. At long-run equilibrium, automation will be strengthened if research devoted to its development becomes more efficient. However, it will be discouraged by an improvement in either the variety creation or labor augmenting technology or by a surge in the rate of time preference.

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1 Introduction

Automation and its impact on our economy has been a highly debated topic over the past few decades. So far, most of the discussion has been focused on the impact of automation on the labor market as this is the area that sees the most direct and immediate effect of the process. On the one hand, there have been concerns on potential job losses as robots will take over many human jobs (Autor et al. (2003), Goos & Manning (2007), Michaels et al. (2014)). For example, as pointed out by Cortes et al. (2017), the most vulnerable jobs in the US market in the last 35 years have been those of routine nature (i.e. those that can be performed by well-defined instructions and procedures). On the other hand, optimistic economists consider automation as a type of innovation that increases capital inputs which, in turn, boosts the demand for labor. In backing up their argument, many researchers point to historical evidence of new jobs, which never existed before, being introduced during each industrial revolution (Autor (2015), Acemoglu & Restrepo (2018*a*), Acemoglu & Restrepo (2018*b*)). Particularly, in our modern time of Industry 4.0, according to Acemoglu & Restrepo (2018*b*), while industrial robots replace labor, new jobs like audio-visual specialists, data analysts, meeting planners and computer support specialists are emerging. Besides automating tasks, technological progress also creates new product lines that recruit labor at first. Although the effect of automation on labor employment is important, this process should be considered more broadly to give us a more comprehensive picture of its large scale impact. For example, it can be considered in conjunction with the way human society moves forward with it such as how we communicate with each other or which political party we vote for in an election. From an economic point of view, that could be the economic development process in which automation significantly changes the way we conduct our production activity ranging from why, what, how we produce. There are several questions that require our immediate attention: (i) How does automation of production alter pattern of long-run growth and factor income distribution? (ii) Is the effect on workers different across those of different skill levels? (iii) Do these answers change when automation occurs in parallel with other types of innovation such as those that create new varieties or improve labor productivity?

In an attempt to answer these questions, we develop an endogenous growth model with automation. There is a unique final consumption good that is produced by combining a large range of intermediate products (varieties). Automation occurs only in the intermediate good sector in which robots/machines take over the role of unskilled workers in production of

some product varieties.[1] This automating process exists in parallel with a second line of technology that is aimed at creating new varieties which, in turn, generates new employment opportunities for unskilled labor. In the competition with robots, unskilled workers can make use of a source of technological change that helps improve its productivity[2] (i.e. labor augmenting technology).[3] These technologies, besides competing with each other, are the forces that drive the competition between robots and unskilled workers through determining relative profitability of intermediate firms. Unlike unskilled workers that are under threat of being knocked out by robots, human capital is the key production factor that is involved in every activity of the economy ranging from production of varieties (e.g. managing or monitoring service) to doing R&D in research labs (for developing new technologies) and education. Within this framework, we investigate conditions under which there exists a long run equilibrium with either no automation, full automation or partial automation. We also study factors affecting the extent of innovation and the direction of technological change.

We obtain several interesting results based on the above setting. For a start, we examine a static model where all production factors are fixed. There is no change in the variety range or labor productivity. The only innovation that occurs is an automating process that replaces unskilled labor with robots. We find that if factor prices are too high, the economy either ends up in full automation (with excessive unskilled labor wage) or no automation (with excessive rental rate). Otherwise, there always exists a unique equilibrium at which either robots or unskilled labor are employed to produce a fixed number of varieties, alongside human capital. A change in either the number of available varieties or capital - labor ratio will affect the effective income shares of unskilled labor and robots but not that of human

[1] In this paper, by robots or machines, we generally refer to artificial intelligence (AI) or computer algorithms that can work or run automatically by themselves. These are primarily made up of physical capital. This capital is quite different from the traditional capital as it has a very high degree of substitutability with unskilled labor. To make it simple, we abstract from considering the traditional capital and assume that all physical capital is used to produce these robots. Throughout the paper, we use the terms 'robots' and 'machines' interchangeably.

[2] According to Ong & Nee (2004), augmented reality technologies using interactive interfaces to increase workers' ability to perceive or control objects can provide support to workers in production or integrated design tasks. They enhance workers' productivity.

[3] We can think of this line of technology as production techniques embedded in traditional physical capital in the form of machines, factory buildings that require skilled labor input to produce. As discussed above, because including both kinds of capital could make our model heavy and distract our attention from the other capital that used in producing robots, we do not model the traditional capital here.

capital.

When this framework is extended to a dynamic structure with an accumulation of physical capital and human capital as well as an evolution of automating technology and variety expansion, we characterize conditions under which a balanced growth path can be reached. We find that if the labor augmenting technology is entirely exogenous, its rate of growth will dictate that of output, consumption, physical capital and human capital. However, when this factor is fully endogenized in the sense that its revolution is determined by the amount of human capital devoted to its R&D, all important economic variables will grow at the same rate as human capital.

Although there is relatively little current research on the question of how automation impacts on growth, a literature was devoted to studying related issues previously. Zeira (1998) is probably the first paper in this line that models exogenous increases in total factor productivity as the key factor that encourages the substitution of capital for labor. More recently, Berg et al. (2018) examine how automation affects economic growth. However, the evolution of robots in Berg et al. (2018) is totally exogenous as treated in Zeira (1998) (this is also pointed out by Hanley (2018)). Another work by Aghion et al. (2017) provide a detailed discussion on how to incorporate automation into an endogenous growth setting. Nevertheless, they do not consider variety expansion as our paper does. In that respect, the work by Acemoglu & Restrepo (2018*b*) is perhaps the one that has closest modelling framework to our theoretical work as it also studies automation in an endogenous growth setting with variety expansion as per Romer (1990) and Grossman & Helpman (1991). Despite that fact, our analysis differs to theirs along several dimensions, with notable distinction in modelling framework. In particular, in their paper, labor productivity is either improved exogenously or as a result of a learning-by-doing process. In our paper, this factor evolves endogenously thanking to purposeful research lab investment using high skilled labor input. Human capital is also the key production factor that takes part in all other activities in this economy ranging from R&D activities in research labs to education and training and production of intermediate products (alongside unskilled labor or machines). This production factor evolves continuously while it is fixed in Acemoglu & Restrepo (2018*b*).

The paper proceeds as follows. Section 2 provides basic setting of the model. Section 3 is devoted to charaterizing the static equilibrium of the economy. Section 4 extends the model to a dynamic framework, starting with an exogenous labor augmenting technology then an endogenous one that can be improved by employing more human capital. Section 5 ends the paper with some concluding remarks.

2 The environment

Consider an economy with a representative infinitely-lived consumer having the following constant relative risk aversion (CRRA) preferences:

$$U = \int_0^\infty \frac{C_t^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt, \tag{1}$$

where C_t is consumption, $\rho > 0$ is the rate of time preference and $\theta \geq 1$ is the coefficient of relative risk aversion (or the inverse intertemporal elasticity of substitution). The budget constraint of the consumer is:

$$C_t + \dot{K}_t \leq Y_t, \tag{2}$$

where $\dot{K}_t$ denotes investment and Y_t denotes the final consumption good whose price is chosen as *numeraire*. Assume that this final consumption good is produced by a large number of competitive firms using a bundle of many varieties (or intermediate products) such that:

$$Y_t = \left(\int_{N_t - Z}^{N_t} y_{it}^{\alpha} di \right)^{\frac{1}{\alpha}}, \tag{3}$$

where y_{it} denotes the quantity of a particular production variety i used in production at time t and $\alpha \in (0,1)$.[4] In this formulation, given that N_t denotes the most complex variety and $N_t - Z$ denotes the least complex one, the set of varieties available for production of final consumption good is always of a constant mass Z (note that Z is not time indexed to signal that it is a constant). This also means that each time when a new variety is created, it will replace the lowest indexed variety in the final consumption good production.[5]

Varieties are produced by monopolists using human capital in combination with either unskilled labor or machines/robots. Assume that there exists $I_t \in [N_t - Z, N_t]$ that divides varieties into two different groups. In particular, if $i \leq I_t$, varieties are technologically automated and will be produced by machines (made up of capital, i.e. k), alongside human capital (i.e. h).

