Iterating the Cuntz-Nica-Pimsner construction for product systems

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Iterating the Cuntz–Nica–Pimsner construction for product systems

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Declaration

I, James Edward Fletcher, declare that this thesis submitted in fulfilment of the requirements for the conferral of the degree Doctor of Philosophy, from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

James Edward Fletcher
January 6, 2017
Abstract

In this thesis we study how decompositions of a quasi-lattice ordered group \((G, P)\) relate to decompositions of the Nica–Toeplitz algebra \(\mathcal{N}T_X\) and Cuntz–Nica–Pimsner algebra \(\mathcal{N}O_X\) of a compactly aligned product system \(X\) over \(P\). In particular, we are interested in the situation where \((G, P)\) may be realised as the semidirect product of quasi-lattice ordered groups. Our results generalise Deaconu’s work on iterated Toeplitz and Cuntz-Pimsner algebras [13]. As a special case we consider when \(P = \mathbb{N}^k\) and \(X\) is the product system associated to a finitely aligned higher-rank graph, and \(\mathbb{N}^k\) is decomposed as \(\mathbb{N}^{k-1} \times \mathbb{N}\).
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Chapter 1

Introduction

One motivation for studying $C^*$-algebras is as a natural setting for unitary representation theory for groups [12, 15], and this has led to the program of studying $C^*$-algebraic representations of other algebraic and analytic objects. For example, $C^*$-algebras have been associated to semigroups [42, 47, 48], rings [11, 41], graphs [18, 56], and dynamical systems [7, 44, 54], in such a way as to encode the structure of the underlying object in the algebra itself. A variety of techniques have also been developed to construct new $C^*$-algebras from old ones, such as crossed products [39, 68, 70], and associating $C^*$-algebras to Hilbert bimodules [27, 34, 53]. At the same time, flexible but tractable models for $C^*$-algebras have proved useful in the study of a number of key examples arising in applications outside of $C^*$-algebra theory.

The idea is to study the structure of these $C^*$-algebras using the structure of the underlying mathematical objects and the basic $C^*$-algebras used in their construction. The classical example is crossed products of $C(X)$ by $\mathbb{Z}$, where the primitive-ideal space of the crossed product can be identified with the quasi-orbit space for the induced $\mathbb{Z}$-action on $X$, and the $K$-theory of the crossed product can be related to the topological $K$-theory of $X$ using the Pimsner–Voiculescu six-term exact sequence. In this thesis we focus on the $C^*$-algebras associated to two mathematical structures called higher-rank graphs and product systems. We prove that these algebras can be constructed iteratively by decomposing their underlying mathematical objects. This procedure has potential applications in computations of $K$-theory and in studying other structural properties of these $C^*$-algebras.

1.1 Historical background

In [9], Cuntz investigated the $C^*$-algebra $O_n$, generated by $n \geq 2$ isometries with mutually orthogonal ranges that sum to the identity, as a $C^*$-algebraic analogue of a type II factor. These Cuntz algebras provided the first examples of separable, unital, simple and purely infinite $C^*$-algebras. Subsequently, Evans showed that $O_n$,
may be realised as the quotient of the Toeplitz algebra $\mathcal{T}_n$, generated by the creation operators on the Fock space $\mathcal{F}_n$ associated to an $n$-dimensional Hilbert space, by the compact operators $\mathcal{K}(\mathcal{F}_n)$ [21].

Motivated by applications to symbolic dynamics based on a strong relationship between $O_n$ and the full shift on an $n$-letter alphabet, Cuntz and Krieger subsequently generalised the Cuntz algebras to the Cuntz–Krieger algebras [10]. Given an $n \times n$ matrix $A$ with entries in $\{0, 1\}$, $O_A$ is the universal $C^*$-algebra generated by $n$ partial isometries with mutually orthogonal range projections, subject to relations between the range and source projections of the partial isometries encoded by the matrix $A$.

Following this, Enomoto and Watatani showed that the information in the matrix $A$ may be represented by a directed graph, leading to a definition of graph algebras [18]. Graph algebras provide a rich supply of examples of $C^*$-algebras — the class of directed graph algebras includes, up to Morita equivalence, the Cuntz algebras [9], Cuntz–Krieger algebras [10, 36], all simple Kirchberg algebras with torsion free $K_1$-group [69], and all AF-algebras [16]. They have also been used as tractable models for the $C^*$-algebras of continuous functions on certain quantum spheres and quantum complex projective spaces [29]. Graph algebras are powerful models because properties of the $C^*$-algebra can be readily extracted from the underlying graphical data — for example the ideal structure of the algebra [4, 3, 30, 37], or its $K$-theory (which, due to some deep theorems, can be easily computed using the adjacency matrix of the graph) [3, 22, 50, 60].

In [53], Pimsner showed how to associate $C^*$-algebras to Hilbert bimodules. Loosely speaking a Hilbert bimodule $X$ consists of a complex vector space equipped with left and right actions of a $C^*$-algebra $A$, which we call the coefficient algebra of $X$, along with an $A$-valued inner-product (an excellent overview is given in [40]). The Toeplitz algebra $\mathcal{T}_X$ of $X$, is the $C^*$-algebra generated by the creation operators on the Fock space $\mathcal{F}_X$ of $X$, generalising the classical Toeplitz algebra $\mathcal{T}_n$. The Cuntz–Pimsner algebra $O_X$ of $X$ is then defined to be the quotient of $\mathcal{T}_X$ by the ideal of generalised compact operators on $\mathcal{F}_X$, and generalises both the Cuntz–Krieger algebras and crossed products by $\mathbb{Z}$.

Pimsner showed that $\mathcal{T}_X$ and $O_X$ each enjoy a universal property, and proved that $\mathcal{T}_X$ is $KK$-equivalent to the coefficient algebra $A$. He used this $KK$-equivalence to relate the $K$-theories of $O_X$ and $A$ via a six-term exact sequence in $K$-theory that generalises the Pimsner–Voiculescu exact sequence for crossed products by $\mathbb{Z}$ [52].

Fowler and Raeburn in [27] investigated representations of the Toeplitz algebra in more detail, relaxing Pimsner’s hypotheses to allow for nonfaithful left actions and Hilbert bimodules that are not full. They also proved an analogue of Coburn’s Theorem giving sufficient conditions for a representation of $\mathcal{T}_X$ on a Hilbert space.
to be faithful. Later, Katsura provided what has come to be accepted as the correct
definition of a Cuntz–Pimsner algebra for an arbitrary Hilbert bimodule [34], gener-
alising Pimsner’s earlier definition. Katsura’s Cuntz–Pimsner algebra, also denoted
by $\mathcal{O}_X$, has two notable advantages over Pimsner’s:

(i) $\mathcal{O}_X$ always contains a faithful copy of the coefficient algebra $A$;

(ii) Any representation of $\mathcal{O}_X$ that is faithful on $A$, is faithful on the fixed-point
algebra for its canonical gauge action $\gamma : \mathbb{T} \to \text{Aut}(\mathcal{O}_X)$.

Katsura used these two properties to prove a gauge-invariant uniqueness theorem
for his Cuntz–Pimsner algebras: any representation of $\mathcal{O}_X$ that is faithful on $A$ and
carries a gauge-action is injective. In the same paper, Katsura also proves
results relating the nuclearity and exactness of $\mathcal{O}_X$ and $\mathcal{T}_X$ to the nuclearity and
exactness of the coefficient algebra. In [33], Katsura went on to investigate the
ideal structure of his Cuntz–Pimsner algebras. His results show that the gauge-
invariant ideals of $\mathcal{O}_X$ are in one-to-one correspondence with certain pairs of ideals
in the coefficient algebra. Subsequently, Katsura’s $C^*$-algebras have been widely
studied, and they unify several interesting classes of $C^*$-algebras under one umbrella,
including all graph algebras, all topological graph algebras [32], crossed products by
Hilbert bimodules [1], and all $C^*$-algebras of topological quivers [46].

Higher-rank graphs, and their $C^*$-algebras, were introduced by Kumjian and
Pask as a generalisation of directed graphs, and their $C^*$-algebras [35]. Observing
that the path space of a directed graph can be viewed as a (small) category, they
defined a higher-rank graph of rank $k \in \mathbb{N} \cup \{0\}$ (or simply a $k$-graph) to be a
countable small category $\Lambda$ together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the
following factorisation property: for any $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$,
there exist unique $\mu, \nu \in \Lambda$ with $d(\mu) = m$ and $d(\nu) = n$ such that $\lambda = \mu \nu$.

In the same paper, Kumjian and Pask showed how to associate a $C^*$-algebra to
each row finite higher-rank graph with no sources, by associating a partial isometry
to each path in the graph. This $C^*$-algebra is often called the Cuntz–Krieger algebra
of the higher-rank graph. They also showed that the Cuntz–Krieger algebra of a
higher-rank graph can be realised as a groupoid $C^*$-algebra, proved a gauge-invariant
uniqueness theorem, and derived a sufficient condition for the Cuntz–Krieger algebra
to be simple.

In [59], Raeburn, Sims, and Yeend, showed how to relax the hypotheses of [35],
and defined the Cuntz–Krieger algebra for an arbitrary finitely aligned higher-rank
graph. They proved gauge invariant and Cuntz–Krieger uniqueness theorems for
these higher-rank graph algebras. Sims subsequently defined relative Cuntz–Krieger
algebras for finitely aligned higher-rank graphs, which includes the class of Toeplitz–
Cuntz–Krieger algebras as a special case [65, 66].
Higher-rank graph algebras provide an even richer supply of $C^*$-algebras than directed graph algebras, generalising the higher-rank Cuntz–Krieger algebras of Robertson and Steger that arose from the study of group actions on the vertices of affine buildings [35, 63], and providing examples of Kirchberg algebras that have torsion in their $K_1$-groups [20]. In fact, every simple Kirchberg algebra is Morita equivalent to a direct limit of higher-rank graph algebras, which was used to prove that all Kirchberg algebras have nuclear dimension one [64]. Furthermore, higher-rank graph algebras give examples, up to Morita equivalence, of AT-algebras with real rank zero [51], such as the Bunce–Deddens algebras and the irrational rotation algebras. These algebras cannot arise as directed graph algebras since they are simple but neither AF nor purely infinite [36].

In [5], Burgstaller showed that the Toeplitz–Cuntz–Krieger algebra of a higher-rank graph has the same $K$-theory as the space of continuous functions on its vertex set. In [19, 20], Evans used a homological spectral sequence to derive expressions for the $K$-theory of the Cuntz–Krieger algebra of a 2-graph in terms of the graph’s adjacency matrices. Unfortunately, Evans’ techniques do not generalise to $k \geq 3$, and it remains an open problem to find nice formulae for the $K$-groups of higher-rank graphs in terms of just their graphical data.

We also note that there have recently been generalisations of higher-rank graphs and their associated $C^*$-algebras. Yeend has defined topological higher-rank graphs [71, 72], simultaneously generalising the work of Kumjian and Pask, and of Katsura. Brownlowe, Sims, and Vittadello have also introduced $P$-graphs, which extends the notion of a $k$-graph by allowing the degree functor to take values in a semigroup $P$ other than $\mathbb{N}$.

In [26], Fowler introduced product systems of Hilbert bimodules, generalising the continuous product systems of Hilbert spaces studied by Arveson [2] and their discrete analogues studied by Dinh [14]. Loosely speaking a product system of Hilbert bimodules over a semigroup $P$ consists of a collection of Hilbert bimodules $\{X_p : p \in P\}$ with common coefficient algebra $A$, such that the balanced tensor product is compatible with the semigroup structure in the following sense

$$X_p \otimes_A X_q \cong X_{pq} \quad \text{for each } p, q \in P \setminus \{e\}.$$  

Fowler was interested in compactly aligned product systems of essential Hilbert bimodules over quasi-lattice ordered groups (first introduced by Nica [48]). To each product system he associates a twisted crossed product algebra $B_P \rtimes_{\tau,X} P$, and examines a subalgebra $\mathcal{N}T_X$ (denoted by $T_{cov}(X)$ in [26]) generated by what he calls a Nica covariant Toeplitz representation of $X$. The main result of [26] characterises the faithful representations of $B_P \rtimes_{\tau,X} P$ on Hilbert spaces.
Fowler also proposed a notion of Cuntz–Pimsner covariance for representations of compactly aligned produced systems, and an associated universal $C^*$-algebra, denoted by $O_X$, which generalises the Cuntz–Pimsner algebra for Hilbert bimodules defined in [53]. In general Fowler’s Cuntz–Pimsner algebra need not contain a faithful copy of $A$, and examples in the appendix of [59] also show that a representation of $O_X$ that is faithful on $A$ need not be faithful on $O_X^\delta$ (where $\delta$ is the canonical coaction of $G$ on $O_X$). In contrast with the Cuntz–Pimsner algebras associated to Hilbert bimodules by both Pimsner [53] and Katsura [34], Fowler’s Cuntz–Pimsner algebra need not be a quotient of $\mathcal{NT}_X$.

To overcome these issues, Sims and Yeend defined a $C^*$-algebra $NO_X$ generated by a universal Cuntz–Nica–Pimsner representation of $X$ [67]. They show that $NO_X$ coincides with Kastura’s Cuntz–Pimsner algebra when $P = N_0$, and coincides with Fowler’s $O_X$ whenever $A$ acts faithfully and compactly on each $X_p$ and each pair of elements in $P$ has a common upper bound. Furthermore, Sims and Yeend show that their Cuntz–Pimsner algebras generalise the Cuntz–Krieger algebras associated to finitely aligned higher-rank graphs [59], and the boundary quotients of Toeplitz algebras studied by Crisp and Laca [8]. Subsequently, Carlsen, Larsen, Sims, and Vittadello proved a gauge-invariant uniqueness theorem for $NO_X$ by viewing it as a co-universal algebra [6].

The object of this thesis is to study how the $C^*$-algebras $\mathcal{NT}_X$ and $NO_X$ associated to a compactly aligned product system $X$ over a quasi-lattice ordered group $(G,P)$ decompose given a decomposition of $(G,P)$. In the first part of the thesis we focus our attention on product systems arising from finitely aligned $k$-graphs, and the decomposition of the semigroup $P = \mathbb{N}^k$ is given by $\mathbb{N}^{k-1} \times \mathbb{N}$.

### 1.2 Outline of the thesis

In Chapter 2 we investigate the $C^*$-algebras associated to higher-rank graphs. Our goal is to show that the $C^*$-algebras associated to a $k$-graph $\Lambda$ can be realised as the Toeplitz algebras and/or Cuntz–Pimsner algebras of suitable Hilbert bimodules over the $C^*$-algebras associated to the $(k-1)$-graph $\Lambda'$ formed by removing all edges of a fixed degree from $\Lambda$. Since there are numerous well-known results relating the structure of Toeplitz and Cuntz–Pimsner algebras of a Hilbert bimodule $X$ over $A$ with the structure of the coefficient algebra $A$ (see Section 1.1), and the structure of $k$-graph algebras is well-understood when $k \leq 2$, we hope to be able to use our results to gain structural information about $k$-graph algebras when $k \geq 3$.

We begin in Section 2.1 and Section 2.2 by providing the necessary background material on Hilbert bimodules and their $C^*$-algebras. Following [40], we define Hilbert modules, adjointable and compact operators, and Hilbert bimodules. We
also look at the (balanced) tensor product of Hilbert modules, and discuss induced representations, which were first studied by Rieffel [62].

As in [27], we define Toeplitz representations of Hilbert bimodules and use Loring’s work on universal $C^*$-algebras [43] to provide a concise proof that there exists a $C^*$-algebra generated by a universal Toeplitz representation. We conclude Section 2.2 by recapping Katsura’s work on Cuntz–Pimsner algebras [34]. In Section 2.3, we summarise the necessary background material on higher-rank graphs and their $C^*$-algebras.

After discussing some basic category theory, we define finitely aligned higher-rank graphs, Toeplitz–Cuntz–Krieger and Cuntz–Krieger families, and the universal $C^*$-algebras that such families generate. We end the section by discussing the uniqueness theorems for Toeplitz–Cuntz–Krieger algebras and Cuntz–Krieger algebras associated to higher-rank graphs.

In Section 2.4 we investigate Toeplitz–Cuntz–Krieger algebras associated to finitely aligned higher-rank graphs. Our main result in this section (Theorem 2.4.8) shows that the Toeplitz–Cuntz–Krieger algebra of a finitely aligned $k$-graph $\Lambda$ may be realised as an iterated Toeplitz algebra of Hilbert bimodules. In particular, we show that there is a Hilbert bimodule $X$, whose coefficient algebra is $TC^*(\Lambda')$, such that $TC^*(\Lambda)$ and $TX$ are isomorphic. Combining this result with ([53], Theorem 4.4), we deduce in Section 2.5 that the Toeplitz–Cuntz–Krieger algebra of a higher-rank graph is $KK$-equivalent to the space of continuous functions vanishing at infinity on the vertex set $\Lambda^0$. This result then provides an alternate proof to ([5], Theorem 2.9), concerning the $K$-theory of Toeplitz–Cuntz–Krieger algebras.

In Section 2.6, we extend our results from Section 2.4 to Cuntz–Krieger algebras of finitely aligned graphs. The main result of this section (Theorem 2.6.12) shows that, under relatively mild hypotheses, the Cuntz–Krieger algebra of a finitely aligned $k$-graph $\Lambda$ may be realised as an iterated Cuntz–Pimsner algebra of Hilbert bimodules. In particular, there is a Hilbert bimodule $X$, with coefficient algebra $C^*(\Lambda')$, such that $C^*(\Lambda)$ and $OX$ are isomorphic. We hope that this may have applications to computations of $K$-theory of $k$-graph algebras when $k \geq 3$ via the Pimsner–Voiculescu exact sequence for Cuntz–Pimsner algebras ([34], Theorem 8.6) and known results describing the $K$-theory of $k$-graph algebras when $k \leq 2$ ([3], Theorem 6.1; [20], Proposition 3.16).

In Chapter 3 we investigate the $C^*$-algebras associated to compactly aligned product systems over quasi-lattice ordered groups. Our aim is to show how a decomposition of the underlying quasi-lattice group as a semidirect product of quasi-lattice ordered groups gives a decomposition of the associated $C^*$-algebras. Our motivation comes from Deaconu’s work on iterated Toeplitz and Cuntz–Pimsner algebras [13]. We show how his iterative procedure can be extended to quasi-lattice ordered groups.
that are more general than $(\mathbb{Z}^2, \mathbb{N}^2)$.

In Section 3.1 we review the necessary background material for product systems and their associated $C^*$-algebras, and present some results from the literature that we will make use of in the rest of the chapter. We begin by recapping the definitions of product systems, representations, compact alignment and Nica covariance. As motivating examples we examine the Fock representation of a product system and show how finitely aligned higher-rank graphs may be viewed as determining compactly aligned product systems. Next we present the definition of Cuntz–Pimsner covariance developed by Sims and Yeend [67] and show how it relates to Fowler’s notion of Cuntz–Pimsner covariance in [26].

In Section 3.3 we examine the Nica–Toeplitz algebra associated to a product system $\mathbb{Z}$ over a quasi-lattice ordered group of the form $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$, where $(G, P)$ and $(H, Q)$ are themselves quasi-lattice ordered groups. If we let $X$ be the product system corresponding to the fibres of $\mathbb{Z}$ associated to the semigroup $P$, then our main result (Theorem 3.3.17) shows that there exists a product system $Y$ over $(H, Q)$, whose coefficient algebra is the Nica–Toeplitz algebra of $X$, such that the Nica–Toeplitz algebras of $Y$ and $Z$ coincide. This result generalises Theorem 2.4.8, and ([13], Lemma 4.1).

In Section 3.4, we extend the results from Section 3.3 to Cuntz–Nica–Pimsner algebras associated to product systems. Our main result, Theorem 3.4.21, shows that there exists a product system $Y$ over $(H, Q)$, whose coefficient algebra is $\mathcal{NO}X$, such that the Cuntz–Nica–Pimsner algebras of $Y$ and $Z$ coincide. This result generalises Theorem 2.6.12 and the second part of ([13], Lemma 4.2). The main difficulty in establishing Theorem 3.4.21 is finding sufficient conditions to ensure that $\mathcal{NO}Y$ acts faithfully on each fibre of $Y$, which makes the Cuntz–Pimsner covariance relation in $\mathcal{NO}Y$ tractable. We deal with this difficulty in Proposition 3.4.12.

To conclude Chapter 3 we examine what we call relative Cuntz–Nica–Pimsner algebras. In particular, we consider the Cuntz–Nica–Pimsner algebra of the product system defined in Section 3.3, as well as the Nica–Toeplitz algebra of the product system defined in Section 3.4. The main result of Section 3.5 (Theorem 3.5.10) generalises the first part of ([13], Lemma 4.2).

In Chapter 4 we show how the iterated Toeplitz and Cuntz–Pimsner algebras investigated by Deaconu in [13] can be viewed in terms of the procedure developed in Chapter 3. We highlight what we believe are some deficiencies in Deaconu’s arguments and explain how these problems can be overcome. In doing so we will see which of the hypotheses used in [13] are necessary — many can be relaxed or removed entirely.

Finally in Appendix A, we present a detailed and relatively self-contained proof of a uniqueness theorem for Nica–Toeplitz algebras, which we need in Sections 3.3
and 3.5. Our proof is heavily based on Fowler’s proof of ([24], Theorem 7.2). Unlike in Fowler’s original paper on product systems of Hilbert bimodules, we do not view the Nica–Toeplitz algebra as a subalgebra of a twisted semigroup crossed product. We also remove Fowler’s hypothesis that each fibre of the product system is essential. This is needed for our applications in Chapter 3, and also aligns our result with the uniqueness theorem for Toeplitz algebras of Hilbert bimodules proved by Fowler and Raeburn ([27], Theorem 2.1), which did not require the bimodule to be essential.
Chapter 2

Motivation from higher-rank graph algebras

Our motivation comes from higher-rank graphs. The basic premise is that given a $k$-graph (subject to some mild hypotheses), the Toeplitz–Cuntz–Krieger algebra of the graph can be realised as the Toeplitz algebra of a Hilbert bimodule whose coefficient algebra is the Toeplitz–Cuntz–Krieger algebra of any one of the graph’s $(k - 1)$-subgraphs formed by deleting all edges of a fixed degree. Moreover, the Cuntz–Krieger algebra of the graph can be realised as the Cuntz–Pimsner algebra of a Hilbert bimodule whose coefficient algebra is the Cuntz–Krieger algebra of any one of the these $(k - 1)$-subgraphs. Firstly, we begin by recapping some of the necessary background material for higher-rank graphs, Hilbert (bi)modules, and the $C^*$-algebras associated to them.

2.1 Hilbert bimodules

We begin by giving a brief summary of the theory of Hilbert modules — first introduced for unital commutative $C^*$-algebras by Kaplansky [31] and later for arbitrary $C^*$-algebras by Paschke [49] and Rieffel [62]. Most of the treatment here follows [40].

Definition 2.1.1. Let $A$ be a $C^*$-algebra. A (right) inner-product $A$-module is a complex vector space $X$ equipped with a map $\langle \cdot, \cdot \rangle_A : X \times X \to A$, linear in its second argument, and a right action of $A$, such that for any $x,y \in X$ and $a \in A$, we have

(i) $\langle x, y \rangle_A = \langle y, x \rangle_A^*$;

(ii) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$;

(iii) $\langle x, x \rangle_A \geq 0$ in $A$; and
(iv) \( \langle x, x \rangle_A = 0 \) if and only if \( x = 0 \).

It follows from ([40], Proposition 1.1) that the formula \( \|x\|_X := \|\langle x, x \rangle_A\|_A^{\frac{1}{2}} \) defines a norm on \( X \). If \( X \) is complete with respect to this norm, we say that \( X \) is a (right) Hilbert \( A \)-module.

We like to think of Hilbert modules as generalisations of Hilbert spaces, with the field of complex numbers replaced by a \( C^* \)-algebra. Whilst Hilbert modules behave in a similar way to Hilbert spaces, with many of the following definitions and results motivated by the theory of Hilbert spaces, there are, as we shall see later, some key differences.

**Example 2.1.2.** Let \( X \) be a compact Hausdorff space and \( \mathcal{H} \) a Hilbert space. For each \( t \in X \), let \( \mathcal{H}_t \) be a closed subspace of \( \mathcal{H} \). Define

\[
E := \{ \xi : X \to \mathcal{H} : \xi \text{ is continuous, } \xi(t) \in \mathcal{H}_t \text{ for each } t \in X \}.
\]

Then \( E \) has the structure of a right \( C(X) \)-module: for any \( \xi \in E \), \( f \in C(X) \), we define \( \xi \cdot f \) by

\[
(\xi \cdot f)(t) := \xi(t)f(t) \text{ for each } t \in X.
\]

Moreover, \( E \) has a \( C(X) \)-valued inner-product: for any \( \xi, \eta \in E \), we define \( \langle \xi, \eta \rangle_{C(X)} \) by

\[
\langle \xi, \eta \rangle_{C(X)}(t) = \langle \xi(t), \eta(t) \rangle_{\mathcal{H}} \text{ for each } t \in X.
\]

With this structure, \( E \) becomes a Hilbert \( C(X) \)-module.

**Example 2.1.3.** Let \( A \) be a \( C^* \)-algebra. Set \( X := A \) and let \( A \) act on the right of \( X \) by right multiplication. Equipping \( X \) with the \( A \)-valued inner-product \( \langle a, b \rangle_A := a^*b \) for each \( a, b \in A \), \( X \) turns into a \( A \) Hilbert \( A \)-module, which we denote by \( A^A \).

The following result shows that not only are right Hilbert \( A \)-modules nondegenerate (in the sense that \( \operatorname{span}\{x \cdot a : x \in X, \ a \in A\} = X \)), but elements of \( X \) can be factorised in a very precise form.

**Proposition 2.1.4** ([61], Proposition 2.31). Let \( X \) be a (right) Hilbert \( A \)-module. Then for every \( x \in X \), there exists a unique \( x' \in X \) such that \( x = x' \cdot \langle x', x' \rangle_A \). We call this the Hewitt–Cohen–Blanchard factorisation of \( x \).

We now introduce an important class of operators on Hilbert modules, analogous to the bounded linear operators on Hilbert spaces.

**Definition 2.1.5.** Let \( X \) and \( Y \) be (right) Hilbert \( A \)-modules. We say that a map \( T : X \to Y \) is adjointable if there exists a map \( S : Y \to X \) such that for each \( x \in X \) and \( y \in Y \), we have \( \langle Tx, y \rangle_A = \langle x, Sy \rangle_A \). We write \( \mathcal{L}_A(X, Y) \) for the set of all adjointable operators from \( X \) to \( Y \). Additionally, we write \( \mathcal{L}_A(X) \) for \( \mathcal{L}_A(X, X) \).
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Remark 2.1.6. Given an adjointable operator \( T : X \to Y \), one can show that there is a unique map \( S : Y \to X \), such that \( \langle Tx, y \rangle_A = \langle x, Sy \rangle_A \) for each \( x \in X \), \( y \in Y \). We write \( T^* \) for \( S \) and call it the adjoint of \( T \). If \( T \) is adjointable, it is automatically both complex linear and \( A \)-linear. Furthermore, it follows from an application of the Banach-Steinhaus Theorem that adjointable operators are bounded. Clearly, \( \mathcal{L}_A(X,Y) \) is closed under taking linear combinations and \( \mathcal{L}_A(X) \) is closed under taking products and adjoints. Moreover, when equipped with the operator norm \( \mathcal{L}_A(X) \) is a \( C^* \)-algebra (for more information, see Chapter 1 of [40]).

In contrast with Hilbert spaces, closed subspaces of Hilbert modules need not be orthogonally complementable. Moreover, bounded linear maps between Hilbert modules need not be adjointable. The next example is due to Lance [40].

Example 2.1.7. Let \( X \) be a compact Hausdorff space and \( Y \) a nonempty closed subset of \( X \) with dense complement. Let \( A := C(X) \) and \( F := A_A \). Consider the closed submodule \( E := \{ f \in C(X) : f|_Y = 0 \} \) of \( F \). Then the orthogonal complement \( E^\perp := \{ y \in F : \langle x, y \rangle_A = 0 \text{ for all } x \in E \} \) is just \( \{0\} \). Therefore, \( E \oplus E^\perp = E \neq F \) and \( E \subset F = E^{\perp\perp} \). Moreover, the inclusion map \( i : E \to F \) is bounded and linear, but not adjointable.

Proof. Since \( Y \) is a closed subset of \( X \), it follows that \( E \) is closed in \( F \) and hence a Hilbert \( C(X) \)-module. Suppose \( y \in E^\perp \). Thus, \( \langle x, y \rangle_A = 0 \) for any \( x \in E \), and so \( \overline{x(t)} y(t) = 0 \) for any \( t \in X \). By the Tietze extension theorem, for each \( t \in X \setminus Y \) there exists \( x \in E \) such that \( x(t) = 1 \), and therefore \( y(t) = \overline{x(t)} y(t) = 0 \). Hence, \( y|_{X \setminus Y} = 0 \). As \( Y \) has dense complement in \( X \) and \( y \in C(X) \), we conclude that \( y = 0 \). Next, looking for a contradiction, suppose \( i \) is adjointable. Then for any \( f \in E \) and \( t \in X \),

\[
\overline{f(t)} = i(f)(t)1(t) = \langle i(f), 1 \rangle_{C(X)} = \langle f, i^*(1) \rangle_{C(X)} = \overline{f(t)} i^*(1)(t).
\]

Thus, \( i^*(1)|_{X \setminus Y} = 1 \), and so \( i^*(1) = 1 \), which is impossible since \( 1 \notin E \) (as \( Y \) is nonempty). Hence, \( i \) is not adjointable. \( \square \)

Next, we introduce a class of operators analogous to the finite-rank operators on a Hilbert space.

Definition 2.1.8. Let \( X \) and \( Y \) be (right) Hilbert \( A \)-modules. For each \( x \in X \) and \( y \in Y \), we define \( \Theta_{x,y} : Y \to X \) by

\[
\Theta_{x,y}(z) = x \cdot \langle y, z \rangle_A \quad \text{for each } z \in Y.
\]

We call operators of this form rank one operators. We also define

\[
\mathcal{FR}_A(Y,X) := \text{span}\{ \Theta_{x,y} : x \in X, \ y \in Y \}
\]
and call elements of this set finite-rank operators from $Y$ to $X$. For simplicity, we write $\mathcal{FR}_A(X)$ for $\mathcal{FR}_A(X,X)$.

The finite-rank operators have the following properties.

**Proposition 2.1.9.** Let $X,Y,Z$ be (right) Hilbert $A$-modules and fix $x \in X$ and $y \in Y$. Then

(i) $\Theta_{x,y} \in \mathcal{L}_A(Y,X)$ and $\Theta^*_{x,y} = \Theta_{y,x}$;

(ii) $\Theta_{x,y} \Theta_{u,v} = \Theta_{x \cdot \langle y,u \rangle_A, v} = \Theta_{x, v \cdot \langle u, y \rangle_A}$ for any $u \in Y$, $v \in Z$;

(iii) $T \Theta_{x,y} = \Theta_{T x, y} \in \mathcal{FR}_A(Y,Z)$ for any $T \in \mathcal{L}_A(X,Z)$;

(iv) $\Theta_{x,y} T = \Theta_{x, T^* y} \in \mathcal{FR}_A(Z,X)$ for any $T \in \mathcal{L}_A(Z,Y)$;

**Proof.** Observe that for any $w \in Y$ and $z \in X$, we have

$$\langle \Theta_{x,y}(w), z \rangle_A = \langle x \cdot \langle y, w \rangle_A, z \rangle_A = \langle y, w \rangle_A \langle x, z \rangle_A$$

$$= \langle w, y \rangle_A \langle x, z \rangle_A = \langle w, y \cdot \langle x, z \rangle_A \rangle_A = \langle w, \Theta_{y,x}(z) \rangle_A.$$ 

Hence, $\Theta^*_{x,y} = \Theta_{y,x}$, which proves (i).

Next observe that for any $z \in Z$, we have

$$\Theta_{x,y} \Theta_{u,v}(z) = \Theta_{x,y} (u \cdot \langle v, z \rangle_A) = x \cdot \langle y, u \cdot \langle v, z \rangle_A \rangle_A$$

$$= x \cdot (\langle y, u \rangle_A \langle v, z \rangle_A) = (x \cdot \langle y, u \rangle_A) \cdot \langle v, z \rangle_A = \Theta_{x \cdot \langle y, u \rangle_A, v}(z).$$

However, we also see that

$$x \cdot (\langle y, u \rangle_A \langle v, z \rangle_A) = x \cdot (\langle v \cdot \langle u, y \rangle_A, z \rangle_A) = \Theta_{x,v \cdot \langle u, y \rangle_A}(z).$$

Thus, $\Theta_{x,y} \Theta_{u,v} = \Theta_{x \cdot \langle y, u \rangle_A, v} = \Theta_{x, v \cdot \langle u, y \rangle_A}$, which proves (ii).

For any $T \in \mathcal{L}_A(X,Z)$, since $T$ is automatically $A$-linear, we must have

$$T \Theta_{x,y}(w) = T(x \cdot \langle y, w \rangle_A) = (Tx) \cdot \langle y, w \rangle_A = \Theta_{T x, y}(w)$$

for any $w \in Y$. Thus, $T \Theta_{x,y} = \Theta_{T x, y}$, which proves (iii).

Combining (i) and (iii), we see that

$$\Theta_{x,y} T = (T^* \Theta^*_{x,y})^* = (T^* \Theta_{y,x})^* = \Theta_{T^* y, x} = \Theta_{x, T^* y}$$

for any $T \in \mathcal{L}_A(Z,Y)$, which proves (iv).

Motivated by the situation for Hilbert spaces, we define the compact operators to be the closure of the finite-rank operators in the set of adjointable operators.
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**Definition 2.1.10.** Let $X$ and $Y$ be (right) Hilbert $A$-modules. We define

$$\mathcal{K}_A(Y, X) := \overline{\mathcal{F}R_A(Y, X)} = \text{span}\{\Theta_{x,y} : x \in X, y \in Y\}$$

and call elements of this set compact operators from $Y$ to $X$. We write $\mathcal{K}_A(X)$ for $\mathcal{K}_A(X, X)$.

**Remark 2.1.11.** It follows from Proposition 2.1.9 (iii) and (iv) that for any Hilbert $A$-modules $X$ and $Y$,

$$\mathcal{L}_A(X, Y)\mathcal{K}_A(X)\mathcal{L}_A(Y, X) \subseteq \mathcal{K}_A(Y).$$

In particular, $\mathcal{K}_A(X)$ is an ideal of the $C^*$-algebra $\mathcal{L}_A(X)$.

**Remark 2.1.12.** Given any $C^*$-algebra $A$, the map $\Theta_{a,b} \mapsto ab^*$ extends to an isomorphism $\mathcal{K}_A(A_A) \cong A$. Furthermore, $\mathcal{L}_A(A_A) \cong \mathcal{M}(A)$, where $\mathcal{M}(A)$ is the multiplier algebra of $A$ ([61], Theorem 2.47).

**Remark 2.1.13.** Whilst we refer to the elements of $\mathcal{K}_A(Y, X)$ as compact operators, viewed as operators between the Banach spaces $Y$ and $X$ they need not be compact (i.e. the closure of the image of the closed ball in $Y$ need not be a compact subset of $X$). Indeed for any unital $C^*$-algebra $A$, we have $\text{id}_A = \Theta_{1_A, 1_A} \in \mathcal{K}_A(A_A)$, but the identity operator on $A$ (viewed as a Banach space) is compact if and only if $A$ is finite-dimensional.

We are interested in Hilbert modules equipped with a left action by adjointable operators. We call these structures Hilbert bimodules.

**Definition 2.1.14.** Let $A$ and $B$ be $C^*$-algebras. A (right) Hilbert $A$–$B$ bimodule consists of a (right) Hilbert $B$-module $X$ and a $*$-homomorphism $\phi : A \to \mathcal{L}_B(X)$. We usually think of $\phi$ as implementing a left action of $A$ on $X$, and write $a \cdot x$ for $\phi(a)x$. Since each $\phi(a)$ is automatically $B$-linear, we have $a \cdot (x \cdot b) = (a \cdot x) \cdot b$ for each $x \in X$, $a \in A$, and $b \in A$. When $A = B$ we will call $X$ a Hilbert $B$-bimodule.

**Example 2.1.15.** Let $A$ and $B$ be $C^*$-algebras, and $\alpha : A \to B$ a $*$-homomorphism. We define $\phi : A \to \mathcal{L}_B(B_B)$ by $\phi(a)(b) := \alpha(a)b$ for each $a \in A$ and $b \in B$. With this additional structure the Hilbert $B$-module $B_B$ becomes a Hilbert $A$–$B$ bimodule, which we denote by $_AB_B$. When $A = B$ and $\alpha$ is the identity map, we write $B_B$ for this Hilbert $B$-bimodule.

We are also interested in structure-preserving maps between Hilbert bimodules.

**Definition 2.1.16.** We say that two Hilbert $A$–$B$-modules $X$ and $Y$ are isomorphic if there exists an adjointable map $T \in \mathcal{L}_B(X, Y)$, such that
(i) $T$ is surjective;

(ii) $T$ is left $A$-linear, i.e. $T(a \cdot x) = a \cdot T(x)$ for each $a \in A, x \in X$;

(iii) $T$ is inner-product preserving, i.e. $\langle T(x), T(x') \rangle_B = \langle x, x' \rangle_B$ for each $x, x' \in X$.

**Remark 2.1.17.** If $X$ and $Y$ are Hilbert $A$-modules and $T : X \rightarrow Y$ is a surjective inner-product preserving map, then $T$ is necessarily adjointable. To see this, observe that since $T$ is inner-product preserving, it is injective. As $T$ is also surjective, there is a well defined inverse $T^{-1} : Y \rightarrow X$. Hence, for any $x \in X$ and $y \in Y$,

$$\langle Tx, y \rangle_A = \langle Tx, T(T^{-1}y) \rangle_A = \langle x, T^{-1}y \rangle_A,$$

which shows that $T$ is adjointable with adjoint $T^{-1}$.

Earlier we saw that every Hilbert module is (right) nondegenerate. However, (left) nondegeneracy of a Hilbert bimodule is not automatic.

**Definition 2.1.18.** We say a Hilbert $A$–$B$ bimodule $X$ is nondegenerate if

$$X = A \cdot X := \overline{\text{span}\{a \cdot x : a \in A, x \in X\}}.$$

One way of combining two Hilbert bimodules is to take their (balanced) tensor product — a process that generalises taking tensor products of Hilbert spaces.

**Definition 2.1.19** ([40], Proposition 4.5). Let $X$ be a Hilbert $A$–$B$ bimodule and $Y$ be a Hilbert $B$–$C$ bimodule. We form the balanced tensor product $X \otimes_B Y$ as follows. Let $X \otimes Y$ be the algebraic tensor product of $X$ and $Y$ as complex vector spaces. Next, let $X \otimes_B Y$ be the quotient of $X \otimes Y$ by the subspace

$$N := \text{span}\{x \cdot b \otimes y - x \otimes b \cdot y : x \in X, y \in Y, b \in B\}$$

(we write $x \otimes_B y$ for the coset $x \otimes y + N$). Then $X \otimes_B Y$ is an inner-product $C$-module with right action and inner-product given on simple tensors by the formulas

$$(x \otimes_B y) \cdot c := x \otimes_B (y \cdot c)$$

and

$$(x \otimes_B y, w \otimes_B z)_C := \langle y, \langle x, w \rangle_B \cdot z \rangle_C,$$

where $x, w \in X, y, z \in Y, c \in C$. We define $X \otimes_B Y$ to be the completion of $X \otimes_B Y$ with respect to the norm induced by this inner-product. The tensor product $X \otimes_B Y$ also carries a left action of $A$ by adjointable operators — on simple tensors it is given by

$$a \cdot (x \otimes_B y) := (a \cdot x) \otimes_B y,$$
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where \( x \in X, y \in Y, a \in A \). We call the Hilbert \( A-C \) bimodule \( X \otimes_B Y \) the balanced tensor product of \( X \) and \( Y \).

**Example 2.1.20.** Let \( X \) be a Hilbert \( A \)-bimodule. Define \( X^{\otimes 0} := AA \). For each \( n \geq 1 \), we define \( X^{\otimes n} \) inductively by \( X^{\otimes n} := X \otimes_A X^{\otimes (n-1)} \). Then each \( X^{\otimes n} \) has the structure of a Hilbert \( A \)-bimodule.

**Remark 2.1.21.** Let \( X \) be a Hilbert \( A \)-module and \( Y \) a Hilbert \( A-B \) bimodule. For each adjointable operator \( S \in \mathcal{L}_A(X) \), there exists an adjointable operator \( S \otimes_A \text{id}_Y \in \mathcal{L}_A(X \otimes_A Y) \) (with adjoint \( S^* \otimes_A \text{id}_Y \)) such that

\[
(S \otimes_A \text{id}_Y)(x \otimes_A y) = (Sx) \otimes_A y
\]

for each \( x \in X, y \in Y \).

The next result shows that tensoring a compact operator by the identity gives a compact operator on the balanced tensor product provided the action on the second factor is by compacts.

**Proposition 2.1.22 ([40], Proposition 4.7).** Let \( X \) be a Hilbert \( A \)-module and \( Y \) a Hilbert \( A-B \) bimodule. Suppose that \( A \) acts compactly on \( Y \). If \( S \in \mathcal{K}_A(X) \), then \( S \otimes_A \text{id}_Y \in \mathcal{K}_B(X \otimes_A Y) \).

One reason to be interested in tensor products of Hilbert bimodules is the theory of induced representations — given a Hilbert \( A-B \) bimodule we can construct representations of \( A \) from representations of \( B \).

**Proposition 2.1.23 ([61], Proposition 2.66).** Let \( A \) and \( B \) be \( C^* \)-algebras and \( X \) a Hilbert \( A-B \) bimodule. Given a nondegenerate representation \( \pi : B \to \mathcal{B}(\mathcal{H}) \) of \( B \) on a Hilbert space \( \mathcal{H} \), there exists a representation \( X-\text{Ind}^A_B \pi : A \to \mathcal{B}(X \otimes_B \mathcal{H}) \) such that

\[
(X-\text{Ind}^A_B \pi)(a)(x \otimes_B h) = (a \cdot x) \otimes_B h
\]

for each \( a \in A, x \in X, h \in \mathcal{H} \).

### 2.2 \( C^* \)-algebras associated to Hilbert bimodules

Associated to each Hilbert bimodule are a number of \( C^* \)-algebras. To discuss the first of these we need the notion of a Toeplitz representation.

**Definition 2.2.1.** Let \( X \) be a Hilbert \( A \)-bimodule. A Toeplitz representation \((\psi, \pi)\) of \( X \) in a \( C^* \)-algebra \( B \) consists of a linear map \( \psi : X \to B \) and a *-homomorphism \( \pi : A \to B \) satisfying

...
We can construct a Toeplitz representation of a balanced tensor product of Hilbert bimodules from Toeplitz representations of the component bimodules.

**Proposition 2.2.2** ([27], Proposition 1.8). Let \( A \) be a \( \mathcal{C}^* \)-algebra with \( X \) and \( Y \) Hilbert \( A \)-bimodules. If \( (\psi, \pi) \) and \( (\mu, \pi) \) are Toeplitz representations of \( X \) and \( Y \) respectively in a common \( \mathcal{C}^* \)-algebra \( B \), then there exists a Toeplitz representation \( (\psi \otimes_A \mu, \pi) \) of the Hilbert \( A \)-bimodule \( X \otimes_A Y \) in \( B \), such that

\[
(\psi \otimes_A \mu)(x \otimes_A y) = \psi(x)\mu(y)
\]

for each \( x \in X, \ y \in Y \).

**Proof.** We check that there is a well-defined linear map \( \psi \otimes_A \mu : X \otimes_A Y \to B \) satisfying (2.1). Observe that if \( x, w \in X \) and \( y, z \in Y \), then

\[
(\psi(x)\mu(y))^* \psi(w)\mu(z) = \mu(y)^*\psi(x)^*\psi(w)\mu(z) \\
= \mu(y)^*\pi(\langle x, w \rangle_A)\mu(z) \\
= \mu(y)^*\mu(\langle x, w \rangle_A \cdot z) \\
= \pi(\langle x, w \rangle_A \cdot z) \\
= \pi(\langle x \otimes_A y, w \otimes_A z \rangle_A)
\]

since both \( (\psi, \pi) \) and \( (\mu, \pi) \) are Toeplitz representations. Thus, if \( \sum_j x_j \otimes y_j \) is a finite sum of simple tensors in \( X \otimes_A Y \), then

\[
\left\| \sum_j \psi(x_j)\mu(y_j) \right\|_B^2 = \left\| \left( \sum_j \psi(x_j)\mu(y_j) \right)^* \left( \sum_j \psi(x_j)\mu(y_j) \right) \right\|_B \\
\leq \left\| \sum_{j,k} \pi(\langle x_j \otimes_A y_j, x_k \otimes_A y_k \rangle_A) \right\|_B \\
\leq \left\| \sum_{j,k} \langle x_j \otimes_A y_j, x_k \otimes_A y_k \rangle_A \right\|_A \\
= \left\| \sum_j x_j \otimes_A y_j \right\|_{X \otimes_A Y}^2.
\]

In particular, if \( \sum_j x_j \otimes_A y_j = \sum_k x_k' \otimes_A y_k' \) are finite sums of simple tensors in
\( X \otimes_A Y \), then
\[
\left\| \sum_j \psi(x_j) \mu(y_j) - \sum_k \psi(x_k') \mu(y_k') \right\|_B \leq \left\| \sum_j x_j \otimes_A y_j - \sum_k x_k' \otimes_A y_k' \right\|_{X \otimes_A Y} = 0.
\]

Hence, there is a well-defined norm-decreasing map \( \psi \circ A \mu \) satisfying (2.1) on \( \text{span} \{ x \otimes_A y : x \in X, y \in Y \} \), which extends to \( X \otimes_A Y \) by continuity. We now check that \((\psi \circ A \mu, \pi)\) is a Toeplitz representation.

For any \( x \in X \), \( y \in Y \), and \( a \in A \), we see that
\[
(\psi \circ A \mu) (a \cdot (x \otimes_A y)) = \psi(a \cdot x) \circ A \mu(y) = \pi(a) \psi(x) \circ A \mu(y) = \psi(a)(\psi \circ A \mu)(x \otimes_A y)
\]
since \((\psi, \pi)\) satisfies (T1). Thus, \((\psi \circ A \mu, \pi)\) satisfies (T1).

For any \( x \in X \), \( y \in Y \), and \( a \in A \), we see that
\[
(\psi \circ A \mu) ((x \otimes_A y) \cdot a) = (\psi \circ A \mu) (x \otimes_A (y \cdot a)) = \psi(x) \circ A \mu(y \cdot a) = \psi(x) \mu(y) \pi(a) = (\psi \circ A \mu)(x \otimes_A y) \pi(a)
\]
since \((\mu, \pi)\) satisfies (T2). Hence, \((\psi \circ A \mu, \pi)\) satisfies (T2).

From our earlier calculation, we see that if \( x, w \in X \) and \( y, z \in Y \), then
\[
((\psi \circ A \mu)(x \otimes_A y))^* (\psi \circ A \mu)(w \otimes_A z)) = (\psi(x) \mu(y))^* \psi(w) \mu(z) = \pi((x \otimes_A y, w \otimes_A z)_A),
\]
and so \((\psi \circ A \mu, \pi)\) satisfies (T3). We conclude that \((\psi \circ A \mu, \pi)\) is a Toeplitz representation of \( X \otimes_A Y \) in \( B \).

\[ \square \]

Remark 2.2.3. Given any Toeplitz representation \((\psi, \pi)\) of a Hilbert \( A \)-bimodule \( X \) in a \( C^* \)-algebra \( B \), we may apply Proposition 2.2.2 to the tensor powers of \( X \) as defined in Example 2.1.20. We define \( \psi^{\otimes 0} := \pi : X^{\otimes 0} = A \to B \), and inductively define \( \psi^{\otimes n} := \psi \otimes_A \psi^{\otimes n-1} : X^{\otimes n} \to B \). Then \((\psi^{\otimes n}, \pi)\) is a Toeplitz representation of \( X^{\otimes n} \) in \( B \) for each \( n \geq 0 \).

The Toeplitz algebra of a Hilbert bimodule \( X \) is defined to be the \( C^* \)-algebra generated by a universal Toeplitz representation of \( X \). As the name suggests, these \( C^* \)-algebras generalise the classical Toeplitz algebra (the \( C^* \)-algebra generated by
the unilateral shift on the Hilbert space $\ell^2(\mathbb{N})$.

**Theorem 2.2.4.** Let $X$ be a Hilbert $A$-bimodule. Then there exists a $C^*$-algebra $T_X$, which we call the Toeplitz algebra of $X$, and a Toeplitz representation $(i_X, i_A)$ of $X$ in $T_X$ that are universal in the following sense:

(i) $T_X$ is generated by $i_X(X) \cup i_A(A)$;

(ii) given any Toeplitz representation $(\psi, \pi)$ of $X$ in a $C^*$-algebra $B$, there exists a *-homomorphism $\psi \times_T \pi : T_X \to B$ such that $(\psi \times_T \pi) \circ i_X = \psi$ and $(\psi \times_T \pi) \circ i_A = \pi$.

**Proof.** Given any Toeplitz representation $(\psi, \pi)$ of $X$ in a $C^*$-algebra $B$, since $\pi$ is a *-homomorphism, $\|\pi(a)\|_B \leq \|a\|_A$ for each $a \in A$. Moreover, relation (T3) implies that

$$\|\psi(x)\|^2_B = \|\psi(x)^*\psi(x)\|_B = \|\pi(\langle x, x \rangle)\|_B \leq \|\langle x, x \rangle\|_A = \|x\|_X^2$$

for any $x \in X$. Hence the relations (T1), (T2), and (T3) are compact $C^*$-relations in the sense of Loring ([43], Definition 2.3). By ([43], Theorem 2.10), there exists a universal Toeplitz representation $(i_X, i_A)$ of $X$ that generates $T_X$. □

The next result shows that $T_X$ has a nice spanning family.

**Proposition 2.2.5.** Let $X$ be a Hilbert $A$-bimodule and $(\psi, \pi)$ a Toeplitz representation of $X$. Then

$$C^*(\psi(X) \cup \pi(A)) = \text{span} \left\{ \psi^m(x)\psi^n(y)^* : m, n \geq 0, x \in X^\otimes m, y \in X^\otimes n \right\}.$$

In particular, since $T_X$ is generated by $i_X(X) \cup i_A(A)$,

$$T_X = \text{span} \left\{ i_X^m(x)i_A^n(y)^* : m, n \geq 0, x \in X^\otimes m, y \in X^\otimes n \right\}.$$

**Proof.** Firstly, we check that for any $u \in X^\otimes p$ and $v \in X^\otimes q$,

$$\psi^p(u)^*\psi^q(v) \in \text{span} \left\{ \psi^m(v)\psi^n(y)^* : m, n \geq 0, x \in X^\otimes m, y \in X^\otimes n \right\}.$$

By taking adjoints, we may assume without loss of generality that $q \geq p$. We can approximate $v$ by a sum of elementary tensors, and in particular by sums of elements of the form $w \otimes_A z$ where $w \in X^\otimes p$ and $z \in X^\otimes q-p$. Choosing $z' \in X^\otimes q-p$ so that $z = z' \cdot \langle z', z' \rangle_A$ by the Hewitt–Cohen–Blanchard factorisation theorem, we have

$$\psi^p(u)^*\psi^q(w \otimes_A z) = \psi^p(u)^*\psi^p(w)\psi^{q-p}(z' \cdot \langle z', z' \rangle_A)$$

$$= \psi^{q-p}(\langle u, w \rangle_A)\psi^{q-p}(z' \psi^{q-p}(\langle z', z' \rangle_A))$$

$$= \psi^{q-p}(\langle u, w \rangle_A \cdot z')\psi^{q-p}(\langle z', z' \rangle_A)^*.$$
Since $\psi^\otimes q$ is norm-decreasing (see the proof of Theorem 2.2.4), and multiplication in $B$ is continuous, we conclude that

$$\psi^\otimes p(u)^*\psi^\otimes q(v) \in \text{span} \left\{ \psi^\otimes q(v)\psi^\otimes n(y)^* : m, n \geq 0, \ x \in X^\otimes m, \ y \in X^\otimes n \right\}.$$ 

Thus, $\text{span} \left\{ \psi^\otimes m(x)\psi^\otimes n(y)^* : m, n \geq 0, \ x \in X^\otimes m, \ y \in X^\otimes n \right\}$ is closed under taking

linear combinations, products, adjoints, and in norm, and so is a $C^*$-subalgebra of $C^* (\psi(X) \cup \pi(A))$.

Hence, to prove the two sets are equal, it suffices to show that the $C^*$-subalgebra $\text{span} \left\{ \psi^\otimes m(x)\psi^\otimes n(y)^* : m, n \geq 0, \ x \in X^\otimes m, \ y \in X^\otimes n \right\}$ contains all of the generators of $C^* (\psi(X) \cup \pi(A))$. If $a \in A$, then $a$ can be written as $bc$ for some $b, c \in A$, and so

$$\pi(a) = \pi(b)\pi(c) = \psi^\otimes 0(b)\psi^\otimes 0(c)^*.$$ 

If $x \in X$, we can choose $x' \in X$ so that $x = x' \cdot \langle x', x' \rangle_A$ by the Hewitt–Cohen–Blanchard factorisation theorem. Thus,

$$\psi(x) = \psi^\otimes 1(x')\psi^\otimes 0(\langle x', x' \rangle_A).$$ 

Putting all of this together, we conclude that

$$C^* (\psi(X) \cup \pi(A)) = \text{span} \left\{ \psi^\otimes m(x)\psi^\otimes n(y)^* : m, n \geq 0, \ x \in X^\otimes m, \ y \in X^\otimes n \right\}.$$ 

The Toeplitz algebra of a Hilbert bimodule also carries an action of the circle group.

**Proposition 2.2.6.** Let $X$ be a Hilbert $A$-bimodule. Then there exists a strongly continuous gauge action, $\gamma^T : \mathbb{T} \to \text{Aut}(\mathcal{T}_X)$ such that $\gamma_z(i_X(x)) = zi_X(x)$ and $\gamma_z(i_A(a)) = i_A(a)$ for each $z \in \mathbb{T}$, $x \in X$, and $a \in A$.

**Proof.** Fix $z \in \mathbb{T}$. Define $\psi : X \to \mathcal{T}_X$ by $\psi(x) := zi_X(x)$ for each $x \in X$. We claim that $(\psi, i_A)$ is a Toeplitz representation of $X$ in $\mathcal{T}_X$. Observe that for any $a \in A$, $x \in X$, we have

$$\psi(a \cdot x) = zi_X(a \cdot x) = zi_A(a)zi_X(x) = i_A(a)\psi(x).$$ 

Thus, $(\psi, i_A)$ satisfies (T1). Furthermore, for any $a \in A$, $x \in X$, we see that

$$\psi(x \cdot a) = zi_X(x \cdot a) = zi_X(x)i_A(a) = \psi(x)i_A(a),$$

$$\pi(a) = \pi(b)\pi(c) = \psi^\otimes 0(b)\psi^\otimes 0(c)^*.$$
which shows that \((\psi, i_A)\) satisfies (T2). Finally, since \(z \in \mathbb{T}\), we have
\[
\psi(x)^*\psi(y) = (zi_X(x))^* (zi_X(y)) = \overline{z}zi_X(x)^* i_X(y) = |z|^2 i_A(\langle x, y \rangle_A) = i_A(\langle x, y \rangle_A)
\]
for any \(x, y \in X\). Hence, \((\psi, i_A)\) satisfies (T3). The universal property of \(\mathcal{T}_X\) now provides a \(*\)-homomorphism \(\gamma_z := \psi \times \tau i_A : \mathcal{T}_X \to \mathcal{T}_X\) such that \(\gamma_z(i_X(x)) = zi_X(x)\) and \(\gamma_z(i_A(a)) = i_A(a)\) for each \(x \in X\), and \(a \in A\).

Next we check that \(z \mapsto \gamma_z\) is a homomorphism from \(\mathbb{T}\) to \(\text{Aut}(\mathcal{T}_X)\). Observe that if \(z, w \in \mathbb{T}\) and \(x, y \in X\), then \(a \in A\), then
\[
(\gamma_z \circ \gamma_z)(i_X(x)) = \gamma_z(wi_X(x)) = zwi_X(x) = \gamma_{zw}(i_X(x))
\]
and
\[
(\gamma_z \circ \gamma_z)(i_A(a)) = \gamma_z(i_A(a)) = i_A(a) = \gamma_{zw}(i_A(a))
\]
Thus, \(\gamma_z \circ \gamma_z\) and \(\gamma_{zw}\) agree on \(i_X(X) \cup i_A(A)\). Since \(i_X(X) \cup i_A(A)\) generates \(\mathcal{T}_X\) and \(\gamma_z, \gamma_{w}, \gamma_{zw}\) are all \(*\)-homomorphisms, we conclude that \(\gamma_z \circ \gamma_z = \gamma_{zw}\). Furthermore, this implies that
\[
\gamma_z \circ \gamma_z = \gamma_{z^2} = \gamma_1 = \text{id}_{\mathcal{T}_X}
\]
for any \(z \in \mathbb{T}\). Hence, \(\gamma_z \in \text{Aut}(\mathcal{T}_X)\) for each \(z \in \mathbb{T}\). Putting all of this together, we see that \(\mathbb{T} \ni z \mapsto \gamma_z \in \text{Aut}(\mathcal{T}_X)\) is a group homomorphism, which we denote by \(\gamma\).

Finally, we check that \(\gamma\) is strongly continuous. Fix \(a \in A\), \(z \in \mathbb{T}\), and let \(\varepsilon > 0\) be given. By Proposition 2.2.5, we may choose \(b := \sum_{j=1}^n i_X^\otimes m_j (x_j)^\otimes n_j (y_j)^* \in \mathcal{T}_X\) such that \(|a - b| < \frac{\varepsilon}{3}\). Since scalar multiplication is continuous, the map
\[
\mathbb{T} \ni w \mapsto \gamma_w(b) = \sum_{j=1}^n w^{m_j-n_j} i_X^\otimes m_j (x_j)^\otimes n_j (y_j)^*
\]
is continuous. Hence, there exists a \(\delta > 0\) such that, whenever \(w \in \mathbb{T}\) and \(|z - w| < \delta\), it follows that \(|\gamma_w(b) - \gamma_z(b)| < \frac{\varepsilon}{3}\). Since \(\gamma_z, \gamma_w \in \text{Aut}(\mathcal{T}_X)\) are isometric, it follows that, whenever \(w \in \mathbb{T}\) and \(|z - w| < \delta\), we have
\[
|\gamma_z(a) - \gamma_w(a)| \leq |\gamma_w(a - b)| + |\gamma_w(a) - \gamma_z(b)| + |\gamma_z(a - b)| < 3 \left(\frac{\varepsilon}{3}\right) = \varepsilon.
\]
Thus, the map \(\mathbb{T} \ni z \mapsto \gamma_z(a) \in \mathcal{T}_X\) is continuous for each \(a \in A\). We conclude that \(\gamma\) is strongly continuous.

The second \(C^*\)-algebra associated to a Hilbert bimodule that we are interested in is universal for Toeplitz representations satisfying an additional constraint, called Cuntz–Pimsner covariance. Originally formulated by Pimsner for Hilbert bimodules with faithful left actions, the Cuntz–Pimsner algebra \(\mathcal{O}_X\) of a Hilbert bimodule \(X\)
was defined to be the quotient of $\mathcal{T}_X$ by the ideal of generalised compact operators on the Fock space $\mathcal{F}_X$ [53]. Subsequently, Katsura defined what has come to be accepted as the correct notion of a Cuntz–Pimsner algebra for a Hilbert bimodule with non faithful left action [34]. It is Katsura’s definition of Cuntz–Pimsner covariance that we use in this thesis. Before we give the definition, we need some more background material.

**Proposition 2.2.7** ([56], Proposition 8.11). Given a Toeplitz representation $(\psi, \pi)$ of a Hilbert $A$-bimodule $X$ in a $C^*$-algebra $B$, there exists a $\ast$-homomorphism $(\psi, \pi)^{(1)} : \mathcal{K}_A(X) \to B$ such that

$$(\psi, \pi)^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*$$

for each $x, y \in X$.

**Definition 2.2.8.** For an ideal $I$ of a $C^*$-algebra $A$, we define

$$I^\perp := \{ a \in A : ab = 0 \text{ for all } b \in I \}.$$ 

We now present Katsura’s notion of Cuntz–Pimsner covariance.

**Definition 2.2.9.** Let $X$ be a Hilbert $A$-bimodule. We say that a Toeplitz representation $(\psi, \pi)$ of $X$ is Cuntz–Pimsner covariant if $(\psi, \pi)^{(1)}(\phi(a)) = \pi(a)$ for any $a \in \phi^{-1}(\mathcal{K}_A(X)) \cap (\ker(\phi))^\perp$.

The Cuntz–Pimsner algebra of a Hilbert $A$-bimodule $X$ is defined to be the $C^*$-algebra generated by a universal Cuntz–Pimsner covariant Toeplitz representation of $X$. When the left action on $X$ is faithful, this coincides with the notion of Cuntz–Pimsner algebra defined by Pimsner in [53]. In general, Katsura’s Cuntz–Pimsner algebra will be the quotient of $\mathcal{T}_X$ by an ideal smaller than the ideal of generalised compact operators on the Fock space $\mathcal{F}_X$. Katsura’s Cuntz–Pimsner algebra has two notable advantages over Pimsner’s algebra:

(i) $\mathcal{O}_X$ will always contains a faithful copy of $A$;

(ii) A representation of $\mathcal{O}_X$ that is faithful on $A$, will be faithful on the fixed-point algebra for the gauge action $\gamma : \mathbb{T} \to \text{Aut}(\mathcal{O}_X)$.

**Theorem 2.2.10.** Let $X$ be a Hilbert $A$-bimodule. Then there exists a $C^*$-algebra $\mathcal{O}_X$, which we call the Cuntz–Pimsner algebra of $X$, and a Cuntz–Pimsner covariant Toeplitz representation $(j_X, j_A)$ of $X$ in $\mathcal{O}_X$ that are universal in the following sense:

(i) $\mathcal{O}_X$ is generated by $j_X(X) \cup j_A(A)$;
(ii) given any Cuntz–Pimsner covariant Toeplitz representation $\psi : X \to B$ of $X$, there exists a $*$-homomorphism $\psi \times_\mathcal{O} \pi : \mathcal{O}_X \to B$ such that $(\psi \times_\mathcal{O} \pi) \circ j_X = \psi$ and $(\psi \times_\mathcal{O} \pi) \circ j_A = \pi$.

Proof. As in the proof of Theorem 2.2.4, ([43], Theorem 2.10) ensures the existence of a universal Cuntz–Pimsner covariant Toeplitz representation $(j_X, j_A)$ of $X$ that generates $\mathcal{O}_X$. $\square$

Proposition 2.2.11. Let $X$ be a Hilbert $A$-bimodule. Then there exists a strongly continuous gauge action, $\gamma : T \to \text{Aut}(\mathcal{O}_X)$ such that $\gamma_z(j_X(x)) = zj_X(x)$ and $\gamma_z(j_A(a)) = j_A(a)$ for each $z \in T$, $x \in X$, and $a \in A$.

Proof. Fix $z \in T$. Define $\psi : X \to \mathcal{O}_X$ by $\psi(x) := zj_X(x)$ for each $x \in X$. The exact same working as in the proof of Proposition 2.2.6 will be sufficient to prove the result, provided the Toeplitz representation $(\psi, j_A)$ of $X$ in $\mathcal{O}_X$ is Cuntz–Pimsner covariant. Observe that if $x, y \in X$, then

$$(\psi, j_A)^{(1)}(\Theta_{x,y}) = \psi(x)^* \psi(y)^* = zj_X(x)^* (zj_X(y)) = z\overline{z} j_X(x) j_X(y)^* = |z|^2 (j_X, j_A)^{(1)}(\Theta_{x,y}) = (j_X, j_A)^{(1)}(\Theta_{x,y}).$$

Since $(\psi, j_A)^{(1)}$ and $(j_X, j_A)^{(1)}$ are $*$-homomorphisms, $(\psi, j_A)^{(1)} = (j_X, j_A)^{(1)}$. The Cuntz–Pimsner covariance of $(\psi, j_A)$ then follows from the Cuntz–Pimsner covariance of $(j_X, j_A)$. $\square$

2.3 Background material for higher-rank graphs and their $C^*$-algebras

Higher-rank graphs are defined using the language of category theory. To begin, we summarise the relevant background material.

Definition 2.3.1. A small category $\mathcal{C}$ is a sextuple $(\text{Obj}(\mathcal{C}), \text{Hom}(\mathcal{C}), \text{dom}, \text{cod}, \text{id}, \circ)$ consisting of

(i) sets $\text{Obj}(\mathcal{C})$ and $\text{Hom}(\mathcal{C})$, elements of which are called the objects and morphisms of $\mathcal{C}$ respectively;

(ii) the domain and codomain functions $\text{dom}, \text{cod} : \text{Hom}(\mathcal{C}) \to \text{Obj}(\mathcal{C})$;

(iii) the identity function $\text{id} : \text{Obj}(\mathcal{C}) \to \text{Hom}(\mathcal{C})$;

(iv) the composition function $\circ : \text{Hom}(\mathcal{C}) \times_{\text{Obj}(\mathcal{C})} \text{Hom}(\mathcal{C}) \to \text{Hom}(\mathcal{C})$, where

$$\text{Hom}(\mathcal{C}) \times_{\text{Obj}(\mathcal{C})} \text{Hom}(\mathcal{C}) := \{(f, g) \in \text{Hom}(\mathcal{C}) \times \text{Hom}(\mathcal{C}) : \text{dom}(f) = \text{cod}(g)\}$$
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\[ \text{is the set of composable pairs of morphisms in } C, \]

satisfying

(i) \( \text{dom}(\text{id}(x)) = \text{cod}(\text{id}(x)) = x \) for each \( x \in \text{Obj}(C) \);

(ii) \( \text{dom}(f \circ g) = \text{dom}(g) \) and \( \text{cod}(f \circ g) = \text{cod}(f) \) for each composable pair \( (f, g) \in \text{Hom}(C) \times_{\text{Obj}(C)} \text{Hom}(C) \);

(iii) \( \text{id}(\text{cod}(f)) \circ f = f \circ \text{id}(\text{dom}(f)) = f \) for each \( f \in \text{Hom}(C) \);

(iv) \( (f \circ g) \circ h = f \circ (g \circ h) \) whenever \( (f, g), (g, h) \in \text{Hom}(C) \times_{\text{Obj}(C)} \text{Hom}(C) \).

We say that \( C \) is a countable if \( \text{Hom}(C) \) is a countable set.

We tend to think of a category as a collection of vertices (the objects) and arrows between them (the morphisms). An arrow then points from its domain object to its codomain object, and composition of morphisms is concatenation of arrows.

**Example 2.3.2.** A unital semigroup \( S \) may be viewed as a small category \( C \) with one object \( x \) as follows: \( \text{Obj}(C) := \{x\} \), \( \text{Hom}(C) := S \), \( \text{dom}(s) = \text{cod}(s) = x \) for each \( s \in S \), \( \text{id}(x) = e_S \), \( s \circ t := st \) for each \( s, t \in S \).

In this thesis, the most important category arising from a unital semigroup is \( \mathbb{N}^k \) equipped with coordinate-wise addition (where \( \mathbb{N} := \{0, 1, 2, \ldots\} \)).

**Remark 2.3.3.** The identity function for a category \( C \) is injective — if \( \text{id}(x) = \text{id}(y) \) for some \( x, y \in \text{Obj}(C) \), then \( x = \text{dom}(\text{id}(x)) = \text{dom}(\text{id}(y)) = y \). For this reason, we frequently identify an object with its image under the identity function, and view the category as consisting solely of morphisms.

We are also interested in structure preserving maps between categories called (covariant) functors. Informally, viewing categories as collections of vertices and arrows, functors are maps that preserve connectivity.

**Definition 2.3.4.** Let \( C \) and \( D \) be small categories. A (covariant) functor \( F \) from \( C \) to \( D \) is a mapping that

(i) associates to each object \( x \in \text{Obj}(C) \) an object \( F(x) \in \text{Obj}(D) \);

(ii) associates to each morphism \( f \in \text{Hom}(C) \) a morphism \( F(f) \in \text{Hom}(D) \),

such that

(i) \( \text{dom}(F(f)) = F(\text{dom}(f)) \) and \( \text{cod}(F(f)) = F(\text{cod}(f)) \) for each \( f \in \text{Hom}(C) \);

(ii) \( F(\text{id}(x)) = \text{id}(F(x)) \) for each \( x \in \text{Obj}(C) \);
(iii) \( F(f \circ g) = F(f) \circ F(g) \) for each \( (f, g) \in \text{Hom}(C) \times \text{Obj}(C) \text{ Hom}(C) \).

We are now ready to define higher-rank graphs.

**Definition 2.3.5.** A higher-rank graph of rank \( k \) (also known simply as a \( k \)-graph) consists of a countable small category \( \Lambda \) and a functor \( d : \Lambda \to \mathbb{N}^k \) satisfying the following factorisation property: for any \( m, n \in \mathbb{N}^k \) and \( \lambda \in \Lambda \) with \( d(\lambda) = m + n \), there exist unique \( \mu, \nu \in \Lambda \) with \( d(\mu) = m \) and \( d(\nu) = n \) such that \( \lambda = \mu \circ \nu \).

The factorisation property allows us to identify the elements in \( \Lambda \) that have degree zero with the identity morphisms in the category.

**Lemma 2.3.6.** Let \( \Lambda \) be a \( k \)-graph. Then \( d^{-1}(\{0\}) = \{id(x) : x \in \text{Obj}(\Lambda)\} \).

**Proof.** Let \( x \in \text{Obj}(\Lambda) \). As \( d \) is a functor, it follows that 
\[
d(id(x)) = d(id(x) \circ id(x)) = 2d(id(x)).
\]
Hence, \( d(id(x)) = 0 \). Thus, \( \{id(x) : x \in \text{Obj}(\Lambda)\} \subseteq d^{-1}(\{0\}) \). For the reverse inclusion, fix \( \lambda \in d^{-1}(\{0\}) \). Since \( \lambda = id(\text{cod}(\lambda)) \circ \lambda = \lambda \circ id(\text{dom}(\lambda)) \) and \( d(id(\text{cod}(\lambda))) = d(id(\text{dom}(\lambda))) = 0 \) (from the first part of the proof), the factorisation property forces
\[
\lambda = id(\text{cod}(\lambda)) = id(\text{dom}(\lambda)) \in \{id(x) : x \in \text{Obj}(\Lambda)\}.
\]

Next, we fix some notation for higher-rank graphs.

**Notation.** Let \( (\Lambda, d) \) be a \( k \)-graph.

(i) For \( 1 \leq i \leq k \), we write \( e_i \) for the \( i \)th generator of \( \mathbb{N}^k \). For \( n \in \mathbb{N}^k \), we write \( n_i \) for the \( i \)th coordinate of \( n \). We use \( \leq \) for the partial order on \( \mathbb{N}^k \) given by \( m \leq n \iff m_i \leq n_i \) for all \( i \). For any \( m, n \in \mathbb{N}^k \), we write \( m \lor n \) for the coordinate-wise maximum of \( m \) and \( n \). Furthermore, for any finite subset \( E := \{m_1, \ldots, m_n\} \subseteq \mathbb{N}^k \), we write \( \lor E \) for \( m_1 \lor \cdots \lor m_n \) (and define \( \lor \emptyset := 0 \)).

(ii) For each \( \lambda, \mu \in \Lambda \) with \( \text{dom}(\lambda) = \text{cod}(\mu) \), we write \( \lambda \mu \) for \( \lambda \circ \mu \).

(iii) For each \( n \in \mathbb{N}^k \), we define \( \Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\} \).

(iv) For each \( \lambda \in \Lambda \), define \( r(\lambda) := id(\text{cod}(\lambda)) \in \Lambda^0 \) and \( s(\lambda) := id(\text{dom}(\lambda)) \in \Lambda^0 \).

The maps \( r, s : \Lambda \to \Lambda^0 \) are called the range and source maps of \( \Lambda \) respectively.

(v) For any subset \( E \subseteq \Lambda \) and \( \lambda \in \Lambda \), we define \( \lambda E := \{\lambda \mu : \mu \in E, \ s(\lambda) = r(\mu)\} \)
and \( E \lambda := \{\mu \lambda : \mu \in E, \ r(\lambda) = s(\mu)\} \). In particular, for each \( v \in \Lambda^0 \) and \( n \in \mathbb{N}^k \), we have \( v \Lambda^n := \{\lambda \in \Lambda : r(\lambda) = v, \ d(\lambda) = n\} \).
(vi) For each $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $m \leq n \leq d(\lambda)$, the factorisation property in $\Lambda$ implies that there exists unique $\mu, \nu, \eta \in \Lambda$ with $\lambda = \mu \nu \eta$ and $d(\mu) = m$, $d(\nu) = n - m$, $d(\eta) = d(\lambda) - n$. We define $\lambda(0, m) := \mu$, $\lambda(m, n) =: \nu$, $\lambda(n, d(\lambda)) := \eta$.

We visualise a $k$-graph by drawing its 1-skeleton, which is the directed graph with vertex set $\Lambda^0$, edge set $\bigsqcup_{i=1}^k \Lambda^e_i$, and range and source maps inherited from $\Lambda$. Edges of different degrees are distinguished using $k$ different colours. In general a $k$-graph need not be completely determined by its 1-skeleton — there exist 1-skeletons that do no correspond to any $k$-graph, whilst distinct $k$-graphs may have the same 1-skeleton. For more details, see Section 2 of [58].

Before we look at associating $C^*$-algebras to higher-rank graphs, we need to discuss the concept of common extensions and minimal common extensions of paths.

**Definition 2.3.7.** Let $\Lambda$ be a $k$-graph. For $\mu, \nu \in \Lambda$ we write

$$CE(\mu, \nu) := \{ \lambda \in \Lambda : \lambda = \mu \alpha = \nu \beta \text{ for some } \alpha, \beta \in \Lambda \} = \mu \Lambda \cap \nu \Lambda.$$  

We call elements of $CE(\mu, \nu)$ common extensions of $\mu$ and $\nu$. We also define

$$MCE(\mu, \nu) := \{ \lambda \in \Lambda^{d(\mu) \lor d(\nu)} : \lambda = \mu \alpha = \nu \beta \text{ for some } \alpha, \beta \in \Lambda \} = CE(\mu, \nu) \cap \Lambda^{d(\mu) \lor d(\nu)},$$

and call elements of $MCE(\mu, \nu)$ minimal common extensions of $\mu$ and $\nu$. We also write

$$\Lambda_{\min}(\mu, \nu) := \{ (\alpha, \beta) : \mu \alpha = \nu \beta \in MCE(\mu, \nu) \} = \{ (\lambda(d(\mu), d(\mu) \lor d(\nu)), \lambda(d(\nu), d(\mu) \lor d(\nu))) : \lambda \in MCE(\mu, \nu) \}.$$  

Informally speaking, a common extension of two paths $\mu, \nu \in \Lambda$ is another path that starts with both $\mu$ and $\nu$. Since a path starting with $\mu$ must have degree at least that of $\mu$, and a path starting with $\nu$ must have degree at least that of $\nu$, any common extension of $\mu$ and $\nu$ must have degree no less than $d(\mu) \lor d(\nu)$. Common extensions with exactly this degree are called minimal common extensions. Elements in $\Lambda_{\min}(\mu, \nu)$ are then ordered pairs of paths that when appended to $\mu$ and $\nu$ respectively give a minimal common extension of $\mu$ and $\nu$. It follows from the factorisation property that if $\lambda$ is a common extension of the paths $\mu$ and $\nu$, then $\lambda(0, d(\mu) \lor d(\nu))$ is a minimal common extension and

$$(\lambda(d(\mu), d(\mu) \lor d(\nu)), \lambda(d(\nu), d(\mu) \lor d(\nu))) \in \Lambda_{\min}(\mu, \nu).$$
We would like to be able to work with $k$-graphs that have sources or infinite receivers, whilst still ensuring that the associated $C^*$-algebras are spanned by collections of partial isometries of a standard form. To avoid convergence issues (in particular infinite sums of projections), we introduce a restriction on the type of $k$-graphs that we will look at.

**Definition 2.3.8.** We say a $k$-graph $\Lambda$ is finitely aligned if $\Lambda^\min(\mu, \nu)$ is finite (possibly empty) for all $\mu, \nu \in \Lambda$.

We now introduce the notion of a Toeplitz–Cuntz–Krieger family for a finitely aligned $k$-graph.

**Definition 2.3.9.** Let $\Lambda$ be a finitely aligned $k$-graph. A Toeplitz–Cuntz–Krieger $\Lambda$-family is a collection $\{q_\lambda : \lambda \in \Lambda\}$ of elements of a $C^*$-algebra such that

1. (TCK1) $\{q_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections;
2. (TCK2) $q_\mu q_\nu = q_{\mu \nu}$ for all $\mu, \nu \in \Lambda$ with $s(\mu) = r(\nu)$;
3. (TCK3) $q_\mu^* q_\nu = \sum_{(\alpha, \beta) \in \Lambda^\min(\mu, \nu)} q_\alpha q_\beta^*$ for all $\mu, \nu \in \Lambda$, where the empty sum is interpreted as zero.

Next we show that relation (TCK3) ensures that a Toeplitz–Cuntz–Krieger family consists of partial isometries.

**Proposition 2.3.10.** If $\Lambda$ is a finitely aligned $k$-graph and $\{q_\lambda : \lambda \in \Lambda\}$ is a Toeplitz–Cuntz–Krieger $\Lambda$-family, then

$$q_\lambda^* q_\mu = \delta_{\lambda, \mu} q_{s(\lambda)}$$

for each $\lambda, \mu \in \Lambda$ with $d(\lambda) = d(\mu)$.

*Proof.* Let $\lambda, \mu \in \Lambda$ with $d(\lambda) = d(\mu)$. If $\lambda = \mu$, then $\Lambda^\min(\mu, \nu) = \{s(\lambda)\}$ and so $q_\lambda^* q_\mu = q_\lambda^* q_\lambda = q_{s(\lambda)}$ by (TCK3). If $\lambda \neq \mu$, then $\Lambda^\min(\lambda, \mu)$ is empty. To see this, observe that if $(\eta, \nu) \in \Lambda^\min(\lambda, \mu)$, then $d(\eta) = d(\lambda) \lor d(\mu) - d(\lambda) = 0$ and $d(\nu) = d(\lambda) \lor d(\mu) - d(\mu) = 0$. Hence, $\eta = s(\lambda)$ and $\nu = s(\mu)$, but this then forces $\lambda = \lambda s(\lambda) = \mu s(\mu) = \mu$, which is impossible. Thus, (TCK3) tells us that $q_\lambda^* q_\mu = 0$. \qed

Given any finitely aligned $k$-graph, we can always get a Toeplitz–Cuntz–Krieger family in the following way.
Example 2.3.11. Suppose \( \Lambda \) is a finitely aligned \( k \)-graph. Let \( \{ \xi_\mu : \mu \in \Lambda \} \) denote the standard orthonormal basis for \( \ell^2(\Lambda) \). For each \( \lambda \in \Lambda \), there exists \( w_\lambda \in B(\ell^2(\Lambda)) \) such that

\[
 w_\lambda \xi_\mu = \begin{cases} 
 \xi_\lambda \mu & \text{if } r(\mu) = s(\lambda) \\
 0 & \text{otherwise}
\end{cases} \quad \text{for each } \mu \in \Lambda. 
\tag{2.2}
\]

Then \( \{ w_\lambda : \lambda \in \Lambda \} \) is a Toeplitz–Cuntz–Krieger \( \Lambda \)-family in \( B(\ell^2(\Lambda)) \).

**Proof.** Firstly we show that for each \( \lambda \in \Lambda \), there exists \( w_\lambda \in B(\ell^2(\Lambda)) \) satisfying (2.2). Fix \( \lambda \in \Lambda \) and let \( f \in \ell^2(\Lambda) \). Define \( w_\lambda(f) : \Lambda \to \mathbb{C} \) by

\[
 (w_\lambda(f))(\nu) := \begin{cases} 
 f(\tau) & \text{if } \nu = \lambda \tau \text{ for some } \tau \in \Lambda \\
 0 & \text{otherwise}
\end{cases}
\]

for each \( \nu \in \Lambda \). Since

\[
 \sum_{\nu \in \Lambda} |(w_\lambda(f))(\nu)|^2 = \sum_{\tau \in s(\lambda) \Lambda} |f(\tau)|^2 \leq \sum_{\tau \in \Lambda} |f(\tau)|^2 = \|f\|_{\ell^2(\Lambda)}^2,
\]

we see that \( w_\lambda(f) \in \ell^2(\Lambda) \). Thus, \( w_\lambda \in B(\ell^2(\Lambda)) \) with \( \|w_\lambda\|_{B(\ell^2(\Lambda))} \leq 1 \). Observe that for any \( \mu, \nu \in \Lambda \),

\[
 (w_\lambda \xi_\mu)(\nu) = \begin{cases} 
 \xi_\mu(\tau) & \text{if } \nu = \lambda \tau \text{ for some } \tau \in \Lambda \\
 0 & \text{otherwise}
\end{cases}
\]

\[
 = \begin{cases} 
 1 & \text{if } \nu = \lambda \tau \text{ for some } \tau \in \Lambda \text{ and } \mu = \tau \\
 0 & \text{otherwise}.
\end{cases}
\]

Thus, if \( r(\mu) \neq s(\lambda) \), then \( (w_\lambda \xi_\mu)(\nu) = 0 \) for each \( \nu \in \Lambda \). On the other hand, if \( r(\mu) = s(\lambda) \), then

\[
 (w_\lambda \xi_\mu)(\nu) = \begin{cases} 
 1 & \text{if } \nu = \lambda \mu \\
 0 & \text{otherwise}
\end{cases}
\]

\[
 = \xi_{\lambda \mu}(\nu).
\]

Hence,

\[
 w_\lambda \xi_\mu = \begin{cases} 
 \xi_{\lambda \mu} & \text{if } r(\mu) = s(\lambda) \\
 0 & \text{otherwise},
\end{cases}
\]

and so (2.2) holds.
Next, we claim that $w_\lambda^* w_\mu$ is given by

$$w_\lambda^* w_\mu = \begin{cases} \xi_\eta & \text{if } \nu = \lambda \eta \text{ for some } \eta \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

for any $\nu \in \Lambda$. (2.3)

To see this, fix $\mu, \nu \in \Lambda$. If $\nu \not\in \lambda \Lambda$, then

$$\langle w_\lambda^* w_\mu, \xi_\nu \rangle = \begin{cases} \langle \xi_\lambda^\mu, \xi_\nu \rangle & \text{if } r(\mu) = s(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \delta_{\lambda \mu, \nu} & \text{if } r(\mu) = s(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

$$= 0,$$

and so $w_\lambda^* \xi_\nu = 0$. On the other hand, if $\nu \in \lambda \Lambda$ (say $\nu = \lambda \eta$), then

$$\langle w_\lambda^* w_\mu, \xi_\nu \rangle = \begin{cases} \delta_{\lambda \mu, \lambda \eta} & \text{if } r(\mu) = s(\lambda) \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta_{\mu, \eta}$$

$$= \langle \xi_\mu, \xi_\eta \rangle.$$  

Hence, $w_\lambda^* \xi_\nu = \xi_\eta$, which establishes (2.3).

We now check that $\{w_\lambda : \lambda \in \Lambda\}$ satisfies the Toeplitz–Cuntz–Krieger relations. If $v, u \in \Lambda^0$ and $\mu \in \Lambda$, then

$$w_\lambda^* w_v w_\mu = \begin{cases} w_\lambda^* \xi_\mu & \text{if } r(\mu) = u \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \xi_\mu & \text{if } v = r(\mu) = u \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta_{v, u} w_\lambda^* \xi_\mu.$$  

Thus, $w_\lambda^* w_v = \delta_{v, u} w_v$. Hence, $\{w_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections, and so $\{w_\lambda : \lambda \in \Lambda\}$ satisfies (TCK1).
Next, observe that for any \( \mu, \nu, \lambda \in \Lambda \) with \( r(\nu) = s(\mu) \), we have

\[
w_\mu w_\nu \xi_\lambda = \begin{cases} w_\mu \xi_\nu & \text{if } r(\lambda) = s(\nu) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \xi_\mu \nu & \text{if } r(\lambda) = s(\nu) \\ 0 & \text{otherwise} \end{cases} = w_\mu \xi_\lambda.
\]

Thus, \( w_\mu w_\nu = w_\mu \nu \) for any \( \mu, \nu \in \Lambda \) with \( r(\nu) = s(\mu) \), and so \( \{ w_\lambda : \lambda \in \Lambda \} \) satisfies (TCK2).

It remains to check that \( \{ w_\lambda : \lambda \in \Lambda \} \) satisfies (TCK3). Observe that for any \( \mu, \lambda \in \Lambda \), we have

\[
w^*_\mu w_\mu \xi_\lambda = \begin{cases} w^*_\mu \xi_\mu & \text{if } r(\lambda) = s(\mu) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \xi_\lambda & \text{if } r(\lambda) = s(\mu) \\ 0 & \text{otherwise} \end{cases} = w_{s(\mu)} \xi_\lambda.
\]

Thus \( w^*_\mu w_\mu = w_{s(\mu)} \). Now fix \( \mu, \nu, \lambda \in \Lambda \). If there exists \( \eta \in \text{MCE}(\mu, \nu) \) such that \( \lambda \in \eta \Lambda \), then \( \lambda \in \mu \Lambda \cap \nu \Lambda = \text{CE}(\mu, \nu) \) and \( \eta = \lambda(0, d(\eta)) = \lambda(0, d(\mu) \lor d(\nu)) \). Alternatively, if for all \( \eta \in \text{MCE}(\mu, \nu) \) we have \( \lambda \notin \eta \Lambda \), then \( \lambda \notin \mu \Lambda \cap \nu \Lambda \). Thus,

\[
\left( \sum_{\eta \in \text{MCE}(\mu, \nu)} w_\eta w^*_\eta \right) \xi_\lambda = \sum_{\{ \eta \in \text{MCE}(\mu, \nu) : \lambda \in \eta \Lambda \}} \xi_\lambda = \begin{cases} \xi_\lambda & \text{if } \lambda \in \mu \Lambda \cap \nu \Lambda \\ 0 & \text{otherwise} \end{cases} = w^*_\mu w_\mu w^*_\nu w_\nu \xi_\lambda.
\]

Thus,

\[
w^*_\mu w_\mu w^*_\nu w_\nu = \sum_{\eta \in \text{MCE}(\mu, \nu)} w_\eta w^*_\eta.
\]
Hence,

\[ w^*_\mu w_\nu = w_{s(\mu)}^* w_\mu^* w_\nu^* w_{s(\nu)} = w^*_\mu w_\mu^* w_\nu^* w_\nu^* w_\nu^* w_\nu \\
= w^*_\mu \left( \sum_{\eta \in \text{MCE}(\mu, \nu)} w_\eta^* w_\eta^* \right) w_\nu \\
= w^*_\mu \left( \sum_{(\alpha, \beta) \in \Lambda_{\min}(\mu, \nu)} w_{\mu \alpha}^* w_{\nu \beta} \right) w_\nu \\
= \sum_{(\alpha, \beta) \in \Lambda_{\min}(\mu, \nu)} w_{s(\mu)}^* w_\alpha^* \left( w_{s(\nu)}^* w_\beta \right)^* \\
= \sum_{(\alpha, \beta) \in \Lambda_{\min}(\mu, \nu)} w_\alpha^* w_{\beta}^* , \]

and so \( \{w_\lambda : \lambda \in \Lambda\} \) satisfies (TCK3).

The next result shows that for each finitely aligned higher-rank graph there exists a \( C^* \)-algebra generated by a universal Toeplitz–Cuntz–Krieger family.