[4] Actually, $\alpha = \frac{\epsilon - 1}{\epsilon}$ where $\epsilon > 1$ is the elasticity of substitution between inputs.

[5] According to The Street (2010), lamplighter and pinsetter are among jobs that have disappeared over the last decades. In the meantime, there will be prospective jobs created in the future such as flight instructor or personal internet of things security repair person (Crimson Education (2018)). In addition, jobs requiring labor can emerge from within firms using AI, for example, trainers to train AI system, explainers to communicate between AI systems and customers, or sustainers to maintain the performance of AI systems (Acemoglu & Restrepo (2018*a*)).

However, for $i > I_t$, varieties will be produced using unskilled labor (i.e. l) instead, alongside human capital.[6] The production of varieties, hence, can be described as follows:

$$y_{it} = \begin{cases} k_{it}^{\beta} h_{it}^{1-\beta}, & \text{if } i \in [N_t - Z, I_t], \\ (A_{it} l_{it})^{\beta} h_{it}^{1-\beta}, & \text{if } i \in (I_t, N_t], \end{cases} \tag{4}$$

where $\beta \in (0, 1)$ and A_{it} denotes labor-augmenting technology used in producing a variety that is strictly increasing in i.[7] Embedded in this equation is the notion that once a variety is automated, the manual work of unskilled labor will be totally replaced by the work of robots. Based on this production function, the production cost of a variety can be derived as:

$$c_{it} = \begin{cases} y_{it} R_t^{\beta} W_{Ht}^{1-\beta} \left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta} \right], & \text{if } i \in [N_t - Z, I_t], \\ y_{it} \left(\frac{W_{Lt}}{A_{it}}\right)^{\beta} W_{Ht}^{1-\beta} \left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta} \right], & \text{if } i \in (I_t, N_t], \end{cases}$$

where W_{Ht}, W_{Lt} and R_t denote factor prices for human capital, H_t, unskilled labor, L_t, and capital, K_t, respectively.

3 Static Equilibrium

Denote p_{it} as price of a particular variety i (for $i \in [N_t - Z, N_t]$). Given that the final consumption good is a *numeraire*, firms in the final good sectors solve the following maximization problem:

$$\max_{y_{it}} Y_t - \int_{N_t - Z}^{N_t} p_{it} y_{it} di,$$

taking the price of all varieties as given. The first order condition for this profit maximization gives the demand for each variety as follows:

$$y_{it} = \left(\frac{1}{p_{it}}\right)^{\frac{1}{1-\alpha}} Y_t. \tag{5}$$

[6]The presence of human capital in the production function is for capturing essential professional service required for this production process such as monitoring, management or coordination. This service, which is the domain of high-skilled workers, is needed regardless of whether unskilled labor or machines is being used.

[7]A_{it} can also be considered as the productivity of labor employed in that production line.

Plugging this into (3) and simplifying, we obtain the ideal price index condition that reads:

$$\left(\int_{N_t-Z}^{N_t} p_{it}^{-\frac{\alpha}{1-\alpha}} di\right)^{\frac{1}{\alpha}} = 1. \tag{6}$$

Meanwhile, the profit function of a representative monopolist supplying a variety is given by:

$$\pi_{it} = \begin{cases} p_{it}y_{it} - R_t k_{it} - W_{Ht}h_{it}, & \text{if } i \in [N_t - Z, I_t], \\ p_{it}y_{it} - W_{Lt}l_{it} - W_{Ht}h_{it}, & \text{if } i \in (I_t, N_t]. \end{cases}$$

The profit maximization problem for the monopolist delivers each variety price, p_{it}, as a mark-up over its production cost, c_{it}:

$$p_{it} = \begin{cases} \frac{1}{\alpha} R_t^{\beta} W_{Ht}^{1-\beta} \left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta}\right], & \text{if } i \in [N_t - Z, I_t], \\ \frac{1}{\alpha}\left(\frac{W_{Lt}}{A_{it}}\right)^{\beta} W_{Ht}^{1-\beta} \left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta}\right], & \text{if } i \in (I_t, N_t]. \end{cases} \tag{7}$$

Substituting the above results into the equation for variety demand given in (5), we get:

$$y_{it} = \begin{cases} \alpha^{\frac{1}{1-\alpha}} \left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta}\right]^{-\frac{1}{1-\alpha}} \left(\frac{1}{R_t}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{1}{W_{Ht}}\right)^{\frac{1-\beta}{1-\alpha}} Y_t, & \text{if } i \in [N_t - Z, I_t], \\ \alpha^{\frac{1}{1-\alpha}} \left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta}\right]^{-\frac{1}{1-\alpha}} \left(\frac{A_{it}}{W_{Lt}}\right)^{\frac{\beta}{1-\alpha}} \left(\frac{1}{W_{Ht}}\right)^{\frac{1-\beta}{1-\alpha}} Y_t, & \text{if } i \in (I_t, N_t]. \end{cases} \tag{8}$$

Hence, profit to the representative monopolist will be:

$$\pi_{it} = \begin{cases} \mathit{\Omega}\left(\frac{1}{R_t}\right)^{\frac{\alpha\beta}{1-\alpha}} \left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}} Y_t, & \text{if } i \in [N_t - Z, I_t], \\ \mathit{\Omega}\left(\frac{A_{it}}{W_{Lt}}\right)^{\frac{\alpha\beta}{1-\alpha}} \left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}} Y_t, & \text{if } i \in (I_t, N_t], \end{cases} \tag{9}$$

where $\mathit{\Omega} = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta}\right]^{-\frac{\alpha}{1-\alpha}}$. Based on this, the demand for each production factor by the monopolist can be derived as:

$$l_{it} = \frac{\beta \mathit{\Omega}}{1-\alpha} \frac{A_{it}^{\frac{\alpha\beta}{1-\alpha}}}{W_{Lt}^{\frac{\alpha\beta}{1-\alpha}+1}} \left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}} Y_t,$$

$$k_{it} = \frac{\beta\Omega}{1-\alpha}\left(\frac{1}{R_t}\right)^{\frac{\alpha\beta}{1-\alpha}+1}\left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}} Y_t,$$

$$h_{it} = \frac{(1-\beta)\Omega}{1-\alpha}\left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}+1}\left(\frac{1}{R_t^{\frac{\alpha\beta}{1-\alpha}}} + \frac{A_{it}^{\frac{\alpha\beta}{1-\alpha}}}{W_{Lt}^{\frac{\alpha\beta}{1-\alpha}}}\right) Y_t.$$

Using (9), we can now calculate the profit ratio between an automated firm and a non-automated one:

$$\frac{\pi_{xi,i\in[N_t-Z,I_t]}}{\pi_{xi,i\in(I_t,N_t]}} = \left(\frac{1}{R_t}\right)^{\frac{\alpha\beta}{1-\alpha}}\left(\frac{W_{Lt}}{A_{it}}\right)^{\frac{\alpha\beta}{1-\alpha}}. \tag{10}$$

In principle, a variety can only be technologically automated if it is cheaper to be produced with machines than with unskilled labor. In other words, automation will take place on variety i only if $R_t \leq \frac{W_{Lt}}{A_{it}}$. This condition is satisfied for the automated range or $i \in [N_t - Z, I_t]$. However, for $i \in (I_t, N_t]$, $R_t > \frac{W_{Lt}}{A_{it}}$. This profit ratio implies that the incentive for a monopolist to automate a variety within each sector is increasing in unskilled labor costs but decreasing in the rental price of machines and the levels of labor augmenting technology.