**Theorem 2.3.12.** Let \( \Lambda \) be a finitely aligned \( k \)-graph. Then there exists a \( C^* \)-algebra \( \mathcal{T}C^*(\Lambda) \), called the Toeplitz–Cuntz–Krieger algebra of \( \Lambda \), and a Toeplitz–Cuntz–Krieger \( \Lambda \)-family \( \{t_\lambda : \lambda \in \Lambda\} \) in \( \mathcal{T}C^*(\Lambda) \) such that

(i) \( \mathcal{T}C^*(\Lambda) \) is generated by \( \{t_\lambda : \lambda \in \Lambda\} \), i.e. \( \mathcal{T}C^*(\Lambda) = C^*(\{t_\lambda : \lambda \in \Lambda\}) \);

(ii) \( \mathcal{T}C^*(\Lambda) \) has the following universal property: given any other Toeplitz–Cuntz–Krieger \( \Lambda \)-family \( \{q_\lambda : \lambda \in \Lambda\} \) in a \( C^* \)-algebra \( B \), there exists a homomorphism \( \pi_q : \mathcal{T}C^*(\Lambda) \to B \) that carries \( t_\lambda \) to \( q_\lambda \) for each \( \lambda \in \Lambda \).

**Proof.** If \( \{q_\lambda : \lambda \in \Lambda\} \) is a Toeplitz–Cuntz–Krieger \( \Lambda \)-family in a \( C^* \)-algebra \( B \), then Proposition 2.3.10 tells us that

\[ \|q_\lambda\|^2_B = \|q_\lambda^* q_\lambda\|_B = \|q_{s(\lambda)}\|_B \]

for any \( \lambda \in \Lambda \). Moreover, since \( q_v \) is a projection for each \( v \in \Lambda^0 \), it follows that \( \|q_\lambda\|_B \leq 1 \) for each \( \lambda \in \Lambda \). Hence, relations (TCK1), (TCK2), and (TCK3) are compact \( C^* \)-relations in the sense of ([43], Definition 2.3), and so ([43], Theorem 2.10) guarantees the existence of a universal Toeplitz–Cuntz–Krieger \( \Lambda \)-family that generates \( \mathcal{T}C^*(\Lambda) \).

**Remark 2.3.13.** We will reserve the notation \( \{t_\lambda : \lambda \in \Lambda\} \) for the universal Toeplitz–Cuntz–Krieger \( \Lambda \)-family. If there is potential confusion over which \( k \)-graph the family \( \{t_\lambda : \lambda \in \Lambda\} \) is universal for, we will write \( t_\lambda^\Lambda \) for \( t_\lambda \).
CHAPTER 2. MOTIVATION FROM HIGHER-RANK GRAPH ALGEBRAS

Remark 2.3.14. Consider the Toeplitz–Cuntz–Krieger Λ-family exhibited in Example 2.3.11. For any \( \lambda \in \Lambda \), we see that \( w_{\lambda} \xi_{s(\lambda)} = \xi_{\lambda} \). Hence, \( w_{\lambda} \neq 0 \). Since the \(*\)-homomorphism \( \pi_{w} : TC^*(\Lambda) \to B(\ell^2(\Lambda)) \) maps \( t_{\lambda} \) to \( w_{\lambda} \) and is norm-decreasing, we conclude that each \( t_{\lambda} \) in the universal Toeplitz–Cuntz–Krieger Λ-family is nonzero.

As expected the Toeplitz–Cuntz–Krieger algebra of a \( k \)-graph is spanned by a collection of partial isometries.

Proposition 2.3.15. Let \( \Lambda \) be a finitely aligned \( k \)-graph and \( \{q_{\lambda} : \lambda \in \Lambda\} \) a Toeplitz–Cuntz–Krieger Λ-family. Then

\[
C^*\{q_{\lambda} : \lambda \in \Lambda\} = \text{span}\{q_{\lambda}q_{\mu}^* : \lambda, \mu \in \Lambda\}.
\]

In particular, since \( TC^*(\Lambda) \) is generated by \( \{t_{\lambda} : \lambda \in \Lambda\} \),

\[
TC^*(\Lambda) = \text{span}\{t_{\lambda}t_{\mu}^* : \lambda, \mu \in \Lambda\}.
\]

Proof. Clearly, \( \text{span}\{q_{\lambda}q_{\mu}^* : \lambda, \mu \in \Lambda\} \) is closed under linear combinations, under adjoints, and in norm. Furthermore, if \( \lambda, \mu, \nu, \eta \in \Lambda \), then (TCK2) and (TCK3) tell us that

\[
(q_{\mu}q_{\nu}^*)(q_{\nu}q_{\eta}^*) = q_{\lambda}(q_{\mu}q_{\nu}q_{\eta}q_{\nu}^*) = q_{\lambda}\left(\sum_{(\alpha, \beta) \in \Lambda_{\min}(\mu, \nu)} q_{\alpha}q_{\beta}^*\right)q_{\eta}^*
\]

\[
= \sum_{(\alpha, \beta) \in \Lambda_{\min}(\mu, \nu)} q_{\lambda}q_{\alpha}(q_{\eta}q_{\beta}^*)
\]

\[
= \sum_{(\alpha, \beta) \in \Lambda_{\min}(\mu, \nu)} q_{\lambda}q_{\alpha}q_{\beta}^*.
\]

Hence, \( \text{span}\{q_{\lambda}q_{\mu}^* : \lambda, \mu \in \Lambda\} \) is closed under taking products as well, and so is a \( C^\ast \)-subalgebra of \( C^*(\{q_{\lambda} : \lambda \in \Lambda\}) \). Observe that for each \( \lambda \in \Lambda \), \( q_{\lambda} = q_{\lambda}q_{s(\lambda)}^* \). Thus, \( \text{span}\{q_{\lambda}q_{\mu}^* : \lambda, \mu \in \Lambda\} \) contains all of the generators of \( C^*(\{q_{\lambda} : \lambda \in \Lambda\}) \), and hence the two sets are equal. \( \square \)

Of course we would like to know when the induced homomorphism from the universal property of the Toeplitz–Cuntz–Krieger algebra is injective. First we need some more definitions.

Definition 2.3.16. Let \( \Lambda \) be a finitely aligned \( k \)-graph and \( q := \{q_{\lambda} : \lambda \in \Lambda\} \) a Toeplitz–Cuntz–Krieger Λ-family. For any \( v \in \Lambda^0 \) and any finite subset \( E \) of \( v\Lambda \), we define

\[
\Delta(q)^E := \prod_{\lambda \in E} (q_v - q_{\lambda}).
\]
**Definition 2.3.17.** Let $\Lambda$ be a $k$-graph, $v \in \Lambda^0$ and $E \subseteq v\Lambda$. We say that $E$ is $v$-exhaustive in $\Lambda$ if, for every $\lambda \in v\Lambda$, there exists $\mu \in E$ such that $\text{MCE}(\lambda, \mu) \neq \emptyset$ (equivalently $\Lambda_{\text{min}}(\lambda, \mu) \neq \emptyset$).

We have the following version of Coburn’s Theorem for Toeplitz–Cuntz–Krieger algebras to tell us when the induced homomorphism of Theorem 2.3.12 is injective.

**Theorem 2.3.18.** Let $\Lambda$ be a finitely aligned $k$-graph and $q := \{q_\lambda : \lambda \in \Lambda\}$ a Toeplitz–Cuntz–Krieger $\Lambda$-family. Then the induced homomorphism $\pi_q$ is injective if and only if $q_v \neq 0$ for each $v \in \Lambda^0$ and $\Delta(q)^E \neq 0$ whenever $E \subseteq v\Lambda \setminus \{v\}$ is finite and $v$-exhaustive in $\Lambda$.

**Proof.** This is a special case of ([57], Theorem 8.1) for finitely aligned product systems of graphs over the semigroup $\mathbb{N}^k$.  

In addition to the Toeplitz–Cuntz–Krieger algebra, there is another $C^*$-algebra with a universal property associated to each finitely aligned higher-rank graph. To discuss this $C^*$-algebra we need the notion of a Cuntz–Krieger family.

**Definition 2.3.19.** Let $\Lambda$ be a finitely aligned $k$-graph. A Cuntz–Krieger $\Lambda$-family is a collection $\{q_\lambda : \lambda \in \Lambda\}$ of elements of a $C^*$-algebra satisfying (TCK1), (TCK2), (TCK3), and

\[(\text{CK}) \prod_{\lambda \in E}(q_v - q_\lambda q_\lambda^*) = 0 \text{ for each } v \in \Lambda^0 \text{ and each finite } v\text{-exhaustive set } E \subseteq v\Lambda.\]

Associated to each finitely aligned higher-rank graph there is a $C^*$-algebra generated by a universal Cuntz–Krieger family.

**Theorem 2.3.20.** Let $\Lambda$ be a finitely aligned $k$-graph. Then there exists a $C^*$-algebra $C^*(\Lambda)$, called the Cuntz–Krieger algebra of $\Lambda$, and a Cuntz–Krieger $\Lambda$-family $\{s_\lambda : \lambda \in \Lambda\}$ in $C^*(\Lambda)$ such that

(i) $C^*(\Lambda)$ is generated by $\{s_\lambda : \lambda \in \Lambda\}$, i.e. $C^*(\Lambda) = C^*\{s_\lambda : \lambda \in \Lambda\}$;

(ii) $C^*(\Lambda)$ has the following universal property: given any other Cuntz–Krieger $\Lambda$-family $\{q_\lambda : \lambda \in \Lambda\}$ in a $C^*$-algebra $B$, there exists a $*$-homomorphism $\pi_q : C^*(\Lambda) \rightarrow B$ that carries $s_\lambda$ to $q_\lambda$ for each $\lambda \in \Lambda$.

**Proof.** As in the proof of Theorem 2.3.12, relations (TCK1), (TCK2), (TCK3), and (CK) are compact $C^*$-relations in the sense of ([43], Definition 2.3). Thus, ([43], Theorem 2.10) ensures the existence of a universal Cuntz–Krieger $\Lambda$-family that generates $C^*(\Lambda)$.  

**Remark 2.3.21.** We will reserve the notation $\{s_\lambda : \lambda \in \Lambda\}$ for the universal Cuntz–Krieger $\Lambda$-family. If there is potential confusion over which $k$-graph the family $\{s_\lambda : \lambda \in \Lambda\}$ is universal for, we will write $s_\lambda^\Lambda$ for $s_\lambda$. 
Remark 2.3.22. It follows from ([59], Proposition 2.12) that $s_v \neq 0$ for each $v \in \Lambda^0$. Proposition 2.3.10 implies that $s_{s(\lambda)} = s^*_\lambda s_\lambda$ for each $\lambda \in \Lambda$, from which it follows that $s_\lambda \neq 0$.

Remark 2.3.23. The universal property of $C^*(\Lambda)$ shows that for each $z \in T^k$ there exists an endomorphism $\gamma_z$ of $C^*(\Lambda)$ such that $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ for each $\lambda \in \Lambda$ (where $z^m := \prod_{i=1}^k z^{m_i}$ for each $m \in \mathbb{N}^k$). It is straightforward to check on generators that $\gamma_z \gamma_w = \gamma_{zw}$ and $\gamma_1 = \text{id}_{C^*(\Lambda)}$. Thus the map $z \mapsto \gamma_z$ is an action of $T^k$ on $C^*(\Lambda)$ by automorphisms, which we denote by $\gamma$ and call the gauge action. We write $\gamma^\Lambda$ for $\gamma$ if there is ambiguity over which $k$-graph we are talking about. An $\xi^3$ argument shows that $\gamma$ is strongly continuous.

We are interested in the gauge action because it helps tell us when representations of $C^*(\Lambda)$ are faithful.

Theorem 2.3.24 ([59], Theorem 4.2). Let $\Lambda$ be a finitely aligned $k$-graph and suppose that $\pi : C^*(\Lambda) \to B$ is a representation of $C^*(\Lambda)$ in a $C^*$-algebra $B$. Suppose that there is a strongly continuous action $\theta : T^k \to \text{Aut}(C^*(\{\pi(s_\lambda) : \lambda \in \Lambda\}))$ such that $\theta_z \circ \pi = \pi \circ \gamma_z$ for each $z \in T^k$. If $\pi(s_v) \neq 0$ for each $v \in \Lambda^0$, then $\pi$ is injective.

2.4 Realising $\mathcal{T}C^*(\Lambda)$ as a Toeplitz algebra

Given a $k$-graph $\Lambda$ (with $k \geq 1$), there exist $k$ $(k-1)$-graphs formed by removing all edges of a fixed degree and restricting the degree functor.

Definition 2.4.1. Let $\Lambda$ be a $k$-graph, with $k \geq 1$. For each $i \in \{1, \ldots, k\}$, we define

$$\Lambda^i := \{\lambda \in \Lambda : d(\lambda)_i = 0\}.$$

In this section we show how, given any $k$-graph $\Lambda$ and any $i \in \{1, \ldots, k\}$, the Toeplitz–Cuntz–Krieger algebra $\mathcal{T}C^*(\Lambda)$ can be realised as the Toeplitz algebra of a Hilbert $\mathcal{T}C^*(\Lambda^i)$-bimodule. We will define the Hilbert $\mathcal{T}C^*(\Lambda^i)$-bimodule that we are interested in to be a certain closed subspace of $\mathcal{T}C^*(\Lambda)$. To equip this set with left and right actions of $\mathcal{T}C^*(\Lambda^i)$ we want a $*$-homomorphism from $\mathcal{T}C^*(\Lambda^i)$ to $\mathcal{T}C^*(\Lambda)$. Moreover, to ensure that we have a $\mathcal{T}C^*(\Lambda^i)$ valued inner-product, we need to know that this homomorphism is injective.

Proposition 2.4.2. Let $\Lambda$ be a finitely aligned $k$-graph, with $k \geq 1$. For each $i \in \{1, \ldots, k\}$, there exists an injective $*$-homomorphism $\phi : \mathcal{T}C^*(\Lambda^i) \to \mathcal{T}C^*(\Lambda)$ such that $\phi(t^\Lambda_\lambda) = t^\Lambda_\lambda$ for each $\lambda \in \Lambda^i$. 
Proof. Clearly, \( \{ t^\lambda : \lambda \in \Lambda^i \} \) satisfies (TCK1) and (TCK2). To see that \( \{ t^\lambda : \lambda \in \Lambda^i \} \) also satisfies (TCK3), it suffices to show that \( \Lambda_{\min} \{ \langle \cdot, \cdot \rangle_{\mathcal{T}C^*(\Lambda^i)} \} \leq 0 \) for any \( \mu, \nu \in \Lambda^i \). To see this, observe that for any \( (\alpha, \beta) \in \Lambda_{\min} \{ \langle \cdot, \cdot \rangle_{\mathcal{T}C^*(\Lambda^i)} \} \) we have

\[
d(\alpha)_i = (d(\mu) \vee d(\nu) - d(\mu))_i = \max\{d(\mu)_i, d(\nu)_i\} - d(\mu)_i = 0
\]

and

\[
d(\beta)_i = (d(\mu) \vee d(\nu) - d(\nu))_i = \max\{d(\mu)_i, d(\nu)_i\} - d(\nu)_i = 0,
\]

and so \( (\alpha, \beta) \in \Lambda_{\min} \{ \langle \cdot, \cdot \rangle_{\mathcal{T}C^*(\Lambda^i)} \} \). Thus, \( \{ t^\lambda : \lambda \in \Lambda^i \} \) is a Toeplitz–Cuntz–Krieger \( \Lambda^i \)-family in \( \mathcal{T}C^*(\Lambda) \), and so by the universal property of \( \mathcal{T}C^*(\Lambda^i) \), there exists a \( * \)-homomorphism \( \phi : \mathcal{T}C^*(\Lambda^i) \rightarrow \mathcal{T}C^*(\Lambda) \) such that \( \phi(t^\lambda) = t^\lambda \) for each \( \lambda \in \Lambda^i \).

We now argue that \( \phi \) is injective. Firstly, we know that \( t^\lambda \neq 0 \) for each \( \lambda \in \Lambda^i \) by Remark 2.3.14. Therefore, by Theorem 2.3.18, it suffices to check that \( \Delta(t^\lambda)^E \neq 0 \) whenever \( \lambda \in \Lambda^i \) and \( E \subseteq v\Lambda^i \setminus \{v\} \) is finite and \( v \)-exhaustive in \( \Lambda^i \).

A simple calculation shows that for each \( \mu \in \Lambda \),

\[
\Delta(w)^E \xi_{\mu} = \begin{cases} 
\xi_{\mu} & \text{if } r(\mu) = v \text{ and } \mu \not\in \lambda \Lambda \text{ for all } \lambda \in E \\
0 & \text{otherwise,}
\end{cases}
\]

where \( \{ w_\lambda : \lambda \in \Lambda \} \) is defined as in Example 2.3.11. Since \( v \not\in E \), \( \Delta(w)^E \xi_v = \xi_v \). As the \( * \)-homomorphism \( \pi_w : \mathcal{T}C^*(\Lambda) \rightarrow \mathcal{B}(\ell^2(\Lambda)) \) carries \( \Delta(t^\lambda)^E \) to \( \Delta(w)^E \) and is norm-decreasing, we conclude that \( \Delta(t^\lambda)^E \neq 0 \), and so \( \phi \) is injective. \( \square \)

Using the injective \( * \)-homomorphism from the previous proposition, we define a collection of Hilbert \( \mathcal{T}C^*(\Lambda^i) \)-bimodules.

**Proposition 2.4.3.** Let \( \Lambda \) be a finitely aligned \( k \)-graph, with \( k \geq 1 \). Fix \( i \in \{1, \ldots, k\} \). For each \( n \geq 0 \), define

\[
X_n := \operatorname{span} \{ t^\lambda t^\mu_* : \lambda, \mu \in \Lambda, \ d(\lambda)_i = n, \ d(\mu)_i = 0 \} \subseteq \mathcal{T}C^*(\Lambda),
\]

taking the closure with respect to the norm on \( \mathcal{T}C^*(\Lambda) \). Then \( X_n \) carries a right action of \( \mathcal{T}C^*(\Lambda^i) \) such that

\[
x \cdot a = x\phi(a)
\]

for each \( x \in X_n \) and \( a \in \mathcal{T}C^*(\Lambda^i) \). There is a \( \mathcal{T}C^*(\Lambda^i) \)-valued inner-product \( \langle \cdot, \cdot \rangle^E_{\mathcal{T}C^*(\Lambda^i)} \) on \( X_n \) such that

\[
\langle x, y \rangle^E_{\mathcal{T}C^*(\Lambda^i)} = \phi^{-1}(x^*y)
\]

for each \( x, y \in X_n \). Then \( X_n \) has the structure of a (right) Hilbert \( \mathcal{T}C^*(\Lambda^i) \)-module. Additionally, there exists a \( * \)-homomorphism \( \psi_n : \mathcal{T}C^*(\Lambda^i) \rightarrow \mathcal{L}_{\mathcal{T}C^*(\Lambda^i)}(X_n) \) such
Recall that
\[ \psi_n(a)(x) = \phi(a)x \]
for each \( x \in X_n \) and \( a \in \mathcal{T}C^*(\Lambda^i) \). Thus, \( X_n \) has the structure of a Hilbert \( \mathcal{T}C^*(\Lambda^i) \)-bimodule.

**Proof.** Firstly, we check that the right action of \( \mathcal{T}C^*(\Lambda^i) \) on \( X_n \) is well-defined. Recall that
\[ \mathcal{T}C^*(\Lambda^i) = \text{span} \left\{ t_{\mu}^{\Lambda} t_{\rho}^{\Lambda^*} : \eta, \rho \in \Lambda^i \right\} \]
and
\[ X_n = \text{span} \left\{ t_{\mu}^{\Lambda} : \lambda, \mu \in \Lambda, \ d(\lambda)_i = n, \ d(\mu)_i = 0 \right\} . \]
Also recall that \( \phi \) is isometric and multiplication is continuous in \( \mathcal{T}C^*(\Lambda) \). Thus, it suffices to show that for any \( \lambda \in \Lambda \) with \( d(\lambda)_i = n \) and \( \eta, \rho, \mu \in \Lambda^i \) we have
\[ t_{\mu}^{\Lambda} t_{\rho}^{\Lambda^*} \phi \left( t_{\eta}^{\Lambda} t_{\rho}^{\Lambda^*} \right) \in X_n. \]
If \( (\alpha, \beta) \in \Lambda_{\min}(\mu, \eta) \), then
\[ d(\lambda \alpha)_i = d(\lambda)_i + d(\alpha)_i = n + d(\alpha)_i = n + \max\{d(\mu)_i, d(\eta)_i\} - d(\mu)_i = n \]
and
\[ d(\rho \beta)_i = d(\rho)_i + d(\beta)_i = \max\{d(\mu)_i, d(\eta)_i\} - d(\eta)_i = 0. \]
So, making use of relation (TCK3), we see that
\[ t_{\mu}^{\Lambda} t_{\rho}^{\Lambda^*} \phi \left( t_{\eta}^{\Lambda} t_{\rho}^{\Lambda^*} \right) = t_{\mu}^{\Lambda} t_{\rho}^{\Lambda^*} t_{\eta}^{\Lambda} t_{\rho}^{\Lambda^*} = \sum_{(\alpha, \beta) \in \Lambda_{\min}(\mu, \eta)} t_{\lambda \alpha}^{\Lambda} t_{\rho \beta}^{\Lambda^*} \in X_n. \]
We now show that there is a \( \mathcal{T}C^*(\Lambda^i) \)-valued inner-product \( \langle \cdot , \cdot \rangle_n^{\mathcal{T}C^*(\Lambda^i)} \) on \( X_n \) such that
\[ \langle x, y \rangle_n^{\mathcal{T}C^*(\Lambda^i)} = \phi^{-1}(x^*y) \]
for each \( x, y \in X_n \). We begin by showing that for each \( x, y \in X_n \), the product \( x^*y \in \phi(\mathcal{T}C^*(\Lambda^i)) = \text{span} \left\{ t_{\eta}^{\Lambda} t_{\rho}^{\Lambda^*} : \eta, \rho \in \Lambda^i \right\} \). Fix \( \lambda, \lambda' \in \Lambda \) with \( d(\lambda)_i = d(\lambda')_i = n \), and \( \mu, \mu' \in \Lambda^i \). If \( (\alpha, \beta) \in \Lambda_{\min}(\lambda, \lambda') \), then
\[ d(\mu \alpha)_i = d(\mu)_i + d(\alpha)_i = d(\alpha)_i = \max\{d(\lambda)_i, d(\lambda')_i\} - d(\lambda)_i = n - n = 0, \]
\[ d(\mu' \beta)_i = d(\mu')_i + d(\beta)_i = d(\beta)_i = \max\{d(\lambda)_i, d(\lambda')_i\} - d(\lambda')_i = n - n = 0. \]
Hence,

\[(t_{\lambda}A_{\mu})^* (t_{\lambda}A_{\mu}) = t_{\mu}A_{\lambda}^* t_{\lambda}A_{\mu}^* = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\lambda')} t_{\mu}^\alpha t_{\lambda}^\beta = \phi(\mathcal{T}C^*(\Lambda)) \, . \]

Since the adjoint and multiplication are both continuous on $\mathcal{T}C^*(\Lambda)$, we conclude that $x^*y \in \phi(\mathcal{T}C^*(\Lambda))$ for each $x, y \in X_n$. Routine calculations show that the map $\langle \cdot, \cdot \rangle_{\mathcal{T}C^*(\Lambda)} : X_n \times X_n \to \mathcal{T}C^*(\Lambda)$, defined by $\langle x, y \rangle_{\mathcal{T}C^*(\Lambda)} := \phi^{-1}(x^*y)$ for each $x, y \in X_n$, is complex linear and $\mathcal{T}C^*(\Lambda)$-linear in its second argument, conjugate symmetric, and positive definite. We also see that the norm $\|\cdot\|_{X_n}$ induced by this inner-product is given by

\[
\|x\|_{X_n}^2 = \|\langle x, x \rangle_{\mathcal{T}C^*(\Lambda)}\|_{\mathcal{T}C^*(\Lambda)} = \|x^*x\|_{\mathcal{T}C^*(\Lambda)} = \|\phi^{-1}(x^*x)\|_{\mathcal{T}C^*(\Lambda)} = \|x\|_{\mathcal{T}C^*(\Lambda)}^2,
\]

for any $x \in X_n$, since $\phi$ is isometric. As $X_n$ is closed in $\mathcal{T}C^*(\Lambda)$ with respect to $\|\cdot\|_{\mathcal{T}C^*(\Lambda)}$ by definition, we see that $X_n$ is complete with respect to the induced norm $\|\cdot\|_{X_n}$, and so $(X_n, \langle \cdot, \cdot \rangle_{\mathcal{T}C^*(\Lambda)})$ is a Hilbert $\mathcal{T}C^*(\Lambda)$-module. Henceforth, we will simply write $\|\cdot\|$ for both $\|\cdot\|_{X_n}$ and $\|\cdot\|_{\mathcal{T}C^*(\Lambda)}$.

It remains to show that the Hilbert $\mathcal{T}C^*(\Lambda)$-module $X_n$ carries a left action of $\mathcal{T}C^*(\Lambda)$ by adjointable operators. In particular, we will show that there exists a *-homomorphism $\psi_n : \mathcal{T}C^*(\Lambda) \to L_{\mathcal{T}C^*(\Lambda)}(X_n)$ such that $\psi_n(a)(x) = \phi(a)x$ for each $x \in X_n$ and $a \in \mathcal{T}C^*(\Lambda)$. We begin by showing that if $x \in X_n$ and $a \in \mathcal{T}C^*(\Lambda)$, then $\phi(a)x \in X_n$. Fix $\lambda \in \Lambda$ with $d(\lambda)_i = n$ and $\eta, \rho, \mu \in \Lambda^i$. If $(\alpha, \beta) \in \Lambda^{\min}(\rho, \lambda)$, then

\[
d(\eta \alpha)_i = d(\eta)_i + d(\alpha)_i = d(\alpha)_i = \max\{d(\rho)_i, d(\lambda)_i\} - d(\rho)_i = d(\lambda)_i - 0 = n
\]

and

\[
d(\mu \beta)_i = d(\mu)_i + d(\beta)_i = d(\beta)_i = \max\{d(\rho)_i, d(\lambda)_i\} - d(\lambda)_i = d(\lambda)_i - d(\lambda)_i = 0.
\]

Thus, an application of relation (TCK3) shows that

\[
\phi(t_{\eta}A_{\rho}) t_{\lambda}A_{\mu}^* = t_{\mu}A_{\lambda}^* t_{\lambda}A_{\mu}^* = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\rho,\lambda)} t_{\eta}^\alpha t_{\lambda}^\beta = \phi(\mathcal{T}C^*(\Lambda)) \in X_n.
\]

By linearity and continuity it follows that $\phi(a)x \in X_n$ for every $a \in \mathcal{T}C^*(\Lambda)$ and $x \in X_n$. We now show that for each $a \in \mathcal{T}C^*(\Lambda)$ the map $\psi_n(a) : X_n \to X_n$ defined by $\psi_n(a)(x) := \phi(a)x$ is adjointable (with adjoint $\psi_n(a^*)$). Since $\phi$ is a
\* \text{-homomorphism, for any } x, y \in X_n, \text{ we have}
\langle \psi_n(a)(x), y \rangle_{T\mathcal{C}^*(\Lambda')} = \phi^{-1} \left( (\phi(a)x)^*y \right) = \phi^{-1} \left( x^*(\phi(a^*)y) \right) = \langle x, \psi_n(a^*)(y) \rangle_{T\mathcal{C}^*(\Lambda')} .

Finally, since \( \phi \) is a \* -homomorphism, it follows that \( \psi_n : T\mathcal{C}^*(\Lambda') \rightarrow \mathcal{L}_{T\mathcal{C}^*(\Lambda')}(X_n) \) is a \* -homomorphism.

Our aim is to show that the Toeplitz algebra of the Hilbert \( T\mathcal{C}^*(\Lambda') \)-bimodule \( X := X_1 \) is isomorphic to the Toeplitz–Cuntz–Krieger algebra of \( \Lambda \). Before we do this we need to analyse the tensor powers of \( X \). Firstly, we need a lemma telling us how, given paths \( \eta, \rho \in \Lambda \), we can factorise elements of \( \Lambda_{\min}(\eta, \rho) \).

\textbf{Lemma 2.4.4.} Let \( \Lambda \) be a finitely aligned \( k \)-graph. For each \( \eta, \rho \in \Lambda \) and \( m \in \mathbb{N}^k \) with \( m \leq d(\rho) \), we have
\[ \Lambda_{\min}(\eta, \rho) = \{ (\alpha_\gamma, \delta) : (\alpha, \beta) \in \Lambda_{\min}(\eta, \rho(0, m)), (\gamma, \delta) \in \Lambda_{\min}(\beta, \rho(m, d(\rho))) \}. \]

\textbf{Proof.} To start we prove that
\[ \{ (\alpha_\gamma, \delta) : (\alpha, \beta) \in \Lambda_{\min}(\eta, \rho(0, m)), (\gamma, \delta) \in \Lambda_{\min}(\beta, \rho(m, d(\rho))) \} \subseteq \Lambda_{\min}(\eta, \rho). \]

Fix \( (\alpha, \beta) \in \Lambda_{\min}(\eta, \rho(0, m)) \) and \( (\gamma, \delta) \in \Lambda_{\min}(\beta, \rho(m, d(\rho))) \). Then
\[ \eta_\alpha_\gamma = \rho(0, m) \beta_\gamma = \rho(0, m) \rho(m, d(\rho)) \delta = \rho_\delta, \]
which shows that \( \eta_\alpha_\gamma = \rho_\delta \in \text{CE}(\eta, \rho) \). Now we show that the common extension \( \eta_\alpha_\gamma = \rho_\delta \) of the paths \( \eta \) and \( \rho \) is minimal by computing the degree of \( \rho_\delta \). Since \( (\gamma, \delta) \in \Lambda_{\min}(\beta, \rho(m, d(\rho))) \), we see that
\[ d(\rho_\delta) = d(\rho(0, m) \rho(m, d(\rho)) \delta) = d(\rho(0, m)) + d(\rho(m, d(\rho)) \delta) = m + d(\beta) \vee d(\rho(m, d(\rho))) = m + d(\beta) \vee (d(\rho) - m). \]

Since \( (\alpha, \beta) \in \Lambda_{\min}(\eta, \rho(0, m)) \), this must be the same as
\[ m + (d(\eta) \vee d(\rho(0, m))) \vee (d(\rho) - m) = m + (d(\eta) \vee m - m) \vee (d(\rho) - m). \]
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Fix $i \in \{1, \ldots, k\}$. If $d(\eta)_i \geq m_i$, then

$$\left( m + (d(\eta) \lor m - m) \lor (d(\rho) - m) \right)_i = m_i + \max\{d(\eta)_i - m_i, d(\rho)_i - m_i\}$$

$$= m_i + \max\{d(\eta)_i, d(\rho)_i\} - m_i$$

$$= \max\{d(\eta)_i, d(\rho)_i\}$$

$$= (d(\eta) \lor d(\rho))_i.$$  

On the other hand, suppose that $d(\eta)_i < m_i$. Using that fact that $d(\eta)_i < m_i \leq d(\rho)_i$ for the penultimate equality, we see that

$$\left( m + (d(\eta) \lor m - m) \lor (d(\rho) - m) \right)_i = m_i + \max\{0, d(\rho)_i - m_i\}$$

$$= m_i + d(\rho)_i - m_i$$

$$= d(\rho)_i$$

$$= \max\{d(\rho)_i, d(\eta)_i\}$$

$$= (d(\eta) \lor d(\rho))_i.$$  

Thus, $d(\rho\delta) = d(\eta) \lor d(\rho)$, and we conclude that $(\alpha\gamma, \delta) \in \Lambda^{\min}(\eta, \rho)$.

Next we check that

$$\Lambda^{\min}(\eta, \rho) \subseteq \{(\alpha\gamma, \delta) : (\alpha, \beta) \in \Lambda^{\min}(\eta, \rho(0, m)), (\gamma, \delta) \in \Lambda^{\min}(\beta, \rho(m, d(\rho)))\}.$$  

Fix $(\lambda, \tau) \in \Lambda^{\min}(\eta, \rho)$. Let $\alpha := \lambda(0, d(\eta) \lor m - d(\eta))$, $\beta := (\rho\tau)(m, d(\eta) \lor m)$, $\gamma := \lambda(d(\eta) \lor m - d(\eta), d(\lambda))$, and $\delta := \tau$. By construction, $(\alpha\gamma, \delta) = (\lambda, \tau)$. Thus it remains to show that $(\alpha, \beta) \in \Lambda^{\min}(\eta, \rho(0, m))$ and $(\gamma, \delta) \in \Lambda^{\min}(\beta, \rho(m, d(\rho)))$.

Since $\eta\lambda = \rho\tau$, we see that

$$\eta\alpha = \eta\lambda(0, d(\eta) \lor m - d(\eta)) = (\eta\lambda)(0, d(\eta) \lor m) = (\rho\tau)(0, d(\eta) \lor m)$$

$$= (\rho\tau)(0, m)(\rho\tau)(m, d(\eta) \lor m).$$  

As $m \leq d(\rho)$, this must be the same as

$$\rho(0, m)(\rho\tau)(m, d(\eta) \lor m) = \rho(0, m)\beta.$$  

Hence, $\eta\alpha = \rho(0, m)\beta \in \text{CE}(\eta, \rho(0, m))$. Since

$$d(\eta\alpha) = d(\eta) + d(\alpha) = d(\eta) + d(\eta) \lor m - d(\eta) = d(\eta) \lor m = d(\eta) \lor d(\rho(0, m))$$
we conclude that \((\alpha, \beta) \in \Lambda^{\min}(\eta, \rho(0, m))\). Since \(d(\eta) \leq d(\eta) \vee m\), we see that

\[
\begin{align*}
\beta \gamma &= (\rho \tau)(m, d(\eta) \vee m) \lambda(d(\eta) \vee m - d(\eta), d(\lambda)) \\
&= (\eta \lambda)(m, d(\eta) \vee m) \lambda(d(\eta) \vee m - d(\eta), d(\lambda)) \\
&= (\eta \lambda)(m, d(\eta) \vee m) (\eta \lambda)(d(\eta) \vee m, d(\eta \lambda)) \\
&= (\eta \lambda)(m, d(\eta \lambda)).
\end{align*}
\]

As \(\eta \lambda = \rho \tau\) and \(m \leq d(\rho)\) this equals

\[
(\rho \tau)(m, d(\rho \tau)) = \rho(m, d(\rho)) \tau = \rho(m, d(\rho)) \delta.
\]

Thus, \(\beta \gamma = \rho(m, d(\rho)) \delta \in \text{CE}(\beta, \rho(m, d(\rho)))\). Since \((\lambda, \tau) \in \Lambda^{\min}(\eta, \rho)\), we have

\[
d(\beta \gamma) = d(\beta) + d(\gamma) = d(\eta) \vee m - m + d(\lambda) - d(\eta) \vee m + d(\eta)
= d(\lambda) + d(\eta) - m
= d(\eta) \vee d(\rho) - d(\eta) + d(\eta) - m
= (d(\eta) \vee d(\rho)) - m.
\]

As \(m \leq d(\rho)\), this is the same as

\[
d(\eta) \vee m \vee d(\rho) - m = (d(\eta) \vee m - m) \vee (d(\rho) - m)
= d(\beta) \vee (d(\rho) - m)
= d(\beta) \vee d(m(m, d(\rho))))
\]

which shows that \((\gamma, \delta) \in \Lambda^{\min}(\beta, \rho(m, d(m)))\).

\[\square\]

**Proposition 2.4.5.** Let \(\Lambda\) be a finitely aligned \(k\)-graph, with \(k \geq 1\). Fix \(i \in \{1, \ldots, k\}\). Define \(X_n\) as in Proposition 2.4.3 and set \(X := X_1\). Then for each \(n \in \mathbb{N} \cup \{0\}\), there exists a Hilbert \(\mathcal{T}C^*(\Lambda^i)\)-bimodule isomorphism \(\Omega_n : X_n \to X^\otimes n\) such that \(\Omega_0 = \phi^{-1}\) and, for \(n \geq 1\),

\[
\Omega_n \left( t^\Lambda_{\lambda, \mu} \right) = t^\Lambda_{\lambda(0, \epsilon_i)} \otimes_{\mathcal{T}C^*(\Lambda^i)} \Omega_{n-1} \left( t^\Lambda_{\lambda(\epsilon_i, d(\lambda))} \right) t^\Lambda_{\mu} \quad (2.4)
\]

for each \(\lambda, \mu \in \Lambda\) with \(d(\lambda)_i = n\) and \(d(\mu)_i = 0\).

**Proof.** Define \(\Omega_0 : X_0 \to X^\otimes 0 = \mathcal{T}C^*(\Lambda^i)\) to be \(\phi^{-1}\). Clearly, \(\Omega_0\) is a Hilbert \(\mathcal{T}C^*(\Lambda^i)\)-bimodule isomorphism. For \(n \geq 1\), we claim that there exists a Hilbert \(\mathcal{T}C^*(\Lambda^i)\)-bimodule isomorphism \(\Omega_n : X_n \to X^\otimes n\) satisfying (2.4). We will define this collection of maps inductively.

Fix \(n \geq 0\) and suppose that \(\Omega_n : X_n \to X^\otimes n\) is a Hilbert \(\mathcal{T}C^*(\Lambda^i)\)-bimodule isomorphism satisfying (2.4). Let \(\lambda, \mu, \nu, \eta \in \Lambda\) with \(d(\lambda)_i = d(\nu)_i = n + 1\) and
\[ d(\mu) = d(\eta) = 0. \] Using the fact that \( \Omega_n \) is left \( TC^*(\Lambda') \)-linear for the second equality, we see that

\[
\langle t^{A}_{\lambda(0,e_i)} \otimes TC^*(\Lambda') \Omega_n \left( t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\eta} \right), t^{A\ast}_{\nu(0,e_i)} \otimes TC^*(\Lambda') \Omega_n \left( t^{A\ast}_{\nu(e_i,d(\lambda))} t^{A\ast}{\eta} \right) \rangle = \\
\langle \Omega_n \left( t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu} \right), \Omega_n \left( t^{A\ast}_{\nu(e_i,d(\lambda))} t^{A\ast}{\eta} \right) \rangle \\
= \langle \Omega_n \left( t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu} \right), \Omega_n \left( t^{A\ast}_{\nu(e_i,d(\lambda))} t^{A\ast}{\eta} \right) \rangle.
\]

Since \( \Omega_n \) is inner-product preserving, this is equal to

\[
\langle t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu}, t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu} \rangle_{TC^*(\Lambda')} = \\
\langle \phi^{-1} \left( t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu} \right) \cdot t^{A\ast}_{\nu(e_i,d(\lambda))} t^{A\ast}{\eta} \rangle_{TC^*(\Lambda')} = \\
\langle t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu}, t^{A\ast}_{\nu(e_i,d(\lambda))} t^{A\ast}{\eta} \rangle_{TC^*(\Lambda')} = \\
\langle t^{A\ast}_\lambda, t^{A\ast}_\nu \rangle_{TC^*(\Lambda')}. \\
\]

Thus, there exists a well-defined norm-decreasing map

\[
\sum a_{\lambda,\mu} t^{A\ast}_{\lambda(0,e_i)} \otimes TC^*(\Lambda') \Omega_n \left( t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu} \right) \\
\text{on span } \{ t^{A\ast}_{\lambda(0,e_i)} : \lambda, \mu \in \Lambda, d(\lambda) = n + 1, d(\mu) = 0 \}, \text{ which extends to } X_{n+1} \text{ by continuity. We denote this extension by } \Omega_{n+1}. \text{ The previous calculation then shows that } \Omega_{n+1} \text{ is inner-product preserving.}
\]

We now show that \( \Omega_{n+1} \) is left \( TC^*(\Lambda') \)-linear. For any \( \lambda, \mu, \nu, \eta \in \Lambda \) with \( d(\lambda) = n + 1 \) and \( d(\mu) = d(\eta) = 0 \), we have

\[
t^{A\ast}_\nu t^{A\ast}_\eta \cdot \Omega_{n+1} \left( t^{A\ast}_{\lambda(0,e_i)} \otimes TC^*(\Lambda') \Omega_n \left( t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu} \right) \right) = \\
t^{A\ast}_\nu t^{A\ast}_\eta \cdot \Omega_{n+1} \left( t^{A\ast}_{\lambda(0,e_i)} \otimes TC^*(\Lambda') \Omega_n \left( t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu} \right) \right) = \\
\sum_{(\alpha,\beta) \in \Lambda^{\min}(\eta,\lambda(0,e_i))} t^{A\ast}_{\nu(0,e_i)} t^{A\ast}_\beta \otimes TC^*(\Lambda') \Omega_n \left( t^{A\ast}_{\lambda(e_i,d(\lambda))} t^{A\ast}{\mu} \right). \tag{2.5}
\]

To simplify this expression, observe that if \( (\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda(0, e_i)) \), then

\[ d(\nu \alpha) = d(\nu) + \max \{ d(\eta), d(\lambda(0, e_i)) \} - d(\eta) = 1 \]

and

\[ d(\beta) = \max \{ d(\eta), d(\lambda(0, e_i)) \} - d(\lambda(0, e_i)) = 0. \]
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Thus, since $\Omega_n$ is left $\mathcal{T}C^*(\Lambda')$-linear, we see that (2.5) is equal to

$$\sum_{(\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda(0, e_i))} t^A_{(\nu, \alpha)(0, e_i)} \otimes_{\mathcal{T}C^*(\Lambda')} t^{A_{\mu}}_{(\nu, \alpha)(e_i, d(\nu))} t^{A_{\ast}}_{(\nu, \alpha)(e_i, d(\nu))} \cdot \Omega_n \left(t^A_{(\nu, \alpha)(e_i, d(\nu))} t^A_{(\nu, \alpha)(e_i, d(\nu))} \right)$$

$$= \sum_{(\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda(0, e_i))} t^A_{(\nu, \alpha)(0, e_i)} \otimes_{\mathcal{T}C^*(\Lambda')} \Omega_n \left(t^A_{(\nu, \alpha)(e_i, d(\nu))} t^{A_{\ast}}_{(\nu, \alpha)(e_i, d(\nu))} \right)$$

$$= \sum_{(\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda(0, e_i))} t^A_{(\nu, \alpha)(0, e_i)} \otimes_{\mathcal{T}C^*(\Lambda')} \Omega_n \left(t^A_{(\nu, \alpha)(e_i, d(\nu))} t^{A_{\ast}}_{(\nu, \alpha)(e_i, d(\nu))} \right).$$

Since $d(\nu) \geq e_i$, we have

$$(\nu \alpha)(0, e_i) = (\nu \alpha \gamma)(0, e_i)$$

and

$$(\nu \alpha)(e_i, d(\nu)) \gamma = (\nu \alpha \gamma)(e_i, d(\nu \gamma)).$$

Assembling these arguments and using Lemma 2.4.4 for the third equality, we have

$$t^A_{(\nu \alpha \gamma) \mu \delta} \cdot \Omega_{n+1} \left(t^{A_{\mu \delta}}_{(\nu \alpha \gamma)} \right) = \sum_{(\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda(0, e_i))} t^A_{(\nu, \alpha)(0, e_i)} \otimes_{\mathcal{T}C^*(\Lambda')} \Omega_n \left(t^A_{(\nu \alpha \gamma)(e_i, d(\nu \gamma))} t^{A_{\mu \delta}}_{(\nu \alpha \gamma)(e_i, d(\nu \gamma))} \right)$$

$$= \sum_{(\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda(0, e_i))} \Omega_{n+1} \left(t^A_{(\nu \gamma \mu \delta)} \right)$$

$$= \sum_{(\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda(0, e_i))} \Omega_{n+1} \left(t^A_{(\nu \gamma \mu \delta)} \right)$$

$$= \sum_{(\alpha, \beta) \in \Lambda^{\min}(\eta, \lambda(0, e_i))} \Omega_{n+1} \left(t^A_{(\nu \gamma \mu \delta)} \right)$$

Since

$$X_{n+1} = \text{span} \left\{ t^A_{(\lambda, \mu)} : \lambda, \mu \in \Lambda, d(\lambda)_i = n + 1, d(\mu)_i = 0 \right\}$$

and

$$\mathcal{T}C^*(\Lambda') = \text{span} \left\{ t^A_{(\nu, \eta)} : \nu, \eta \in \Lambda^i \right\},$$

we conclude, by linearity and continuity, that $\Omega_{n+1}$ is left $\mathcal{T}C^*(\Lambda')$-linear.

Next, we show that $\Omega_{n+1} : X_{n+1} \rightarrow X_{n+1}^{\otimes n+1}$ is surjective. Fix $\lambda, \mu, \nu, \eta \in \Lambda$ with $d(\lambda)_i = 1, d(\nu)_i = n$, and $d(\mu)_i = d(\eta)_i = 0$. Using the left $\mathcal{T}C^*(\Lambda')$-linearity of $\Omega_n$
for the last equality, we see that

\[ \Omega_{n+1} \left( \sum_{(\alpha,\beta) \in \Lambda^{\min}} t_{\lambda \alpha}^{\Lambda} t_{\eta \beta}^{\Lambda*} \right) = \sum_{(\alpha,\beta) \in \Lambda^{\min}} t_{\lambda(0,e_i)}^{\Lambda} \otimes \mathcal{T}C^{*}(\Lambda) \Omega_n \left( t_{\lambda(0,e_i),d(\lambda)}^{\Lambda} t_{\eta \beta}^{\Lambda*} \right) = \sum_{(\alpha,\beta) \in \Lambda^{\min}} t_{\lambda(0,e_i)}^{\Lambda} \otimes \mathcal{T}C^{*}(\Lambda) \Omega_n \left( t_{\lambda(0,e_i),d(\lambda)}^{\Lambda} t_{\eta \beta}^{\Lambda*} \right) = t_{\lambda(0,e_i)}^{\Lambda} \otimes \mathcal{T}C^{*}(\Lambda) \Omega_n \left( t_{\lambda(0,e_i),d(\lambda)}^{\Lambda} t_{\eta \beta}^{\Lambda*} \right) = t_{\lambda(0,e_i)}^{\Lambda} \otimes \mathcal{T}C^{*}(\Lambda) \Omega_n \left( t_{\lambda(0,e_i),d(\lambda)}^{\Lambda} t_{\eta \beta}^{\Lambda*} \right) \in X \otimes \mathcal{T}C^{*}(\Lambda) X^{\otimes n}. \]

Since \( X_m = \sum_{\lambda,\mu} \{ t_{\lambda(0,e_i)}^{\Lambda} \otimes \mathcal{T}C^{*}(\Lambda) \Omega_n \left( t_{\lambda(0,e_i),d(\lambda)}^{\Lambda} t_{\eta \beta}^{\Lambda*} \right) \} \) for each \( m \geq 0 \) and the map \( \Omega_n : X_n \to X^{\otimes n} \) is surjective, we conclude that \( \Omega_{n+1} \) is surjective.

We have now shown that \( \Omega_{n+1} \) is inner-product preserving and surjective. Remark 2.1.17 tells us that \( \Omega_{n+1} \) is adjointable. Since \( \Omega_{n+1} \) is also left \( \mathcal{T}C^{*}(\Lambda) \)-linear, we conclude that \( \Omega_{n+1} \) is a \( \mathcal{T}C^{*}(\Lambda) \)-bimodule isomorphism from \( X_{n+1} \) to \( X^{\otimes n+1} \) as required.

We now work towards showing that the Toeplitz algebra of the Hilbert \( \mathcal{T}C^{*}(\Lambda) \)-bimodule \( X \) is isomorphic to the Toeplitz–Cuntz–Krieger algebra of \( \Lambda \). The idea is to use the universal properties of \( \mathcal{T}_X \) and \( \mathcal{T}C^{*}(\Lambda) \) to get \( * \)-homomorphisms between the two \( C^{*} \)-algebras, and then argue that these maps are mutually inverse. Firstly, we need a result telling us how the Hilbert \( \mathcal{T}C^{*}(\Lambda) \)-bimodule isomorphisms from Proposition 2.4.5 interact with the tensor product.

**Lemma 2.4.6.** Let \( \{ \Omega_n : n \geq 0 \} \) be the collection of Hilbert \( \mathcal{T}C^{*}(\Lambda) \)-bimodule isomorphisms defined in Proposition 2.4.5. Then for any \( m, n \geq 0 \) and \( x \in X_m, y \in X_n \),

\[ \Omega_m (x) \otimes \mathcal{T}C^{*}(\Lambda) \Omega_n (y) = \Omega_{m+n} (xy). \] (2.6)

In particular, if \( \lambda, \mu \in \Lambda \) with \( r(\mu) = s(\lambda) \), then

\[ \Omega_{d(\lambda)} \left( t_{\lambda(0,e_i)}^{\Lambda} \right) \otimes \mathcal{T}C^{*}(\Lambda) \Omega_{d(\mu)} \left( t_{\mu(0,e_i)}^{\Lambda} \right) = \Omega_{d(\lambda) + d(\mu)} \left( t_{\lambda(0,e_i)}^{\Lambda} t_{\mu(0,e_i)}^{\Lambda} \right). \]

**Proof.** We will use induction on \( m \). The \( m = 0 \) case is equivalent to the left \( \mathcal{T}C^{*}(\Lambda) \)-linearity of \( \Omega_n \), which we proved in Proposition 2.4.5. Now suppose that (2.6) holds for some \( m \geq 0 \). Let \( n \geq 0 \) and fix \( \lambda, \mu, \nu, \tau \in \Lambda \) with \( d(\lambda)_i = m + 1, d(\nu)_i = n \), and


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\( d(\mu)_i = d(\tau)_i = 0 \). Applying the inductive hypothesis, we see that

\[
\Omega_{m+1} \left( t_{\lambda}^A t_{\mu}^{A^*} \right) \otimes_{\mathcal{T}C^*(\Lambda')} \Omega_n \left( t_{\nu}^A t_{\tau}^{A^*} \right)
= t_{\lambda(0,e_i)}^A \otimes_{\mathcal{T}C^*(\Lambda')} \Omega_m \left( t_{\lambda(e_i,d(\lambda))}^A t_{\mu}^{A^*} \right) \otimes_{\mathcal{T}C^*(\Lambda')} \Omega_n \left( t_{\nu}^A t_{\tau}^{A^*} \right)
= t_{\lambda(0,e_i)}^A \otimes_{\mathcal{T}C^*(\Lambda')} \Omega_{m+n} \left( t_{\lambda(e_i,d(\lambda))}^A t_{\mu}^{A^*} t_{\nu}^A t_{\tau}^{A^*} \right) \tag{2.7}
\]

where the last equality follows from applying relation (TCK3) in \( \mathcal{T}C^*(\Lambda) \). Since \( d(\lambda) \geq (m+1)e_i \geq e_i \), (2.7) must equal

\[
\sum_{(\alpha,\beta) \in \Lambda^{min}(\mu,\nu)} t_{(\lambda\alpha)^{(0,e_i)}}^A \otimes_{\mathcal{T}C^*(\Lambda')} \Omega_{m+n} \left( t_{(\lambda\alpha)(e_i,d(\lambda\alpha))}^A t_{\tau\beta}^{A^*} \right)
= \sum_{(\alpha,\beta) \in \Lambda^{min}(\mu,\nu)} \Omega_{m+n+1} \left( t_{\lambda\alpha}^A t_{\tau\beta}^{A^*} \right)
= \Omega_{m+n+1} \left( t_{\lambda\mu}^A t_{\nu}^A t_{\tau}^{A^*} \right).
\]

Since \( X_j = \text{span}\{ t_{\lambda\mu}^A t_{\nu}^{A^*} : \lambda, \mu \in \Lambda, d(\lambda)_i = j, d(\mu)_i = 0 \} \) for each \( j \geq 0 \), we conclude that (2.6) holds for \( m+1 \) as well.

We now get a *-homomorphism from \( \mathcal{T}C^*(\Lambda) \) to \( \mathcal{T}_X \) by exhibiting a Toeplitz–Cuntz–Krieger \( \Lambda \)-family in \( \mathcal{T}_X \).

**Proposition 2.4.7.** Let \( \Lambda \) be a finitely aligned \( k \)-graph, with \( k \geq 1 \). Fix \( i \in \{1,\ldots,k\} \). Define \( X_n \) as in Proposition 2.4.3 and set \( X := X_1 \). Consider the collection of Hilbert \( \mathcal{T}C^*(\Lambda^i) \)-bimodule isomorphisms \( \{ \Omega_n : n \geq 0 \} \) defined in Proposition 2.4.5. For each \( \lambda \in \Lambda \), define \( u_\lambda \in \mathcal{T}_X \) by

\[
u_\lambda := i_X^{\otimes d(\lambda)_i} \left( \Omega_{d(\lambda)_i} \left( t_{\lambda}^A \right) \right).
\]

Then \( \{ u_\lambda : \lambda \in \Lambda \} \) is a Toeplitz–Cuntz–Krieger \( \Lambda \)-family in \( \mathcal{T}_X \). Hence, there exists a *-homomorphism \( \pi_u : \mathcal{T}C^*(\Lambda) \to \mathcal{T}_X \) such that \( \pi_u \left( t_{\lambda}^A \right) = u_\lambda \) for each \( \lambda \in \Lambda \).

**Proof.** Firstly, we check that \( \{ u_\lambda : \lambda \in \Lambda \} \) satisfies (TCK1). For any \( v \in \Lambda^0 \), we see that

\[
v := i_X^{\otimes d(v)} \left( \Omega_{d(v)_i} \left( t_{\lambda}^A \right) \right) = i_X^{\otimes \emptyset} \left( \Omega_0 \left( t_{\lambda}^A \right) \right) = i_{\mathcal{T}C^*(\Lambda^i)} (\phi^{-1 \rightarrow} \left( t_{\lambda}^A \right)) = i_{\mathcal{T}C^*(\Lambda^i)} \left( t_{\nu}^A \right).
\]

Since \( i_{\mathcal{T}C^*(\Lambda^i)} \) is a *-homomorphism and \( \left\{ t_{\nu}^A : v \in \Lambda^0 \right\} \) is a collection of mutually orthogonal projections, it follows that \( \{ u_v : v \in \Lambda^0 \} \) consists of mutually orthogonal projections.
Next we check that \( \{ u_\lambda : \lambda \in \Lambda \} \) satisfies (TCK2). Fix \( \lambda, \mu \in \Lambda \) with \( r(\mu) = s(\lambda) \). Making use of Lemma 2.4.6, we see that

\[
u_\lambda u_\mu = \iota_X^{\otimes d(\lambda)} \left( \Omega_{d(\lambda)} \left( t^\Lambda_{\lambda} \right) \right) \iota_X^{\otimes d(\mu)} \left( \Omega_{d(\mu)} \left( t^\Lambda_{\mu} \right) \right) = \iota_X^{\otimes (d(\lambda) + d(\mu))} \left( \Omega_{d(\lambda)} \left( t^\Lambda_{\lambda} \right) \otimes TC^*(\Lambda^i) \Omega_{d(\mu)} \left( t^\Lambda_{\mu} \right) \right) = \iota_X^{\otimes d(\mu)} \left( \Omega_{d(\mu)} \left( t^\Lambda_{\mu} \right) \right) = u_{\mu} \lambda.
\]

Finally we check that \( \{ u_\lambda : \lambda \in \Lambda \} \) satisfies (TCK3). Let \( \lambda, \mu \in \Lambda \). Suppose that \( d(\mu)_i \geq d(\lambda)_i \). As

\[
\Omega_{d(\mu)} \left( t^\Lambda_{\mu} \right) = \Omega_{d(\lambda)} \left( t^\Lambda_{\mu(0,d(\lambda)_i,e_i)} \right) \otimes TC^*(\Lambda^i) \Omega_{d(\mu)} \left( t^\Lambda_{\lambda} \right)
\]

by Lemma 2.4.6, it follows that

\[
u_\lambda u_\mu^* = \iota_X^{\otimes d(\lambda)} \left( \Omega_{d(\lambda)} \left( t^\Lambda_{\lambda} \right) \right)^* \iota_X^{\otimes d(\mu)} \left( \Omega_{d(\mu)} \left( t^\Lambda_{\mu} \right) \right) = \iota_X^{\otimes d(\mu)_i - d(\lambda)_i} \left( \left( \Omega_{d(\lambda)} \left( t^\Lambda_{\lambda} \right), \Omega_{d(\lambda)} \left( t^\Lambda_{\mu(0,d(\lambda)_i,e_i)} \right) \right) \right) \cdot \Omega_{d(\mu)_i - d(\lambda)_i} \left( t^\Lambda_{\mu(0,d(\lambda)_i,e_i,d(\mu))} \right).
\]

As \( \Omega_{d(\lambda)} \) preserves inner-products and \( \Omega_{d(\mu)_i - d(\lambda)_i} \) is left \( TC^*(\Lambda^i) \)-linear, this must be the same as

\[
\iota_X^{\otimes d(\mu)_i - d(\lambda)_i} \left( \Omega_{d(\mu)_i - d(\lambda)_i} \left( \left( t^\Lambda_{\lambda}, t^\Lambda_{\mu(0,d(\lambda)_i,e_i)} \right) \right) \right) \cdot \Omega_{d(\mu)_i - d(\lambda)_i} \left( t^\Lambda_{\mu(0,d(\lambda)_i,e_i,d(\mu))} \right) = \iota_X^{\otimes d(\mu)_i - d(\lambda)_i} \left( \Omega_{d(\mu)_i - d(\lambda)_i} \left( \sum_{(\alpha, \beta) \in \Lambda_{\min}(\lambda, \mu)} t^\Lambda_{\alpha} t^\Lambda_{\beta} \right) \right),
\]

where the last equality comes from the fact that \( \{ t^\Lambda_{\lambda} : \lambda \in \Lambda \} \) satisfies (TCK3). Moreover, if \( (\alpha, \beta) \in \Lambda_{\min}(\lambda, \mu) \), then

\[
d(\alpha)_i = \max \{ d(\lambda)_i, d(\mu)_i \} - d(\lambda)_i = d(\mu)_i - d(\lambda)_i
\]

and

\[
d(\alpha)_i = \max \{ d(\lambda)_i, d(\mu)_i \} - d(\mu)_i = 0.
\]
Thus, as $\Omega_{d(\mu)_{i} - d(\lambda)_{i}}$ is right $T C^{\ast}(\Lambda)^{\ast}$-linear,

$$u_{\lambda}^{\ast} u_{\mu} = \sum_{(\alpha, \beta) \in \Lambda^{\min}} i_{X}^{\otimes d(\mu)_{i} - d(\lambda)_{i}} \left( \Omega_{d(\mu)_{i} - d(\lambda)_{i}} \left( t_{\alpha}^{L} \right) \cdot t_{\beta}^{L^{\ast}} \right)$$

$$= \sum_{(\alpha, \beta) \in \Lambda^{\min}} \Omega_{d(\alpha)_{i}} \left( t_{\alpha}^{L} \right) \cdot i_{X}^{\otimes d(\beta)_{i}} \left( \Omega_{d(\beta)_{i}} \left( t_{\beta}^{L^{\ast}} \right) \right)^{\ast}$$

$$= \sum_{(\alpha, \beta) \in \Lambda^{\min}} u_{\alpha} u_{\beta}^{\ast}.$$ 

If $d(\lambda)_{i} \geq d(\mu)_{i}$, we can apply the previous working to $(u_{\lambda}^{\ast} u_{\mu})^{\ast} = u_{\mu}^{\ast} u_{\lambda}$. This completes the proof that $\{ u_{\lambda} : \lambda \in \Lambda \}$ satisfies (TCK3). Hence, $\{ u_{\lambda} : \lambda \in \Lambda \}$ is a Toeplitz–Cuntz–Krieger $\Lambda$-family in $T_{X}$. The universal property of $T C^{\ast}(\Lambda)$ induces a $\ast$-homomorphism $\pi_{u} : T C^{\ast}(\Lambda) \to T_{X}$ such that $\pi_{u} \left( t_{\lambda}^{L} \right) = u_{\lambda}$ for each $\lambda \in \Lambda$. \hfill \qed

It is considerably easier to get a $\ast$-homomorphism from $T_{X}$ to $T C^{\ast}(\Lambda)$. Once we have it, there is still some work left to show that it is the inverse of the $\ast$-homomorphism $\pi_{u} : T C^{\ast}(\Lambda) \to T_{X}$ from Proposition 2.4.7.

**Theorem 2.4.8.** Let $\Lambda$ be a finitely aligned $k$-graph, with $k \geq 1$. Fix $i \in \{ 1, \ldots, k \}$. Define $X_{n}$ as in Proposition 2.4.3 and set $X := X_{1}$. Let $\iota : X \to T C^{\ast}(\Lambda)$ denote the inclusion map. Then $(\iota, \phi)$ is a Toeplitz representation of $X$ in $T C^{\ast}(\Lambda)$, and hence, by the universal property of $T_{X}$, there exists a $\ast$-homomorphism $\iota \times_{T} \phi : T_{X} \to T C^{\ast}(\Lambda)$ such that $(\iota \times_{T} \phi) \circ i_{X} = \iota$ and $(\iota \times_{T} \phi) \circ i_{T C^{\ast}(\Lambda^{\ast})} = \phi$. Moreover, $\pi_{u}$ and $\iota \times_{T} \phi$ are mutually inverse. Thus, $T C^{\ast}(\Lambda) \cong T_{X}$.

**Proof.** It is elementary to check that $(\iota, \phi)$ is a Toeplitz representation of $X$ in $T C^{\ast}(\Lambda)$. Observe that for any $x \in X$ and $a \in A$, we have

$$\iota(a \cdot x) = \iota(a) \cdot x = \phi(a) x = \phi(a) \iota(x)$$

and

$$\iota(x \cdot a) = x \cdot a = x \phi(a) = \iota(x) \phi(a),$$

which proves that $(\iota, \phi)$ satisfies (T1) and (T2). If $x, y \in X$, then

$$\iota(x)^{\ast} \iota(y) = x^{\ast} y = \phi \left( \phi^{-1}(x^{\ast} y) \right) = \phi \left( \langle x, y \rangle_{T C^{\ast}(\Lambda^{\ast})}^{1} \right),$$

and so $(\iota, \phi)$ satisfies (T3).

It remains to check that $\iota \times_{T} \phi$ and $\pi_{u}$ are mutually inverse. Fix $\lambda \in \Lambda$. If
\[ d(\lambda)_i = 0, \text{ then} \]
\[
((\iota \times T \phi) \circ \pi_u)(t^A_\lambda) = (\iota \times T \phi)(u_\lambda) = (\iota \times T \phi)\left(i_X^{\otimes d(\lambda)_i} (\Omega_{d(\lambda)_i}, (t^A_\lambda))\right) \\
= (\iota \times T \phi)\left(i_{TC^*(\Lambda^i)} (t^A_\lambda)\right) = \phi(t^{A_i}_\lambda) = t^{A_i}_\lambda.
\]

If \( d(\lambda)_i = 1, \) then
\[
((\iota \times T \phi) \circ \pi_u)(t^A_\lambda) = (\iota \times T \phi)(u_\lambda) = (\iota \times T \phi)\left(i_X^{\otimes d(\lambda)_i} (\Omega_{d(\lambda)_i}, (t^A_\lambda))\right) \\
= (\iota \times T \phi)(i_X (t^A_\lambda)) = \iota (t^A_\lambda) = t^A_\lambda.
\]

If \( d(\lambda)_i \geq 2, \) then
\[
((\iota \times T \phi) \circ \pi_u)(t^A_\lambda) = (\iota \times T \phi)(u_\lambda) \\
= (\iota \times T \phi)\left(i_X^{\otimes d(\lambda)_i} (\Omega_{d(\lambda)_i}, (t^A_\lambda))\right) \\
= (\iota \times T \phi)\left(i_{TC^*(\Lambda^i)} (t^A_\lambda) \otimes TC^*(\Lambda^i) \cdots \otimes TC^*(\Lambda^i) \otimes (\lambda_{(d(\lambda)_i-1)e_i,d(\lambda)_i)}\right) \\
= (\iota \times T \phi) (i_X (t^A_{\lambda_{(0,e_i)})} \cdots i_X (t^A_{\lambda_{(d(\lambda)_i-1)e_i,d(\lambda)_i)}) \\
= \iota (t^A_{\lambda_{(0,e_i)})} \cdots \iota (t^A_{\lambda_{(d(\lambda)_i-1)e_i,d(\lambda)_i)}) \\
= t^A_{\lambda_{((0,e_i))})} \cdots t^A_{\lambda_{((d(\lambda)_i-1)e_i,d(\lambda)_i)}} \\
= t^A_\lambda.
\]

Since \( TC^*(\Lambda) \) is generated by \( \{t^A_\lambda : \lambda \in \Lambda\} \), we conclude that \( (\iota \times T \phi) \circ \pi_u = \text{id}_{TC^*(\Lambda)}. \)

We now show that \( \pi_u \circ (\iota \times T \phi) = \text{id}_{T_X}. \) If \( \mu \in \Lambda^i \), then
\[
(\pi_u \circ (\iota \times T \phi))(i_{TC^*(\Lambda^i)} (t^A_{\mu})) = \pi_u \phi(t^{A_i}_{\mu}) = \pi_u (t^A_{\mu}) = u_\mu \\
= i_X^{\otimes d(\lambda)_i} (\Omega_{d(\lambda)_i}, (t^A_{\mu})) = i_{TC^*(\Lambda^i)} (t^A_{\mu}).
\]

For any \( \lambda \in \Lambda \) with \( d(\lambda)_i = 1 \) and \( \mu \in \Lambda^i \), we see that
\[
(\pi_u \circ (\iota \times T \phi))(i_X (t^A_{\lambda t^{A_i}})) = \pi_u (\iota (t^A_{\lambda t^{A_i}})) \\
= \pi_u (t^A_{\lambda t^{A_i}}) \\
= u_\lambda u^*_\mu \\
= i_X^{\otimes d(\lambda)_i} (\Omega_{d(\lambda)_i}, (t^A_{\lambda})) i_X^{\otimes d(\mu)_i} (\Omega_{d(\mu)_i}, (t^A_{\mu}))^* \\
= i_X (t^A_{\lambda}) i_{TC^*(\Lambda^i)} (t^A_{\mu})^* \\
= i_X (t^A_{\lambda t^{A_i}}).
\]

Since \( T_X \) is generated by \( i_X(X) \cup i_{TC^*(\Lambda^i)}(TC^*(\Lambda^i)) \), whilst \( TC^*(\Lambda^i) \) is generated by \( \{t^{A_i}_\mu : \mu \in \Lambda^i\} \) and \( X = \text{span} \{t^{A_i t^{A_i}}_{\lambda} : \lambda, \mu \in \Lambda, \ d(\lambda)_i = 1, d(\mu)_i = 0\} \), we conclude
that \( \pi_u \circ (\iota \times_T \phi) = \text{id}_{T_X} \). Thus, \( \iota \times_T \phi \) and \( \pi_u \) are mutually inverse.

### 2.5 A \( KK \)-equivalence between \( \mathcal{T}C^*(\Lambda) \) and \( c_0(\Lambda^0) \)

In this section we use Theorem 2.4.8 to show that the Toeplitz–Cuntz–Krieger algebra of a finitely aligned \( k \)-graph is \( KK \)-equivalent to the continuous functions that vanish at infinity on its vertex set. As a corollary, we compute the \( K \)-theory of Toeplitz–Cuntz–Krieger algebras associated to finitely aligned \( k \)-graphs. As a first step, we look at the \( C^* \)-algebras associated to a zero graph.

**Proposition 2.5.1.** Let \( \Lambda \) be a 0-graph. Then \( C^*(\Lambda) \cong \mathcal{T}C^*(\Lambda) \cong c_0(\Lambda^0) \).

**Proof.** Firstly, since \( \Lambda \) contains nothing apart from vertices it is finitely aligned and \( \text{FE}(\Lambda) = \emptyset \). For \( v \in \Lambda^0 = \Lambda \), define \( w_v \in c_0(\Lambda) \) by \( w_v(u) := \delta_{v,u} \) for all \( u \in \Lambda \).

It is elementary to verify that \( \{ w_v : v \in \Lambda \} \) is a Toeplitz–Cuntz–Krieger \( \Lambda \)-family. The induced homomorphism \( \pi_w : \mathcal{T}C^*(\Lambda) \to c_0(\Lambda) \) is obviously surjective. Since each \( w_v \neq 0 \), we conclude that \( \pi_w \) is injective by Theorem 2.3.18. Moreover, since \( \Lambda \) contains only vertices, any Toeplitz–Cuntz–Krieger \( \Lambda \) automatically satisfies relation (CK) of Definition 2.3.19, and we conclude that \( C^*(\Lambda) \cong \mathcal{T}C^*(\Lambda) \).

The following result due to Pimsner shows the relationship between the Toeplitz algebra of a Hilbert bimodule and its coefficient algebra.

**Theorem 2.5.2** ([53], Theorem 4.4). Let \( X \) be a Hilbert \( A \)-bimodule. Suppose \( A \) is separable and \( X \) is countably generated as a right \( A \)-module. Then \( T_X \) and \( A \) are \( KK \)-equivalent. That is there exist \( \alpha \in KK(A,T_X) \) and \( \beta \in KK(T_X,A) \) such that

\[
\alpha \otimes_{T_X} \beta = 1_A, \quad \beta \otimes_A \alpha = 1_{T_X},
\]

where, for any \( C^* \)-algebra \( B \), \( 1_B \) is the multiplicative identity in the ring \( KK(B,B) \), given by \( [(B,\text{id}B,B),0]\).

**Remark 2.5.3.** We note that if \( A \) is nonseparable or \( X \) is not countably generated, then ([34], Proposition 8.2) ensures that the inclusion of \( A \) in \( T_X \) still induces an isomorphism at the level of \( K \)-theory.

**Proposition 2.5.4.** Let \( \Lambda \) be a finitely aligned \( k \)-graph. Then \( \mathcal{T}C^*(\Lambda) \cong_{KK} c_0(\Lambda^0) \).

**Proof.** We will use induction on \( k \). If \( k = 0 \) then \( \mathcal{T}C^*(\Lambda) \) and \( c_0(\Lambda^0) \) are isomorphic by Proposition 2.5.1. Now suppose that \( k \geq 1 \), and that the result holds for any finitely aligned \( (k-1) \)-graph. By Theorem 2.4.8 we may, for any \( i \in \{1,\ldots,k\} \), realise the Toeplitz–Cuntz–Krieger algebra of \( \Lambda \) as the Toeplitz algebra of the Hilbert \( \mathcal{T}C^*(\Lambda^i) \)-bimodule \( X := X_1 \) defined in Proposition 2.4.3.
Since $k$-graphs are countable categories by definition, $\mathcal{TC}^*(\Lambda^i)$ is separable and $X$ is finitely generated as a right $\mathcal{TC}^*(\Lambda^i)$-module. Thus, by Theorem 2.5.2, we have $\mathcal{TC}^*(\Lambda) \cong_{KK} \mathcal{TC}^*(\Lambda^i)$. Since $\Lambda^i$ is a $(k - 1)$-graph, the inductive hypothesis then tells us that $\mathcal{TC}^*(\Lambda^i) \cong_{KK} c_0(\Lambda^0)$. As $KK$-equivalence is transitive, we conclude that $\mathcal{TC}^*(\Lambda) \cong_{KK} c_0(\Lambda^0)$. □

Remark 2.5.5. In Pimsner’s proof of ([53], Theorem 4.4), $\alpha \in KK(A, \mathcal{T}_X)$ is given by the class of the inclusion of $A$ in $\mathcal{T}_X$. Hence, the inductive argument in the proof of Proposition 2.5.4 shows that the $KK$-equivalence of $c_0(\Lambda^0)$ and $\mathcal{TC}^*(\Lambda)$ is given by the inclusion of $c_0(\Lambda^0)$ in $\mathcal{TC}^*(\Lambda)$.

Remark 2.5.6. Since $KK$-equivalence is transitive, Proposition 2.5.4 is equivalent to the fact that the Toeplitz–Cuntz–Krieger algebra of any finitely aligned $k$-graph is $KK$-equivalent to the Toeplitz–Cuntz–Krieger algebra of any of its $2^k$ subgraphs obtained by deleting all edges of degree $\{e_i : i \in F\}$ for some $F \subseteq \{1, \ldots, k\}$.

Since $KK$-equivalent $C^*$-algebras have the same $K$-theory, Proposition 2.5.4 tells us that $\mathcal{TC}^*(\Lambda)$ and $c_0(\Lambda^0)$ have the same $K$-theory. Thus, we have an alternate proof to ([5], Theorem 2.9) describing the $K$-theory of Toeplitz–Cuntz–Krieger algebras.

Corollary 2.5.7. Let $\Lambda$ be a finitely aligned $k$-graph. Then $K_0(\mathcal{TC}^*(\Lambda)) \cong \bigoplus_{v \in \Lambda^0} \mathbb{Z}$ and $K_1(\mathcal{TC}^*(\Lambda)) \cong 0$.

2.6 Realising $C^*(\Lambda)$ as a Cuntz–Pimsner algebra

In Section 2.4 we showed how the Toeplitz–Cuntz–Krieger algebra of a finitely aligned $k$-graph $\Lambda$ can be realised as the Toeplitz algebra of a Hilbert $\mathcal{TC}^*(\Lambda^i)$-bimodule. As a consequence $\mathcal{TC}^*(\Lambda) \cong_{KK} \mathcal{TC}^*(\Lambda^i)$, and so by an inductive argument $\mathcal{TC}^*(\Lambda) \cong_{KK} \mathcal{TC}^*(\Lambda^0) = c_0(\Lambda^0)$. In this section we prove an analogous result for Cuntz–Krieger algebras, defining a Hilbert $C^*(\Lambda^i)$-bimodule $X$ and showing that the Cuntz–Pimsner algebra of $X$ is isomorphic to $C^*(\Lambda)$.

Our methodology is very similar to that of Section 2.4. As before, the first step is to find sufficient conditions for the inclusion of $\Lambda^i$ in $\Lambda$ to induce an injective $*$-homomorphism from $C^*(\Lambda^i)$ to $C^*(\Lambda)$. Before we do this, we need some definitions and preliminary results.

Definition 2.6.1. Let $\Lambda$ be a $k$-graph. For any $E \subseteq \Lambda$ and $\mu \in \Lambda$ we define

$$\text{Ext}_{\Lambda}(\mu; E) := \bigcup_{\lambda \in E} \{\alpha \in \Lambda : \mu \alpha \in \text{MCE}(\mu, \lambda)\}.$$ 

Informally speaking, $\text{Ext}_{\Lambda}(\mu; E)$ is the set of paths in $\Lambda$ that when appended to $\mu$ give a minimal common extension of $\mu$ with something in $E$. 
Lemma 2.6.2 ([59], Lemma C.5.). Let $\Lambda$ be a finitely aligned $k$-graph. Fix $v \in \Lambda^0$ and let $E \subseteq v\Lambda$ be a finite $v$-exhaustive set in $\Lambda$. Then for any $\mu \in v\Lambda$, the set $\operatorname{Ext}_{\Lambda}(\mu; E) \subseteq s(\mu)\Lambda$ is finite and $s(\mu)$-exhaustive in $\Lambda$.

Proof. Firstly, we check that $\operatorname{Ext}_{\Lambda}(\mu; E)$ is finite. For each $\lambda \in E$, since $\Lambda$ is finitely aligned, the set $\{\alpha \in \Lambda : \mu\alpha \in \operatorname{MCE}(\mu, \lambda)\}$ is finite. As $E$ is finite, $\operatorname{Ext}_{\Lambda}(\mu; E)$ is the finite union of finite sets, and so finite. It remains to verify that $\operatorname{Ext}_{\Lambda}(\mu; E)$ is $s(\mu)$-exhaustive in $\Lambda$. Fix $\sigma \in s(\mu)\Lambda$. Since $\mu\sigma \in v\Lambda$ and $E \subseteq v\Lambda$ is $v$-exhaustive in $\Lambda$, there exists $\lambda \in E$ and $\alpha, \beta \in \Lambda$ such that $\mu\sigma\alpha = \lambda\beta \in \operatorname{MCE}(\lambda, \mu\sigma)$. Let $$\tau := (\sigma\alpha)(0, d(\lambda) \lor d(\mu) - d(\mu)),$$ which is well-defined since $$d(\lambda) \lor d(\mu) - d(\mu) \leq d(\lambda) \lor d(\mu) - d(\mu) = d(\mu) - d(\mu) = d(\lambda).$$ Then $$\mu\tau = \mu(\sigma\alpha)(0, d(\lambda) \lor d(\mu) - d(\mu)) = (\mu\sigma\alpha)(0, d(\lambda) \lor d(\mu))$$ $$= (\lambda\beta)(0, d(\lambda) \lor d(\mu))$$ $$= \lambda\beta(0, d(\lambda) \lor d(\mu) - d(\lambda))$$ $$\in \Lambda^{d(\mu) \lor d(\lambda)},$$ which shows that $\mu\tau \in \operatorname{MCE}(\mu, \lambda)$. As $\lambda \in E$, we see that $\tau \in \operatorname{Ext}_{\Lambda}(\mu; E)$. Furthermore, $$\tau(\sigma\alpha)(d(\lambda) \lor d(\mu) - d(\mu), d(\mu)) = \sigma\alpha,$$ which shows that $\operatorname{CE}(\tau, \sigma) \neq \emptyset$, and so $\operatorname{MCE}(\tau, \sigma) \neq \emptyset$. Therefore, $\operatorname{Ext}_{\Lambda}(\mu; E)$ is $s(\mu)$-exhaustive in $\Lambda$. \hfill $\square$

Lemma 2.6.3. Let $\Lambda$ be a $k$-graph such that $\Lambda^i$ is finitely aligned and has no sources. Fix $v \in \Lambda^0$ and let $E \subseteq v\Lambda^i$ be finite and $v$-exhaustive in $\Lambda^i$. Then $E$ is finite and $v$-exhaustive in $\Lambda$.

Proof. We need only show that $E$ is $v$-exhaustive in $\Lambda$. Fix $\lambda \in v\Lambda$ and write $\lambda = \lambda'\lambda_i$ with $\lambda' \in \Lambda^i$ and $\lambda_i \in \Lambda^{\text{Nes}}$. Let $$N := \bigvee \{d(\mu) : \mu \in \operatorname{Ext}(\lambda'; E)\},$$ which exists since $\operatorname{Ext}_{\Lambda^i}(\lambda'; E)$ is finite by Lemma 2.6.2. Since $N_i = 0$ and $\Lambda^i$ has no sources we can choose $\tau \in s(\lambda_i)\Lambda^N \subseteq \Lambda^i$. Using the factorisation property we
write $\lambda_\tau \tau = \tau' \lambda'_\tau$ where $\tau' \in \Lambda^i$ and $\lambda'_\tau \in \Lambda^N\tau$. Since $r(\tau') = r(\lambda_\tau) = s(\lambda')$ and $\text{Ext}_{\Lambda^i}(\lambda'; E) \subseteq s(\lambda')\Lambda^i$ is $s(\lambda')$-exhaustive in $\Lambda^i$ by Lemma 2.6.2, there exists $\mu \in \text{Ext}(\lambda'; E)$ such that $\text{MCE}(\mu, \tau') \neq \emptyset$. As $N$ is maximal, $d(\tau') = d(\tau) = N \geq d(\mu)$. Thus, $d(\mu) \wedge d(\tau') = N = d(\tau)$, which forces $\tau' = \mu \beta$ for some $\beta \in \Lambda^i$. Moreover, since $v \in \text{Ext}_{\Lambda^i}(\lambda'; E)$, we know that $\lambda' \mu = s\xi \in \text{MCE}(\lambda', \mu)$ for some $\sigma \in E$ and $\xi \in \Lambda^i$. Therefore,

$$\sigma \xi \beta \lambda'_\tau = \lambda' \mu \beta \lambda'_\tau = \lambda' \tau' \lambda'_\tau = \lambda' \lambda_\tau = \lambda \tau.$$ 

Thus, $\text{CE}(\sigma, \lambda) \neq \emptyset$, and so $\text{MCE}(\sigma, \lambda) \neq \emptyset$. As $\sigma \in E$, we conclude that $E$ is $v$-exhaustive in $\Lambda$. 

Making use of the previous lemma, we now show that the inclusion of $\Lambda^i$ in $\Lambda$ induces an injective $*$-homomorphism from the Cuntz–Krieger algebra of $\Lambda^i$ to the Cuntz–Krieger algebra of $\Lambda$.

**Proposition 2.6.4.** Let $\Lambda$ be a finitely aligned $k$-graph with $k \geq 1$. Fix $i \in \{1, \ldots, k\}$ and suppose that $\Lambda^i$ has no sources. Then there exists an injective $*$-homomorphism $\phi : C^*(\Lambda^i) \to C^*(\Lambda)$ carrying $s^\Lambda_{\lambda} \tau$ to $s^\Lambda_{\lambda}$ for each $\lambda \in \Lambda^i$.

**Proof.** We claim that $\{s^\Lambda_{\lambda} : \lambda \in \Lambda^i\} \subseteq C^*(\Lambda)$ is a Cuntz–Krieger $\Lambda^i$-family. The same argument as in the proof of Proposition 2.4.2 shows that $\{s^\Lambda_{\lambda} : \lambda \in \Lambda^i\}$ satisfies (TCK1), (TCK2), and (TCK3), so we need only worry about checking that relation (CK) holds. With this in mind, fix $v \in (\Lambda^i)^0 = \Lambda^0$ and suppose $E \subseteq v\Lambda$ is finite and $v$-exhaustive in $\Lambda^i$. By Lemma 2.6.3, $E$ is finite and $v$-exhaustive in $\Lambda$. As $\{s^\Lambda_{\lambda} : \lambda \in \Lambda\}$ satisfies (CK), we conclude that $\{s^\Lambda_{\lambda} : \lambda \in \Lambda^i\}$ does as well. The universal property of $C^*(\Lambda^i)$ then induces a $*$-homomorphism $\phi$ from $C^*(\Lambda^i)$ to $C^*(\Lambda)$ such that $\phi(s^\Lambda_{\lambda}) = s^\Lambda_{\lambda}$ for each $\lambda \in \Lambda^i$.

The injectivity of $\phi$ follows from an application of Theorem 2.3.24. To begin with, Remark 2.3.22 tells us that $\phi(s^\Lambda_{\lambda}) = s^\Lambda_{\lambda} \neq 0$ for each $v \in \Lambda^0$. Moreover, restricting the gauge action $\gamma^\Lambda$ of $\mathbb{T}^k$ on $C^*(\Lambda)$ to $\mathbb{T}^{k-1}$, gives an action of $\mathbb{T}^{k-1}$ on $C^*(\{s^\Lambda_{\lambda} : \lambda \in \Lambda^i\}) = C^*(\{s^\Lambda_{\lambda} : \lambda \in \Lambda^i\}) \subseteq C^*(\Lambda)$ that intertwines $\phi$ and the gauge action $\gamma^\Lambda$ of $\mathbb{T}^{k-1}$ on $C^*(\Lambda^i)$. 

We are now ready to define the collection of Hilbert $C^*(\Lambda^i)$-bimodules that we are interested in.

**Proposition 2.6.5.** Let $\Lambda$ be a finitely aligned $k$-graph with $k \geq 1$. Fix $i \in \{1, \ldots, k\}$ and suppose that $\Lambda^i$ has no sources. For each $n \geq 0$, define

$$X_n := \overline{\text{span}} \left\{ s^\Lambda_{\lambda} s^\Lambda_{\mu}^* : \lambda, \mu \in \Lambda^i, d(\lambda)_i = n, d(\mu)_i = 0 \right\} \subseteq C^*(\Lambda),$$
taking the closure with respect to the norm on \( C^*(\Lambda) \). Then \( X_n \) carries a right action of \( C^*(\Lambda^i) \) such that
\[
x \cdot a := x \phi(a)
\]
for each \( x \in X_n \) and \( a \in C^*(\Lambda^i) \). There is a \( C^*(\Lambda^i) \)-valued inner-product \( \langle \cdot, \cdot \rangle_{C^*(\Lambda^i)}^n \) on \( X_n \) such that
\[
\langle x, y \rangle_{C^*(\Lambda^i)}^n := \phi^{-1}(x^* y)
\]
for each \( x, y \in X_n \). Then \( X_n \) has the structure of a \((right)\) Hilbert \( C^*(\Lambda^i)\)-module. Additionally, there exists a \(*\)-homomorphism \( \psi_n : C^*(\Lambda^i) \to \mathcal{L}_{C^*(\Lambda^i)}(X_n) \) such that
\[
\psi_n(a)(x) = \phi(a)x
\]
for each \( x \in X_n \) and \( a \in C^*(\Lambda^i) \). Thus, \( X_n \) has the structure of a Hilbert \( C^*(\Lambda^i)\)-bimodule.

**Proof.** By Proposition 2.6.4, there is an injective \(*\)-homomorphism \( \phi \) from \( C^*(\Lambda^i) \) to \( C^*(\Lambda) \) carrying \( s^A_\lambda \) to \( s^A_\lambda \) for each \( \lambda \in \Lambda^i \). Exactly the same argument as in the proof of Proposition 2.4.3 shows that each \( X_n \) is a well-defined Hilbert \( C^*(\Lambda^i)\)-bimodule. \( \square \)

**Proposition 2.6.6.** Let \( \Lambda \) be a finitely aligned \( k \)-graph with \( k \geq 1 \). Fix \( i \in \{1, \ldots, k\} \) and suppose that \( \Lambda^i \) has no sources, so that the collection of Hilbert \( C^*(\Lambda^i)\)-bimodules \( \{X_n : n \geq 0\} \) from Proposition 2.6.5 exists.

Set \( X := X_1 \). Define \( \Omega_0 : X_0 \to X^0 = C^*(\Lambda^i) \) to be \( \phi^{-1} \). For \( n \geq 1 \), define \( \Omega_n : X_n \to X^\otimes n \) inductively by
\[
\Omega_n (s^A_\lambda s^A_\mu) := s^A_{\lambda(0,e_i)} \otimes_{C^*(\Lambda^i)} \Omega_{n-1} (s^A_{\lambda(e_i,d(\lambda))} s^A_\mu)
\]
for each \( \lambda, \mu \in \Lambda \) with \( d(\lambda)_i = n \) and \( d(\mu)_i = 0 \). Then for each \( n \geq 0 \), \( \Omega_n \) is a Hilbert \( C^*(\Lambda^i)\)-bimodule isomorphism. Moreover, for any \( m, n \geq 0 \) and \( x \in X_m \), \( y \in X_n \),
\[
\Omega_m(x) \otimes_{C^*(\Lambda^i)} \Omega_n(y) = \Omega_{m+n}(xy).
\]

**Proof.** The proof is identical to that of Proposition 2.4.5 and Lemma 2.4.6. \( \square \)

Shortly, we will investigate the Cuntz–Pimsner algebra of the Hilbert \( C^*(\Lambda^i)\)-bimodule \( X \). The next result gives sufficient conditions for the \(*\)-homomorphism \( \psi := \psi_1 : C^*(\Lambda^i) \to \mathcal{L}_{C^*(\Lambda^i)}(X) \) that implements the left action of \( C^*(\Lambda^i) \) on \( X \) to be faithful.

**Lemma 2.6.7.** Let \( \Lambda \) be a finitely aligned \( k \)-graph \((k \geq 1)\) with no sources. Then the \(*\)-homomorphism \( \psi : C^*(\Lambda^i) \to \mathcal{L}_{C^*(\Lambda^i)}(X) \) is injective.
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Proof. Since \( \Lambda \) has no sources, \( \Lambda^i \) also has no sources, and the Hilbert \( C^*(\Lambda^i) \)-bimodule \( X := X_1 \) from Proposition 2.6.5 exists. We want to apply Theorem 2.3.24 to show that \( \psi \) is injective. Firstly, we check that \( \psi \left( s^A_v \right) \neq 0 \) for each \( v \in \Lambda^0 \). Given \( v \in \Lambda^0 \), since \( \Lambda \) has no sources, we can choose \( \lambda \in v\Lambda^e \). Then \( s^A_{\lambda} = s^A_{\lambda} s^A_{\lambda}^* s^A_{\lambda} \in X \) and
\[
\psi \left( s^A_v \right) s^A_{\lambda} = \psi \left( s^A_v \right) s^A_{\lambda} = s^A_{\lambda} \neq 0
\]
by Remark 2.3.22, we conclude that \( \psi \left( s^A_v \right) \neq 0 \).