From the ideal price index condition given in (6), we arrive at:

$$\frac{\Omega}{1-\alpha}\left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}}\left(\frac{m_t}{R_t^{\frac{\alpha\beta}{1-\alpha}}} + \frac{A_t^{\frac{\alpha\beta}{1-\alpha}}}{W_{Lt}^{\frac{\alpha\beta}{1-\alpha}}}\right) = 1, \tag{11}$$

where $A_t^{\frac{\alpha\beta}{1-\alpha}} = \int_{I_t}^{N_t} A_{it}^{\frac{\alpha\beta}{1-\alpha}} di$ representing the aggregate level of labor augmenting technology and $m_t = I_t - (N_t - Z)$ denotes the range of varieties that are produced with robots.[8] By summing up across intermediate firms, the aggregate demand for each production factor can be calculated as:

$$\frac{\beta\Omega}{1-\alpha}\left(\frac{1}{W_{Lt}}\right)^{\frac{\alpha\beta}{1-\alpha}+1}\left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}} Y_t A_t^{\frac{\alpha\beta}{1-\alpha}} = L_t, \tag{12}$$

$$\frac{(1-\beta)\Omega}{1-\alpha}\left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}+1}\left(\frac{m_t}{R_t^{\frac{\alpha\beta}{1-\alpha}}} + \frac{A_t^{\frac{\alpha\beta}{1-\alpha}}}{W_{Lt}^{\frac{\alpha\beta}{1-\alpha}}}\right) Y_t = H_{yt}, \tag{13}$$

[8] Within a certain range of available varieties, there is a one to one mapping between I and m. However, we opt to use two different notations in which I refers to the index of variety while m refers to the range of varieties under automation. This distinction is important especially in the context of a dynamic equilibrium considered later on in which I (i.e. the index) varies with time while m (i.e. the range) is constant.

$$\frac{\beta \Omega}{1-\alpha}\left(\frac{1}{R_t}\right)^{\frac{\alpha\beta}{1-\alpha}+1}\left(\frac{1}{W_{Ht}}\right)^{\frac{\alpha(1-\beta)}{1-\alpha}} Y_t m_t = K_t. \tag{14}$$

where $H_{yt} = \int_{N_t-Z}^{N_t} h_{it} di$ denotes total human capital devoted to production of varieties. Because all human capital is used for producing varieties in this setting, we have $H_{yt} = H_t$. From (12)-(14), we get $W_{Lt}L_t = \Gamma_{Lt}Y_t$, where $\Gamma_{Lt} = \frac{\beta\Omega}{1-\alpha}.\frac{A_t^{\frac{\alpha\beta}{1-\alpha}}}{W_{Lt}^{\frac{\alpha\beta}{1-\alpha}}W_{Ht}^{\frac{\alpha(1-\beta)}{1-\alpha}}}$, and $R_tK_t = \Gamma_{Kt}Y_t$, where $\Gamma_{Kt} = \frac{\beta\Omega}{1-\alpha}.\frac{m_t}{R_t^{\frac{\alpha\beta}{1-\alpha}}W_{Ht}^{\frac{\alpha(1-\beta)}{1-\alpha}}}$. This, together with (11), gives $W_{Ht}H_{yt} = \Gamma_{Ht}Y_t$, where $\Gamma_{Ht} = 1 - \Gamma_{Lt} - \Gamma_{Kt}$. This implies:

$$W_{Lt}L_t + R_tK_t + W_{Ht}H_{yt} = Y_t. \tag{15}$$

It can be seen that Γ_{Lt}, Γ_{Kt} and Γ_{Ht} play a role as effective shares of unskilled labor, machines and human capital respectively in the final output. Further investigation on (12)-(14) reveals that $\Gamma_{Ht} = 1-\beta$ and $\Gamma_{Kt} + \Gamma_{Lt} = \beta$. This means that by construction human capital always account for a constant share in final output leaving the rest for unskilled labor and machines. Which of these latter two factors enjoys a bigger share will depend on the dynamics of the technologies as well as the factor prices. Results obtained imply that the aggregate final output described in (3) can be expressed as:

$$Y_t = \alpha\left(m_t^{\frac{1-\alpha}{1-\alpha+\alpha\beta}}K_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} + A_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}}L_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}}\right)^{\frac{1-\alpha+\alpha\beta}{\alpha}} H_{yt}^{1-\beta}. \tag{16}$$

This shows that final output is a function of capital and unskilled labor, which are augmented by automation and labor augmenting technology respectively, as well as human capital.

Now we focus our attention on the case of a static equilibrium which is defined by the capital stock, K_t, human capital stock, H_t, unskilled labor force, L_t, the variety range, Z, the automated range, m_t, and the level of labor augmenting technology, A_{it}. In this equilibrium, things are time invariant so we suppress the time subscript t on the variables. Because N is constant, to simplify notation, we set $N \equiv Z$ so that the index range will be between 0 and Z. As a result, we have $I \equiv m$. To characterize the equilibrium, we need to work out the output level, factor prices, factor utilization and the equilibrium range of automated products. Note that, in this static equilibrium, we always have $C = Y$ (because in this static equilibrium $\dot{K}_t = 0$). From the representative household's optimization problem, we have $R = \rho + \theta g$ where g is the rate of growth of consumption. In this static economy, $g = 0$ making $R = \rho$.

We first start with the case of full automation. In this case, it is always cheaper (and, thus, more profitable) for monopolists to produce varieties with machines than with unskilled labor so all unskilled workers are made redundant. Production of varieties only uses human capital and machines. In other words, the full range of products is automated so $m = Z$. Because, A_i is strictly increasing in i, the condition for full automation to occur will be $\rho < \frac{W_L}{A_N}$.

By contrast, for the case of no automation ($m = 0$), the rental rate is so high that unskilled labor is preferred to machines in the production of varieties. All unskilled workers will have a job while all machines are made redundant. The condition for having no automation will, therefore, be $\rho > \frac{W_L}{A_0}$.

The proposition below summarises possibilities for which extreme cases regarding automation can happen:

Proposition 1. *In a static economy with automation replacing unskilled labor with robots in production of varieties, the economy may fall into one of the following extreme cases:*

- *If $\rho < \frac{W_L}{A_N}$, full automation will occur: all intermediates are produced by robots and human capital while unskilled labor is made redundant.*
- *If $\rho > \frac{W_L}{A_0}$, no automation will be conducted: all intermediates are produced by unskilled labor and human capital while robots are made redundant.*

More interesting will be the case in which both robots and unskilled labor co-exist and compete with each other. In order to have both automated (i.e. produced with machines) and non-automated varieties (i.e. produced with unskilled labor) or $m \in (0, Z)$, it is essential that $\frac{W_L}{A_N} < R < \frac{W_L}{A_{N-Z}}$. Therefore, I^* will be the one that satisfies this equation:

$$\frac{W_L}{A_I} = R, \tag{17}$$

or in the nearest left-sided neighborhood of that value. From (12) and (14), we have:

$$\left(\frac{R}{W_L}\right)^{\frac{\alpha\beta}{1-\alpha}+1} \frac{A^{\frac{\alpha\beta}{1-\alpha}}}{m} = \frac{L}{K}. \tag{18}$$

To enhance the tractability of our model, in what follows, we make an important assumption on the form of the labor augmenting technology:

Assumption 1. *Static labor augmenting technology takes the following form:*

$$A_i = i^{\frac{1-\alpha}{\alpha\beta}}, i \in (0, Z)$$

For $\alpha, \beta \in (0, 1)$, this assumption implies that each variety in line is a step up in terms of technology from the previous one. As a result, the condition that $\frac{W_L}{A_N} < R = \frac{W_L}{A_I} < \frac{W_L}{A_{N-Z}}$ is automatically satisfied. In addition, the assumption allows us to arrive at $A^{\frac{\alpha\beta}{1-\alpha}} = \int_m^Z idi = \frac{Z^2-m^2}{2}$. As a result, we can work out that $\frac{A^{\frac{\alpha\beta}{1-\alpha}}}{A_I^{\frac{\alpha\beta}{1-\alpha}}} = \frac{(Z-m)(Z+m)}{2m}$. Inserting this into the above equation after using (17), we obtain:

$$m^2 + \frac{2L}{K}.m^{\frac{1-\alpha+2\alpha\beta}{\alpha\beta}} = Z^2. \tag{19}$$

This provides us with one equation in one unknown variable, i.e. m. We have the following lemma for this equation:

Lemma 1. *Equation (19) yields a unique solution $m^* \in (0, Z)$. Other things equal, m^* will be higher if there is an increase in either the capital - labor ratio, $\frac{K}{L}$, or the range of available varieties, Z.*

Proof. Note that for $m \in [0, Z]$, the left hand side (LHS) of equation (19) is increasing in m. The range of value for the LHS is $[0; Z^2 + \frac{2L}{K}Z^{\frac{1-\alpha+2\alpha\beta}{\alpha\beta}}]$. In the meantime, the right hand side (RHS) of the equation is a positive constant and equal to Z^2. This implies that there exists a unique $m^* \in (0, Z)$ that solves this equation due to the single crossing property.