It remains to show that there exists an action \( \theta : T^{k-1} \to \text{Aut}(\psi(C^*(\Lambda^i))) \) such that \( \theta_z \circ \psi = \psi \circ \gamma_z^A \) for each \( z \in T^{k-1} \). Fix \( z \in T^{k-1} \). We now show that the formula
\[
s^A_{\lambda}(a) \mapsto s^A_{\lambda}(\gamma_z^A(a)) \ (\lambda \in \Lambda^e, \ a \in C^*(\Lambda^i))
\]
extends by linearity and continuity to all of \( X = \text{span} \left\{ s^A_{\lambda}(a) : \lambda \in \Lambda^e, \ a \in C^*(\Lambda^i) \right\} \).

Let \( m \in \mathbb{N} \) and fix \( \lambda_1, \ldots, \lambda_m \in \Lambda^e \) and \( a_1, \ldots, a_m \in C^*(\Lambda^i) \). Then
\[
\left\| \sum_{j=1}^m s^A_{\lambda_j}(\gamma_z^A(a_j)) \right\|_X^2 = \left\langle \sum_{j=1}^m s^A_{\lambda_j}(\gamma_z^A(a_j)), \sum_{j=1}^m s^A_{\lambda_j}(\gamma_z^A(a_j)) \right\|_{C^*(\Lambda^i)}
\]
\[
= \left\| \sum_{j=1}^m \phi^{-1} \left( s^A_{\lambda_j}(\gamma_z^A(a_j)) \right)^* \left( s^A_{\lambda_j}(\gamma_z^A(a_j)) \right) \right\|_{C^*(\Lambda^i)}
\]
\[
= \left\| \sum_{j=1}^m \gamma_z^A(a_j)^* \phi^{-1} \left( s^A_{\lambda_j} s^A_{\lambda_j}^* \right) \gamma_z^A(a_j) \right\|_{C^*(\Lambda^i)}.
\]

However, since \( \lambda_1, \ldots, \lambda_m \in \Lambda^e \), Proposition 2.3.10 tells us that this equals
\[
\left\| \sum_{j=1}^m \delta_{\lambda_j, \lambda} \gamma_z^A(a_j) \phi^{-1}(s^A_{\lambda_j}) \gamma_z^A(a_j) \right\|_{C^*(\Lambda^i)} = \left\| \sum_{j=1}^m \delta_{\lambda_j, \lambda} \gamma_z^A(a_j) \gamma_z^A(s^A_{\lambda_j}) \gamma_z^A(a_j) \right\|_{C^*(\Lambda^i)}
\]
\[
= \left\| \gamma_z^A \left( \sum_{j=1}^m \delta_{\lambda_j, \lambda} a_j^* s^A_{\lambda_j} a_j \right) \right\|_{C^*(\Lambda^i)}
\]
\[
= \left\| \sum_{j=1}^m \delta_{\lambda_j, \lambda} a_j^* s^A_{\lambda_j} a_j \right\|_{C^*(\Lambda^i)},
\]
where the last equality follows from the fact that \( \gamma_z \) is an automorphism, and hence isometric. Finally, this is the same as
\[
\left\| \sum_{j=1}^m a_j^* \phi^{-1}(s^A_{\lambda_j} s^A_{\lambda_j}) a_j \right\|_{C^*(\Lambda^i)} = \left\langle \sum_{j=1}^m s^A_{\lambda_j}(a_j), \sum_{j=1}^m s^A_{\lambda_j}(a_j) \right\|_{C^*(\Lambda^i)}
\]
\[
= \left\| \sum_{j=1}^m s^A_{\lambda_j}(a_j) \right\|^2_X.
\]
Thus, the formula formula \( s_A^\Lambda \phi(a) \leftrightarrow s_A^\Lambda \phi \left( \gamma_z^A(a) \right) \) extends to an isometry, which we denote by \( U_z \), on all of \( X \). The isometry \( U_z \) is surjective since, for any \( \lambda \in \Lambda^i \) and \( a \in C^*(\Lambda^i) \), we have

\[
U_z \psi(a) U_z^{-1} = \psi \left( \gamma_z^A(a) \right) \in \psi \left( C^*(\Lambda^i) \right) \subseteq \mathcal{L}_{C^*(\Lambda^i)}(X).
\]

To see this, fix \( \eta, \rho, \nu, \tau \in \Lambda^i \) and \( \lambda \in \Lambda^i \). Then

\[
\psi \left( s_\eta^A s_\rho^{A*} \right) \left( s_\nu^A s_\tau^{A*} \right) = s_\eta^A s_\rho^A s_\nu^A s_\tau^A.
\]

Hence,

\[
U_z \psi \left( s_\eta^A s_\rho^{A*} \right) U_z^{-1} = \psi \left( \gamma_z^A \left( s_\nu^A s_\tau^{A*} \right) \right).
\]

\[
= U_z \psi \left( s_\eta^A s_\rho^{A*} \right) \left( s_\eta^A \phi \left( \gamma_z^A \left( s_\nu^A s_\tau^{A*} \right) \right) \right)
\]

\[
= z^{d(\tau) - d(\nu)} U_z \psi \left( s_\eta^A s_\rho^{A*} \right) \left( s_\eta^A \phi \left( s_\nu^A s_\tau^{A*} \right) \right)
\]

\[
= z^{d(\tau) - d(\nu)} \sum_{(\alpha, \beta) \in \Lambda^{\min}(\rho, \lambda)} s_{\gamma(z)}^A \left( s_{\rho}(e_i, d(\rho)) s_{\gamma(z)}^A \right) s_{\gamma(z)}^A \left( s_{\rho}(e_i, d(\rho)) s_{\gamma(z)}^A \right)
\]

\[
= z^{d(\tau) - d(\nu)} \sum_{(\alpha, \beta) \in \Lambda^{\min}(\rho, \lambda)} z^{d((\rho)(e_i, d(\rho))) - d(\tau \beta)} s_{\gamma(z)}^A \left( s_{\rho}(e_i, d(\rho)) s_{\gamma(z)}^A \right)
\]

\[
= \sum_{(\alpha, \beta) \in \Lambda^{\min}(\rho, \lambda)} z^{d(\tau) - d(\nu)} + d((\rho)(e_i, d(\rho))) - d(\tau \beta) s_{\gamma(z)}^A \left( s_{\rho}(e_i, d(\rho)) s_{\gamma(z)}^A \right)
\]

However, if \( (\alpha, \beta) \in \Lambda^{\min}(\rho, \lambda) \), then

\[
d(\tau) - d(\nu) + d((\rho)(e_i, d(\rho))) - d(\tau \beta) = d(\eta) + d(\alpha) - e_i - d(\nu) - d(\beta)
\]

\[
= d(\eta) + d(\rho) \lor d(\lambda \nu) - d(\rho) - e_i
\]

\[
- d(\nu) \lor d(\lambda \nu) + d(\lambda \nu)
\]

\[
= d(\eta) - d(\rho) - e_i + d(\lambda)
\]

\[
= d(\eta) - d(\rho).
\]
Since we also see that \( \lambda \)

Firstly, we show that \( \mathrm{Proof.} \)

Let \( \psi \) be a group homomorphism, so is \( \psi \). Since \( \gamma^A \) is a group homomorphism, so is \( \theta \).

We also need to know which elements of \( C^*(\Lambda^i) \) act compactly on \( X \). The next lemma gives sufficient conditions for all of \( C^*(\Lambda^i) \) to act compactly on \( X \).

\textbf{Lemma 2.6.8.} Let \( \Lambda \) be a finitely aligned \( k \)-graph, such that \( \Lambda^i \) has no sources, so that the Hilbert \( C^*(\Lambda^i) \)-bimodule \( X := X_1 \) from Proposition 2.6.5 exists. If \( v \Lambda^{e_i} \) is finite for each \( v \in \Lambda^0 \), then \( \psi(C^*(\Lambda^i)) \subseteq K_{C^*(\Lambda^i)}(X) \).

\textbf{Proof.} Firstly, we show that \( \psi \left( s_{A}^{v} \right) \in K_{C^*(\Lambda^i)}(X) \) for each \( v \in \Lambda^0 \). Fix \( v \in \Lambda^0 \), \( \lambda \in \Lambda^{e_i} \), \( a \in C^*(\Lambda^i) \). Since \( v \Lambda^{e_i} \) is finite, it follows that \( \sum_{\mu \in v \Lambda^{e_i}} \Theta_{s_{A}^{\lambda},s_{A}^{\mu}} \subseteq K_{C^*(\Lambda^i)}(X) \).

We also see that

\[
\left( \sum_{\mu \in v \Lambda^{e_i}} \Theta_{s_{A}^{\lambda},s_{A}^{\mu}} \right) (s_{A}^{\mu} \phi(a)) = \sum_{\mu \in v \Lambda^{e_i}} s_{A}^{\lambda} \cdot \left( s_{A}^{\mu} s_{A}^{\lambda} \phi(a) \right)_{C^*(\Lambda^i)} = \sum_{\mu \in v \Lambda^{e_i}} s_{A}^{\lambda} s_{A}^{\mu} s_{A}^{\lambda} \phi(a).
\]

Proposition 2.3.10 then tells us that \( s_{A}^{\lambda} s_{A}^{\mu} = \delta_{\mu,\lambda} s_{A}^{\lambda} \) for each \( \mu \in v \Lambda^{e_i} \), and so the previous line equals

\[
\delta_{\mu,v(\lambda)} s_{A}^{\lambda} \phi(a) = s_{A}^{\lambda} s_{A}^{\mu} \phi(a) = \phi \left( s_{A}^{\lambda} \right) s_{A}^{\mu} \phi(a) = \psi \left( s_{A}^{\lambda} \right) \left( s_{A}^{\mu} \phi(a) \right).
\]

Hence, \( \psi \left( s_{A}^{v} \right) = \sum_{\mu \in v \Lambda^{e_i}} \Theta_{s_{A}^{\lambda},s_{A}^{\mu}} \subseteq K_{C^*(\Lambda^i)}(X) \). Since

\[
\begin{align*}
C^*(\Lambda^i) &= \text{span} \left\{ s_{A}^{\lambda} s_{A}^{\mu} : \lambda, \mu \in \Lambda^i \right\} \\
&= \text{span} \left\{ s_{A}^{v} s_{A}^{\lambda} s_{A}^{\mu} : v \in \Lambda^0, \lambda, \mu \in \Lambda^i \right\} \\
&= \text{span} \left\{ s_{A}^{v} : v \in \Lambda^0 \right\} \text{span} \left\{ s_{A}^{\lambda} s_{A}^{\mu} : \lambda, \mu \in \Lambda^i \right\}.
\end{align*}
\]
and \( \mathcal{K}_{C^*_{\Lambda^v}}(X) \) is an ideal of \( \mathcal{L}_{C^*_{\Lambda^v}}(X) \), we conclude that \( \psi(C^*(\Lambda^v)) \subseteq \mathcal{K}_{C^*_{\Lambda^v}}(X) \) as claimed.

In the previous lemma we assumed that \( v\Lambda^{e_i} \) was finite for each \( v\Lambda^{e_i} \) to ensure that \( C^*(\Lambda^v) \) acts compactly on \( X \). We now show that if \( \Lambda \) is locally convex (in the sense that whenever \( \mu \in \Lambda^{e_i} \) and \( \nu \in \Lambda^{e_i} \) with \( i \neq j \) and \( r(\mu) = r(\nu) \), we have \( s(\mu)\Lambda^{e_i} \neq \emptyset \) and \( s(\nu)\Lambda^{e_i} \neq \emptyset \)), then relation (CK) in \( C^*(\Lambda) \) enables us to write the vertex projection \( s^*_v \), for any \( v \in \Lambda^0 \) with \( v\Lambda^{e_i} \) finite and nonempty, as the sum of the range projections coming from the edges in \( v\Lambda^{e_i} \). Firstly, we need a lemma.

**Lemma 2.6.9.** Let \( \Lambda \) be a a finitely aligned \( k \)-graph and \( \{r_\lambda : \Lambda \in \Lambda \} \) a Toeplitz–Cuntz–Krieger \( \Lambda \)-family. If \( n \in \mathbb{N}^k \) and \( F \) is a nonempty finite subset of \( v\Lambda^n \), then

\[
\prod_{\lambda \in F}(r_v - r_\lambda r^*_\lambda) = r_v - \sum_{\lambda \in F} r_\lambda r^*_\lambda. \tag{2.8}
\]

**Proof.** We will prove the result using induction on \( |F| \). Clearly, (2.8) holds when \( |F| = 1 \), so let \( m \geq 1 \) and suppose (2.8) holds whenever \( F \) contains \( m \) elements. Fix \( F' \subseteq v\Lambda^n \) with \( |F'| = m + 1 \). Thus, for any \( \mu \in F' \), applying the inductive hypothesis to \( F' \setminus \{\mu\} \) we see that

\[
\prod_{\lambda \in F'}(r_v - r_\lambda r^*_\lambda) = (r_v - r_\mu r^*_\mu) \prod_{\lambda \in F'\setminus\{\mu\}}(r_v - r_\lambda r^*_\lambda)
= (r_v - r_\mu r^*_\mu) \left(r_v - \sum_{\lambda \in F'\setminus\{\mu\}} r_\lambda r^*_\lambda\right)
= \left(r^2_v - \sum_{\lambda \in F'\setminus\{\mu\}} r_v r_\lambda r^*_\lambda\right) - \left(r_\mu r^*_\mu r_v - \sum_{\lambda \in F'\setminus\{\mu\}} r_\mu r^*_\mu r_\lambda r^*_\lambda\right).
\]

Since every path in \( F' \) has degree \( n \), Proposition 2.3.10 tells us that \( r^*_\mu r_\lambda = 0 \) for each \( \lambda \in F' \setminus \{\mu\} \). Hence, the previous line equals

\[
\left(r_v - \sum_{\lambda \in F'\setminus\{\mu\}} r_\lambda r^*_\lambda\right) - r_\mu r^*_\mu = r_v - \sum_{\lambda \in F'} r_\lambda r^*_\lambda,
\]
as required. \( \square \)

**Proposition 2.6.10.** Let \( \Lambda \) be a finitely aligned \( k \)-graph and \( \{r_\lambda : \Lambda \in \Lambda \} \) a Cuntz–Krieger \( \Lambda \)-family. Suppose \( \Lambda \) is locally convex. Then, for each \( v \in \Lambda^0 \) with \( v\Lambda^{e_i} \) nonempty and finite, we have

\[
r_v = \sum_{\lambda \in v\Lambda^{e_i}} r^*_\lambda r_\lambda.
\]
Proof. We claim that \( v\Lambda^{e_i} \) is \( v \)-exhaustive subset of \( v\Lambda \) for each \( v \in \Lambda^0 \). To see this, suppose that \( \lambda \in v\Lambda \). We need to show there exists \( \mu \in v\Lambda^{e_i} \) such that \( \Lambda^{\min}(\lambda, \mu) \neq \emptyset \). If \( \lambda = v \) then, for any \( \mu \in v\Lambda^{e_i} \), we have

\[
\{(\mu, s(\mu))\} = \Lambda^{\min}(\lambda, \mu).
\]

If \( d(\lambda) \neq 0 \), then with \( \mu := \lambda(0, e_i) \in v\Lambda^{e_i} \) we have

\[
\{(s(\lambda), \lambda(e_i, d(\lambda)))\} = \Lambda^{\min}(\lambda, \mu).
\]

If \( d(\lambda) = 0 \), then the local convexity of \( \Lambda \) allows us to choose \( \nu \in s(\lambda)\Lambda^{e_i} \). With \( \mu := (\lambda \nu)(0, e_i) \in v\Lambda^{e_i} \), we have

\[
(\nu, (\lambda \nu)(e_i, d(\lambda \nu))) \in \Lambda^{\min}(\lambda, \mu).
\]

Thus, \( v\Lambda^{e_i} \) is \( v \)-exhaustive. Since \( v\Lambda^{e_i} \) is finite by assumption, relation (CK) tells us that \( \prod_{\lambda \in v\Lambda^{e_i}} (r_v - r_\lambda r_\lambda^*) = 0 \). Applying Lemma 2.6.9 with \( F = v\Lambda^{e_i} \), we conclude that \( r_v = \sum_{\lambda \in v\Lambda^{e_i}} r_\lambda r_\lambda^* \).

\[\Box\]

Remark 2.6.11. It follows from Lemma 2.6.10 that \( \psi(s^\Lambda_v) = \sum_{\lambda \in v\Lambda^{e_i}} \Theta s^\Lambda_{s^\Lambda_v} \) for each \( v \in \Lambda^0 \) with \( v\Lambda^{e_i} \) nonempty and finite. To see this observe that for any \( \mu \in \Lambda^{e_i} \) and \( a \in C^*(\Lambda^i) \),

\[
\psi\left(s^\Lambda_v\right) \left(s^\Lambda_{s^\Lambda_v} \phi(a)\right) = s^\Lambda_{s^\Lambda_v} \phi(a) = \sum_{\lambda \in v\Lambda^{e_i}} s^\Lambda_{s^\Lambda_v} s^\Lambda_{\lambda} \phi(a)
\]

\[
= \sum_{\lambda \in v\Lambda^{e_i}} s^\Lambda_{s^\Lambda_v} \cdot \left(s^\Lambda_{\lambda} \phi(a)\right) = \left(\sum_{\lambda \in v\Lambda^{e_i}} \Theta s^\Lambda_{s^\Lambda_v} \right) \left(s^\Lambda_v \phi(a)\right).
\]

We are now ready to prove that the Cuntz–Pimsner algebra of \( X \) is isomorphic to the Cuntz–Krieger algebra of \( \Lambda \).

**Theorem 2.6.12.** Let \( \Lambda \) be a \( a \) finitely aligned \( k \)-graph with no sources, so that the Hilbert \( C^*(\Lambda^i) \)-bimodule \( X := X_1 \) from Proposition 2.6.5 exists. Suppose that \( v\Lambda^{e_i} \) is finite for each \( v \in \Lambda^0 \). Then \( \mathcal{O}_X \cong C^*(\Lambda) \).

**Proof.** The proof is very similar to the analogous statement for Toeplitz algebras in Theorem 2.4.8. We claim that \( (\iota, \phi) \), with \( \iota : X \to C^*(\Lambda) \) the inclusion map, is a Cuntz–Pimsner covariant Toeplitz representation of \( X \) in \( C^*(\Lambda) \). Exactly the same argument as in the proof of Theorem 2.4.8 shows that \( (\iota, \phi) \) is a Toeplitz representation. It remains to check that \( (\iota, \phi) \) is Cuntz–Pimsner covariant, i.e. \( (\iota, \phi)^{(1)}(\psi(a)) = \phi(a) \) for each \( a \in \psi^{-1}(\mathcal{K} C^*(\Lambda^i)(X)) \cap (\ker(\psi))^\perp \).

Since \( \Lambda \) has no sources (in particular \( \Lambda \) is locally convex) and \( v\Lambda^{e_i} \) is finite for each \( v \in \Lambda^0 \), Lemma 2.6.7 and Lemma 2.6.8 tell us that the intersection...
Therefore, 

\( \psi^{-1}(K_{C^*}(\Lambda^i)(X) \cap (\ker(\psi))^\perp) \) is all of \( C^*(\Lambda^i) \), which is generated by the Cuntz–Krieger \( \Lambda^i \)-family \( \{ s_{\lambda}^i : \lambda \in \Lambda^i \} \). Thus, it suffices to show that \( (\iota, \phi)^{(1)}(\psi(s_{\lambda}^i)) = \phi(s_{\lambda}^i) \) for each \( \lambda \in \Lambda^i \). With this in mind, fix \( \lambda \in \Lambda^i \).

Then

\[
\psi(s_{\lambda}^i) = \psi(s_{\lambda}^i) \psi(s_{\lambda}^i) = \psi(s_{\lambda}^i) \left( \sum_{\mu \in s(\lambda) \Lambda^i} \Theta_{s_{\mu}^i, s_{\lambda}^i}^{s_{\lambda}^i} \right) = \sum_{\mu \in s(\lambda) \Lambda^i} \Theta_{s_{\mu}^i, s_{\lambda}^i}^{s_{\lambda}^i} = \sum_{\mu \in s(\lambda) \Lambda^i} \Theta_{s_{\mu}^i, s_{\lambda}^i}^{s_{\lambda}^i}.
\]

Therefore,

\[
(\iota, \phi)^{(1)}(\psi(s_{\lambda}^i)) = (\iota, \phi)^{(1)} \left( \sum_{\mu \in s(\lambda) \Lambda^i} \Theta_{s_{\mu}^i, s_{\lambda}^i}^{s_{\lambda}^i} \right) = \sum_{\mu \in s(\lambda) \Lambda^i} \iota(s_{\mu}^i) \iota(s_{\lambda}^i) = \sum_{\mu \in s(\lambda) \Lambda^i} s_{\lambda}^i \iota(s_{\mu}^i) = s_{\lambda}^i \iota(s_{\mu}^i) = s_{\lambda}^i = \phi(s_{\lambda}^i).
\]

Thus, \( (\iota, \phi) \), is a Cuntz–Pimsner covariant Toeplitz representation of \( X \) in \( C^*(\Lambda) \). So there exists a \(*\)-homomorphism \( \iota \times_\phi : O_X \rightarrow C^*(\Lambda) \) such that \( \iota \times_\phi \circ j_X = \iota \) and \( (\iota \times_\phi \circ j_{C^*(\Lambda)}) = \phi \), where \( (j_X, j_{C^*(\Lambda)}) \) is the universal Cuntz–Pimsner covariant Toeplitz representation of \( X \).

Next, we show that the collection \( \{ u_{\lambda} : \lambda \in \Lambda \} \subseteq O_X \) of partial isometries defined by \( u_{\lambda} := j_X^{\odot d(\lambda)}(\Omega_{d(\lambda)}(s_{\lambda}^i)) \) is a Cuntz–Krieger \( \Lambda \)-family. The proof of Theorem 2.4.8 shows that \( \{ u_{\lambda} : \lambda \in \Lambda \} \) is a Toeplitz–Cuntz–Krieger \( \Lambda \)-family. Thus, it remains to check that \( \{ u_{\lambda} : \lambda \in \Lambda \} \) satisfies (CK). By ([59], Theorem C.1), it suffices to show that if \( v \in \Lambda^0 \) and \( E \subseteq \bigcup_{j=1}^k v \Lambda^{e_j} \) is a finite \( v \)-exhaustive set in \( \Lambda \), then

\[
\prod_{\lambda \in E} (u_v - u_{\lambda} u_{\lambda}^*) = 0.
\]

It is not immediately clear how to combine relation (CK) in \( C^*(\Lambda^i) \) and \( C^*(\Lambda) \) with the Cuntz–Pimsner covariance of \( (j_X, j_{C^*(\Lambda)}) \) to show that \( \prod_{\lambda \in E} (u_v - u_{\lambda} u_{\lambda}^*) \) is zero. If \( E \cap \Lambda^{e_i} = \emptyset \), then \( E = E \cap \Lambda^i \) is a \( v \)-exhaustive set in \( \Lambda^i \). Applying relation (CK) in \( C^*(\Lambda^i) \), we see that

\[
\prod_{\lambda \in E} (u_v - u_{\lambda} u_{\lambda}^*) = \prod_{\lambda \in E \cap \Lambda^i} (u_v - u_{\lambda} u_{\lambda}^*) = j_{C^*(\Lambda^i)} \left( \prod_{\lambda \in E \cap \Lambda^i} (s_{\lambda}^i - s_{\lambda}^i s_{\lambda}^i u_{\lambda}^*) \right) = 0.
\]

On the other hand, if \( E \cap \Lambda^{e_i} \neq \emptyset \), we can apply Lemma 2.6.9 with \( F = E \cap \Lambda^{e_i} \) to
see that

\[ 0 = \prod_{\lambda \in E} \left( s^A_v - s^A_\lambda s^*_\lambda \right) = \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \prod_{\mu \in E \cap \Lambda^{e_i}} \left( s^A_v - s^A_\lambda s^*_\lambda \right) = \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) \left( s^A_v - \sum_{\mu \in E \cap \Lambda^{e_i}} s^A_\mu s^*_\mu \right) = \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) - \sum_{\mu \in E \cap \Lambda^{e_i}} \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) s^A_\mu s^*_\mu , \]

where the first equality comes from applying relation (CK) in \( C^*(\Lambda) \). Therefore,

\[ \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) = \sum_{\mu \in E \cap \Lambda^{e_i}} \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) s^A_\mu s^*_\mu , \]

and so in \( \mathcal{L}_{C^*(\Lambda^i)}(X) \) we have

\[ \psi \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) = \sum_{\mu \in E \cap \Lambda^{e_i}} \Theta \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) s^A_\mu s^*_\mu . \]

Since \( \psi \) is injective by Lemma 2.6.7, it follows that \( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \) is again applying Lemma 2.6.9 to the Toeplitz–Cuntz–Krieger \( \Lambda \)-family \( \{ u_\lambda : \lambda \in \Lambda \} \) and the nonempty finite set \( E \cap \Lambda^{e_i} \), we see that

\[ \prod_{\lambda \in E} \left( u_v - u_\lambda u^*_\lambda \right) = \prod_{\lambda \in E \cap \Lambda^i} \left( u_v - u_\lambda u^*_\lambda \right) - \sum_{\mu \in E \cap \Lambda^{e_i}} \left( \prod_{\lambda \in E \cap \Lambda^i} \left( u_v - u_\lambda u^*_\lambda \right) \right) u_\mu u^*_\mu \]

\[ = j_{C^*(\Lambda^i)} \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) \]

\[ - \sum_{\mu \in E \cap \Lambda^{e_i}} \left( j_{C^*(\Lambda^i)} \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) \right) j_X \left( s^A_\mu \right) j_X \left( s^A_\mu \right)^* \]

\[ = j_{C^*(\Lambda^i)} \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) \]

\[ - \sum_{\mu \in E \cap \Lambda^{e_i}} j_X \left( \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) s^A_\mu \right) j_X \left( s^A_\mu \right)^* \]

\[ = j_{C^*(\Lambda^i)} \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) \]

\[ - \left( j_X j_{C^*(\Lambda^i)} \right)^{(1)} \left( \sum_{\mu \in E \cap \Lambda^{e_i}} \Theta \left( \prod_{\lambda \in E \cap \Lambda^i} \left( s^A_v - s^A_\lambda s^*_\lambda \right) \right) s^A_\mu s^*_\mu \right) . \]
However, since \((j_X, j_{C^*(\Lambda)})\) is Cuntz–Pimsner covariant,

\[
(j_X, j_{C^*(\Lambda)})^{(1)} \left( \sum_{\mu \in E \cap \Lambda_i} \Theta(\Pi_{\lambda \in E \cap \Lambda_i} (s^0_{\lambda} - s^*_\lambda s^0_{\lambda}^\ast)) \right) 
= (j_X, j_{C^*(\Lambda)})^{(1)} \left( \psi \left( \prod_{\lambda \in E \cap \Lambda_i} \left( s^A_{\lambda} - s^*_\lambda s^A_{\lambda}^\ast \right) \right) \right) 
= j_{C^*(\Lambda)} \left( \prod_{\lambda \in E \cap \Lambda_i} \left( s^A_{\lambda} - s^*_\lambda s^A_{\lambda}^\ast \right) \right).
\]

Therefore,

\[
\prod_{\lambda \in E} (u_\lambda - u_\lambda u_\lambda^*) = 0.
\]

Thus, \(\{u_\lambda : \lambda \in \Lambda\}\) is a Cuntz–Krieger \(\Lambda\)-family in \(O_X\). The universal property of \(C^*(\Lambda)\) then induces a \(*\)-homomorphism \(\pi_u : C^*(\Lambda) \to O_X\) such that \(\pi_u (s^\Lambda_{\lambda}) = u_\lambda\) for each \(\lambda \in \Lambda\). Exactly the same argument as in Theorem 2.4.8 shows that \(\pi_u\) and \(\iota \times_{C^*} \phi\) are mutually inverse. We conclude that \(C^*(\Lambda) \cong O_X\).

Moving forward, it would be interesting to see if the hypotheses of Theorem 2.6.12 can be relaxed. The assumption that \(\Lambda\) has no sources and \(v\Lambda^e_i\) is finite were put in place to ensure that \(C^*(\Lambda^i)\) acts faithfully and compactly on \(X\). In particular, we would like to know whether we can weaken the assumption that \(\Lambda\) has no sources (say to \(\Lambda\) being locally convex) and still ensure that \(\psi\) is faithful.

It would also be interesting to see what analysis of \(O_X\) can be performed when \(C^*(\Lambda^i)\) does not act faithfully and/or compactly on \(X\). Unfortunately, it seems rather difficult to get a handle on what the ideal \(\psi^{-1}(K_{C^*(\Lambda^i)}(X)) \cap \ker(\psi)^\perp\) looks like in this situation. One possible approach to tackling this problem would be to make use of Katsura’s work on gauge invariant ideals of \(O_X\) (see ([33], Theorem 8.6)) — as the next result shows, the ideal \(\psi^{-1}(K_{C^*(\Lambda^i)}(X)) \cap \ker(\psi)^\perp\) is gauge-invariant.

**Proposition 2.6.13.** Let \(\Lambda\) be a finitely aligned \(k\)-graph such that \(\Lambda^i\) has no sources, so that the Hilbert \(C^*(\Lambda^i)\)-bimodule \(X := X_1\) from Proposition 2.6.5 exists. Then \(\ker(\psi)^\perp\) and \(\psi^{-1}(K_{C^*(\Lambda^i)}(X))\) are gauge-invariant ideals of \(C^*(\Lambda^i)\).

**Proof.** Recall from the proof of Lemma 2.6.7 that for each \(z \in T^{k-1}\) there exists a bijective linear map \(U_z : X \to X\) such that

\[
U_z \left( s^\Lambda_{\lambda} \phi(a) \right) = s^\Lambda_{\lambda} \phi \left( \gamma^A_z(a) \right)
\]

for each \(\lambda \in \Lambda^e_i\) and \(a \in C^*(\Lambda^i)\). Moreover,

\[
\psi \left( \gamma^A_z(a) \right) = U_z \psi(a) U_z^{-1}
\]
for each \( a \in C^*(\Lambda^i) \).

We now show that \( \ker(\psi) \) is gauge-invariant. If \( a \in \ker(\psi) \) and \( z \in \mathbb{T}^{k-1} \), then

\[
\psi \left( \gamma^A_z(a) \right) = U_z \psi(a) U_z^{-1} = 0.
\]

Hence, \( \gamma^A_z(a) \in \ker(\psi) \). From this we also see that \( \ker(\psi)^\perp \) is gauge invariant: if \( a \in \ker(\psi)^\perp \), \( b \in \ker(\psi)^\perp \), and \( z \in \mathbb{T}^{k-1} \), then

\[
\gamma^A_z(a) b = \gamma^A_z(a \gamma^A_z(b)) = \gamma^A_z(0) = 0.
\]

Next we show that \( \psi^{-1}(\mathcal{K}_{C^*(\Lambda^i)}(X)) \) is gauge-invariant. Since

\[
\psi \left( \gamma^A_z(a) \right) = U_z \psi(a) U_z^{-1}
\]

for each \( a \in C^*(\Lambda^i) \) and \( z \in \mathbb{T}^{k-1} \), it suffices to show that \( U_z \Theta_{x,y} U_z^{-1} \in \mathcal{K}_{C^*(\Lambda^i)}(X) \) for each \( x, y \in X \subseteq C^*(\Lambda) \). We claim that

\[
U_z \Theta_{x,y} U_z^{-1} = \Theta_{\gamma^A_z(x), \gamma^A_z(y)} \in \mathcal{K}_{C^*(\Lambda^i)}(X),
\]

where \( \tilde{z} \in \mathbb{T}^k \) is defined by

\[
\tilde{z}_j := \begin{cases} 
z_j & \text{if } 1 \leq j < i \\
0 & \text{if } j = i \\
z_{j-1} & \text{if } i < j \leq k. \end{cases}
\]

To see this, fix \( \lambda, \nu, \eta \in \Lambda^{e_i} \) and \( \mu, \tau, \rho \in \Lambda^i \). Then

\[
\left( U_z \Theta_{s^A_{\lambda \eta}^*, s^A_{\nu \eta}^*} U_z^{-1} \right) \left( s^A_{\eta \rho}^* s^A_{\mu} \right) = U_z \left( \left( z^{d(\mu) - d(\eta) + e_i} s^A_{\lambda \eta} s^A_{\rho} \right) \right)
\]

\[
= U_z \left( z^{d(\mu) - d(\eta) + e_i} s^A_{\lambda \eta} s^A_{\rho} \right) \left( \sum_{(\alpha, \beta) \in \Lambda^{\min}(\nu, \eta), (\gamma, \delta) \in \Lambda^{\min}(\mu, \tau)} s^A_{\gamma \tau} s^A_{\rho \delta} \right)
\]

\[
= z^{d(\mu) - d(\eta) + e_i} \sum_{(\alpha, \beta) \in \Lambda^{\min}(\nu, \eta), (\gamma, \delta) \in \Lambda^{\min}(\mu, \tau)} z^{d(\lambda \gamma) - d(\eta \delta) - d(\rho \delta)} s^A_{\lambda \gamma} s^A_{\rho \delta} \cdot
\]

\[
= \sum_{(\alpha, \beta) \in \Lambda^{\min}(\nu, \eta), (\gamma, \delta) \in \Lambda^{\min}(\mu, \tau)} z^{d(\lambda \gamma) - d(\beta \delta) - d(\eta \delta)} s^A_{\lambda \gamma} s^A_{\rho \delta}.
\]
However, for any \((\alpha, \beta) \in \Lambda_{\min}(\nu, \eta)\) and \((\gamma, \delta) \in \Lambda_{\min}(\mu, \tau\alpha)\), we have

\[
d(\lambda\gamma) - d(\beta\delta) - d(\eta) = d(\lambda) + (d(\mu) \lor d(\tau\alpha) - d(\mu)) - (d(\nu) \lor d(\eta) - d(\eta)) \\
- (d(\mu) \lor d(\tau\alpha) - d(\tau\alpha)) - d(\eta) \\
= d(\lambda) - d(\mu) - d(\nu) \lor d(\eta) + d(\tau) + d(\alpha) \\
= d(\lambda) - d(\mu) - (d(\nu) \lor d(\eta) + d(\tau) + (d(\nu) \lor d(\eta) - d(\nu)) \\
= d(\lambda) - d(\mu) + d(\tau) - d(\nu).
\]

Hence,

\[
\left( U_{z} \Theta_{s_{\lambda}^{\Lambda} s_{\mu}^{\Lambda^{*}}, s_{\nu}^{\Lambda} s_{\eta}^{\Lambda^{*}} U_{z}^{-1}} \right) (s_{\eta}^{\Lambda}, s_{\rho}^{\Lambda^{*}}) = \sum \left( (\alpha, \beta) \in \Lambda_{\min}(\nu, \eta), \right) \left( (\gamma, \delta) \in \Lambda_{\min}(\mu, \tau\alpha) \right) z^{d(\lambda) - d(\mu) + d(\tau) - d(\nu)} s_{\lambda\gamma}^{\Lambda} s_{\mu\beta\delta}^{\Lambda^{*}} \\
= \sum z^{d(\lambda) - d(\mu) + d(\tau) - d(\nu)} s_{\lambda\gamma}^{\Lambda} s_{\mu\beta\delta}^{\Lambda^{*}} \\
= \Theta_{s_{\lambda}^{\Lambda} s_{\mu}^{\Lambda^{*}}, s_{\nu}^{\Lambda} s_{\eta}^{\Lambda^{*}}} (s_{\eta}^{\Lambda}, s_{\rho}^{\Lambda^{*}}).
\]

\[
= \Theta_{\gamma_{\lambda}^{\Lambda} (s_{\lambda}^{\Lambda} s_{\mu}^{\Lambda^{*}}), \gamma_{\delta}^{\Lambda} (s_{\nu}^{\Lambda} s_{\eta}^{\Lambda^{*}})} (s_{\rho}^{\Lambda} s_{\mu}^{\Lambda^{*}}).
\]
Chapter 3

Factorisation of product systems over quasi-lattice ordered groups, and their associated $C^*$-algebras

In this chapter, we generalise the results of Chapter 2 from finitely aligned higher-rank graphs to arbitrary compactly aligned product systems. The results in this chapter also extend to more-general collections of Hilbert bimodules the work of Deaconu [13] on iterated Toeplitz and Cuntz–Pimsner algebras associated to pairs of commuting Hilbert bimodules. In Section 3.1, we recap the necessary background for product systems and their associated $C^*$-algebras, as well as listing some of the key results that we will need later. After this we investigate iterated Nica–Toeplitz algebras (Section 3.3) and iterated Cuntz–Nica–Pimsner algebras (Section 3.4). Finally, in Section 3.5 we consider what we call relative Cuntz–Nica–Pimsner algebras.

3.1 Background on product systems and their associated $C^*$-algebras

Product systems of Hilbert bimodules over arbitrary semigroups were first introduced by Fowler [24], as a variation of Arveson’s continuous tensor product systems over $(0, \infty)$ [2].

Definition 3.1.1. Let $A$ be a $C^*$-algebra and $P$ a semigroup with identity $e$. A product system over $P$ with coefficient algebra $A$ is a semigroup $X = \bigsqcup_{p \in P} X_p$ such that

(i) $X_p \subseteq X$ is a Hilbert $A$-bimodule for each $p \in P$ ;

(ii) $X_e$ is equal to the Hilbert $A$-bimodule $AA$;
Remark 3.1.2. Let $X$ be a product system over $P$ with coefficient algebra $A$.

(i) For each $p \in P$, we write $\phi_p : A \to \mathcal{L}_A(X_p)$ for the $*$-homomorphism that implements the left action of $A$ on $X_p$, i.e. $\phi_p(a)(x) = a \cdot x = ax$ for each $a \in A$ and $x \in X_p$. Also, for each $p \in P$, we write $\langle \cdot, \cdot \rangle^p_A$ for the $A$-valued inner-product on $X_p$.

(ii) As $X$ satisfies (ii) and (iv) of Definition 3.1.1, there exist $A$-linear inner-product preserving maps $M_{p,e} : X_p \otimes_A X_e \to X_p$ and $M_{e,p} : X_e \otimes_A X_p \to X_p$ such that $M_{p,e}(x \otimes_A a) = xa = x \cdot a$ and $M_{e,p}(a \otimes_A x) = ax = a \cdot x$ for each $p \in P$, $a \in X_e = A$, and $x \in X_p$. By the Hewitt–Cohen–Blanchard factorisation theorem, each $M_{p,e}$ is automatically an $A$-bimodule isomorphism. On the other hand, the maps $M_{e,p}$ need not be isomorphisms, since we do not require that each $X_p$ is (left) nondegenerate (i.e. $M_{e,p}$ need not be surjective).

(iii) Since $X$ is a semigroup, multiplication in $X$ is associative. In particular, $\phi_{pq}(a)(xy) = (\phi_p(a)x)y$ for all $p, q \in P$, $a \in A$, $x \in X_p$, and $y \in X_q$.

As the next example shows, product systems generalise Hilbert bimodules.

Example 3.1.3. Let $X$ be a Hilbert $A$-bimodule. Define $X_n := X^\otimes n$, for each $n \in \mathbb{N}$. As in Remark 2.1.20, each $X_n$ is a Hilbert $A$-bimodule. Moreover, using the tensor product as multiplication, $X := \bigsqcup_{n \in \mathbb{N}} X_n$ has the structure of a semigroup. For each $m, n \in \mathbb{N} \setminus \{0\}$, we have an $A$-bimodule isomorphism $M_{m,n} : X_m \otimes_A X_n \to X_{m+n}$ given by the identity map. Thus, $X$ is a product system over $\mathbb{N}$.

The Hilbert bimodule isomorphisms $M_{p,q}$ allows us to move adjointable operators between the various fibres of $X$.

Remark 3.1.4. Given $p \in P \setminus \{e\}$ and $q \in P$, the $A$-bimodule isomorphism $M_{p,q} : X_p \otimes_A X_q \to X_{pq}$ enables us to define a $*$-homomorphism $\iota^p_{pq} : \mathcal{L}_A(X_p) \to \mathcal{L}_A(X_{pq})$ by

$$\iota^p_{pq}(S) := M_{p,q} \circ \left( S \otimes_A \text{id}_{X_q} \right) \circ M_{p,q}^{-1}$$

for each $S \in \mathcal{L}_A(X_p)$. Equivalently, the $*$-homomorphism $\iota^p_{pq}$ is characterised by the formula $\iota^p_{pq}(S)(xy) = (Sx)y$ for each $S \in \mathcal{L}_A(X_p)$, $x \in X_p$, $y \in X_q$. Since $X_e \otimes_A X_q$ need not in general be isomorphic to $X_q$, we cannot necessarily define a map from
\( \mathcal{L}_A(X_e) \) to \( \mathcal{L}_A(X_q) \) using the above procedure. However, as \( \mathcal{K}_A(X_e) = \mathcal{K}_A(AA_A) \cong A \), we can define \( \iota'_q : \mathcal{K}_A(X_e) \to \mathcal{L}_A(X_q) \) by \( \iota'_q(a) := \phi_q(a) \). For notational purposes, we define \( \iota'_p : \mathcal{L}_A(X_p) \to \mathcal{L}_A(X_r) \) to be the zero map whenever \( p, r \in P \) and \( r \neq pq \) for all \( q \in P \).

The next result shows what happens when we compose the \( \ast \)-homomorphisms \( \iota^q_p \).

**Lemma 3.1.5.** Let \( X \) be a product system over \( P \) with coefficient algebra \( A \). For any \( p, q, r \in P \), we have

\[
\iota^pqr \circ \iota^pq = \iota^pqr.
\]

**Proof.** If \( p = e \), we need to show that \( \iota^pqr \circ \iota^p = \iota^q \circ \iota^e_p \) and \( \iota^pqr = \iota^e \) agree on \( \mathcal{K}_A(X_e) \) for \( a, b \in A \). Part (iii) of Remark 3.1.2 shows that

\[
\left( \iota^q_e \circ \iota^e_p \right)(\phi_{a,b}) = \iota^q_e(\phi_q(ab^*)) = \phi_q(ab^*) = \iota^q_e(\phi_{a,b}).
\]

Since \( \iota^q_e \circ \iota^e_p \) and \( \iota^e \) are \( \ast \)-homomorphisms, we conclude that \( \iota^q_e \circ \iota^e_p = \iota^e q \) as required.

On the other hand, if \( p \neq e \), then elements of \( X_{pq} \) can be approximated by linear combinations of elements of the form \( uvw \) with \( u \in X_p \), \( v \in X_q \), and \( w \in X_r \). For any \( T \in \mathcal{L}_A(X_p) \), we have

\[
(\iota^pqr \circ \iota^p)(T)(uvw) = \iota^pqr(\iota^p(T)(uvw)) = \iota^p(T)(uvw)w = ((Tv)w)w = (Tu)(vw).
\]

Since \( (\iota^pqr \circ \iota^p)(T), \iota^pqr(T) \in \mathcal{L}_A(X_{pqr}) \) are linear and continuous, we conclude that

\[
(\iota^pqr \circ \iota^p)(T) = \iota^pqr(T).
\]

Hence, \( \iota^pqr \circ \iota^p = \iota^pqr \). \( \square \)

**Proposition 3.1.6.** Let \( X \) be a product system over \( P \) with coefficient algebra \( A \). Fix \( p, q \in P \) and \( x \in X_p \). Then there exists an adjointable map \( \Theta_x \in \mathcal{L}_A(X_q, X_{pq}) \) defined by \( \Theta_x(y) := xy \) for each \( y \in X_q \). If \( p = e \), then \( x \in X_e = A \) and \( \Theta^*_x = \phi_q(x^*) = \Theta_{x^*} \). If \( p \neq e \), then the adjoint is determined by the formula

\[
\Theta^*_x(uv) = \langle x, u \rangle_A^p \cdot v
\]

for each \( u \in X_p \) and \( v \in X_q \).

**Proof.** If \( p = e \), then \( x \in X_e \) and so \( \Theta_x = \phi_q(x) \), which we already know is an adjointable operator on \( X_q \) with adjoint \( \phi_q(x^*) = \Theta_{x^*} \). Now suppose that \( p \neq e \).
For any \( u \in X_p \) and \( v, y \in X_q \), since \( M_{p,q} : X_p \otimes_A X_q \to X_{pq} \) is inner product preserving,

\[
\langle \Theta_p(y), uv \rangle_A^{pq} = \langle x, uv \rangle_A^{pq} = \langle x \otimes_A y, u \otimes_A v \rangle_A = \langle y, (x u)_{A}^p \cdot v \rangle_A^q.
\]

Since \( X_{pq} = \overline{\text{span}}\{uv : x \in X_p, v \in X_q\} \), it follows that \( \Theta_p \) is adjointable and \( \Theta_p^*(uv) = \langle x, u \rangle_{A}^p : v \) for each \( u \in X_p \) and \( v \in X_q \).

To associate \( C^* \)-algebras to product systems we introduce the notion of a representation of a product system.

**Definition 3.1.7.** Let \( X \) be a product system over a semigroup \( P \) with coefficient algebra \( A \). A representation of \( X \) in a \( C^* \)-algebra \( B \) is a map \( \psi : X \to B \) such that:

1. (T1) each \( \psi_p := \psi|_{X_p} \) is a linear map, and \( \psi_e \) is a \( C^* \)-homomorphism;
2. (T2) \( \psi_p(x)\psi_q(y) = \psi_{pq}(xy) \) for all \( p, q \in P \) and \( x \in X_p, y \in X_q \);
3. (T3) \( \psi_p(x)^*\psi_p(y) = \psi_e(\langle x, y \rangle_{A}^p) \) for all \( p \in P \) and \( x, y \in X_p \).

**Example 3.1.8.** Let \( X \) be a Hilbert \( A \)-bimodule and \( (\varphi, \pi) \) a Toeplitz representation of \( X \) in a \( C^* \)-algebra \( B \). Let \( X := \bigsqcup_{n \in \mathbb{N}} X^\otimes_n \) be the product system over \( \mathbb{N} \) associated to \( X \) (see Example 3.1.3). If we define \( \psi : X \to B \) by \( \psi_n := \varphi^\otimes_n \) for each \( n \in \mathbb{N} \), then \( \psi \) is a representation of \( X \) in \( B \).

**Proposition 3.1.9.** Let \( X \) be a product system over a semigroup \( P \) with coefficient algebra \( A \), and \( \psi : X \to B \) a representation of \( X \). Then \( \psi \) is norm-decreasing. Moreover, \( \psi \) is isometric if and only if \( \psi_e \) is injective.

**Proof.** Since \( \psi_e \) is a \( C^* \)-homomorphism, it is norm-decreasing. So relation (T3) of Definition 3.1.7 gives

\[
\|\psi_p(x)\|_B^2 = \|\psi_p(x)^*\psi_p(x)\|_B = \|\psi_e(\langle x, x \rangle_{A}^p)\|_B \leq \|\langle x, x \rangle_{A}^p\|_A = \|x\|_{X_p}
\]

for each \( x \in X_p \). If \( \psi_e \) is injective, then it is isometric, and equality holds throughout the above expression. Clearly if \( \psi \) is isometric, then \( \psi_e = \psi|_{X_e} \) is isometric, and hence injective.

**Remark 3.1.10.** If \( \psi : X \to B \) is a representation, then \( (\psi_p, \psi_e) \) is a Toeplitz representation of the Hilbert \( A \)-bimodule \( X_p \) for each \( p \in P \). For each \( p \in P \), we define \( \psi^{(p)} : K_A(X_p) \to B \) to be \( (\psi_p, \psi_e)^{(1)} \) as in Proposition 2.2.7; that is \( \psi^{(p)}(\Theta_{x,y}) = \psi_p(x)\psi_p(y)^* \) for all \( x, y \in X_p \).

Associated to each product system over a (left cancellative) semigroup, there exists a canonical representation called the Fock representation. Before we discuss this representation, we need to know about direct sums of Hilbert bimodules.
Definition 3.1.11. Let \( \{ X_j : j \in J \} \) be a collection of Hilbert \( A \)-bimodules indexed by some set \( J \). We define \( \bigoplus_{j \in J} X_j \) to be the set of all sequences \( (x_j)_{j \in J} \), with \( x_j \in X_j \), such that \( \sum_{j \in J} \langle x_j, x_j \rangle_A \) converges in \( A \). For \( (x_j)_{j \in J}, (y_j)_{j \in J} \in \bigoplus_{j \in J} X_j \), we define \( \langle (x_j)_{j \in J}, (y_j)_{j \in J} \rangle_A := \sum_{j \in J} \langle x_j, y_j \rangle_A \). To see that this sum converges, observe that for any finite set \( F \subseteq J \), ([40], Proposition 1.1) tells us that

\[
\left\| \sum_{j \in F} \langle x_j, y_j \rangle_A \right\|_A^2 \leq \left\| \sum_{j \in F} \langle x_j, x_j \rangle_A \right\|_A \left\| \sum_{j \in F} \langle y_j, y_j \rangle_A \right\|_A.
\]

It follows that \( \bigoplus_{j \in J} X_j \) is complete with respect to the norm induced by this inner product. The algebraic direct sum \( \bigodot_{j \in J} X_j \) is dense in \( \bigoplus_{j \in J} X_j \), a fact that we will frequently make use of when working with (infinite) direct sums of Hilbert bimodules. The pointwise left and right actions of \( A \) on \( \bigodot_{j \in J} X_j \) extend to left and right actions of \( A \) on \( \bigoplus_{j \in J} X_j \), under which \( \bigoplus_{j \in J} X_j \) is a Hilbert \( A \)-bimodule.

Example 3.1.12. Let \( X \) be a product system over a left cancellative semigroup \( P \) with coefficient algebra \( A \). Let \( \mathcal{F}_X := \bigoplus_{p \in P} X_p \), which we call the Fock space of \( X \). Then there exists an isometric representation \( l : X \to \mathcal{L}_A(\mathcal{F}_X) \) such that, for each \( p \in P \),

\[
l_p(x)(y_q)_{q \in P} = (\Theta_x(y_q))_{q \in P}
\]

for each \( x \in X_p \) and \( (y_q)_{q \in P} \in \mathcal{F}_X \). We call \( l \) the Fock representation of \( X \).

**Proof.** Fix \( p \in P \) and \( x \in X_p \). If \( y \in X_q \) for some \( q \in P \), then

\[
\langle \Theta_x(y), \Theta_x(y) \rangle_A^{p,q} = \langle x, x \rangle_A^{p,q} = \langle M_{p,q}(x \otimes_A y), M_{p,q}(x \otimes_A y) \rangle
\]

\[
= \langle x \otimes_A y, x \otimes_A y \rangle_A
\]

\[
= \langle y, (x, x)^p_A \cdot y \rangle^q_A
\]

\[
= \left\langle \left( (x, x)^p_A \right)^{\frac{1}{2}}, \left( (x, x)^p_A \right)^{\frac{1}{2}} \cdot y \right\rangle^q_A.
\]

Thus, an application of ([40], Proposition 1.2) shows that

\[
\langle \Theta_x(y), \Theta_x(y) \rangle_A^{p,q} \leq \phi_q \left( \left\langle \left( (x, x)^p_A \right)^{\frac{1}{2}} \right\rangle \right)_{\mathcal{L}_A(X_q)} \langle y, y \rangle_A^q \leq \left\langle \left( (x, x)^p_A \right)^{\frac{1}{2}} \right\rangle^2_{\mathcal{L}_A(X_q)} \langle y, y \rangle_A^q
\]

\[
\leq \left\| (x, x)^p_A \right\|_A \langle y, y \rangle_A^q
\]

\[
\leq \left\| x \right\|_A^2 \langle y, y \rangle_A^q.
\]

Hence, we may define a map \( l_p(x) : \mathcal{F}_X \to \mathcal{F}_X \) by \( l_p(x)(y_q)_{q \in P} := (\Theta_x(y_q))_{q \in P} \) for each \( (y_q)_{q \in P} \in \mathcal{F}_X \). We now show that \( l_p(x) \) is an adjointable operator. If \( u \in X_p \),
and \( v \in X_q \), then ([40], Proposition 1.2) implies that
\[
\langle \Theta^*_x(uv), \Theta^*_x(uv) \rangle^q_A \leq \| \Theta^*_x \|_{\mathcal{L}_A(X_p; X_q)} \langle uv, uv \rangle^{pq}_A.
\]

Hence, we can define a map \( S_x : F_X \rightarrow F_X \) by the linear extension of the formula
\( S_x(uv) = \Theta^*_x(uv) \) for \( u \in X_p, z \in X \), and \( S_x|_{X_q} = 0 \) if \( q \not\in pP \). We will show that \( S_x \) is the adjoint of \( l_p(x) \). Using the fact that \( S_x \) acts as \( \Theta^*_x \) on the summands \( X_q \) of \( F_X \) with \( q \in pP \) and zero on the summands with \( q \not\in pP \), we see that for any \((y_q)_{q \in P}, (z_q)_{q \in P} \in F_X \),
\[
\langle l_p(x)(y_q)_{q \in P}, (z_q)_{q \in P} \rangle_A = \left( \langle \Theta_x(y_q)_{q \in P}, (z_q)_{q \in P} \rangle \right)_A
= \sum_{q \in P} \langle \Theta_x(y_q), z_{pq} \rangle^q_A
= \sum_{q \in P} \langle y_q, \Theta^*_x(z_{pq}) \rangle^q_A
= \langle (y_q)_{q \in P}, S_x(z_q)_{q \in P} \rangle_A.
\]

Hence, \( l_p(x) \) is adjointable, with adjoint \( S_x \) as claimed.

With this in mind, we may define \( l : X \rightarrow L_A(F_X) \) by \( X_p \ni x \mapsto l_p(x) \). We claim that \( l \) is a representation. Each \( l_p \) is linear, and \( l_e \) is a *-homomorphism (since each \( \phi_p \) is a *-homomorphism). Hence, \( l \) satisfies (T1).

Next we check that \( l \) satisfies (T2). Let \( p, q \in P, x \in X_p, y \in X_q, \) and \((z_r)_{r \in P} \in F_X \). Since multiplication in \( X \) is associative,
\[
l_p(x)l_q(y)(z_r)_{r \in P} = (xy z_r)_{r \in P} = ((xy) z_r)_{r \in P} = l_{pq}(xy)(z_r)_{r \in P},
\]
which shows that \( l_p(x)l_q(y) = l_{pq}(xy) \). Hence, \( l \) satisfies (T2).

Next we check that \( l \) satisfies (T3). Fix \( p \in P, x, y \in X_p, \) and \((z_r)_{r \in P} \in F_X \). We have
\[
l_p(x)^*l_p(y)(z_r)_{r \in P} = l_p(x)^* (\Theta(g)(z_r))_{r \in P} = l_p(x)^* (yz_r)_{r \in P} = (\Theta^*_x(yz_r))_{r \in P}
= ((x,y)_{A} \cdot z_r)_{r \in P}
= (\Theta_{x,y}^p(z_r))_{r \in P}
= l_e ((x,y)_{A}^p) (z_r)_{r \in P}.
\]

Thus, \( l_p(x)^*l_p(y) = l_e ((x,y)_{A}^p) \), and so \( l \) satisfies (T3). We conclude that \( l \) is a representation of \( X \) in \( L_A(F_X) \).

Finally, we need to show that \( l \) is isometric. By Proposition 3.1.9, it suffices to check that \( l_e \) is injective. We have \( l_e = \bigoplus_{p \in P} \phi_p \), so \( l_e \) is injective because \( \phi_e \) is
injective (it is just left multiplication on $X_e = A$ by elements of $A$).

We are also interested in situations where the underlying semigroup possesses some additional order structure. In particular we focus on the quasi-lattice ordered groups first introduced by Nica [48].

**Definition 3.1.13.** A quasi-lattice ordered group $(G, P)$ consists of a group $G$ and a subsemigroup $P$ of $G$ such that $P \cap P^{-1} = \{ e \}$, and with respect to the partial order on $G$ induced by

$$p \leq q \Leftrightarrow p^{-1}q \in P,$$

any two elements $p, q \in G$ which have a common upper bound in $P$ have a least common upper bound in $P$.

**Remark 3.1.14.** Since $P \cap P^{-1} = \{ e \}$, it follows that the relation $\leq$ on $G$ is antisymmetric. Hence, whenever two elements in $G$ have a least common upper bound in $P$, this least common upper bound is unique. We write $p \vee q$ for the least common upper bound of $p, q \in G$ if it exists. For $p, q \in G$, we write $p \vee q = \infty$ if $p$ and $q$ have no common upper bound in $P$, and $p \vee q < \infty$ otherwise. If $p \vee q < \infty$ for every $p, q \in P$, we say that $(G, P)$ is directed.

**Remark 3.1.15.** The concept of least upper bounds in $(G, P)$ can be extended from pairs of elements in $P$ to finite subsets of $P$. We define $\bigvee \emptyset := e$, $\bigvee \{ p \} := p$ for any $p \in P$, and for any $n \geq 2$ and $C := \{ p_1, \ldots, p_n \} \subseteq P$ we define $\bigvee C := p_1 \vee \cdots \vee p_n$ (since $\leq$ is antisymmetric, this is well-defined).

**Example 3.1.16.** Let $G := \mathbb{Q}_+^*$ be the group of positive rationals with multiplication, and $P := \mathbb{N}_+$ the subsemigroup of positive integers. Then $(G, P)$ is a quasi-lattice ordered group.

**Proof.** If $x \in P \cap P^{-1}$, then $x, \frac{1}{x} \in P = \mathbb{N}_+$, which forces $x = 1 = e_G$. Hence, $P \cap P^{-1} = \{ e_G \}$. Next, suppose that $x, y \in G$ have a common upper bound $z \in P$. Thus, $\frac{z}{x}, \frac{z}{y} \in P$. Hence, $\{ mx = ny : m, n \in \mathbb{N}_+ \}$ is a nonempty subset of $\mathbb{N}$, and so has a least element $w$ by the well-ordering principle. It is now straightforward to check that $w$ is the least common upper bound of $x$ and $y$.

**Remark 3.1.17.** The quasi-lattice ordered group $(G, P)$ from Example 3.1.16 is in fact directed — for any $x, y \in P$, the set $\{ mx = ny : m, n \in \mathbb{N}_+ \}$ is always nonempty as it contains $yx = xy$. The minimal element of this set (i.e. the least common multiple of $x$ and $y$) is then the least upper bound for $x$ and $y$.

Given a quasi-lattice ordered group $(G, P)$, we can now ask for representations of product systems over $P$ to respect this additional semi-lattice like structure by satisfying a suitable covariance relation, which we call Nica covariance. Fowler
and Raeburn first investigated representations satisfying this covariance relation in [26], but only for product systems of Hilbert spaces (and representations on Hilbert spaces). In [23], Fowler noted that in order to discuss Nica covariance of representations in arbitrary $C^*$-algebras, we must restrict our attention to so-called compactly aligned product systems.

**Definition 3.1.18.** Let $(G, P)$ be a quasi-lattice ordered group and $X$ a product system over $P$. We say that $X$ is compactly aligned if, whenever $S \in K_A(X_p)$ and $T \in K_A(X_p)$ for some $p, q \in P$ with $p \vee q < \infty$, we have $\iota_p^{p \vee q}(S) \iota_q^{p \vee q}(T) \in K_A(X_{p \vee q})$.

**Remark 3.1.19.** This condition does not imply that either $\iota_p^{p \vee q}(S)$ or $\iota_q^{p \vee q}(T)$ is compact.

**Example 3.1.20.** A finitely aligned $k$-graph $\Lambda$ gives rise to a compactly aligned product system $X(\Lambda)$ over the quasi-lattice ordered group $(\mathbb{Z}_k, \mathbb{N}_k)$ with coefficient algebra $c_0(\Lambda^0)$ as follows. For each $n \in \mathbb{N}_k$, $X(\Lambda)_n$ is the completion of $c_c(\Lambda^0)$ with respect to the norm induced by the $c_0(\Lambda^0)$-valued inner-product given by

$$\langle f, g \rangle_{c_0(\Lambda^0)}^n(v) := \sum_{\lambda \in \Lambda^n v} f(\lambda)g(\lambda)$$

for each $f, g \in c_c(\Lambda^n)$, $v \in \Lambda^0$. The left and right actions of $c_0(\Lambda^0)$ on $X(\Lambda)_n$ are given by

$$(a \cdot f \cdot b)(\lambda) := a(r(\lambda))f(\lambda)b(s(\lambda))$$

for each $f \in X(\Lambda)_n$, $a, b \in c_0(\Lambda^0)$, $\lambda \in \Lambda^n$. For each $m, n \in \mathbb{N}_k$, we get an isomorphism $M_{m,n} : X(\Lambda)_m \otimes_A X(\Lambda)_n \to X(\Lambda)_{m+n}$ from the factorisation property in $\Lambda$, given by

$$(M_{m,n}(f \otimes_A g))(\lambda) = f(\lambda(0, m))g(\lambda(m, m + n))$$

for each $f \in X(\Lambda)_m$, $g \in X(\Lambda)_n$, $\lambda \in \Lambda^{m+n}$.

We can now ask that a representation of a compactly aligned product system respects the order structure on the underlying semigroup.

**Definition 3.1.21.** Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$. We say that a representation $\psi : X \to B$ is Nica covariant if, for any $p, q \in P$ and $S \in K_A(X_p)$, $T \in K_A(X_q)$, we have

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p \vee q)}(\iota_p^{p \vee q}(S)\iota_q^{p \vee q}(T)) & \text{if } p \vee q < \infty \\ 0 & \text{otherwise.} \end{cases}$$
It may not be immediately clear why we would want our product systems to be compactly aligned and their representations to be Nica covariant. In Proposition 2.2.5, we saw that given a Toeplitz representation \((\psi, \pi)\) of a Hilbert bimodule, products of the form \(\psi^p(u)^*\psi^q(v)\) can be approximated by linear combinations of products of the form \(\psi^{\infty_m}(x)\psi^{\infty_n}(y)^*\). It follows from this fact that the \(C^*\)-algebra generated by \((\psi, \pi)\) has a nice spanning family. In contrast, the \(C^*\)-algebra generated by an arbitrary representation of a product system need not have these properties.

The next result shows that the Nica covariance of a representation \(\psi\) enables us to pass elements of the form \(\psi_p(x)^*\) from the left to the right of elements of the form \(\psi_q(y)\).

**Lemma 3.1.22** ([24], Proposition 5.10). Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\). Suppose \(\psi : X \to B\) is a Nica covariant representation of \(X\) in a \(C^*\)-algebra \(B\). Let \(p, q \in P\) and \(x \in X_p, y \in X_q\). If \(p \lor q = \infty\), then \(\psi_p(x)^*\psi_q(y) = 0\). If \(p \lor q < \infty\), then

\[
\psi_p(x)^*\psi_q(y) \in \text{span} \left\{ \psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* \right\}.
\]

**Proof.** Choose \(x' \in X_p\) and \(y' \in X_q\) so that \(x = x' \cdot (x', x')_A^p\) and \(y = y' \cdot (y', y')_A^q\) by the Hewitt–Cohen–Blanchard factorisation theorem. If \(p = e\), then \(x \in X_e = A\), and so

\[
\psi_p(x)^*\psi_q(y) = \psi_q(x^* \cdot y) = \psi_q(x^* \cdot y') \psi_e((y', y')_A^q)
\]

\[
= \psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* \in \text{span} \left\{ \psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* \right\}.
\]

If \(q = e\), then \(y \in X_e = A\), and so

\[
\psi_p(x)^*\psi_q(y) = \psi_p(y^* \cdot x)^* = \psi_e((x', x')_A^p) \psi_p(y^* \cdot x')^*
\]

\[
= \psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* \in \text{span} \left\{ \psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* \right\}.
\]

Hence, we may as well assume that \(p, q \neq e\). Since \(\psi\) is a representation, we have

\[
\psi_p(x)^*\psi_q(y) = \psi_p(x' \cdot (x', x')_A^p)^* \psi_q(y' \cdot (y', y')_A^q)
\]

\[
= (\psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* \psi_q(y' \cdot (y', y')_A^q)
\]

\[
= (\psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* \psi_q(y' \cdot (y', y')_A^q)
\]

\[
= \psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* \psi_{q_{\infty -1}}(X_{q_{\infty -1}})^*.
\]

If \(p \lor q = \infty\), then \(\psi_{p_{\infty -1}}(X_{p_{\infty -1}})\psi_{q_{\infty -1}}(X_{q_{\infty -1}})^* = 0\) since \(\psi\) is Nica covariant, and so
\[ \psi_p(x)^* \psi_q(y) = 0. \] If \( p \lor q < \infty \), then
\[ i^\nu_{pq}(\Theta_{x',x'}) i^\nu_{pq}(\Theta_{y',y'}) \in \mathcal{K}_A \left( X_{pq} \right), \]
and so can be approximated in norm by a finite sum of rank one operators \( \Theta_{\mu,\nu} \) where \( \mu, \nu \in X_{pq} \). Thus,
\[ \psi(p)(\Theta_{x',x'}) \psi(q)(\Theta_{y',y'}) = \psi(p)(\Theta_{x',x'}) \psi(q)(\Theta_{y',y'}) \]
can be approximated by finite sums of the form \( \psi_{pq}(\mu) \psi_{pq}(\nu)^* \). Since \( p \neq e, \mu \) can be approximated by finite sums of products \( \sigma \tau \) where \( \sigma \in X_p \) and \( \tau \in X_{p^{-1}(pq)} \). Similarly, since \( q \neq e, \nu \) can be approximated by finite sums of products \( \eta \rho \) where \( \eta \in X_q \) and \( \rho \in X_{q^{-1}(pq)} \). Hence,
\[ \psi_p(x')^* \psi_p(\sigma) \psi_{pq}(\tau) \psi_{pq}(\rho)^* \psi_q(\eta)^* \psi_q(y') = \psi_p(x')^* \psi_p(\sigma) \psi_{pq}(\tau) \psi_{pq}(\rho)^* \psi_q(\eta)^* \psi_q(y') \]
can be approximated by finite sums of the form
\[ \psi_p(x')^* \psi_p(\sigma)^1 \psi_{pq}(\tau)^1 \psi_{pq}(\rho)^1 \psi_q(\eta)^1 \psi_q(y') \]
\[ = \psi_{pq}(\sigma)^1 \psi_{pq}(\tau)^1 \psi_{pq}(\rho)^1 \psi_{pq}(\eta)^1 \psi_{pq}(\rho)^1 \psi_q(y') \]
\[ \in \overline{\text{span}} \{ \psi_{pq}(X_{pq})(X_{pq})^* \} \]
Since \( \overline{\text{span}} \{ \psi_{pq}(X_{pq})(X_{pq})^* \} \) is closed under taking linear combinations and in norm, we conclude that (3.1) holds. \( \square \)

Generally, when working with Nica covariant representations it suffices to know that \( \psi_p(X_p)^* \psi_q(X_q) \subseteq \psi_{pq}(X_{pq}) (X_{pq})^* \psi_{pq}(X_{pq}) (X_{pq})^* \) whenever \( p \lor q < \infty \). However, for our proofs of Lemma 3.5.7 and Proposition 3.5.9 in Section 3.5 we will need to explicitly write an element of \( \psi_p(X_p)^* \psi_q(X_q) \) as a limit of linear combinations of the form \( \psi_{pq}(u) \psi_{pq}(v)^* \) — the next remark makes this precise.

\textbf{Remark 3.1.23.} Suppose \( x \in X_p \) and \( y \in X_q \) with \( p, q \neq e \) and \( p \lor q < \infty \). Choose \( x' \in X_p \) and \( y' \in X_q \) so that \( x = x' \cdot (x', x')_A^q \) and \( y = x' \cdot (y', y')_A^q \) by the Hewitt–Cohen–Blanchard factorisation theorem. Since \( X \) is compactly aligned and
\[ X_{pq} = \overline{\text{span}} \{ \sigma \tau : \sigma \in X_p, \tau \in X_{p^{-1}(pq)} \} = \overline{\text{span}} \{ \eta \rho : \eta \in X_q, \rho \in X_{q^{-1}(pq)} \}, \]
we can write
\[ i^\nu_{pq}(\Theta_{x',x'}) i^\nu_{pq}(\Theta_{y',y'}) = \lim_{i \to \infty} \sum_{k} \Theta_{\sigma_i \tau_i \eta_i \rho_i} \in \mathcal{K}_A \left( X_{pq} \right), \]
where \( \sigma_j \in \mathbf{X}_p, \tau_j \in \mathbf{X}_{p^{-1}(p \lor q)}, \eta_j \in \mathbf{X}_q, \rho_j \in \mathbf{X}_{q^{-1}(p \lor q)} \). The proof of Lemma 3.1.22 shows that

\[
\psi_p(x)^* \psi_q(y) = \lim_{i \to \infty} \sum_{j_i=1}^{k_i} \psi_{p^{-1}(p \lor q)}(\langle x', \sigma_{j_i} \rangle_A \tau_{j_i}) \psi_{q^{-1}(p \lor q)}(\langle y', \eta_{j_i} \rangle_A \rho_{j_i}).
\]

Moreover, if \( z \in \mathbf{X}_r \) and \( w \in \mathbf{X}_s \), then

\[
\psi_r(z) \psi_p(x)^* \psi_q(y) \psi_s(w)^* = \lim_{i \to \infty} \sum_{j_i=1}^{k_i} \psi_{r^{-1}(p \lor q)}(z \langle x', \sigma_{j_i} \rangle_A \tau_{j_i}) \psi_{s^{-1}(p \lor q)}(w \langle y', \eta_{j_i} \rangle_A \rho_{j_i}).
\]

The next result shows that product systems over totally ordered quasi-lattice ordered groups (in particular over \((\mathbb{Z}, \mathbb{N})\)) are automatically compactly aligned and their representations automatically Nica covariant. Hence, compactly aligned product systems generalise Fock spaces of Hilbert bimodules, and Nica covariant representations generalise Toeplitz representations.

**Proposition 3.1.24** ([24], Proposition 5.8). Let \((G, P)\) be a quasi-lattice ordered group and \( \mathbf{X} \) a product system over \( P \) with coefficient algebra \( A \). If \( A \) acts compactly on each fibre of \( \mathbf{X} \), or \((G, P)\) is totally ordered by \( \leq \), then \( \mathbf{X} \) is compactly aligned. Furthermore, if \((G, P)\) is totally ordered, then any representation \( \psi : \mathbf{X} \to B \) of \( \mathbf{X} \) in a \( C^* \)-algebra \( B \), is Nica covariant.

**Proof.** Firstly, suppose that \( A \) acts compactly on each fibre of \( \mathbf{X} \). We claim that

\[
i_{p}^{pq}(S) \in \mathcal{K}_A(\mathbf{X}_{pq}) \quad \text{for any } p, q \in P, S \in \mathcal{K}_A(\mathbf{X}_p).
\]

Fix \( p, q \in P \) and \( S \in \mathcal{K}_A(\mathbf{X}_p) \). If \( p = e \), then

\[
i_{p}^{pq}(S) = i_{e}^{q}(S) \in \phi_q(A) \subseteq \mathcal{K}_A(\mathbf{X}_q) = \mathcal{K}_A(\mathbf{X}_{pq}).
\]

Now suppose that \( p \neq e \). Since \( S \otimes_A \text{id}_{\mathbf{X}_q} \in \mathcal{K}_A(\mathbf{X}_p \otimes_A \mathbf{X}_q) \) by Proposition 2.1.22, we see that

\[
i_{p}^{pq}(S) = M_{p,q} \circ (S \otimes_A \text{id}_{\mathbf{X}_q}) \circ M_{p,q}^{-1} \in \mathcal{K}_A(\mathbf{X}_{pq})
\]

by parts (iii) and (iv) of Proposition 2.1.9. It is now elementary to show that \( \mathbf{X} \) is compactly aligned. If \( p, q \in P \) with \( p \lor q < \infty \) and \( S \in \mathcal{K}_A(\mathbf{X}_p), T \in \mathcal{K}_A(\mathbf{X}_q) \), then \( i_p^{p\lor q}(S) \) and \( i_q^{p\lor q}(T) \) are both in \( \mathcal{K}_A(\mathbf{X}_{p\lor q}) \), and so \( i_p^{p\lor q}(S)i_q^{p\lor q}(T) \in \mathcal{K}_A(\mathbf{X}_{p\lor q}) \).

Now we suppose that \((G, P)\) is totally ordered. Let \( p, q \in P \). Without loss of generality, we may assume that \( p \geq q \). For any \( S \in \mathcal{K}_A(\mathbf{X}_p) \) and \( T \in \mathcal{K}_A(\mathbf{X}_q) \), since \( \mathcal{K}_A(\mathbf{X}_p) \) is an ideal of \( \mathcal{L}_A(\mathbf{X}_p) \), we see that

\[
i_p^{p\lor q}(S)i_q^{p\lor q}(T) = i_p^{p}(S)i_q^{p}(T) = Si_q^{p}(T) \in \mathcal{K}_A(\mathbf{X}_p) = \mathcal{K}_A(\mathbf{X}_{p\lor q}).
\]
Thus, \( X \) is compactly aligned. We now show that \( \psi \) is Nica covariant. Firstly, suppose that \( q = e \). Then for any \( x, y \in X_p \) and any \( a, b \in A = X_q \), we have

\[
\psi_p^q(x, y) \psi_p^q(a, b) = \psi_p^q(x, y) * \phi_p^q(a, b) = \psi_p^q(x, y) \tag{3.1.25}
\]

Before we prove this, we need a couple of lemmas.

Given a compactly aligned product system, there always exists at least one isometric Nica covariant representation: the Fock representation of Example 3.1.12. Before we prove this, we need a couple of lemmas.

**Lemma 3.1.25.** Let \((G, P)\) be a quasi-lattice ordered group and \( X \) a compactly aligned product system over \( P \). Fix \( p, q \in P \) with \( q \geq p \). For any \( x, y \in X_p \),

\[
\iota_p^q(\Theta_{x,y}) = \Theta_x \circ \Theta_y^*.
\]

**Proof.** If \( p = e \), then \( x, y \in X_e = A \), and so for any \( z \in X_q \), we have

\[
(\Theta_x \circ \Theta_y^*) (z) = \phi_q(xy^*) = \iota_p^q(\Theta_{x,y}) (z).
\]

Thus, \( \Theta_x \circ \Theta_y^* = \iota_p^q(\Theta_{x,y}) \) when \( p = e \). Now suppose that \( p \neq e \). For any \( u \in X_p \) and \( v \in X_{p-1} \), we have

\[
(\Theta_x \circ \Theta_y^*) (uv) = \Theta_x ((y, u)^{p,v}) = x(y, u)^{p,v} = \Theta_{x,y} (u,v) = \iota_p^q(\Theta_{x,y}) (uv).
\]

Since \( X_q = \text{span}\{uv : u \in X_p, v \in X_{p-1} \} \), whilst both \( \Theta_x \circ \Theta_y^* \) and \( \iota_p^q(\Theta_{x,y}) \) are
Lemma 3.1.26. Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\). Let \(l : X \to \mathcal{L}_A(\mathcal{F}_X)\) denote the Fock representation of \(X\) (see Example 3.1.12). Then for any \(p, r \in P\), \(S \in \mathcal{K}_A(X_p)\), and \((z_q)_{q \in P} \in \mathcal{F}_X\), we have
\[
(l^{(p)}(S)(z_q)_{q \in P})_r = l_p^r(S)(z_r).
\]

Proof. Since \(l^{(p)}\) and \(l_p^r\) are \(*\)-homomorphisms, it suffices to prove the result when \(S\) is a rank one operator. Let \(x, y \in X_p\). If \(r \not\geq p\), then \(l^{(p)}\) acts as the zero operator on the summand \(X_r\) of \(\mathcal{F}_X\) and \(l_p^r\) is the zero homomorphism. Hence,
\[
(l^{(p)}(\Theta_{x,y}) (z_q)_{q \in P})_r = 0 = l_p^r(\Theta_{x,y})(z_r).
\]
If \(r \geq p\), then Lemma 3.1.25 gives
\[
(l^{(p)}(\Theta_{x,y}) (z_q)_{q \in P})_r = (l_p(x)l_p(y)^*(z_q)_{q \in P})_r = (\Theta_x \circ \Theta_y^*)(z_r) = l_p^r(\Theta_{x,y})(z_r).
\]
Thus, \((l^{(p)}(\Theta_{x,y}) (z_q)_{q \in P})_r = l_p^r(\Theta_{x,y})(z_r)\) as required.

Proposition 3.1.27 ([24], Lemma 5.3). Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\). Then the Fock representation of \(X\) is Nica covariant.

Proof. Let \(p, q \in P\) and fix \(S \in \mathcal{K}_A(X_p)\), \(T \in \mathcal{K}_A(X_q)\). For any \((z_r)_{r \in P} \in \mathcal{F}_X\) and \(t \in P\), Lemma 3.1.26 gives
\[
(l^{(p)}(S)l^{(q)}(T)(z_r)_{r \in P})_t = l_p^t(S)l_q^t(T)(z_t).
\]
(3.2)
If \(p \lor q = \infty\), then this must be zero since \(t\) cannot be greater than both \(p\) and \(q\). On the other hand, if \(p \lor q < \infty\), then Lemma 3.1.5 and another application of Lemma 3.1.26 tells us that (3.2) is equal to
\[
l_{p\lor q}^t(l_{p\lor q}^{(p\lor q)}(S)l_{q\lor q}^{(p\lor q)}(T))(z_t) = (l_{p\lor q}^{(p\lor q)}(S)l_{q\lor q}^{(p\lor q)}(T))(z_r)_{r \in P}_t.
\]
Since \((z_r)_{r \in P} \in \mathcal{F}_X\) and \(t \in P\) were arbitrary, we conclude that
\[
l^{(p)}(S)l^{(q)}(T) = \begin{cases} l_{p\lor q}^{(p\lor q)}(S)l_{q\lor q}^{(p\lor q)}(T) & \text{if } p \lor q < \infty \\ 0 & \text{otherwise,} \end{cases}
\]
and so the Fock representation is Nica covariant.

The following result shows one way of associating a \(C^*\)-algebra to a compactly aligned product system.
CHAPTER 3. FACTORIZATION OF PRODUCT SYSTEMS

Theorem 3.1.28 ([24], Theorem 6.3). Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\). Then there exists a \(C^*\)-algebra \(\mathcal{N}T_X\), which we call the Nica–Toeplitz algebra of \(X\), and a Nica covariant representation \(i_X : X \to \mathcal{N}T_X\), that are universal in the following sense:

(i) \(\mathcal{N}T_X\) is generated by the image of \(i_X\);

(ii) if \(\psi : X \to B\) is any other Nica covariant representation of \(X\), then there exists a *-homomorphism \(\psi_* : \mathcal{N}T_X \to B\) such that \(\psi_* \circ i_X = \psi\).

Remark 3.1.29. Since \((\mathbb{Z}, \mathbb{N})\) is totally ordered, Proposition 3.1.24 says that the product system \(X := \bigsqcup_{n \in \mathbb{N}} X^\otimes n\) associated to a Hilbert \(A\)-bimodule \(X\) is compactly aligned. Moreover, any representation of \(X\) is Nica covariant. Hence, there exists an isomorphism \(\theta : T_X \to \mathcal{N}T_X\) such that \(\theta \circ i_{X}^{\otimes n} = i_X^n\) for each \(n \in \mathbb{N}\).

Proposition 3.1.30. Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\). Then the universal Nica covariant representation \(i_X\) is isometric.

Proof. From Example 3.1.12 and Example 3.1.27, we know that the Fock representation \(l : X \to \mathcal{L}_A(\mathcal{F}X)\) is isometric. By the universal property of \(\mathcal{N}T_X\), there exists a *-homomorphism \(l_* : \mathcal{N}T_X \to \mathcal{L}_A(\mathcal{F}X)\) such that \(l_* \circ i_X = l\). Thus, for any \(x \in X\),

\[
\|x\|_X = \|l(x)\|_{\mathcal{L}_A(\mathcal{F}X)} = \|l_*(i_X(x))\|_{\mathcal{L}_A(\mathcal{F}X)} \leq \|i_X(x)\|_{\mathcal{N}T_X}.
\]

Since every representation is automatically contractive by Proposition 3.1.9, we conclude that \(i_X\) is isometric.

Due to Nica covariance, Nica–Toeplitz algebras have nice spanning sets.

Proposition 3.1.31. Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\). Then

\[
\mathcal{N}T_X = \overline{\text{span}} \{i_X(x)i_X(y)^* : x, y \in X\}.
\]

Proof. The set \(\overline{\text{span}} \{i_X(x)i_X(y)^* : x, y \in X\}\) is closed under taking linear combinations, adjoints, and in norm. An application of Lemma 3.1.22 shows that it is closed under taking products as well. Hence, \(\overline{\text{span}} \{i_X(x)i_X(y)^* : x, y \in X\}\) is a \(C^*\)-subalgebra of \(\mathcal{N}T_X\).

Thus, to prove the above set equality it suffices to show that the subalgebra \(\overline{\text{span}} \{i_X(x)i_X(y)^* : x, y \in X\}\) contains the generators of \(\mathcal{N}T_X\), i.e. the image of \(i_X\).

To see this, observe that for any \(p \in P\) and \(x \in X_p\), if \(x' \in X_p\) is chosen so that \(x = x' \cdot (x', x')_A^p\) by the Hewitt–Cohen–Blanchard factorisation theorem, then

\[
i_{X_p}(x) = i_{X_p}(x \cdot (x, x)_A^p) = i_{X_p}(x)i_{X_p}((x, x)_A^p)^*.
\]
In Proposition 2.2.6 and Proposition 2.2.11 we saw that the Toeplitz algebra and
Cuntz–Pimsner algebra of a Hilbert bimodule carry actions of $\mathbb{T}$, the dual group of $\mathbb{Z}$. We would like to be able to do something similar for product systems over quasi-lattice ordered groups. Since the groups we are dealing with are often not abelian, and so fail to have dual groups, we need to introduce the notion of coactions. In the following definition, we use an unadorned $\otimes$ to denote the minimal tensor product of $C^*$-algebras.

**Definition 3.1.32.** Let $G$ be a discrete group. The universal property of the group $C^*(G)$ induces a $\ast$-homomorphism $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ such that $\delta_G(i_G(g)) = i_G(g) \otimes i_G(g)$ for each $g \in G$ (where $i_G : G \to C^*(G)$ is the universal unitary representation of $G$). A (full) coaction of $G$ on a $C^*$-algebra $A$, is an injective $\ast$-homomorphism $\delta : A \to A \otimes C^*(G)$ satisfying

(i) $\delta$ is nondegenerate: $A \otimes C^*(G) = \text{span} \{ \delta(A)(1_M(A) \otimes C^*(G)) \}$;

(ii) $\delta$ satisfies the coaction identity $(\delta \otimes \text{id}_{C^*(G)}) \circ \delta = (\text{id}_A \otimes \delta) \circ \delta$.

For those readers interested in learning more about coactions in general, we suggest Appendix A of [17].

**Definition 3.1.33.** Let $G$ be a discrete group and $A$ a $C^*$-algebra. Given a coaction $\delta : A \to A \otimes C^*(G)$, we define the generalised fixed-point algebra of $A$ to be

$$A^\delta := \{ a \in A : \delta(a) = a \otimes i_G(e) \}.$$ 

The next result shows that Nica–Toeplitz algebras carry canonical coactions of their underlying quasi-lattice ordered groups.

**Proposition 3.1.34 ([24], Proposition 4.7).** Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$. Then there exists a coaction $\delta_X : \mathcal{NT}_X \to \mathcal{NT}_X \otimes C^*(G)$ such that $\delta_X(i_{X_p}(x)) = i_{X_p}(x) \otimes i_G(p)$ for each $p \in P$, $x \in X_p$.

**Proof.** Define $\psi : X \to \mathcal{NT}_X \otimes C^*(G)$ by $\psi(x) := i_{X_p}(x) \otimes i_G(p)$ for each $x \in X_p$.

We claim that $\psi$ is a Nica covariant representation of $X$.

Since each $i_{X_p}$ is linear, each $\psi_p$ is linear. Since $i_{X_s}$ is a $\ast$-homomorphism, so is $\psi_e$. Hence, $\psi$ satisfies (T1).

To see that $\psi$ satisfies (T2), observe that for any $p, q \in P$ and $x \in X_p$, $y \in X_q$, we have

$$\psi_p(x)\psi_q(y) = (i_{X_p}(x) \otimes i_G(p))(i_{X_q}(y) \otimes i_G(q)) = i_{X_{pq}}(xy)(i_G(p)i_G(q)) = i_{X_{pq}}(xy)(i_G(pq)) = \psi_{pq}(xy).$$
Thus, $\psi$ satisfies (T2).

We now show that $\psi$ satisfies (T3). If $p \in P$ and $x, y \in X_p$, then

$$\psi_p(x)^*\psi_p(y) = \left(i_{X_p}(x) \otimes i_G(p)\right)^* \left(i_{X_p}(y) \otimes i_G(p)\right)$$

$$= \left(i_{X_p}(x)^* \otimes i_G(p)^*\right) \left(i_{X_p}(y) \otimes i_G(p)\right)$$

$$= i_{X_p}(x)^* i_{X_p}(y) \otimes i_G(p)^* i_G(p)$$

$$= i_{X_p}((x, y)^p_A) \otimes i_G(e)$$

$$= \psi_e((x, y)^p_A).$$

Hence, $\psi$ satisfies (T3), and so $\psi$ is a representation.

It remains to check that $\psi$ is Nica covariant. Firstly, we show that

$$\psi^{(p)}(S) = i^{(p)}(S) \otimes i_G(e)$$

for each $S \in K_A(X_p)$. To see this, observe that for any $x, y \in X_p$, we have

$$\psi^{(p)}(\Theta_{x,y}) = \psi_p(x)\psi_p(y)^* = \left(i_{X_p}(x) \otimes i_G(p)\right) \left(i_{X_p}(y) \otimes i_G(p)\right)^*$$

$$= \left(i_{X_p}(x) \otimes i_G(p)\right) \left(i_{X_p}(y)^* \otimes i_G(p)^*\right)$$

$$= i_{X_p}(x)i_{X_p}(y)^* \otimes i_G(p)^* i_G(p)$$

$$= i^{(p)}(\Theta_{x,y}) \otimes i_G(e).$$

Thus, if $S \in K_A(X_p)$ and $T \in K_A(X_q)$, we have

$$\psi^{(p)}(S)\psi^{(q)}(T) = \left(i^{(p)}(S) \otimes i_G(e)\right) \left(i^{(q)}(T) \otimes i_G(e)\right)$$

$$= i^{(p)}(S)i^{(q)}(T) \otimes i_G(e)$$

$$= \begin{cases} 
  i^{(p\lor q)} \left(i^{(p\lor q)}(S)i^{(p\lor q)}(T)\right) \otimes i_G(e) & \text{if } p \lor q < \infty \\
  0 & \text{otherwise}
\end{cases}$$

$$= \begin{cases} 
  \psi^{(p\lor q)}(i^{(p\lor q)}(S)i^{(p\lor q)}(T)) & \text{if } p \lor q < \infty \\
  0 & \text{otherwise}
\end{cases}.$$

Hence, $\psi$ is a Nica covariant representation of $X$. Thus, the universal property of $\mathcal{N}\mathcal{T}_X$ induces a $*$-homomorphism $\delta_X : \mathcal{N}\mathcal{T}_X \to \mathcal{N}\mathcal{T}_X \otimes C^*(G)$ such that

$$\delta_X(i_{X_p}(x)) = i_{X_p}(x) \otimes i_G(p)$$

for each $p \in P$, $x \in X_p$. It remains to check that $\delta_X$ is a coaction of $G$. To see that $\delta_X$ is injective, observe that $\text{id}_{\mathcal{N}\mathcal{T}_X} = (\text{id}_{\mathcal{N}\mathcal{T}_X} \otimes \varepsilon) \circ \delta_X$ where $\varepsilon : C^*(G) \to \mathbb{C}$ is the $*$-homomorphism induced by the unitary representation of $G$ that sends every
element of $G$ to 1. To see that $\delta_X$ is nondegenerate, firstly observe that

\[ \mathcal{N} T_X \otimes C^*(G) = \text{span} \{ i_{X_p}(x) i_{X_q}(y)^* \otimes i_G(g) : p, q \in P, \ g \in G, \ x \in X_p, \ y \in X_q \}. \]

Since, for any $x \in X_p$, $y \in X_q$, $g \in G$, we have

\[
\delta_X (i_{X_p}(x) i_{X_q}(y)^*) (1_{\mathcal{N} T_X} \otimes i_G (qp^{-1} g)) = (i_{X_p}(x) i_{X_q}(y)^* \otimes i_G (pq^{-1} g)) (1_{\mathcal{N} T_X} \otimes i_G (qp^{-1} g)) = (i_{X_p}(x) i_{X_q}(y)^* \otimes i_G (g)),
\]

we see that $\mathcal{N} T_X \otimes C^*(G) = \text{span} \{ \delta_X (\mathcal{N} T_X) (1_{\mathcal{N} T_X} \otimes i_G (G)) \}$ as required.