An increase in the capital-labor ratio, $\frac{K}{L}$ will shift the graph of the LHS downward while the graph of the RHS (which is a horizontal line) stays unchanged. As a result, the two graphs will intersects at a higher value of m^*.

Similarly, an increase in Z will shift the graph of the RHS upward while keeping that of the LHS stays unchanged. This means that the two graphs will now cross each other at a higher value of m^*. $\square$

This lemma establishes the existence and uniqueness of the solution to equation (19) which is the key equation for determining our static equilibrium. Upon getting m^*, we can calculate A and then Y in (16). After that, we can compute factor prices W_H, W_L and R from (12)-(14). These values characterize our static equilibrium. We can now state the following proposition:

Proposition 2. *Suppose that labor augmenting technology takes the form as specified in Assumption 1. Then the system admits a unique static equilibrium along which either machines or unskilled labor are employed to produce a fixed number of varieties, alongside human capital. Other things equal, at this equilibrium:*

- *An expansion of variety range increases the effective income share of unskilled labor, reduces the effective income share of capital, but has no impact on that of human capital.*
- *An increase in capital-labor ratio reduces the effective income share of unskilled labor, increases the effective income share of capital, but does not affect that of human capital.*

Proof. Note that the first part of this proposition has been proven in the proof of Lemma 1. As a result, in here we only provide proof for the comparative static results.

After substituting $R = \frac{W_L}{m^{\frac{1-\alpha}{\alpha\beta}}}$ and $A^{\frac{\alpha\beta}{1-\alpha}} = \frac{Z^2-m^2}{2}$ into (11), we obtain:

$$\frac{\Omega}{(1-\alpha)} \cdot \frac{1}{W_L^{\frac{\alpha\beta}{1-\alpha}} W_H^{\frac{\alpha(1-\beta)}{1-\alpha}}} = \frac{2}{Z^2+m^2}.$$

Plugging this into the formulas capturing the income shares of unskilled labor, physical capital and human capital yields:

$$\Gamma_L = \frac{\beta}{1 + \frac{K}{L} {m^*}^{\frac{\alpha-1}{\alpha\beta}}},$$

$$\Gamma_K = \frac{\beta}{1 + \frac{L}{K} {m^*}^{\frac{1-\alpha}{\alpha\beta}}},$$

$$\Gamma_H = 1 - \Gamma_L - \Gamma_K = 1 - \beta.$$

In calculating these results, we also make use of (19). An increase in Z, which then leads to an increase in m^* as per Lemma 1, will increase the income share of unskilled labor, Γ_L, while reduce that of capital, Γ_K. However, this change has no impact on the income share of human capital because this income share is always equal to $1-\beta$ regardless of m^*.

Now we examine the impact of an increase in $\frac{K}{L}$ on the income shares. To that end, we make use of (19) to transform the income shares Γ_L, Γ_K from above to obtain (note that Γ_H always receives the value of $1-\beta$):

$$\Gamma_L = \frac{\beta(Z^2 - m^{*2})}{Z^2 + m^{*2}},$$

$$\varGamma_K = \frac{2\beta}{1 + \left(\frac{Z}{m^*}\right)^2}.$$

For a constant Z, an increase in $\frac{K}{L}$ will increase m^* which, in turn, lowers $\varGamma_L$. However, such a change will boost up $\varGamma_K$ while leaving $\varGamma_H$ unchanged. $\square$

The main implication of this proposition is the following. There exists a unique static equilibrium at which either machines or unskilled labor are employed to produce intermediate products alongside human capital. Because human capital is the essential production factor that is engaged in every activity of this production process, its effective income share is unaffected by changes in other factors. However, those changes will affect effective income shares of unskilled labor and robots. In particular, an expansion of variety range unambiguously increases the share of unskilled labor versus that of its machinery counterparts. This is because such a change relatively widens the range of varieties produced with unskilled labor more than that produced with machines, thus creates relatively more demand for labor and increases the wage rate relatively more than the rental rate. By contrast, while an increase in capital - labor ratio makes machines relatively cheaper than unskilled labor, such a reduction in relative factor prices will be less than proportionate to the change of factor endowments.

4 Dynamic Equilibrium

In this section, instead of having a fixed total stock, capital can now evolve over time as a result of households' optimization problem. Additionally, there is a dynamic process associated with each of the following activities: the development of automation technology, the expansion of variety set, the evolution of labor augmenting technology and the accumulation of human capital. In particular, assume that automation and variety expansion both require human capital (or skilled labor). The rate of introducing new automation is:

$$\dot{I}_t = \mu_I \frac{H_{It}}{A_{It}}, \tag{20}$$

where H_{It} denotes the amount of human capital devoted to developing automating technologies and $\mu_I > 0$ is the productivity of this activity. Embedded in this formula is the notion that a variety attached with a higher level of unskilled labor productivity is harder to be automated. It also means that it

costs more to automate such a variety in terms of human capital. Similarly, the rate of introducing new varieties is:

$$\dot{N}_t = \mu_N \frac{H_{Nt}}{A_{Nt}}, \tag{21}$$

where H_{Nt} is the use of human capital for this task and $\mu_N > 0$ is its corresponding productivity. Here, a same argument applies for the deflation of A_{Nt} to capture the increase in research cost.

Assume that at each point in time, skilled workers who possess human capital will have to devote a fraction u_t of their time to improve their skill level. The evolution of human capital is of the following form:

$$\dot{H}_t = \lambda H_{Ht} \equiv \lambda u_t H_t, \tag{22}$$

where $\lambda > 0$ is the productivity of human capital production and $H_{Ht} = u_t H_t$ is amount of human capital devoted to its own production (e.g. through attaining education or on the job training). This is important for skilled workers to keep themselves updated with recent changes in technology so that they can be able to work in research labs and to work with robots (in automated firms) or unskilled labor (in non-automated firms, especially those creating new varieties). In this formulation, u_t is the fraction of human capital employed for human capital accumulating purpose that will be determined endogenously within the model.

Assume that the size of unskilled labor is fixed so that $L_t = L, \forall t$.[9] The factor market clearing conditions imply that equations (12)-(14) hold. In addition, we have:

$$H_{yt} + H_{Ht} + H_{Nt} + H_{It} = H_t. \tag{23}$$

As compared to what was specified in the previous section, human capital is now employed in other activities than variety production. This equation summarizes well that fact.

Now we define $\tilde{y}_t = \frac{Y_t}{m_t A_t}$, $\tilde{k}_t = \frac{K_t}{m_t A_t}$, $\tilde{c}_t = \frac{C_t}{m_t A_t}$ and $\tilde{h}_{yt} = \frac{H_{yt}}{m_t A_t}$ as normalized output, capital, consumption and human capital respectively. Here, the product $A_t m_t$ is used as a measurement of technology in the intermediate good sector. While m_t reflects the level of technology for automated varieties, A_t captures that for non-automated ones as well as the creation of new varieties (note that the appearance of N_t is captured by A_t). Using this

[9] Assuming a growing labor force does not affect the nature of our model results. Rather, it complicates our notations further.

normalized variables, we can rewrite the final output production function as the following:

$$\tilde{y}_t = \alpha \left(m_t^{\frac{1-\alpha}{1-\alpha+\alpha\beta}} \tilde{k}_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} + m_t^{\frac{-\alpha\beta}{1-\alpha+\alpha\beta}} L_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} \right)^{\frac{1-\alpha+\alpha\beta}{\alpha}} \tilde{h_{yt}}^{1-\beta}.$$

From this result, we can then derive the rental rate and the wage rates for both unskilled labor and human capital:

$$R_t = \alpha\beta m_t^{1-\beta} \left(m_t \tilde{k}_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} + L_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} \right)^{\frac{1-2\alpha+\alpha\beta}{\alpha}} \tilde{k}_t^{\frac{\alpha-1}{1-\alpha+\alpha\beta}} \tilde{h_{yt}}^{1-\beta}, \tag{24}$$

$$W_{Lt} = \alpha\beta A_t m_t^{1-\beta} \left(m_t \tilde{k}_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} + L_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} \right)^{\frac{1-2\alpha+\alpha\beta}{\alpha}} L_t^{\frac{\alpha-1}{1-\alpha+\alpha\beta}} \tilde{h_{yt}}^{1-\beta}, \tag{25}$$

$$W_{Ht} = \alpha(1-\beta) m_t^{-\beta} \left(m_t \tilde{k}_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} + L_t^{\frac{\alpha\beta}{1-\alpha+\alpha\beta}} \right)^{\frac{1-\alpha+\alpha\beta}{\alpha}} \tilde{h_{yt}}^{-\beta}. \tag{26}$$

Let us consider the variety that lies in the borderline between automation and non-automation (i.e. variety I). A firm that automates variety I will need to compensate the existing monopolist by paying her the present discounted value of the profit that would otherwise be generated using unskilled labor if not replaced.[10] Let V_{It} denote the value of automating and becoming a monopolist at time t then:

$$V_{It} = \Omega \int_t^{\infty} e^{-\int_0^{\tau} R_s ds} \left[\left(\frac{1}{R_\tau} \right)^{\frac{\alpha\beta}{1-\alpha}} - \left(\frac{A_{I\tau}}{W_{L\tau}} \right)^{\frac{\alpha\beta}{1-\alpha}} \right] \frac{Y_\tau}{W_{H\tau}^{\frac{\alpha(1-\beta)}{1-\alpha}}} d\tau.$$

Now let V_{Nt} capture the value of creating a new variety and becoming a monopolist then:

$$V_{Nt} = \Omega \int_t^{\infty} e^{-\int_0^{\tau} R_s ds} \left[\left(\frac{A_{N\tau}}{W_{L\tau}} \right)^{\frac{\alpha\beta}{1-\alpha}} - \left(\frac{1}{R_\tau} \right)^{\frac{\alpha\beta}{1-\alpha}} \right] \frac{Y_\tau}{W_{H\tau}^{\frac{\alpha(1-\beta)}{1-\alpha}}} d\tau.$$

Embedded in this formula is the notion that a firm that creates the highest indexed (i.e. the most complex) variety will need to compensate the existing technology monopolist who is producing the lowest indexed (i.e. the

[10] Put it differently, if an existing monopolist decides to automate the production process of its variety, her decision will be made based on the discounted accrual of extra profit generated at each date.

least complex) variety (that has been automated) by paying her the present discounted value of profit.

Under the assumption of free entry in the research activities, new research firms will enter the market until all profit opportunities are exhausted. This means that the levels of human capital in each research activity will be determined by the arbitrage condition which equates the marginal cost of an extra unit of human capital to its expected marginal benefit:

$$\mu_I \frac{V_{It}}{A_{It}} = W_{Ht}, \tag{27}$$

$$\mu_N \frac{V_{Nt}}{A_{Nt}} = W_{Ht}. \tag{28}$$

As a result, the technology market will be cleared when:

$$\mu_I \frac{V_{It}}{A_{It}} = \mu_N \frac{V_{Nt}}{A_{Nt}}. \tag{29}$$

An equilibrium in this economy is defined by time paths of output, consumption, capital, human capital, number of automated varieties, number of available varieties $\{Y_t, C_t, K_t, H_t, m_t, N_t\}$, factor prices $\{p_{it}, W_{Lt}, W_{Ht}\}$, value functions of technology monopolists $\{V_{It}, V_{Nt}\}$ and allocation of human capital $\{H_{yt}, H_{Ht}, H_{It}, H_{Nt}\}$ such that all markets clear, all firms maximize their profits, the evolution of I_t and N_t is determined by free entry, and the representative household maximizes its utility.

Now we define the balanced growth path (BGP) equilibrium as an equilibrium path in which the normalized variables $\tilde{c}_t, \tilde{k}_t, \tilde{y}_t, \tilde{h}_t$ are constant. In addition, all other variables grow at constant rates. Along this BGP, final output grows at a constant rate g as consumption; m_t is constant so that both machines and unskilled labor produce a fixed number of varieties within the mass of available varieties. This means that W_{Ht} is constant along the BGP as per (26). From (20) and (21), because m is constant so $\dot{N}_t = \dot{I}_t$. This implies:

$$\mu_I \frac{H_{It}}{A_{It}} = \mu_N \frac{H_{Nt}}{A_{Nt}}. \tag{30}$$

We will now suppress the time arguments for those that do not vary with time to simplify the notations.

From utility function defined in (1), the Euler equation for the rate of growth of consumption reads $g = \frac{R_t - \rho}{\theta}$. This implies that the rental rate is also constant and is equal to:

$$R = \rho + \theta g. \tag{31}$$

Using this to recalculate the value functions yields:

$$V_{It} = \frac{\Omega Y_t}{W_H^{\frac{\alpha(1-\beta)}{1-\alpha}}} \cdot \int_t^{\infty} e^{-\int_t^{\tau}(R-g)ds} \left[R^{-\frac{\alpha\beta}{1-\alpha}} - \left(\omega_{LIt} e^{\int_t^{\tau} g_{\omega LI} ds} \right)^{-\frac{\alpha\beta}{1-\alpha}} \right] d\tau,$$

$$V_{Nt} = \frac{\Omega Y_t}{W_H^{\frac{\alpha(1-\beta)}{1-\alpha}}} \cdot \int_t^{\infty} e^{-\int_t^{\tau}(R-g)ds} \left[\left(\omega_{LNt} e^{\int_t^{\tau} g_{\omega LN} ds} \right)^{-\frac{\alpha\beta}{1-\alpha}} - R^{-\frac{\alpha\beta}{1-\alpha}} \right] d\tau,$$

where $\omega_{LIt} = \frac{W_{Lt}}{A_{It}}$ and $\omega_{LNt} = \frac{W_{Lt}}{A_{Nt}}$ denote productivity-adjusted wages at the margin of either automation or variety expansion respectively and $g_{\omega_{LI}}$ and $g_{\omega_{LN}}$ are their corresponding growth rates. The results for the value functions can, therefore, be further simplified as follows:

$$V_{It} = \frac{\Omega Y_t}{W_H^{\frac{\alpha(1-\beta)}{1-\alpha}}} \left(\frac{R^{-\frac{\alpha\beta}{1-\alpha}}}{R-g} - \frac{\omega_{LIt}^{-\frac{\alpha\beta}{1-\alpha}}}{R-g+g_{\omega LI} \cdot \frac{\alpha\beta}{1-\alpha}} \right), \tag{32}$$

$$V_{Nt} = \frac{\Omega Y_t}{W_H^{\frac{\alpha(1-\beta)}{1-\alpha}}} \left(\frac{\omega_{LNt}^{-\frac{\alpha\beta}{1-\alpha}}}{R-g+g_{\omega LN}} - \frac{R^{-\frac{\alpha\beta}{1-\alpha}}}{R-g} \right). \tag{33}$$