Finally, it remains to check that $\delta_X$ satisfies the coaction identity:

\[
(\delta_X \otimes \text{id}_{C^*(G)}) \circ \delta_X = (\text{id}_{\mathcal{N} T_X} \otimes \delta_G) \circ \delta_X.
\]

Since both maps are $*$-homomorphisms, it suffices to check equality on the generators of $\mathcal{N} T_X$ — if $x \in X_p$, then

\[
((\delta_X \otimes \text{id}_{C^*(G)}) \circ \delta_X) (i_{X_p}(x)) = (\delta_X \otimes \text{id}_{C^*(G)})(i_{X_p}(x) \otimes i_G(p)) = i_{X_p}(x) \otimes i_G(p) \otimes i_G(p) = (\text{id}_{\mathcal{N} T_X} \otimes \delta_G)(i_{X_p}(x) \otimes i_G(p)) = ((\text{id}_{\mathcal{N} T_X} \otimes \delta_G) \circ \delta_X)(i_{X_p}(x)).
\]

We are almost ready to discuss the notion of Cuntz–Pimsner covariance for representations of compactly aligned product systems over quasi-lattice ordered groups. Before this, we need some more preliminary material from [67].

**Definition 3.1.35.** Let $X$ be a Hilbert $A$-bimodule and $I$ an ideal of $A$. Define

\[ X \cdot I := \{ x \cdot a : x \in X, \ a \in I \}. \]

**Definition 3.1.36.** Let $(G, P)$ be a quasi-lattice ordered group and $X$ a product system over $P$. For each $p \in P$, define the ideal $I_p$ of $A$ by

\[
I_p := \begin{cases} A & \text{if } p = e \\ \cap_{e < q \leq p} \ker (\phi_q) & \text{if } p \neq e. \end{cases}
\]
Using this, for each \( p \in P \) we define the Hilbert \( A \)-bimodule

\[
\tilde{X}_p := \bigoplus_{q \leq p} X_q \cdot \mathbb{I}_{q^{-1}p}.
\]

For each \( p \in P \), we define \( \tilde{\phi}_p : A \to \mathcal{L}_A (\tilde{X}_p) \) for the \(*\)-homomorphism defined by

\[
\left( \tilde{\phi}_p(a)(x) \right)_q := \phi_q(a)(x_q)
\]

for each \( a \in A, x \in \tilde{X}_p, \) and \( q \leq p \).

**Remark 3.1.37.** As in Remark 3.1.4, for each \( p, q \in P \) with \( p \neq e \), we have a \(*\)-homomorphism \( \tilde{\iota}_q^p : \mathcal{L}_A (\tilde{X}_p) \to \mathcal{L}_A (\tilde{X}_q) \) characterised by

\[
\left( \tilde{\iota}_q^p(S)(x) \right)_r := \iota_r^p(S)(x_r)
\]

for each \( x \in \tilde{X}_p \) and \( r \leq q \). Additionally, after identifying \( \mathcal{K}_A (\tilde{X}_e) \) with \( A \), we define \( \tilde{\iota}_q^e : \mathcal{K}_A (\tilde{X}_e) \to \mathcal{L}_A (\tilde{X}_q) \) to be \( \tilde{\phi}_q \) for each \( q \in P \).

To give an example of what the Hilbert bimodule \( \tilde{X}_p \) looks like, we return to higher-rank graphs. Given a \( k \)-graph \( \Lambda \), we define

\[
\Lambda^{\leq n} := \{ \lambda \in \Lambda : d(\lambda) \leq n \text{ and, if } d(\lambda)_i < n_i, \text{ then } s(\lambda) \Lambda^{e_i} = \emptyset \}
\]

for each \( n \in \mathbb{N}^k \). Thus, an element of \( \Lambda^{\leq n} \) is a path of degree no greater than \( n \), that cannot be (backwards) extended nontrivially to a path of degree no greater than \( n \).

**Example 3.1.38.** Let \( \Lambda \) be a finitely aligned \( k \)-graph. Define the product system \( X(\Lambda) \) as in Example 3.1.20. Then \( \tilde{X}(\Lambda) \cong c_0(\Lambda_{\leq n}) \) for each \( n \in \mathbb{N} \).

**Proof.** It is straightforward to check that \( X(\Lambda)_n = \text{span} \{ \delta_\lambda : \lambda \in \Lambda^n \} \) for each \( n \in \mathbb{N}^k \), where \( \delta_\lambda \in c_0(\Lambda^n) \) is the point-mass function defined by \( \delta_\lambda(\mu) := \delta_{\lambda, \mu} \) for each \( \mu \in \Lambda^n \). Thus, for each \( n \in \mathbb{N}^k \), we see that \( \ker(\phi_n) = \text{span} \{ \delta_\nu : v\Lambda^n = \emptyset \} \).

Making use of the factorisation property in \( \Lambda \), we see that for any \( n \in \mathbb{N}^k \setminus \{0\} \), we have

\[
I_n = \bigcap_{0 < r \leq n} \ker(\phi_r) = \text{span} \{ \delta_\nu : v\Lambda^r = \emptyset \text{ for all } 0 < r \leq n \}
\]

\[
= \text{span} \{ \delta_\nu : v\Lambda^{e_i} = \emptyset \text{ for all } i \text{ such that } n_i \neq 0 \}.
\]
Therefore, for each \(m, n \in \mathbb{N}_k\) with \(m < n\), we have

\[
X(\Lambda)_m \cdot I_{n-m} = \text{span} \{ \delta_{\lambda} \cdot \delta_v : \lambda, v \in \Lambda^m, v \Lambda^{e_i} = \emptyset \text{ for all } i \text{ such that } (n-m)_i \neq 0 \} = \text{span} \{ \delta_{\lambda} : \lambda \in \Lambda^m, s(\lambda)\Lambda^{e_i} = \emptyset \text{ for all } i \text{ such that } m_i < n_i \}.
\]

Thus,

\[
\tilde{X}(\Lambda)_n = \bigoplus_{m \leq n} X(\Lambda)_m \cdot I_{n-m} = X(\Lambda)_n \oplus \bigoplus_{m < n} X(\Lambda)_m \cdot I_{n-m} = \text{span} \{ \delta_{\lambda} : d(\lambda) \leq n, s(\lambda)\Lambda^{e_i} = \emptyset \text{ for all } i \text{ such that } m_i < n_i \} = \text{span} \{ \delta_{\lambda} : \lambda \in \Lambda^{\leq n} \}.
\]

Whilst the collection of bimodules \(\{ \tilde{X}_p : p \in P \}\) may resemble a product system, we warn that in general \(\tilde{X}_p \otimes_A \tilde{X}_q\) need not be isomorphic to \(\tilde{X}_{pq}\).

**Definition 3.1.39.** Let \((G, P)\) be a quasi-lattice ordered group. We say that a predicate statement \(P(s)\) (where \(s \in P\)) is true for large \(s\) if, given any \(p \in P\), there exists \(q \geq p\), such that \(P(s)\) is true whenever \(s \geq q\).

Finally, we are ready to present the definition of Cuntz–Pimsner covariance originally formulated by Sims and Yeend in [67]. We give a definition only in the situation that all of the \(*\)-homomorphisms \(\tilde{\phi}_p : A \to \mathcal{L}_A(\tilde{X}_p)\) are injective. We will see later in Proposition 3.1.43 that this is automatically true for a large collection of product systems.

**Definition 3.1.40.** Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\) with coefficient algebra \(A\). Suppose that each homomorphism \(\tilde{\phi}_p : A \to \mathcal{L}_A(\tilde{X}_p)\) is injective. We say that a representation \(\psi : X \to B\) is Cuntz–Pimsner covariant if, for any finite set \(F \subseteq P\) and any choice of compact operators \(\{ T_p \in \mathcal{K}_A(\tilde{X}_p) : p \in F \}\) such that \(\sum_{p \in F} \tilde{\iota}_s(T_p) = 0 \in \mathcal{L}_A(\tilde{X}_s)\) for large \(s\), we have \(\sum_{p \in F} \psi^{(p)}(T_p) = 0\).

Note: we say that a representation is Cuntz–Nica–Pimsner covariant if it is both Nica covariant and Cuntz–Pimsner covariant.

**Theorem 3.1.41 ([67], Proposition 3.12).** Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\). Suppose that each homomorphism \(\tilde{\phi}_p : A \to \mathcal{L}_A(\tilde{X}_p)\) is injective. Then there exists a \(C^*\)-algebra \(\mathcal{N}\mathcal{O}_X\), which we call the Cuntz–Nica–Pimsner algebra of \(X\), and a Cuntz–Nica–Pimsner covariant representation \(j_X : X \to \mathcal{N}\mathcal{O}_X\) such that:
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(i) \( \mathcal{NO}_X = \text{span} \{ j_X(x)j_X(y)^* : x, y \in X \} \);

(ii) the pair \( (\mathcal{NO}_X, j_X) \) is universal in the sense that if \( \psi : X \to B \) is any other Cuntz–Nica–Pimsner covariant representation of \( X \), then there exists a \(*\)-homomorphism \( \Pi \psi : \mathcal{NO}_X \to B \) such that \( \Pi \psi \circ j_X = \psi \).

Sims and Yeend showed that their Cuntz–Nica–Pimsner algebras generalise the Cuntz-Pimsner algebras associated to Hilbert bimodules as defined by Katsura.

Proposition 3.1.42 ([67], Proposition 5.3). Let \( X \) be a Hilbert \( A \)-bimodule and \( X := \bigsqcup_{n \in \mathbb{N}} X^\otimes n \) the associated product system over \( \mathbb{N} \). Then there exists an isomorphism \( \theta : O_X \to \mathcal{NO}_X \) such that \( \theta \circ j^\otimes n_X = j^\otimes n_X \) for each \( n \in \mathbb{N} \).

There are a number of situations where the requirement that all of the \(*\)-homomorphisms \( \tilde{\phi}_p \) are injective is automatic.

Proposition 3.1.43 ([67], Lemma 3.15). Let \( (G, P) \) be a quasi-lattice ordered group and \( X \) a compactly aligned product system over \( P \). Then each \( \tilde{\phi}_p \) is injective if each \( \phi_p \) is injective, or every nonempty bounded subset of \( P \) has a maximal element in the following sense:

If \( S \subseteq P \) is nonempty and there exists \( q \in P \) such that \( p \leq q \) for all \( p \in S \), then there exists \( p \in S \) such that \( p \not\leq p' \) for all \( p' \in S \setminus \{p\} \).

Remark 3.1.44. If each \( \phi_p \) is injective, then \( \tilde{X}_p \cong X_p \) for each \( p \in P \). Moreover, this isomorphism intertwines \( \phi_p \) and \( \tilde{\phi}_p \), as well as \( \iota_p^r \) and \( \tilde{\iota}_p^r \). Thus, in this situation a representation \( \psi : X \to B \) is Cuntz–Pimsner covariant provided, whenever \( F \subseteq P \) is finite and \( \{ T_p \in \mathcal{K}_A(X_p) : p \in F \} \) is such that \( \sum_{p \in F} \iota_p^s(T_p) = 0 \in \mathcal{L}_A(X_s) \) for large \( s \), then \( \sum_{p \in F} \psi^p(T_p) = 0 \).

In [24], Fowler defined a representation \( \psi \) of a product system \( X \) over a semigroup \( P \) (with each \( \phi_p \) injective) to be Cuntz–Pimsner covariant if, for every \( p \in P \), the Toeplitz representation \( (\psi_p, \psi_e) \) of \( X_p \) is Cuntz–Pimsner covariant. The following result shows the relationship between Fowler’s notion of Cuntz–Pimsner covariance and the Cuntz–Pimsner covariance we defined for representations of compactly aligned product systems in Definition 3.1.40. For the next proposition, recall Remark 3.1.14 — we say that a quasi-lattice ordered group \( (G, P) \) is directed if \( p \lor q < \infty \) for every \( p, q \in P \).

Proposition 3.1.45 ([67], Proposition 5.1). Let \( (G, P) \) be a directed quasi-lattice ordered group and \( X \) a compactly aligned product system over \( P \). Suppose each \( \phi_p \) is injective, and let \( \psi : X \to B \) be a representation. Then

(i) If \( \psi \) is Cuntz–Pimsner covariant in the sense of Definition 3.1.40, then

\( (\psi_p, \psi_e)^{(1)}(\phi_p(a)) = \psi_e(a) \) for each \( a \in \phi_p^{-1}(\mathcal{K}_A(X_p)) \), \( p \in P \);
(ii) If $\phi_p(A) \subseteq K_A(X_p)$ and $(\psi_p, \psi_e)^{(i)}(\phi_p(a)) = \psi_e(a)$ for each $a \in A$, $p \in P$, then $\psi$ is Cuntz–Pimsner covariant in the sense of Definition 3.1.40.

We will make use of the following result later.

**Theorem 3.1.46** ([67], Theorem 4.1). Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$. Suppose that each homomorphism $\tilde{\phi}_p : A \to \mathcal{L}_A(X_p)$ is injective. Then $j_X$ is isometric.

The next result shows that the canonical coaction of $G$ on $\mathcal{N}_T X$ descends to $\mathcal{N}_O X$.

**Proposition 3.1.47.** Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$. Suppose that each homomorphism $\tilde{\phi}_p : A \to \mathcal{L}_A(X_p)$ is injective. Then there exists a coaction $\nu_X : \mathcal{N}_O X \to \mathcal{N}_O X \otimes C^*(G)$ such that $\nu_X(j_X(x)) = j_X(x) \otimes i_G(p)$ for each $p \in P$, $x \in X_p$.

**Proof.** Define $\psi : X \to \mathcal{N}_O X \otimes C^*(G)$ by $\psi(x) := j_{X_p}(x) \otimes i_G(p)$ for each $x \in X_p$.

The proof of Proposition 3.1.34 shows that $\psi$ is a Nica covariant representation. We show that it is Cuntz–Pimsner covariant as well. Suppose that $F \subseteq P$ is finite and $\{T_p \in K_A(X_p) : p \in F\}$ is such that $\sum_{p \in F} \tilde{\tau}_p(T_p) = 0$ for large $s$. Since $j_X$ is Cuntz–Pimsner covariant, it follows that

$$\sum_{p \in F} \psi^{(p)}(T_p) = \sum_{p \in F} \left( j_X^{(p)}(T_p) \otimes i_G(e) \right) = \left( \sum_{p \in F} j_X^{(p)}(T_p) \right) \otimes i_G(e) = 0 \otimes i_G(e) = 0.$$

Thus, the representation $\psi$ induces a $*$-homomorphism $\nu_X : \mathcal{N}_O X \to \mathcal{N}_O X \otimes C^*(G)$ such that $\nu_X(j_X(x)) = j_X(x) \otimes i_G(p)$ for each $p \in P$, $x \in X_p$. The same reasoning as in the proof of Proposition 3.1.34, shows that $\nu_X$ is a coaction of $G$. \( \square \)

We have the following gauge-invariant uniqueness theorem for Cuntz–Nica–Pimsner algebras.

**Theorem 3.1.48** ([6], Corollary 4.12). Let $(G, P)$ be a quasi-lattice ordered group with $G$ amenable. Let $X$ be a compactly aligned product system over $P$ with either each $\phi_p$ injective, or each $\tilde{\phi}_p$ injective and $(G, P)$ directed. Then a surjective $*$-homomorphism $\phi : \mathcal{N}_O X \to B$ is injective if and only if

(i) $\phi|_{\mathcal{N}_O(A)}$ is injective; and

(ii) there exists a coaction $\beta : B \to B \otimes C^*(G)$ such that $\beta \circ \phi = (\phi \otimes i_{C^*(G)}) \circ \nu_X$.  


3.2 Combining quasi-lattice ordered groups

In the next few sections we are going to investigate how product systems and their associated $C^*$-algebras decompose given a decomposition of the underlying quasi-lattice ordered group. The next result shows that the direct product of quasi-lattice ordered groups is quasi-lattice ordered. Furthermore, it provides sufficient conditions for a semidirect product of quasi-lattice ordered groups to be quasi-lattice ordered.

**Lemma 3.2.1.** Suppose $(G, P)$ and $(H, Q)$ are quasi-lattice ordered groups and $\alpha : H \to \text{Aut}(G)$ is a group homomorphism such that $\alpha_H(P) \subseteq P$. Then the semidirect product $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ is a quasi-lattice ordered group.

**Proof.** Firstly, we check that $P \rtimes_\alpha Q$ is a subsemigroup of $G \rtimes_\alpha H$. If $(g, h), (g', h') \in P \rtimes_\alpha Q$, then
\[(g, h)(g', h') = (g\alpha_h(g'), hh') \in P \rtimes_\alpha Q\]
since $P$ and $Q$ are subsemigroups of $G$ and $H$ respectively, and $\alpha_Q(P) \subseteq \alpha_H(P) \subseteq P$.

Secondly, if $(g, h) \in (P \rtimes_\alpha Q) \cap (P \rtimes_\alpha Q)^{-1}$ then
\[(g, h)^{-1} = (\alpha_{h^{-1}}(g^{-1}), h^{-1}) \in P \rtimes_\alpha Q.\]

Thus, $h, h^{-1} \in Q$, which forces $h = e_H$ since $(H, Q)$ is quasi-lattice ordered. Hence, $g, \alpha_{h^{-1}}(g^{-1}) = g^{-1} \in P$, which forces $g = e_G$ since $(G, P)$ is also quasi-lattice ordered. Therefore, $(g, h) = (e_G, e_H) = e_{G \rtimes_\alpha H}$, and so $(P \rtimes_\alpha Q) \cap (P \rtimes_\alpha Q)^{-1} = \{e_{G \rtimes_\alpha H}\}$.

Finally, we show that the order on $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ is the product order, i.e. $(g, h) \leq (g', h')$ in $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ if and only if $g \leq g'$ in $(G, P)$ and $h \leq h'$ in $(H, Q)$. Let $g, g' \in G$ and $h, h' \in H$. Suppose $(g, h) \leq (g', h')$ in $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$. Then
\[(g, h)^{-1}(g', h') = (\alpha_{h^{-1}}(g^{-1}g'), h^{-1}h') \in P \rtimes_\alpha Q.\]

Hence, $h^{-1}h' \in Q$ and
\[g^{-1}g' = \alpha_h(\alpha_{h^{-1}}(g^{-1}g')) \in \alpha_Q(P) \subseteq \alpha_H(P) \subseteq P.\]

Hence, $h \leq h'$ and $g \leq g'$. Conversely, if $g \leq g'$ and $h \leq h'$, then $h^{-1}h' \in Q$ and $\alpha_{h^{-1}}(g^{-1}g') \in \alpha_H(P) \subseteq P$, and so $(g, h)^{-1}(g', h') = (\alpha_{h^{-1}}(g^{-1}g'), h^{-1}h') \in P \rtimes_\alpha Q$. Thus, $(g, h) \leq (g', h')$. We conclude that $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ is a quasi-lattice ordered group and for any $(g, h), (g', h') \in P \rtimes_\alpha Q$,
\[(g, h) \vee (g', h') = \begin{cases} (g \vee g', h \vee h') & \text{if } g \vee g', h \vee h' < \infty \\ \infty & \text{otherwise.} \end{cases}\]
Whilst the conditions in Lemma 3.2.1 are sufficient for a semidirect product of quasi-lattice ordered groups to be quasi-lattice ordered, they are not necessary, as the following example due to Laca and Raeburn [38] shows.

Example 3.2.2. Let \((G, P) := (\mathbb{Q}, \mathbb{N})\) with addition as the group operation, and \((H, Q) := (\mathbb{Q}^+, \mathbb{N}_+)\) with multiplication as the group operation. Then \((G, P)\) and \((H, Q)\) are quasi-lattice ordered groups. Define \(\alpha : H \to \text{Aut}(G)\) by \(\alpha_h(g) := hg\). Then \(\alpha_H(P) \not\subseteq P\), but \((G \rtimes_\alpha H, P \rtimes_\alpha Q)\) is a quasi-lattice ordered group.

Proof. It is elementary to check that \((G, P)\) is quasi-lattice ordered. Furthermore, \((G, P)\) is directed with \(x \vee y = \max\{x, y\}\) for any \(x, y \in P\). We already showed that \((H, Q)\) is quasi-lattice ordered in Example 3.1.16 and directed in Remark 3.1.17. Clearly, \(\alpha_H(P) \not\subseteq P\), since \(\alpha^{\mathbb{Q}_+}(1) = \frac{1}{2} \not\in P\). The proof of Lemma 3.2.1 shows that \((P \rtimes_\alpha Q) \cap (P \rtimes_\alpha Q)^{-1} = \{(e_G, e_H)\}\). Observe that for any \((x, y), (s, t) \in G \rtimes_\alpha H\), we have

\[(x, y)^{-1}(s, t) = \left(\frac{s - x}{y}, \frac{t}{y}\right)\]

Now suppose that \((x, y), (u, v) \in G \rtimes_\alpha H\) have a common upper bound \((s, t) \in P \rtimes_\alpha Q\). Thus, \(s \in (x + y\mathbb{N}) \cap (u + v\mathbb{N}) \cap \mathbb{N}\) and \(t \in y\mathbb{N}_+ \cap v\mathbb{N}_+ \cap \mathbb{N}_+\). Using the well-ordering principle, we can define \(p := \min((x + y\mathbb{N}) \cap (u + v\mathbb{N}) \cap \mathbb{N})\) and \(q := \min(y\mathbb{N}_+ \cap v\mathbb{N}_+ \cap \mathbb{N}_+)\). We claim that \((p, q) = (x, y) \vee (u, v)\).

It is elementary to check that \((x, y), (u, v) \leq (p, q)\). To see that \((p, q)\) is the least upper bound for \((x, y)\) and \((u, v)\), suppose that \((x, y), (u, v) \leq (a, b) \leq (p, q)\) for some \((a, b) \in P \rtimes_\alpha Q\).

The first inequality tells us that \(a \in (x + y\mathbb{N}) \cap (u + v\mathbb{N}) \cap \mathbb{N}\) and \(b \in y\mathbb{N}_+ \cap v\mathbb{N}_+ \cap \mathbb{N}_+\). By the definition of \(p\) and \(q\), this forces \(p \leq a\) and \(q \leq b\) (using the usual ordering on the reals). On the other hand, the second inequality tells us that \(q \in b\mathbb{N}_+\), which forces \(q \geq b\) (and so \(q = b\)), and \(p \in a + b\mathbb{N}\), which forces \(p \geq a\) (and so \(p = a\)). Hence, \((p, q) = (a, b)\), and so \((p, q) = (x, y) \vee (u, v)\) as claimed. Putting all of this together, we conclude that the semidirect product \((G \rtimes_\alpha H, P \rtimes_\alpha Q)\) is quasi-lattice ordered.

Remark 3.2.3. The order on \((G \rtimes_\alpha H, P \rtimes_\alpha Q)\) in Example 3.2.2 is not the product order. Indeed for any \((x, y), (u, v) \in P \rtimes_\alpha Q\), we have

\[(x, y) \vee (u, v) = (\min((x + y\mathbb{N}) \cap (u + v\mathbb{N})), \min(y\mathbb{N}_+ \cap v\mathbb{N}_+))\]

if \((x + y\mathbb{N}) \cap (u + v\mathbb{N}) \neq \emptyset\), whilst \((x, y) \vee (u, v) = \infty\) otherwise. On the other hand, \(x \vee u = \max\{x, u\}\) in \((G, P)\) and \(y \vee v = \min(y\mathbb{N}_+ \cap v\mathbb{N}_+) = \text{lcm}\{y, v\}\) in \((H, Q)\). This also shows that \((G \rtimes_\alpha H, P \rtimes_\alpha Q)\) is not directed (for example \((0, 2) \vee (1, 2) = \infty\)), despite both of its factors being directed.
3.3 Factorising Nica–Toeplitz algebras

In this section we investigate product systems over semidirect products of quasi-lattice ordered groups of the sort appearing in Lemma 3.2.1. More precisely, given a product system $Z$ (with coefficient algebra $A$) over a quasi-lattice ordered group of the form $(G \rtimes \alpha H, P \times \alpha Q)$, with $(G, P)$ and $(H, Q)$ quasi-lattice ordered groups and $\alpha_H(P) \subseteq P$, we will show that there exists a product system $X$ (also with coefficient algebra $A$) over $(G, P)$ sitting inside $Z$, and a product system $Y^{NT}$ over $(H, Q)$ with coefficient algebra $NTX$, such that the Nica–Toeplitz algebras of $Z$ and $Y^{NT}$ are isomorphic. In passing from the product system $Z$ to the product system $Y^{NT}$, we have in a sense decreased the size of the product system at the expense of increasing the size of the coefficient algebra, without losing any $C^*$-algebraic information in the process.

To help the reader keep track of everything that is going on, we first provide a brief overview of the key results that we will prove. We also provide a pair of commuting diagrams that summarise the various spaces, and the maps between them, that we are going to be working with as we show that the Nica–Toeplitz algebras of $Z$ and $Y^{NT}$ coincide. We hope that if the reader happens to lose their way in Sections 3.3.1 and 3.3.2 they will be able to return to this overview and the two diagrams for assistance.

(1) In Proposition 3.3.1 we define a product system $X \subseteq Z$ over $(G, P)$, and show that the inclusion of $X$ in $Z$ induces a homomorphism $\phi^{NT}_X$ from $NTX$ to $NTZ$ such that $\phi^{NT}_X \circ i_X = i_Z$.

(2) In Proposition 3.3.2 we argue that the $*$-homomorphism $\phi^{NT}_X$ is injective.

(3) In Proposition 3.3.3, we use $\phi^{NT}_X$ to construct a collection $\{Y^{NT}_q : q \in Q\}$ of Hilbert $NTX$-modules inside $NTZ$.

(4) In Proposition 3.3.6, we use $\phi^{NT}_X$ to show that each $Y^{NT}_q$ carries a left action of $NTX$ by adjointable operators.

(5) In Propositions 3.3.7 and 3.3.12, we show that $Y^{NT} := \bigsqcup_{q \in Q} Y^{NT}_q$ is a compactly aligned product system over $(H, Q)$ with coefficient algebra $NTX$.

Once we have the product system $Y^{NT}$, we prove that $NTY^{NT} \cong NTZ$.

(6) In Proposition 3.3.13 we use the universal Nica covariant representations of $Z$ and $Y^{NT}$ to construct a representation $\varphi^{NT}$ of $Z$ in $NTY^{NT}$.

(7) In Proposition 3.3.15, we prove that $\varphi^{NT}$ is Nica covariant, and use the universal property of $NTZ$ to induce a $*$-homomorphism $\Omega^{NT} : NTZ \to NTY^{NT}$ such that $\Omega^{NT} \circ i_{Z_{(p,q)}} = i_{Y^{NT}_q}$. 


(8) In Proposition 3.3.16 we show that the inclusion of $Y^N T$ in $N T Z$ is a Nica covariant representation, and use the universal property of $N T_{Y^N T}$ to induce a $*$-homomorphism $\Omega^N T : N T_{Y^N T} \to N T Z$ such that $\Omega^N T \circ i_{Y^N T}$ is the inclusion of $Y^N T_q$ in $N T Z$.

(9) In Theorem 3.3.17 we prove that $\Omega^N T$ and $\Omega^N T'$ are mutually inverse isomorphisms.

In summary, we will show that for every $p \in P$ and $q \in Q$, the maps in the following two diagrams exist and make the diagrams commute.

**Figure 3.1:** The homomorphism $\phi^N T_X$

$$
\begin{array}{c}
X_p \xrightarrow{i_{z(p \cdot e_H)}} N T Z \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
N T_X = Y^N T_{e_H} \\
\end{array}
$$

**Figure 3.2:** The homomorphisms $\Omega^N T$, $\Omega'^N T$

$$
\begin{array}{c}
\text{Z(p \cdot q)} \xrightarrow{i_{z(p \cdot q)}} Y^N T_q \equiv \text{incl.} \\
\end{array}
$$

### 3.3.1 Constructing the product system

**Standing Hypotheses.** From now until the end of Section 3.5, we will assume that $(G, P)$ and $(H, Q)$ are quasi-lattice ordered groups and $\alpha : H \to \text{Aut}(G)$ is a group homomorphism such that $\alpha_H(P) \subseteq P$. Thus, by Lemma 3.2.1, $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ is a quasi-lattice ordered group. Moreover, $Z$ will be a compactly aligned product system over $(G \rtimes_\alpha H, P \rtimes_\alpha Q)$ with coefficient algebra $A$. 
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Since $G \cong G \rtimes_{\alpha} \{e_H\}$ is a subgroup of $G \rtimes_{\alpha} H$ and $P \cong P \rtimes_{\alpha} \{e_H\}$ is a subsemigroup of $P \rtimes_{\alpha} Q$, we obtain a product system over the quasi-lattice ordered group $(G, P)$ by considering just those fibres of $Z$ coming from $P \rtimes_{\alpha} \{e_H\}$.

**Proposition 3.3.1.** For each $p \in P$, let $X_p := Z_{(p,e_H)}$. Then $X := \bigsqcup_{p \in P} X_p$ is a compactly aligned product system over $(G, P)$ with coefficient algebra $A$, sitting inside $Z$. Moreover, the inclusion of $X$ in $Z$ induces a $*$-homomorphism $\psi^{NT} : NT_X \to NT_Z$ such that $\phi_X^{NT}(i_X(x)) = i_Z(x)$ for each $x \in X$.

**Proof.** For each $p \in P$, define $\psi_p^{NT} : X_p \to NT_Z$ by $\psi_p^{NT} := i_Z_{(p,e_H)}$. We claim that $\psi^{NT}$ is a Nica covariant representation of $X$ in $NT_Z$. Since $i_Z$ is a representation, we know that each $\psi_p^{NT}$ is linear and $\psi_e^{NT}$ is a $*$-homomorphism. If $p, r \in P$ and $x \in X_p$, $z \in X_r$, then

$$\psi_p^{NT}(x)\psi_r^{NT}(z) = i_Z_{(p,e_H)}(x)i_Z_{(r,e_H)}(z) = i_Z_{(p,e_H)(r,e_H)}(xz) = i_Z_{(pr,e_H)}(xz) = \psi_{pr}^{NT}(xz).$$

For $p \in P$ and $x, z \in X_p$, we have

$$\psi_p^{NT}(x)^*\psi_p^{NT}(z) = i_Z_{(p,e_H)}(x)^*i_Z_{(p,e_H)}(z) = i_Z_{(v_g,e_H)}(\langle x, z \rangle^p_A)$$

$$= i_Z_{(v_g,e_H)}(\langle x, z \rangle^p_A)$$

Thus, $\psi^{NT}$ is a representation. It remains to check that $\psi^{NT}$ is Nica covariant. Fix $p, r \in P$ and $S \in K_A(X_p)$, $T \in K_A(X_r)$. Since $\psi^{NT(t)} = i_Z^{((t,e_H))}$ for any $t \in P$, we see that

$$\psi^{NT(p)}(S)\psi^{NT(r)}(T) = i_Z^{((p,e_H))}(S) i_Z^{((r,e_H))}(T).$$

If $(p, e_H) \lor (r, e_H) = \infty$ (which is precisely when $p \lor r = \infty$), Nica covariance of $i_Z$ tells us that $\psi^{NT(p)}(S)\psi^{NT(r)}(T) = 0$. On the other hand, when $p \lor r < \infty$, we have $(p, e_H) \lor (r, e_H) = (p \lor r, e_H) < \infty$, and so

$$\psi^{NT(p)}(S)\psi^{NT(r)}(T) = i_Z^{((p,e_H) \lor (r,e_H))}(S) i_Z^{((r,e_H) \lor (r,e_H))}(T)$$

$$= i_Z^{((p \lor r,e_H))}(S) i_Z^{((r \lor r,e_H))}(T)$$

$$= \psi^{NT(p \lor r)}(i_Z^{((p \lor r,e_H))}(T))$$

Thus, $\psi^{NT}$ is Nica covariant. The universal property of $NT_X$ then induces a $*$-homomorphism $\phi_X^{NT} : NT_X \to NT_Z$ such that $\phi_X^{NT}(i_X(x)) = \psi^{NT}(x) = i_Z(x)$ for each $x \in X$.

We will use the $*$-homomorphism $\phi_X^{NT}$ to define $NT_X$-valued inner-products on
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an appropriate collection of subspaces of $NT_z$ to obtain a product system over $Q$.
Before we do this, we need to know that $\phi^{NT}_X$ is injective.

**Proposition 3.3.2.** Suppose $G$ is an amenable group. Then the $*$-homomorphism $\phi^{NT}_X : NT_X \rightarrow NT_z$ is injective.

**Proof.** Fix a faithful nondegenerate representation $\rho : A \rightarrow B(H)$ of $A$ on a Hilbert space $H$. Let $l_\cdot : NT_z \rightarrow L_A(F_Z)$ denote the Fock representation of $NT_z$. To show that $\phi^{NT}_X$ is faithful, it suffices to show that

$$F_Z \text{-Ind}^{NT_X}_A \rho = (l_\cdot \circ \phi^{NT}_X) \otimes_A \text{id}_H : NT_X \rightarrow B(F_Z \otimes_A H)$$

is faithful. Denote by $\varrho := (l_\cdot \circ \phi^{NT}_X \circ i_X) \otimes_A \text{id}_H$ the Nica covariant representation of $X$ that induces the $*$-homomorphism $(l_\cdot \circ \phi^{NT}_X) \otimes_A \text{id}_H$. Our aim is to prove that

(i) for each $p \in P \setminus \{e_G\}$,

$$A \otimes_A H = X_{e_G} \otimes_A H \subseteq (\varrho_p(X_p)(F_Z \otimes_A H))^\perp;$$

(ii) $A$ acts faithfully (via $\varrho_{e_G}$) on $A \otimes_A H$.

To see why this suffices, suppose that (i) and (ii) hold. Let $P^\rho_t := \text{proj}_{\varrho_t(X_t)(F_Z \otimes_A H)}$ for each $t \in P \setminus \{e_G\}$. Then the representation

$$A \ni a \mapsto \varrho_{e_G}(a) \prod_{t \in K} (1 - P^\rho_t) \in B(F_Z \otimes_A H)$$

is faithful for each finite subset $K \subseteq P \setminus \{e_G\}$. Since $G$ is amenable, Theorem A.15 then implies that $(l_\cdot \circ \phi^{NT}_X) \otimes_A \text{id}_H$ is faithful as required. We now prove (i) and (ii).

If $p \in P$, then

$$\varrho_p(X_p)(F_Z \otimes_A H) = l_\cdot (iz_{(p,e_H)}(Z_{(p,e_H)}))(F_Z) \otimes_A H$$

$$= \bigoplus_{(s,t) \in P \times Q} Z_{(p,e_H)(s,t)} \otimes_A H$$

$$= \bigoplus_{(s,t) \in P \times Q; p \leq s} Z_{(s,t)} \otimes_A H.$$

Now suppose that $p \in P \setminus \{e_G\}$. We suppose $p \leq e_G$ and derive a contradiction. Then $p^{-1} = p^{-1} e_G \in P$, which forces $p = e_G$, since $P \cap P^{-1} = \{e_G\}$. Thus, $p \not\leq e_G$.

Hence for any $a \in X_{e_G} = A$, $z \in Z_{(s,t)}$ with $p \leq s$, and $h, g \in H$, we see that

$$\langle a \otimes_A h, z \otimes_A g \rangle_C = \langle h, \langle a, z \rangle_A \cdot g \rangle_C = 0,$$
since $A = X_{eG}$ and $Z_{(s,t)}$ are orthogonal in $\mathcal{F}_X$. Since the inner-product on $\mathcal{F}_Z \otimes_A H$ is linear and continuous, we conclude that

$$A \otimes_A H \subseteq (\rho_p(X_p)(\mathcal{F}_Z \otimes_A H))^\perp,$$

which proves (i).

It remains to show that $A$ acts faithfully on $A \otimes_A H$. This follows from ([61], Corollary 2.74) since $A$ acts faithfully on itself by left multiplication and the representation $\rho$ is faithful. \hfill \Box

Using the injective *-homomorphism $\phi_X^{NT} : \mathcal{N}T_X \to \mathcal{N}T_Z$, we can construct a Hilbert $\mathcal{N}T_X$-module $Y_q^{NT}$ for each $q \in Q$.

**Proposition 3.3.3.** Suppose $G$ is an amenable group so that the *-homomorphism $\phi_X^{NT}$ is injective by Proposition 3.3.2. Let $Y_{eH}^{NT} := _{\mathcal{N}T_X}(\mathcal{N}T_X)_{\mathcal{N}T_X}$, and for each $q \in Q \setminus \{e_H\}$, define

$$Y_q^{NT} := \text{span}\left\{i_{Z_{(eG,q)}}(x)\phi_X^{NT}(b) : x \in Z_{(eG,q)}, b \in \mathcal{N}T_X\right\} \subseteq \mathcal{N}T_Z.$$

For each $q \in Q \setminus \{e_H\}$, $Y_q^{NT}$ carries a right action of $\mathcal{N}T_X$ given by

$$y \cdot b := y\phi_X^{NT}(b)$$

for each $y \in Y_q^{NT}$ and $b \in \mathcal{N}T_X$. There is an $\mathcal{N}T_X$-valued inner-product on $Y_q^{NT}$ such that

$$\phi_X^{NT}(\langle y, w \rangle^{\mathcal{N}T_X}_q) = y^*w$$

for all $y, w \in Y_q^{NT}$. With this structure, each $Y_q^{NT}$ is a right Hilbert $\mathcal{N}T_X$-module.

**Proof.** Since multiplication in $\mathcal{N}T_Z$ is continuous, it is clear that the right action of $\mathcal{N}T_X$ on $Y_q^{NT}$ is well-defined. Next, we check that $y^*w \in \phi_X^{NT}(\mathcal{N}T_X)$ for any $y, w \in Y_q^{NT}$. Since $\phi_X^{NT}$ is a *-homomorphism, it suffices to check the case where $y = i_{Z_{(eG,q)}}(x)\phi_X^{NT}(b)$ and $w = i_{Z_{(eG,q)}}(z)\phi_X^{NT}(c)$ for some $x, z \in Z_{(eG,q)}$, and $b, c \in \mathcal{N}T_X$. For $i_Z$ is a representation, we see that

$$y^*w = \phi_X^{NT}(b)^*i_{Z_{(eG,q)}}(x)^*i_{Z_{(eG,q)}}(z)\phi_X^{NT}(c)$$

$$= \phi_X^{NT}(b)^*i_{Z_{(eG,q)}}((x, z)_{A_{(eG)}})\phi_X^{NT}(c)$$

$$= \phi_X^{NT}(b^*i_{Z_{(eG,q)}}((x, z)_{A_{(eG)}})c)$$

$$\in \phi_X^{NT}(\mathcal{N}T_X).$$

Thus, since $\phi_X^{NT}$ is injective, we may define $\langle \cdot, \cdot \rangle_q^{\mathcal{N}T_X} : Y_q^{NT} \times Y_q^{NT} \to \mathcal{N}T_X$ by $\langle y, w \rangle_q^{\mathcal{N}T_X} := (\phi_X^{NT})^{-1}(y^*w)$ for each $y, w \in Y_q^{NT}$. Since $\phi_X^{NT}$ is linear, it follows
that \( \langle \cdot, \cdot \rangle_{\mathcal{N}^T X} \) is complex linear in its second argument. If \( y, w \in \mathcal{Y}^q \) and \( b \in \mathcal{N}^T X \), then
\[
\langle y, w \cdot b \rangle_{\mathcal{N}^T X}^q = \langle y, w\phi^q_{X}(b) \rangle_{\mathcal{N}^T X}^q = (\phi^q_{X})^{-1}(y^*w\phi^q_{X}(b)) \\
= (\phi^q_{X})^{-1}(y^*w)b \\
= \langle y, w \rangle_{\mathcal{N}^T X}^q b,
\]
and
\[
\langle y, w \rangle_{\mathcal{N}^T X}^q = (\phi^q_{X})^{-1}(y^*w) = (\phi^q_{X})^{-1}(w^*y)^* = (\langle w, y \rangle_{\mathcal{N}^T X}^q)^*.
\]
Also, if \( y \in \mathcal{Y}^q \), then
\[
\langle y, y \rangle_{\mathcal{N}^T X}^q = (\phi^q_{X})^{-1}(y^*y) \geq 0
\]
since \( y^*y \geq 0 \) in \( \mathcal{N}^T Z \). Moreover, if \( \langle y, y \rangle_{\mathcal{N}^T X}^q = 0 \), then \( y^*y = 0 \) because \( \phi^q_{X} \) is injective, which forces \( y = 0 \). Lastly, we show that the norm induced by \( \langle \cdot, \cdot \rangle_{\mathcal{N}^T X}^q \) is just the norm on \( \mathcal{N}^T Z \) restricted to \( \mathcal{Y}^q \). Since \( \phi^q_{X} \) is isometric, we see that for any \( y \in \mathcal{Y}^q \),
\[
\|y\|_{\mathcal{Y}^q}^2 := \|\langle y, y \rangle_{\mathcal{N}^T X}^q\|_{\mathcal{N}^T X} = \|\langle \phi^q_{X} \rangle^{-1}(y^*y)\|_{\mathcal{N}^T X} = \|y^*y\|_{\mathcal{N}^T Z} = \|y\|_{\mathcal{N}^T Z}^2.
\]
As \( \mathcal{Y}^q \) is, by definition, closed in \( \mathcal{N}^T Z \), we see that \( \mathcal{Y}^q \) is complete with respect to the norm induced by \( \langle \cdot, \cdot \rangle_{\mathcal{N}^T X}^q \). Putting all of this together, we conclude that \( \mathcal{Y}^q \) is a right Hilbert \( \mathcal{N}^T X \)-module. \( \square \)

Remark 3.3.4. In the situation of Proposition 3.3.3, since \( \alpha_H(P) \subseteq P \), it follows that \( \alpha_q(P) = P \) for each \( q \in Q \). Hence, for any \( q \in Q \setminus \{e_H\} \),
\[
\mathcal{Y}^q = \overline{\text{span}} \left\{ i_{Z_{(c, q)}}(Z_{(e_H, q)}) \phi^q_{X}(\mathcal{N}^T X) \right\} \\
= \overline{\text{span}} \left\{ i_{Z_{(c, q)}}(Z_{(e_H, q)}) i_{Z_{(p, e_H)}}(Z_{(p, e_H)})(Z_{(r, e_H)})^* : p, r \in P \right\} \\
= \overline{\text{span}} \left\{ i_{Z_{(c, q)}}(Z_{(e_H, q)}) i_{Z_{(p, e_H)}}(Z_{(p, e_H)})(Z_{(r, e_H)})^* : p, r \in P \right\} \\
= \overline{\text{span}} \left\{ i_{Z_{(c, q)}}(Z_{(p, e_H)})(Z_{(p, e_H)})(Z_{(r, e_H)})^* : p, r \in P \right\}.
\]
Furthermore,
\[
\mathcal{Y}^q_{e_H} = \mathcal{N}^T X \cong \phi^q_{X}(\mathcal{N}^T X) = \overline{\text{span}} \left\{ i_{Z_{(p, e_H)}}(Z_{(p, e_H)})(Z_{(p, e_H)})(Z_{(r, e_H)})^* : p, r \in P \right\}.
\]

Remark 3.3.5. It follows from an application of the Hewitt–Cohen–Blanchard factorisation theorem to \( Z_{(e_H, q)} \) that \( i_{Z_{(c, q)}}(Z_{(e_H, q)}) \subseteq \mathcal{Y}^q \). If \( z \in Z_{(c, q)} \), and
$z' \in \mathbb{Z}_{(e_G,q)}$ is chosen so that $z = z' \cdot (z', z')_{A}^{(e_G,q)}$ by the Hewitt–Cohen–Blanchard factorisation theorem, then

$$i_{z_{e_G,q}}(z) = i_{z_{e_G,q}}(z' \cdot (z', z')_{A}) = i_{z_{e_G,q}}(z') f_{X}^{NT} \left( i_{x_{e_G}}((z', z')_{A}) \right)$$

$$= i_{z_{e_G,q}}(z') \phi_{X}^{NT} \left( i_{x_{e_G}}((z', z')_{A}) \right)$$

$$\in Y_q^{NT}.$$ 

We now show that $Y_q^{NT}$ also carries a left action of $\mathcal{N}T_X$ by adjointable operators for each $q \in Q \setminus \{e_H\}$, and hence is a Hilbert $\mathcal{N}T_X$-bimodule.