4.1 Exogenous labor augmenting technical change

We first start with the case of an exogenous process of labor productivity improvement. To simplify our calculation, we assume that each line of labor-augmenting technology is growing at a same exogenous constant rate a. In addition, we continue to assume that each next variety in the line is a step up in terms of technology from its previous counterpart. Denote $A_{i0} = i^{\frac{1-\alpha}{\alpha\beta}}$ as the initial level of labor augmenting technology associated with variety $i \in [0, Z]$ in the sector. We can write our labor augmenting technological change as in the following assumption:

Assumption 2. *Labor augmenting technology in each variety line evolves exogenously according to:*

$$A_{it} = i^{\frac{1-\alpha}{\alpha\beta}}.e^{at}.$$

The key difference between this assumption and Assumption 1 is that labor augmenting technology can now evolve over time. We also assume a same rate of growth across different existing varieties that are produced with unskilled labor to simplify algebras. This assumption allows us to calculate $A_t = A_0.e^{at}$ where $A_0^{\frac{\alpha\beta}{1-\alpha}} = \int_{I_t}^{N_t} A_{i0}^{\frac{\alpha\beta}{1-\alpha}} di$. In other words, A_t grows at rate a as well. From

this and (25), we can work out that $g_{\omega LI} = g_{\omega LN} = 0$ and $g = a$. This means that W_{Lt} and H_t also grow at rate a. Hence, (22) implies $u = \frac{a}{\lambda}$ and $H_{Ht} = \frac{aH_t}{\lambda}$.

From (29), (31), (32) and (33), and we have:

$$\mu_I. \left[\frac{1}{R^{\frac{\alpha\beta}{1-\alpha}}} - \frac{1}{\omega_{LI}^{\frac{\alpha\beta}{1-\alpha}}} \right] = \mu_N \left(\frac{m}{Z}\right)^{\frac{1-\alpha}{\alpha\beta}} . \left[\frac{Z}{m\omega_{LI}^{\frac{\alpha\beta}{1-\alpha}}} - \frac{1}{R^{\frac{\alpha\beta}{1-\alpha}}} \right], \tag{34}$$

in which $R = (\rho + \theta a)$ as per (31). In working out this equation, we make use of Assumption 2 to obtain $\frac{A_{Nt}}{A_{It}} = \frac{A_{N0}}{A_{I0}} = \left(\frac{Z}{m}\right)^{\frac{1-\alpha}{\alpha\beta}}$.

Note that for $\omega_{LI} \leq R$, there will be no automation at all as it will be more profitable to produce with unskilled labor instead of machines. By contrast, for $\omega_{LN} = \left(\frac{m}{z}\right)^{\frac{1-\alpha}{\alpha\beta}} \omega_{LI} \geq R$, there will be full automation as varieties are cheaper to be produced with machines. This means that partial automation can only occur if $\omega_{LI} \geq R \geq \omega_{LN} = \left(\frac{m}{Z}\right)^{\frac{1-\alpha}{\alpha\beta}} \omega_{LI}$. Within this domain, the above equation can be rearranged to get:

$$\omega_{LI}^{\frac{\alpha\beta}{1-\alpha}} = R^{\frac{\alpha\beta}{1-\alpha}} . \frac{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}-1} Z}{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}}}. \tag{35}$$

A quick check reveals that the fraction on the right hand side is always $\in [1, \frac{Z}{m}]$ for $m \in [0; Z]$. This guarantees that the condition $\omega_{LI} \geq R \geq \omega_{LN}$ is always satisfied for $m \in [0; Z]$. When $m = Z$, we have $\omega_{LI} = R = \omega_{LN}$. When $m = 0$, we have $\omega_{LI} = R > \omega_{LN}$.

Now using (11) and (12) to obtain W_{Ht}. After substituting the results into (27) and making use of (32) and (35), we obtain:

$$\frac{2\mu_I(1-\alpha)LR}{\beta(R-a)} . \frac{1}{Z+m} \left(\frac{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}-1} Z}{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}}} \right)^{\frac{1-\alpha}{\alpha\beta}+1} \frac{\mu_N m^{\frac{1-\alpha}{\alpha\beta}}}{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}-1} Z}$$
$$- \left(\frac{\Omega}{1-\alpha} \right)^{\frac{1-\alpha}{\alpha(1-\beta)}} . \frac{1}{R^{\frac{\beta}{1-\beta}}} \left[m + \frac{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}}}{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}-1} Z} . \frac{Z^2 - m^2}{2m} \right]^{\frac{1-\alpha}{\alpha(1-\beta)}} = 0. \tag{36}$$

This equation allows us to solve for m. This variable will help us determine ω_{LI} from (35) and W_H from (11). From (25) and (26), we can then figure out $\tilde{k}$ and $\tilde{h}_y$ as well as K_t and H_{yt}. According to (30), we have:

$$H_{It} = \frac{\mu_N H_{Nt}}{\mu_I} . \left(\frac{m}{Z}\right)^{\frac{1-\alpha}{\alpha\beta}} \tag{37}$$

This, together with H_{Ht}, H_{yt} obtained above and equation (23) for equilibrium of human capital market, will pin down all allocations of this production factor. To make sure that this equation yields a solution over its defined domain of $[0, Z]$, we make the following assumption:

Assumption 3. *Parameters satisfy the following:*
(i) $\frac{\alpha\beta}{1-\alpha} \geq 1$,
(ii) $\frac{2\mu_I(1-\alpha)L(\rho+\theta a)^{\frac{1}{1-\beta}}}{\beta[\rho+(\theta-1)a]} \cdot \left(\frac{1-\alpha}{\Omega}\right)^{\frac{1-\alpha}{\alpha(1-\beta)}} \cdot \frac{\mu_N}{(\mu_I+\mu_N)(2Z+1)Z^{\frac{1-\alpha}{\alpha(1-\beta)}}} > 1$,
where $\Omega = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta}\right]^{-\frac{\alpha}{1-\alpha}}$.

The first condition is needed to make sure that the productivity-adjusted wage paid to unskilled labor is a decreasing function of m. Indeed, if this condition does not hold, full automation will definitely occur and unskilled labor can never compete with machines due to its cost disadvantage. Meanwhile, the second condition is required so that there exists a solution to equation (36). Clearly, this condition will be automatically satisfied when L is very large.

Under this assumption, we can state the following lemma on the existence (and also the uniqueness) of the solution:

Lemma 2. *As soon as Assumption 3 holds, there exists a unique $m^* \in (0, Z)$ that solves equation (36).*

Proof. Consider the LHS of equation (36) which is a continuous function of m over the domain $(0; Z]$. An important observation is that when m tends to 0, LHS tends to $-\infty$. When $m = Z$, LHS is positive following Assumption 3. This means that the LHS changes its sign over $(0; Z]$. As a result, there exists a value of $m^* \in (0; Z]$ that makes LHS $= 0$.

Now we need to show that m^* is unique over $(0; Z]$. Indeed, m^* also solves equation (34) which can be rearrange to yield:

$$\frac{\mu_I}{R^{\frac{\alpha\beta}{1-\alpha}}} = \mu_N \left(\frac{m}{Z}\right)^{\frac{1-\alpha}{\alpha\beta}} \cdot \left[\frac{1}{\omega_{LN}^{\frac{\alpha\beta}{1-\alpha}}} - \frac{1}{R^{\frac{\alpha\beta}{1-\alpha}}}\right] + \frac{\mu_I}{\omega_{LI}^{\frac{\alpha\beta}{1-\alpha}}}.$$

Given that $R = \rho + \theta a$ is constant, the LHS of this equation is clearly a constant. Because $\frac{\alpha\beta}{1-\alpha} \geq 1$ by assumption, ω_{LI} is decreasing in m. This implies that ω_{LN} is also decreasing in m. As a result, the whole RHS is increasing in m. Because m^* solves this equation, it is the unique solution of this equation. This value is the key for obtaining other important dimensions of the interior BGP. □

From the above analysis, we can now summarize our main results in the proposition below:

Proposition 3. *Suppose that Assumptions 2 and 3 hold. Then there exists a unique interior BGP in which machines and unskilled labor are each employed to produce a fixed range of varieties alongside human capital. Along this BGP, output, consumption, stock of machines and human capital all grow at the same exogenous rate of labor productivity improvement.*

This proposition formally establishes the existence of a unique equilibrium BGP in which there is a fixed range of varieties produced with robots while the rest with unskilled labor, alongside human capital. The interesting thing is that the exogenous rate of growth of labor augmenting technology dictates that of output, consumption, physical capital and human capital. The dynamics of this factor also determine the rental rate as well as the adjusted wages paid to unskilled and skilled labor.

It should be noted that in this dynamic setting, there are three main ongoing forces that affect the system, particularly, the employment of unskilled labor. While automation replaces unskilled labor with machines, new variety creation generates demand for unskilled labor in newly created product lines. Acemoglu & Restrepo (2018*a*) refer to these effects as *displacement effect* and *reinstatement effect* on unskilled labor respectively. In this context, the displacement effect works in an opposite direction with the reinstatement effect. There is also a *productivity effect* triggered by the dynamics of labor augmenting technology. This productivity effect is deemed to somewhat offset the displacement effect of automation. At $m*$, all these effects balance out each other leading to a long run equilibrium for the whole economy. This equilibrium will only change when there is an exogenous shock to the system.

Upon obtaining the unique interior BGP, we move forward to do the comparative statics. Here, we pay particular attention to the impact of a change in some important parameters on the degree of automation in the economy. The main content of that exercise is put in the proposition below:

Proposition 4. *Along the interior BGP, while an increase in the productivity of automation technology, μ_I, increases the automated range of varieties, an increase in the productivity of variety expansion technology, μ_N, does the opposite. An increase in either the rate of time preference ρ, or the rate of growth of labor augmenting technology, a, discourages automation.*

Proof. Note that equation (34) can be rearranged to yield:

$$\frac{\mu_N}{\mu_I}\left(\frac{m}{Z}\right)^{\frac{1-\alpha}{\alpha\beta}} \cdot \left[\left(\frac{\rho+\theta a}{\omega_{LN}}\right)^{\frac{\alpha\beta}{1-\alpha}} - 1\right] + \left(\frac{\rho+\theta a}{\omega_{LI}}\right)^{\frac{\alpha\beta}{1-\alpha}} = 1.$$

As shown in the proof of Lemma (2), the LHS of this equation is an increasing function of m. Meanwhile, its RHS is always a constant. Other things equal, an increase in either μ_N, ρ or a will shift the graph of the LHS upward while that of the RHS stays the same resulting in a lower value of equilibrium m^*. However, an increase in μ_I will shift this graph down resulting in a higher value of m^*. □

The results in this proposition can be explained through market mechanism. Obviously, an increase in μ_I will increase the expected benefit of investing in this line of research so more resources will be pulled towards automation. As a result, the automated range will be expanded. However, an increase in μ_N makes variety creation more attractive and, hence, reduces the range of products produced with robots.

An increase in ρ means consumers relatively prefer present consumption to future consumption so they will lend less money and, thus, the rental rate R will rise. This will make production using robots relatively less profitable than using unskilled labor. Hence, there will be a contraction in the range of automated varieties.

An increase in a will reduce the rate of growth of adjusted wage paid to unskilled labor so unskilled labor becomes relatively more competitive to robots. This implies for a fixed range of available varieties used in production of final consumption good the number of varieties to be produced with robots will be smaller.

4.2 Endogenous labor augmenting technical change

Now assume that labor augmenting technological progress requires human capital investment. To simplify notation, define $B_{it} = A_{it}^{\frac{\alpha\beta}{1-\alpha}}$ and $B_t = A_t^{\frac{\alpha\beta}{1-\alpha}} = \int_{I_t}^{N_t} B_{it}di$. Assume that at any point in time, a non-automated intermediate firm that hires one unit of human capital for researching purpose is successful in improving its labor productivity with a Poisson arrival rate $\eta > 0$. When an innovation is successful, the productivity level is improved as the following:

$$\dot{B}_{it} = \mu_B h_{Bit} {B_t}^{\sigma}, \text{for } i \in (I_t; N_t], \tag{38}$$

where $\mu_B > 0$ denotes the efficiency of this research activity and h_{Bit} is the amount of human capital employed for research. The appearance of B_t, which reflects the aggregate level of unskilled labor productivity, in this formula is to capture any potential knowledge spillovers and σ denotes the degree of such a spillovers process. Aggregating across firms we get:

$$\dot{B}_t = \mu_B H_{Bt} {B_t}^{\sigma}, \tag{39}$$

where $H_{Bt} = \int_{I_t}^{N_t} h_{Bit} di$ is total human capital used for conducting research leading to improvement of unskilled labor productivity. It can be seen that due to symmetry, the amount of human capital employed for conducting research will be the same for every non-automated firm, i.e. $h_{Bit} = \frac{H_{Bt}}{Z-m}, \forall i \in (I_t; N_t]$.

The market clearing condition for human capital will now read:

$$H_{yt} + H_{Ht} + H_{Nt} + H_{It} + H_{Bt} = H_t. \tag{40}$$

Along the BGP, the allocation of human capital to each sector will be a constant fraction of the total human capital stock. Also along the BGP, B_t grows at a constant rate g_B. Thus, using (39), we work out that $g_B = \frac{1}{1-\sigma} g_H$. This, together with (38), implies that all B_{it}, for $i \in (I_t; N_t]$, grow at rate g_B. By definition, for $i \in (I_t; N_t]$, all A_{it} and A_t grow at rate $a = \frac{1-\alpha}{\alpha\beta} g_B = \frac{1-\alpha}{\alpha\beta(1-\sigma)} g_H$. To simplify our calculation, we choose σ in a way that $\frac{1-\alpha}{\alpha\beta(1-\sigma)} = 1$. This is equivalent to $\sigma = \frac{\alpha\beta+\alpha-1}{\alpha\beta}$. The convenience of this choice of parameters is that it makes $a = g_H$ as in the previous sub-section so we do not have to re-derive all equations for $\tilde{y}_t, R_t, W_{Ht}, W_{Lt}$ in their intensive forms. In the meantime, it does not affect our main results. The only difference is that now g_H will be endogenously determined within the model.

Again, we focus our attention on the non-automated firm that lies in the borderline with those automated ones, i.e. firm I. This firm will employ human capital to improve its unskilled labor productivity if the expected benefit outweighs the expected cost of that activity. The optimal level of human capital is determined by the following:

$$\eta V_{Bt} = W_{Ht}. \tag{41}$$

This formulation equates the marginal cost of human capital, W_{Ht}, with its marginal benefit, ηV_{Bt}. Here, V_{Bt} is the value of upgrading unskilled labor productivity that is given by the present discounted value of the incremented profit that would otherwise be generated using unskilled labor before being replaced by machines:

$$V_{Bt} = \frac{\Omega \mu_B Y_t}{W_{Ht}^{\frac{\alpha(1-\beta)}{1-\alpha}}} . \int_t^\infty e^{-\int_t^\tau (R-g) ds} \frac{B_\tau^\sigma}{W_{L\tau}^{\frac{\alpha\beta}{1-\alpha}}} d\tau. \tag{42}$$

This equation is obtained by using (38) and (9) before imposing the BGP conditions. This equation can be further simplified to yield:

$$V_{Bt} = \frac{\Omega \mu_B Y_t}{W_{Ht}^{\frac{\alpha(1-\beta)}{1-\alpha}} A_t} . \frac{\omega_{LIt}^{-\frac{\alpha\beta}{1-\alpha}}}{R} . \frac{(Z-m)(Z+m)}{2m}. \tag{43}$$

This, together with (27) and (41), delivers:

$$\mu_I.\frac{\omega_{LI}^{\frac{\alpha\beta}{1-\alpha}} - R^{\frac{\alpha\beta}{1-\alpha}}}{R^{\frac{\alpha\beta}{1-\alpha}}(R-g_H)} = \frac{\eta\mu_B}{R}\left[\frac{(Z-m)(Z+m)}{2m}\right]^{\frac{\alpha\beta-1+\alpha}{\alpha\beta}}, \tag{44}$$

where $R = \rho + \theta g_H$. In addition, similar to the previous section, the two equations (35) and (36) still hold. As a result, we have three equations in three unknown variables m, g_H and ω_{LI}. Solving this system of equations will give us the BGP for the whole economic system. In particular, from (35) and (44), we obtain:

$$\frac{R}{R-g_H} = \frac{\eta\mu_B}{\mu_I\mu_N}.\left[\frac{(Z-m)(Z+m)}{2m}\right]^{\frac{\alpha\beta-1+\alpha}{\alpha\beta}}.\frac{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}}}{m^{\frac{1-\alpha}{\alpha\beta}-1}(Z-m)}. \tag{45}$$

This result and that in (35) can be substituted into (36) to get an equation for m only:

$$\frac{\mu_B(1-\alpha)L\eta}{\beta}.\left[\frac{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}-1}Z}{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}}}.\frac{2m}{Z^2-m^2}\right]^{\frac{1-\alpha}{\alpha\beta}}$$
$$-\left(\frac{\Omega}{1-\alpha}\right)^{\frac{1-\alpha}{\alpha(1-\beta)}}.\frac{1}{R^{\frac{\beta}{1-\beta}}}\left[m + \frac{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}}}{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}-1}Z}.\frac{Z^2-m^2}{2m}\right]^{\frac{1-\alpha}{\alpha(1-\beta)}} = 0, \tag{46}$$

where $\frac{1}{R} = \frac{1-\theta}{\rho} + \frac{\theta}{\rho}.\frac{\mu_I\mu_N}{\eta\mu_B}.\left(\frac{2}{Z+m}\right)^{\frac{\alpha\beta-1+\alpha}{\alpha\beta}}.\frac{(Z-m)^{\frac{1-\alpha}{\alpha\beta}}}{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}}}$. This shows that R is increasing in m.

Lemma 3. *There always exists a value $m^* \in (0, Z)$ that solves equation (46) and that value is unique.*

Proof. Consider the LHS of equation (46) which is a continuous function of $m \in (0, Z)$. When m tends to z, LHS tends to $+\infty$. However, when $m = 0$, LHS tends to $-\infty$. This means the LHS changes its sign over its domain of $(0, Z)$. In other words, the graph of LHS will intersect with the horizontal axis at least one time. Each point of intersection is an interior BGP of the system.

Note that if m^* is a solution to equation (46), it will also be a solution to equation (34). After rearranging this equation, we obtain:

$$\mu_I = \mu_N\left(\frac{m}{Z}\right)^{\frac{1-\alpha}{\alpha\beta}}.\left[\frac{R^{\frac{\alpha\beta}{1-\alpha}}}{\omega_{LN}^{\frac{\alpha\beta}{1-\alpha}}} - 1\right] + \frac{\mu_I R^{\frac{\alpha\beta}{1-\alpha}}}{\omega_{LI}^{\frac{\alpha\beta}{1-\alpha}}}.$$

Clealy, the LHS of this equation is a constant. Meanwhile the RHS is increasing in m because R is increasing in m and ω_{LI} and ω_{LN} are both decreasing in m. Given that m^* solves this equation, it will be the unique solution to this equation. □

Proposition 5. *Suppose that Assumption 3(i) holds. Then there always exists a unique interior BGP in which the range of automated varieties is endogenously determined by the system.*

The essence of this proposition is to emphasize the existence of a unique BGP at which both machines anf unskilled workers find their own jobs in producing varieties. This is equivalent to what was obtained previously in Proposition 3. However, the key difference lies in the feature that here the dynamics of all activities are determined within the system. In particular, output, consumption, variety expansion, automation and labor productivity all grow at the rate of growth of human capital, the key production factor of the economy. This rate of growth is, in turn, determined within the system by parameters characterizing research activities (the supply side) and households' preferences (the demand side).

Proposition 6. *Other things equal, the range of automated varieties will be expanded further in the long run if either the rate of time preference, ρ, or the research efficiency of labor augmenting technology, μ_B, is lower.*

Proof. We rearrange equation (34) to obtain the following:

$$\frac{\mu_N}{\mu_I}\left(\frac{m}{Z}\right)^{\frac{1-\alpha}{\alpha\beta}} \cdot \left[\frac{R^{\frac{\alpha\beta}{1-\alpha}}}{\omega_{LN}^{\frac{\alpha\beta}{1-\alpha}}} - 1\right] + \frac{R^{\frac{\alpha\beta}{1-\alpha}}}{\omega_{LI}^{\frac{\alpha\beta}{1-\alpha}}} = 1,$$

where $\frac{1}{R} = \frac{1-\theta}{\rho} + \frac{\theta}{\rho} . \frac{\mu_I \mu_N}{\eta\mu_B} . \left(\frac{2}{Z+m}\right)^{\frac{\alpha\beta-1+\alpha}{\alpha\beta}} . \frac{(Z-m)^{\frac{1-\alpha}{\alpha\beta}}}{\mu_I Z^{\frac{1-\alpha}{\alpha\beta}} + \mu_N m^{\frac{1-\alpha}{\alpha\beta}}}$. As shown in the proof of Lemma 2, the LHS of this equation is increasing in m. In addition, clearly, R is increasing in ρ. Hence, other things equal, a decrease in ρ will shift the graph of the LHS down resulting in a higher equilibrium value of m^*. A similar argument applies for the case of a decrease in μ_B. □

The impact of a change in ρ on m can be explained as in Proposition 4. As for μ_B, an increase in μ_B will attract more human capital towards this line of research leaving less of this resource for other activities including research that target displacing unskilled labor with robots. It also reduces the wage rate of unskilled labor and, thus, render this production factor with a comparative advantage over its machinery counterparts. Being discouraged by this change, the range of automated products will contract.

5 Conclusion

In this paper, we have considered a simple model of endogenous growth with automation on the production of intermediate products. Besides automation, there are other lines of ongoing innovation that aim at either creating new product varieties to be used in the production of the final consumption good or improving unskilled labor productivity. While the stock of automating capital is accumulated similarly as physical capital in the form of robots and machines, it acts as a production factor that perfectly substitutes for unskilled labor in the production process for varieties. Within that setting, we first established an equilibrium for the economy in the static case. We conditions that lead to full automation, partial automation or no automation at all. In extending the model to a dynamic setting, we derived balance growth paths under alternative cases of labor augmenting technology: exogenous versus endogenous. We showed that in each case there exists a unique balanced growth path along which each of the production factors, robots and unskilled labor, is employed to produce a fixed range of varieties alongside human capital. When labor augmenting technology is exogenous, its rate of growth dictates that of output, consumption and wages. However, when this factor is endogenously determined within the model, it is the rate of growth of human capital that does the job.

The stock-taking message here is that while automation threatens unskilled labor, it does no harm to skilled labor. Only workers who work in jobs that can be easily done by industrial robots are under pressure. Those who perform in jobs of high human complexity will be relatively relaxed. Automation helps maintain long run growth by forcing unskilled labor to improve its productivity. Thinking in a positive way, industrial robots may serve as an ideal supplement for unskilled labor in countries where this resource is relatively scarce.

There are several interesting theoretical and conceptual issues with which our framework can be further enriched. Particularly important is to allow automation to take over the work performed by skilled workers. Another new dimension might be to explore the possibility of excessive automation and whether robot taxation can help improve economic efficiency and social welfare. Last but not least, given that data on robots becomes more and more accessible, it may be interesting to use them to test theoretical predictions produced in this paper. All these open promising avenues for a fruitful research agenda.

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