**Proposition 3.3.6.** Suppose $G$ is an amenable group so that the Hilbert $\mathcal{N}T_X$-module $Y_q^{NT}$ of Proposition 3.3.3 is defined. For each $q \in Q \setminus \{e_H\}$, there exists a $*$-homomorphism $\phi_{q}^{NT} : \mathcal{N}T_X \to \mathcal{L}_{NT_X} (Y_q^{NT})$ such that

$$\phi_{q}^{NT}(b)(y) = \phi_{X}^{NT}(b)y$$

for each $b \in \mathcal{N}T_X$ and $y \in Y_q^{NT}$.

**Proof.** Firstly, we check that $\phi_{X}^{NT} (\mathcal{N}T_X) Y_q^{NT} \subseteq Y_q^{NT}$. Since $(G \rtimes_{\alpha} H, P \rtimes_{\alpha} Q)$ has the product order, if $s, t \in P$, then

$$(s, e_H)(t, e_H)^{-1} ((t, e_H) \vee (e_G, q)) = (s, e_H)(t^{-1}, e_H)(t, q)$$

$$= (s, q)$$

$$= (e_G, q)(\alpha_{q^{-1}}(s), e_H)$$

and

$$(e_G, q)^{-1} ((t, e_H) \vee (e_G, q)) = (e_G, q^{-1})(t, q) = (\alpha_{q^{-1}}(t), e_H).$$

Thus, for any $s, t \in P$, $x \in X_s$, $y \in X_t$, $z \in \mathbb{Z}_{(e_G,q)}$, and $b \in \mathcal{N}T_X$, applying Lemma 3.1.22 to the Nica covariant representation $i_{z}$, we have

$$\phi_{X}^{NT} \left( i_{x_{e_G}}(x) i_{x_{e_G}}(y^{*}) i_{z_{e_G,q}}(z) \phi_{X}^{NT}(b) \right)$$

$$= i_{z_{(s, e_H)}}(x) i_{z_{(t, e_H)}}(y^{*}) i_{z_{e_G,q}}(z) \phi_{X}^{NT}(b)$$

$$\subseteq \text{span} \left\{ i_{z_{e_G,q}}(Z_{(e_G,q)}) i_{Z_{(\alpha_{q^{-1}}(s), e_H)}} \left( Z_{(\alpha_{q^{-1}}(s), e_H)} \right) i_{Z_{(\alpha_{q^{-1}}(t), e_H)}} \left( Z_{(\alpha_{q^{-1}}(t), e_H)} \right)^{*} \phi_{X}^{NT}(b) \right\}$$

$$= \text{span} \left\{ i_{Z_{(e_G,q)}}(Z_{(e_G,q)}) \phi_{X}^{NT} \left( i_{x_{\alpha_{q^{-1}}(s)}} \left( X_{\alpha_{q^{-1}}(s)} \right) i_{x_{\alpha_{q^{-1}}(t)}} \left( X_{\alpha_{q^{-1}}(t)} \right)^{*} \phi_{X}^{NT}(b) \right) \right\}$$

$$\subseteq \text{span} \left\{ i_{Z_{(e_G,q)}}(Z_{(e_G,q)}) \phi_{X}^{NT} (\mathcal{N}T_X) \right\}$$

$$= Y_q^{NT}.$$
We begin by checking that \( N^T_X = \text{span} \{ i_{X_s} (X_s) i_{X_t} (X_t)^* : s, t \in P \} \) and 

\[
Y^N_T = \text{span} \left\{ i_{Z(e_{G,q})} (Z(e_{G,q})) \phi_X^N (N^T_X) \right\}, \]

whilst \( \phi_X^N \) is a *-homomorphism and multiplication in \( N^T_Z \) is bilinear and continuous, we conclude that \( \phi_X^N (N^T_X) Y^N_T \subseteq Y^N_T \) as required. Thus, for each \( b \in N^T_X \), we may define \( \Phi_q^N (b) : Y_q^N \rightarrow Y_q^N \) by \( \Phi_q^N (b) (y) = \phi_X^N (b) y \) for each \( y \in Y_q^N \). Next, we claim that \( \Phi_q^N (b) \) is adjointable with \( \Phi_q^N (b)^* = \Phi_q^N (b^*) \). To see this, observe that for any \( y, w \in Y_q^N \),

\[
\langle \Phi_q^N (b) (y), w \rangle_{N^T_X}^q = \langle \phi_X^N (b) y, w \rangle_{N^T_X}^q = (\phi_X^N)^{-1} (y^* \phi_X^N (b^*) w) = (y, \phi_X^N (b^*) w)_{N^T_X}^q = (y, \Phi_q^N (b^*) (w))_{N^T_X}^q.
\]

Finally, since \( \phi_X^N \) is linear and multiplicative, the map \( b \mapsto \Phi_q^N (b) \) is also linear and multiplicative. Thus, \( b \mapsto \Phi_q^N (b) \) is a *-homomorphism from \( N^T_X \) to \( L_{N^T_X} (Y_q^N) \).

Next we show that \( Y^N_T := \bigsqcup_{q \in Q} Y_q^N \) can be viewed as a product system over the quasi-lattice ordered group \((H, Q)\) with coefficient algebra \( N^T_X \).

**Proposition 3.3.7.** Suppose \( G \) is an amenable group so that the Hilbert \( N^T_X \)-bimodule \( Y_q^N \) from Propositions 3.3.3 and 3.3.6 is defined. Let \( Y^N_T := \bigsqcup_{q \in Q} Y_q^N \). Then \( Y^N_T \) is a product system over \((H, Q)\) with coefficient algebra \( N^T_X \), and multiplication given by multiplication in \( N^T_Z \).

**Proof.** We already know from Propositions 3.3.3 and 3.3.6 that each \( Y_q^N \) is a Hilbert \( N^T_X \)-bimodule, and \( Y_{e_H}^N = N^T_X (N^T_X)_{N^T_X} \) by definition. If we equip \( Y^N_T \) with the associative multiplication from \( N^T_Z \) and identify \( N^T_X \) with \( \phi_X^N (N^T_X) \subseteq N^T_Z \), then \( Y^N_T \) becomes a semigroup. It is straightforward to check that multiplication in \( Y^N_T \) by elements of \( Y_{e_H}^N = N^T_X \) implements the left and right actions of \( N^T_X \) on each \( Y_q^N \).

For \( Y^N_T \) to be a product system, it remains to show that there exists a Hilbert \( N^T_X \)-bimodule isomorphism \( M_{q,t}^{Y^N_T} : Y_q^N \otimes_{N^T_X} Y_t^N \rightarrow Y_{qt}^N \) for each \( q, t \in Q \setminus \{ e_H \} \) such that

\[
M_{q,t}^{Y^N_T} (y \otimes_{N^T_X} w) = yw \quad \text{for each } y \in Y_q^N, w \in Y_t^N.
\]

We begin by checking that \( Y_q^N Y_t^N \subseteq Y_{qt}^N \) for each \( q, t \in Q \). Making use of
Proposition 3.3.6, we see that

\[
Y_q^{NT} Y_t^{NT} = \text{span} \left\{ i_z(x,q) (Z_{(e_g,q)}) \phi_x^{NT} (N^T X) \right\} \text{span} \left\{ i_z(x,t) (Z_{(e_g,t)}) \phi_x^{NT} (N^T X) \right\}
\]

\[
= \text{span} \left\{ i_z(x,q) (Z_{(e_g,q)}) \phi_x^{NT} (N^T X) i_z(x,t) (Z_{(e_g,t)}) \phi_x^{NT} (N^T X) \right\}
\]

\[
\subseteq \text{span} \left\{ i_z(x,q) (Z_{(e_g,q)}) i_z(x,t) (Z_{(e_g,t)}) \phi_x^{NT} (N^T X) \right\}
\]

\[
\subseteq \text{span} \left\{ i_z(x,q) (Z_{(e_g,q)}(e_g,t)) \phi_x^{NT} (N^T X) \right\}
\]

\[
= \text{span} \left\{ i_z(x,q) (Z_{(e_g,q)}) \phi_x^{NT} (N^T X) \right\}
\]

\[
\subseteq Y_q^{NT}
\]

Next, observe that if \(y, u \in Y_q^{NT}\) and \(w, v \in Y_t^{NT}\), then

\[
\langle y \otimes N^T X w, u \otimes N^T X v \rangle_{N^T X} = \langle w, (y, u)_{N^T X}^* v \rangle_{N^T X}
\]

\[
= \langle w, \phi \left( (\phi_x^{NT})^{-1}(y^* u) \right) v \rangle_{N^T X}
\]

\[
= (\phi_x^{NT})^{-1}(w^* y^* u v)
\]

\[
= \langle yw, uv \rangle_{N^T X}^{qt}
\]

Thus, the map \(y \otimes N^T X w \mapsto yw\) extends by linearity and continuity to a well-defined inner-product preserving map from \(Y_q^{NT} \otimes_{N^T X} Y_t^{NT}\) to \(Y_q^{NT}\), which we denote by \(M_q^{Y^{NT}}\). Clearly, \(M_q^{Y^{NT}}\) is both left and right \(N^T X\)-linear. Surjectivity of \(M_q^{Y^{NT}}\) follows from the fact that \(Z_{(e_g,q)} = Z_{(e_g,q)(e_g,t)} \cong Z_{(e_g,q)} \otimes_A Z_{(e_g,t)}\) (since \(q \neq e_H\)) and \(i_z(x,q) (Z_{(e_g,q)}) \subseteq Y_q^{NT}\):

\[
Y_q^{NT} = \text{span} \left\{ i_z(x,q) (Z_{(e_g,q)}) \phi_x^{NT} (N^T X) \right\}
\]

\[
= \text{span} \left\{ i_z(x,q) (Z_{(e_g,q)}) i_z(x,t) (Z_{(e_g,t)}) \phi_x^{NT} (N^T X) \right\}
\]

\[
= i_z(x,q) (Z_{(e_g,q)}) \text{span} \left\{ i_z(x,t) (Z_{(e_g,t)}) \phi_x^{NT} (N^T X) \right\}
\]

\[
\subseteq Y_q^{NT} Y_t^{NT}
\]

\[
= M_q^{Y^{NT}} (Y_q^{NT} \otimes_{N^T X} Y_t^{NT})
\]

Putting all of this together, we conclude that \(M_q^{Y^{NT}}\) is an \(N^T X\)-bimodule isomorphism.

We now prove that if \(A\) acts faithfully on \(Z_{(e_g,q)}\), then the \(*\)-homomorphism \(\phi_q^{NT} : N^T X \rightarrow L_{N^T X} (Y_q^{NT})\) is injective. We will make use of this result later in Subsection 3.5 when we consider the Cuntz–Nica–Pimsner algebra of the product system \(Y^{NT}\).
Proposition 3.3.8. Suppose $G$ is an amenable group so that the product system $Y^N_T$ from Proposition 3.3.7 is defined. If $q \in Q$ and $A$ acts faithfully on $Z_{(eg,q)}$, then the $*$-homomorphism $\Phi^N_T : N^*_T \rightarrow L_{N^*_T} (Y^N_T)$ is injective.

Proof. When $q = e_H$, the map $\Phi^N_T$ is just left multiplication on $Y^N_T = N^*_T X$ by elements of $N^*_T X$, which is obviously faithful. So suppose that $q \in Q \setminus \{ e_H \}$. Let $\rho : A \rightarrow B(\mathcal{H})$ be a faithful nondegenerate representation of $A$ on a Hilbert space $\mathcal{H}$. To prove that $\Phi^N_T$ is faithful, it suffices to show that the induced representation

$$(Y^N_T \otimes_{N^*_T X} F_X) \text{Ind}_{A}^N \rho = \Phi^N_T \otimes_{N^*_T X} \text{id}_{F_X} \otimes_A \text{id}_{\mathcal{H}} : N^*_T X \rightarrow B(Y^N_T \otimes_{N^*_T X} F_X \otimes_A \mathcal{H})$$

is faithful. Let $\varrho := (\Phi^N_T \otimes_{N^*_T X} \text{id}_{F_X} \otimes_A \text{id}_{\mathcal{H}}) \circ \iota_X$ denote the Nica covariant representation of $X$ that induces $\Phi^N_T \otimes_{N^*_T X} \text{id}_{F_X} \otimes_A \text{id}_{\mathcal{H}}$. Our aim is to prove that

(i) for each $p \in P \setminus \{ e_G \}$,

$$i_{(eg,a)} (Z_{(eg,q)}) \otimes_{N^*_T X} A \otimes \mathcal{A} \mathcal{H} \subseteq (\varrho_p (X_p) (Y^N_T \otimes_{N^*_T X} F_X \otimes_A \mathcal{H}))^\perp;$$

(ii) $A$ acts faithfully (via $\varrho_{eg}$) on $i_{(eg,a)} (Z_{(eg,q)}) \otimes_{N^*_T X} A \otimes \mathcal{A} \mathcal{H}$.

To see that this suffices, suppose for a moment that (i) and (ii) hold. For each $t \in P \setminus \{ e_G \}$, let $P_t^\rho := \text{proj}_{\varrho_t (X_t) (Y^N_T \otimes_{N^*_T X} F_X \otimes_A \mathcal{H})}$.

Then the representation

$$A \ni a \mapsto \varrho_{eg}(a) \prod_{t \in K} (1 - P_t^\rho) \in B \left( Y^N_T \otimes_{N^*_T X} F_X \otimes_A \mathcal{H} \right)$$

is faithful for each finite subset $K \subseteq P \setminus \{ e_G \}$. Since $G$ is amenable, Theorem A.15 then implies that $\Phi^N_T \otimes_{N^*_T X} \text{id}_{F_X} \otimes_A \text{id}_{\mathcal{H}}$ is faithful as required.

We now prove (i) and (ii). If $p \in H$, then

$$\Phi^N_T (i_{X_p} (X_p)) (Y^N_T)$$

$$= \bigoplus \{ i_{(p,e_H)} (Z_{(p,e_H)}) i_{Z_{(s,q)}} (Z_{(s,q)}) i_{Z_{(t,e_H)}} (Z_{(t,s)}}) : s, t \in P \}$$

$$= \bigoplus \{ i_{Z_{(p,q)}} (Z_{(p,q)}) i_{Z_{(t,e_H)}} (Z_{(t,e_H)})^* : s, t \in P \}$$

$$= \bigoplus \{ i_{Z_{(eg,q)}} (Z_{(eg,q)}) i_{Z_{(\alpha_q^{-1} (ps),e_H)}} (Z_{(\alpha_q^{-1} (ps),e_H)}) i_{Z_{(t,e_H)}} (Z_{(t,e_H)})^* : s, t \in P \}$$

$$= i_{Z_{(eg,q)}} (Z_{(eg,q)}) \bigoplus \{ i_{X_{\alpha_q^{-1} (ps)}} (X_{\alpha_q^{-1} (ps)}) i_{X_t} (X_t) : s, t \in P \}.$$
Hence,

\[ q_p(X_p) \left( Y_q^{\mathcal{N}T} \otimes_{\mathcal{N}T_X} \mathcal{F}_X \otimes_A \mathcal{H} \right) = \Phi_q^{\mathcal{N}T} \left( i_{X_p}(X_p) \right) \left( Y_q^{\mathcal{N}T} \right) \otimes_{\mathcal{N}T_X} \mathcal{F}_X \otimes_A \mathcal{H} \]

\[ = i_{Z_{(e_G,q)}}(Z_{(e_G,q)}) \mathcal{N}T_X \text{span} \left\{ i_{X_{a_{q-1}(p)}}(X_{a_{q-1}(p)}) \cdot i_{X_i}(X_i)^* : s, t \in P \right\} \otimes_{\mathcal{N}T_X} \mathcal{F}_X \otimes_A \mathcal{H} \]

\[ = i_{Z_{(e_G,q)}}(Z_{(e_G,q)}) \mathcal{N}T_X \text{span} \left\{ i_{X_{a_{q-1}(p)}}(X_{a_{q-1}(p)}) \cdot i_{X_i}(X_i)^* : s, t \in P \right\} \cdot \mathcal{F}_X \otimes_A \mathcal{H} \]

\[ \subseteq i_{Z_{(e_G,q)}}(Z_{(e_G,q)}) \mathcal{N}T_X \bigoplus_{\{ m \in P : a_{q-1}(p) \leq m \}} X_m \otimes_A \mathcal{H} . \]

Now suppose that \( p \in P \setminus \{ e_G \} \). Looking for a contradiction, suppose \( a_{q-1}(p) \leq e_G \). Thus, \( a_{q-1}(p)^{-1} = e_G \in P \). However, since \( \alpha_H(P) \subseteq P \), we also have that \( a_{q-1}(p)^{-1} \in \alpha_Q(H)^{-1} \subseteq P^{-1} \). Hence, \( a_{q-1}(p)^{-1} \in P \cap P^{-1} = \{ e_G \} \), which forces \( p = e_G \). Hence for any \( z, w \in Z_{(e_G,q)} \), \( a \in X_{e_G} = A \), \( x \in X_m \) with \( a_{q-1}(p) \leq m \), and \( h, g \in \mathcal{H} \), we see that

\[ \left\langle i_{Z_{(e_G,q)}}(z) \otimes_{\mathcal{N}T_X} a \otimes_A h, i_{Z_{(e_G,q)}}(w) \otimes_{\mathcal{N}T_X} x \otimes_A g \right\rangle \mathcal{C} = \left\langle h, \left\langle a, \left\langle i_{Z_{(e_G,q)}}(z), i_{Z_{(e_G,q)}}(w) \right\rangle_{\mathcal{N}T_X} \cdot x \right\rangle_A \cdot g \right\rangle \mathcal{C} \]

\[ = \left\langle h, \left\langle a, i_{X_{e_G}}(\langle z, w \rangle_{A^{(e_G,q)}}) \cdot x \right\rangle_A \cdot g \right\rangle \mathcal{C} \]

\[ = 0 , \]

since \( i_{X_{e_G}}(\langle z, w \rangle_{A^{(e_G,q)}}) \cdot x \in X_m \) which is orthogonal to \( A = X_{e_G} \) in the Fock space \( \mathcal{F}_X \). Since the inner-product on \( Y_q^{\mathcal{N}T} \otimes_{\mathcal{N}T_X} \mathcal{F}_X \otimes_A \mathcal{H} \) is linear and continuous, we conclude that

\[ i_{Z_{(e_G,q)}}(Z_{(e_G,q)}) \mathcal{N}T_X A \otimes_A \mathcal{H} \subseteq \left( q_p \left( X_p \right) \left( Y_q^{\mathcal{N}T} \otimes_{\mathcal{N}T_X} \mathcal{F}_X \otimes_A \mathcal{H} \right) \right) \mathcal{C} . \]

Next we check that \( A \) acts faithfully on \( i_{Z_{(e_G,q)}}(Z_{(e_G,q)}) \mathcal{N}T_X A \otimes_A \mathcal{H} \) via the \( * \)-homomorphism \( \rho_{e_G} = (\Phi_q^{\mathcal{N}T} \circ i_{X_{e_G}}) \mathcal{N}T_X \text{id}_{\mathcal{F}_X} \otimes_A \text{id}_{\mathcal{H}} \). Fix \( a \in A \setminus \{ 0 \} \). Since \( A \) acts faithfully on \( Z_{(e_G,q)} \), we can choose \( z \in Z_{(e_G,q)} \) such that \( a \cdot z \neq 0 \). Since \( \rho \) is faithful, we can then choose \( h \in \mathcal{H} \) such that \( \langle h, \langle a \cdot z, a \cdot z \rangle_{A^{(e_G,q)}} \cdot h \rangle \mathcal{C} \neq 0 \). By the Hewitt–Cohen–Blanchard factorisation theorem, we can write \( z = z' \cdot (z', z')_{A^{(e_G,q)}} \) for some \( z' \in Z_{(e_G,q)} \). Now,

\[ \left\langle i_{Z_{(e_G,q)}}(a \cdot z') \otimes_{\mathcal{N}T_X} \langle z', z' \rangle_{A^{(e_G,q)}} \rangle_A, i_{Z_{(e_G,q)}}(a \cdot z') \otimes_{\mathcal{N}T_X} \langle z', z' \rangle_{A^{(e_G,q)}} \rangle_A \]

\[ = \left\langle \langle z', z' \rangle_{A^{(e_G,q)}}, i_{Z_{(e_G,q)}}(a \cdot z'), i_{Z_{(e_G,q)}}(a \cdot z') \right\rangle_{\mathcal{N}T_X} \cdot \langle z', z' \rangle_{A^{(e_G,q)}} \rangle_A \]

\[ = \left\langle \langle z', z' \rangle_{A^{(e_G,q)}}, i_{X_{e_G}}(\langle a \cdot z', a \cdot z' \rangle_{A^{(e_G,q)}}) \cdot \langle z', z' \rangle_{A^{(e_G,q)}} \rangle_A \right. \]
Lemma 3.3.10. Suppose \( a \cdot z, a \cdot z \in \mathcal{A} \). Let \( M_{\mathcal{T}} \) be aligned. The next result characterises the compact operators on each fibre of \( Y \).

Thus, \( A \) acts faithfully on \( i_{\mathcal{Z}(e_{G,q})}(\mathcal{Z}(e_{G,q})) \otimes_{\mathcal{N}_{\mathcal{T}}A} A \otimes A \mathcal{H} \).

We now work towards showing that the product system \( Y_{q,T} \) is compactly aligned. The next result characterises the compact operators on each fibre of \( Y_{q,T} \).

We need some more notation: given a \( C^* \)-algebra \( B \), for each \( b \in B \), we write \( M_b \in \mathcal{L}_B(B_B) \) for the map defined by \( M_b(c) := bc \) for each \( c \in B \).

\[ \left\| \varrho_{b}(a) \left( i_{\mathcal{Z}(e_{G,q})}(z') \otimes_{\mathcal{N}_{\mathcal{T}}A} (z', z')^{(e_{G,q})} \otimes A \right) \right\|^2 \]

Thus, \( A \) acts faithfully on \( i_{\mathcal{Z}(e_{G,q})}(\mathcal{Z}(e_{G,q})) \otimes_{\mathcal{N}_{\mathcal{T}}A} A \otimes A \mathcal{H} \).

Lemma 3.3.9. Suppose \( G \) is an amenable group so that the Hilbert \( \mathcal{N}_{\mathcal{T}}A \)-bimodule \( Y_{q,T}^{(r)} \) from Propositions 3.3.3 and 3.3.6 is defined. For each \( q \in Q \), if \( b \in \mathcal{N}_{\mathcal{T}}A \) is such that \( M_b \in \mathcal{L}_{\mathcal{N}_{\mathcal{T}}A}(Y_{q,T}^{(r)}) \), then

\[ \left\| M_b \right\|_{\mathcal{L}_{\mathcal{N}_{\mathcal{T}}A}(Y_{q,T}^{(r)})} \leq \left\| b \right\|_{\mathcal{N}_{\mathcal{T}}A} . \]

Proof. Since the norm on \( Y_{q,T}^{(r)} \subseteq \mathcal{N}_{\mathcal{T}}A \) is just the restriction of the norm on \( \mathcal{N}_{\mathcal{T}}A \), and the norm on any \( C^* \)-algebra is submultiplicative, we have

\[ \left\| M_b \right\|_{\mathcal{L}_{\mathcal{N}_{\mathcal{T}}A}(Y_{q,T}^{(r)})} = \sup_{\left\| y \right\|_{Y_{q,T}^{(r)}} \leq 1} \left\| M_b(y) \right\|_{Y_{q,T}^{(r)}} \leq \sup_{\left\| y \right\|_{\mathcal{N}_{\mathcal{T}}A} \leq 1} \left\| y \right\|_{\mathcal{N}_{\mathcal{T}}A} \leq \left\| b \right\|_{\mathcal{N}_{\mathcal{T}}A} . \]

Lemma 3.3.10. Suppose \( G \) is an amenable group so that the product system \( Y_{q,T}^{(r)} \) from Proposition 3.3.7 is defined. For each \( q \in Q \), let

\[ \mathcal{N}_{\mathcal{T}}A^{(r)} := \text{span} \left\{ i_{\mathcal{Z}(p,q)}(\mathcal{Z}(p,q)) i_{\mathcal{Z}(r,q)}(\mathcal{Z}(r,q))^* : p, r \in P \right\} . \]

(i) Let \( q, t \in Q \) and \( T \in \mathcal{K}_{\mathcal{N}_{\mathcal{T}}A}(Y_{q,T}^{(r)}) \). Then \( i_{\mathcal{Z}(p,q)}(T) \in \mathcal{L}_{\mathcal{N}_{\mathcal{T}}A}(Y_{q,T}^{(r)}) \) is left multiplication by an element of \( \mathcal{N}_{\mathcal{T}}A^{(r)} \) on \( Y_{q,T}^{(r)} \subseteq \mathcal{N}_{\mathcal{T}}A \). In particular, if \( y, w \in Y_{q,T}^{(r)} \), then \( yw^* \in \mathcal{N}_{\mathcal{T}}A^{(r)} \), and the rank one operator \( \Theta_{y,w} \in \mathcal{K}_{\mathcal{N}_{\mathcal{T}}A}(Y_{q,T}^{(r)}) \).
Thus, by part (i) of the result, it suffices to consider the case when $q = e_H$. By Lemma 3.1.22, for any $u \in Z_{(r,e_H), t}$, and $b \in N T^q_H$.

Now suppose that $q \neq e_H$. For any $u \in Z_{(e_G,q)}$, $v \in Z_{(e_G,t)}$, and $b \in N T_X$,

$$
\ell_q^t(\Theta_{y,w})(i_{Z_{(e_G,t)}}(uv)) = \ell_q^t(\Theta_{y,w})(i_{Z_{(e_G,q)}}(u))i_{Z_{(e_G,t)}}(v)\phi_{e_G}^X(b).
$$

Thus, $\ell_q^t(\Theta_{y,w}) = M_{\phi_X^c(yw^*)}$. Moreover, $\phi_X^c(yw^*) \in \phi_X^c(N T^c_X) = N T^q_H$.

Now suppose that $q \neq e_H$. For any rank one operator $\Theta_{y,w} \in K_M$, if $q \neq e_H$, then for any $z \in Y^N_T$, we have

$$
\ell_q^t(\Theta_{y,w})(z) = \Phi_{q}^{NT}(yw^*)(z) = \phi_{e_G}^X(yw^*)z = M_{\phi_X^c(yw^*)}(z).
$$


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satisfies

$$
\ell_q^t(\Theta_{y,w}) = \begin{cases} 
M_{yw^*} & \text{if } q \neq e_H \\
M_{\phi_X^c(yw^*)} & \text{if } q = e_H.
\end{cases}
$$

(ii) Let $b \in N T^q_Z$ and $q \neq e_H$. Then $M_b \in K_{N T^q_X}(Y^N_T)$.

Proof. We begin by proving part (i) of the result. Fix $q, t \in Q$ and a rank one operator $\Theta_{y,w} \in K_{N T^q_X}(Y^N_T)$. If $q = e_H$, then for any $z \in Y^N_T$, we have

$$
\ell_q^t(\Theta_{y,w})(z) = \Phi_{q}^{NT}(yw^*)(z) = \phi_{e_G}^X(yw^*)z = M_{\phi_X^c(yw^*)}(z).
$$

Thus, $\ell_q^t(\Theta_{y,w}) = M_{\phi_X^c(yw^*)}$. Moreover, $\phi_X^c(yw^*) \in \phi_X^c(N T^c_X) = N T^q_H$.

Now suppose that $q \neq e_H$. For any $u \in Z_{(e_G,q)}$, $v \in Z_{(e_G,t)}$, and $b \in N T_X$,

$$
\ell_q^t(\Theta_{y,w})(i_{Z_{(e_G,t)}}(uv)) = \ell_q^t(\Theta_{y,w})(i_{Z_{(e_G,q)}}(u))i_{Z_{(e_G,t)}}(v)\phi_{e_G}^X(b).
$$

Since both $\ell_q^t(\Theta_{y,w})$ and $M_{yw^*}$ are linear and continuous, and since

$$
Y^N_T = \overline{\text{span}} \left\{ i_{Z_{(e_G,q)}}(Z_{(e_G,q)})i_{Z_{(e_G,t)}}(Z_{(e_G,t)})\phi_{e_G}^X(N T_X) \right\},
$$

we conclude that $\ell_q^t(\Theta_{y,w}) = M_{yw^*} \in L_{N T^q_X}(Y^N_T)$.

It remains to check that if $y, w \in Y^N_T$, then $yw^* \in N T^q_Z$. Since $N T^q_Z$ is closed in norm and under taking linear combinations, it suffices to consider the case when $y := i_{Z_{(p,q)}}(x)i_{Z_{(r,e_H)}}(z)^*$ and $w := i_{Z_{(m,q)}}(u)i_{Z_{(n,e_H)}}(v)^*$ for some $p, r, m, n \in P$ and $x \in Z_{(p,q)}$, $z \in Z_{(r,e_H)}$, $u \in Z_{(m,q)}$, $v \in Z_{(n,e_H)}$. By Lemma 3.1.22, if $r \neq n = \infty$ then

$$
yw^* = i_{Z_{(p,q)}}(x)i_{Z_{(r,e_H)}}(z)^*i_{Z_{(n,e_H)}}(v)i_{Z_{(m,q)}}(u)^* = 0,
$$

which is certainly in $N T^q_Z$. On the other hand, if $r \neq n < \infty$, then Lemma 3.1.22 tells us that

$$
yw^* = i_{Z_{(p,q)}}(x)i_{Z_{(r,e_H)}}(z)^*i_{Z_{(n,e_H)}}(v)i_{Z_{(m,q)}}(u)^* \\
\in \overline{\text{span}} \left\{ i_{Z_{(p,q),(r-1)\cong r\cong \infty)}}(Z_{(p,q),(r-1)\cong r\cong \infty))i_{Z_{(m,q),(n-1)\cong n\cong \infty)}}(Z_{(m,q),(n-1)\cong n\cong \infty)) \right\}
\subseteq N T^q_Z.
$$
This completes the proof of part (i).

We now prove part (ii) of the result. Let \( b := i_z(p,q)(z)i_z(r,q)(w)^* \in \mathcal{NT}_z^q \) where \( p, r \in P \) and \( z \in Z_{(p,q)}, w \in Z_{(r,q)} \). Then \( i_z(p,q)(z), i_z(r,q)(w) \in Y_q^\mathcal{NT} \) and, so by (i) we see that

\[
M_{i_z(p,q)(z)i_z(r,q)(w)^*} = i_q^q \left( \Theta_{i_z(p,q)(z)i_z(r,q)(w)} \right) = \Theta_{i_z(p,q)(z)i_z(r,q)(w)} \in \mathcal{K}_{\mathcal{NT}_x} \left( Y_q^\mathcal{NT} \right).
\]

Since the map \( b \mapsto M_b \) is linear, and \( \|M_b\|_{\mathcal{L}_{\mathcal{NT}_x}(Y_q^\mathcal{NT})} \leq \|b\|_{\mathcal{NT}_z} \) whenever \( b \in \mathcal{NT}_z \) is such that \( M_b \in \mathcal{L}_{\mathcal{NT}_x} \left( Y_q^\mathcal{NT} \right) \) (by Lemma 3.3.9), we conclude that \( M_b \in \mathcal{K}_{\mathcal{NT}_x} \left( Y_q^\mathcal{NT} \right) \) whenever \( b \in \text{span} \left\{ i_z(p,q)(Z_{(p,q)}) i_z(r,q)(Z_{(r,q)})^* : p, r \in P \right\} = \mathcal{NT}_z^q \).

\[\square\]

**Lemma 3.3.11.** For any \( q, t \in Q \),

\[
\mathcal{NT}_z^q \cap \mathcal{NT}_z^t \subseteq \begin{cases} \mathcal{NT}_z^{qt} & \text{if } q \vee t < \infty \\ \{0\} & \text{otherwise} \end{cases}
\]

In particular, each \( \mathcal{NT}_z^q \) is a subalgebra of \( \mathcal{NT}_z \).

**Proof.** Since multiplication in \( \mathcal{NT}_z \) is bilinear and continuous, it suffices to show that

\[
i_z(p,q)(x)i_z(r,q)(z)^*i_z(m,t)(u)i_z(n,t)(v)^* \in \begin{cases} \mathcal{NT}_z^{qt} & \text{if } q \vee t < \infty \\ \{0\} & \text{otherwise} \end{cases}
\]

whenever \( p, r, m, n \in P \) and \( x \in Z_{(p,q)}, y \in Z_{(r,q)}, u \in Z_{(m,t)}, v \in Z_{(n,t)} \). If \( r \vee m = \infty \) or \( q \vee t = \infty \), then \( (r, q) \vee (m, t) = \infty \), and so Lemma 3.1.22 tells us that

\[
i_z(p,q)(x)i_z(r,q)(z)^*i_z(m,t)(u)i_z(n,t)(v)^* = 0.
\]

On the other hand, if \( r \vee m < \infty \) and \( q \vee t < \infty \), then \( (r, q) \vee (m, t) = (r \vee m, q \vee t) < \infty \), and so

\[
i_z(p,q)(x)i_z(r,q)(z)^*i_z(m,t)(u)i_z(n,t)(v)^* \\
\in \text{span} \left\{ i_z(p_{r-1}(r \vee m), q \vee t) \left( Z_{(p_{r-1}(r \vee m), q \vee t)} \right) i_z(m_{n-1}(r \vee m), q \vee t) \left( Z_{(m_{n-1}(r \vee m), q \vee t)} \right)^* \right\} \\
\subseteq \mathcal{NT}_z^{qt} \]

We are finally ready to prove that the product system \( Y^\mathcal{NT} \) is compactly aligned.

**Proposition 3.3.12.** Suppose \( G \) is an amenable group so that the product system \( Y^\mathcal{NT} \) from Proposition 3.3.7 is defined. Then \( Y^\mathcal{NT} \) is compactly aligned.

**Proof.** Let \( S \in \mathcal{K}_{\mathcal{NT}_x} \left( Y_q^\mathcal{NT} \right) \) and \( T \in \mathcal{K}_{\mathcal{NT}_x} \left( Y_t^\mathcal{NT} \right) \) with \( q \vee t < \infty \). If \( q = t = e_H \),
then
\[ i^q_{q,t}(S)i^q_{q,t}(T) = i^{e_H}_{e_H}(S)i^{e_H}_{e_H}(T) = ST \in \mathcal{K}_{NT_X} (Y_{e_H}^{NT}) = \mathcal{K}_{NT_X} (Y_{q,t}^{NT}). \]

Now suppose that \( q \neq e_H \) or \( t \neq e_H \). Thus, \( q \lor t \neq e_H \). By Lemma 3.3.10, \( i^q_{q,t}(S) = M_b \) and \( i^q_{q,t}(T) = M_c \) for some \( b \in NT^q_Z \) and \( c \in NT^q_Z \). Since \( bc \in NT^{q,t}_Z \) by Lemma 3.3.11, we can use Lemma 3.3.10 again to see that
\[ i^q_{q,t}(S)i^q_{q,t}(T) = M_bM_c = M_{bc} \in \mathcal{K}_{NT_X} (Y_{q,t}^{NT}). \]

3.3.2 Isomorphisms of Nica–Toeplitz algebras

For the compactly aligned product system \( Y^{NT} \) defined in Subsection 3.3.1, we will show that \( NT_{Y^{NT}} \cong NT_Z \). To do this we will use the universal property of each \( C^* \)-algebra to induce a \( * \)-homomorphism to the other, and then check that these \( * \)-homomorphisms are mutually inverse. In summary, we will show that the maps in Figure 3.2 exist and make the diagram commutative.

To make our arguments easier to write down, we will identify the coefficient algebra \( NT_X \) of \( Y^{NT} \) with \( \phi_X^{NT} (NT_X) \subseteq NT_Z \). Thus, every fibre of \( Y^{NT} \) can be viewed as sitting inside \( NT_Z \), and the left and right actions of \( NT_X \cong \phi_X^{NT} (NT_X) \) on each \( Y_q^{NT} \) are just multiplication in \( NT_Z \).

To begin we get a \( * \)-homomorphism from \( NT_Z \) to \( NT_{Y^{NT}} \) by exhibiting a Nica covariant representation of \( Z \) in \( NT_{Y^{NT}} \).

**Proposition 3.3.13.** Suppose \( G \) is an amenable group so that the compactly aligned product system \( Y^{NT} \) from Proposition 3.3.7 is defined. Define \( \varphi^{NT} : Z \to NT_{Y^{NT}} \) by
\[ \varphi^{NT}_{(p,q)} := i_{Y^{NT}} \circ i_{Z_{(p,q)}}, \]
for each \( (p,q) \in P \rtimes \alpha Q \). Then \( \varphi^{NT} \) is a representation of \( Z \).

**Proof.** Firstly, \( \varphi^{NT}_{(e_G,e_H)} = i_{Y^{NT}} \circ i_{Z_{(e_G,e_H)}} \) is the composition of \( * \)-homomorphisms and so is a \( * \)-homomorphism. Similarly, \( \varphi^{NT}_{(p,q)} = i_{Y^{NT}} \circ i_{Z_{(p,q)}} \) is the composition of linear maps, and so is linear for any \( p \in P, q \in Q \). For \( (p,q),(s,t) \in P \rtimes \alpha Q \) and \( z \in Z_{(p,q)}, w \in Z_{(s,t)}, \)
\[ \varphi^{NT}_{(p,q)}(z)\varphi^{NT}_{(s,t)}(w) = i_{Y^{NT}} \left( i_{Z_{(p,q)}}(z) \right) i_{Y^{NT}} \left( i_{Z_{(s,t)}}(w) \right) = i_{Y^{NT}} \left( i_{Z_{(p,q)}}(z) i_{Z_{(s,t)}}(w) \right) \]
since \( i_Y \) is a representation
\[ = i_{Y^{NT}} \left( i_{Z_{(p,q)}}(z) i_{Z_{(s,t)}}(w) \right) \]
since \( i_Z \) is a representation
\[ = i_{Y^{NT}} \left( i_{Z_{(p+q,s+t)}}(zw) \right) \]
Proof. It suffices to prove the result when

\[ \phi \]

Suppose Proposition 3.3.15.

For \((p, q) \in P \rtimes \alpha Q\) and \(z, w \in Z_{(p, q)}\),

\[
\varphi_{(p, q)}^{NT}(z)^* \varphi_{(p, q)}^{NT}(w) = i_{Y^T}(z)^* i_{Y^q}(i_{Z_{(p, q)}}(w)) \\
= i_{Y^T}(z)^* i_{P^q,H}(i_{Z_{(p, q)}}(w))^{q^T} \Theta_{(p, q)}(z, w) \\
= i_{Y^T}(z)^* i_{Z_{(p, q)}}(z, w)^{(p, q)} \\
= \varphi_{(p, q)}^{NT}(z, w)^{(p, q)}.
\]

Hence, \(\varphi^{NT}\) is a representation of \(Z\) in \(N^T_{Y} \).

To show that \(\varphi^{NT}\) is Nica covariant, we need a lemma.

Lemma 3.3.14. Suppose \(G\) is an amenable group so that the compactly aligned product system \(Y^{NT}\) from Proposition 3.3.7 is defined. Let \((p, q) \in P \rtimes \alpha Q\) and \(T \in \mathcal{K}_A Z_{(p, q)}\). Then

\[
\varphi_{(p, q)}^{NT}(T) = i_{Y^T}^{(q)}(M_{i_{Z_{(p, q)}}}(T)).
\]

Proof. It suffices to prove the result when \(T\) is a rank one operator. Let \(T := \Theta_{z, w} \in \mathcal{K}_A Z_{(p, q)}\) for some \(z, w \in Z_{(p, q)}\). Making use of Lemma 3.3.10, we have that

\[
\varphi_{(p, q)}^{(p, q)}(T) = \varphi_{(p, q)}(z)^* \varphi_{(p, q)}(w) = i_{Y^T}(z)^* i_{Y^q}(i_{Z_{(p, q)}}(w))^* \\
= i_{Y^T}^{(q)}(i_{Z_{(p, q)}}(z)^* i_{Z_{(p, q)}}(w)) \\
= i_{Y^T}^{(q)}(M_{i_{Z_{(p, q)}}}(z, w)^{(p, q)}) \\
= i_{Y^T}^{(q)}(M_{i_{Z_{(p, q)}}}(T)).
\]

Proposition 3.3.15. Suppose \(G\) is an amenable group so that the compactly aligned product system \(Y^{NT}\) from Proposition 3.3.7 is defined. Then the representation \(\varphi^{NT}\) is Nica covariant. Hence, by the universal property of \(N^T_{Z}\), there exists a \(*\)-homomorphism \(\Omega^{NT} : N^T_{Z} \rightarrow N^T_{Y}\) such that \(\Omega^{NT} \circ i_{Z_{(p, q)}} = \varphi_{(p, q)}^{NT} = i_{Y^T}^{(q)} \circ i_{Z_{(p, q)}}\) for each \((p, q) \in P \rtimes \alpha Q\).

Proof. Fix \((p, q), (s, t) \in P \rtimes \alpha Q\) and let \(S \in \mathcal{K}_A Z_{(p, q)}\), \(T \in \mathcal{K}_A Z_{(s, t)}\). Using
Lemma 3.3.14 we see that

\[ \varphi^{NT((p,q))}(S)\varphi^{NT((s,t))}(T) = i_{Y^{NT}}^{(q)} \left( M_{\ell(z)}^{((p,q))}(S) \right) i_{Y^{NT}}^{(t)} \left( M_{\ell(z)}^{((s,t))}(T) \right). \]  

(3.4)

Since \( i_{Y^{NT}} \) is Nica covariant, this is zero if \( q \lor t = \infty \). If \( q \lor t < \infty \), then (3.4) gives

\[ \varphi^{NT((p,q))}(S)\varphi^{NT((s,t))}(T) = i_{Y^{NT}}^{(q \lor t)} \left( i_{q}^{(q \lor t)} \left( M_{\ell(z)}^{((p,q))}(S) \right) i_{t}^{(q \lor t)} \left( M_{\ell(z)}^{((s,t))}(T) \right) \right) \]

\[ = i_{Y^{NT}}^{(q \lor t)} \left( M_{\ell(z)}^{((p,q))}(S) M_{\ell(z)}^{((s,t))}(T) \right) \]

(3.5)

Since \( i_{z} \) is also Nica covariant and \( (G \rtimes_{\alpha} H, P \rtimes_{\alpha} Q) \) has the product order, if \( p \lor s = \infty \), then the last line is zero. If \( p \lor s < \infty \), then another application of Lemma 3.3.14 shows that (3.5) yields

\[ \varphi^{NT((p,q))}(S)\varphi^{NT((s,t))}(T) = i_{Y^{NT}}^{(q \lor s, q \lor t)} \left( i_{(p,q)}^{(q \lor s,q \lor t)} \left( M_{\ell(z)}^{((p,q)\lor (s,t))}(S) \right) i_{(s,t)}^{(q \lor s,q \lor t)}(T) \right) \]

\[ = \varphi^{NT((p,q))}(S)\varphi^{NT((s,t))}(T) \]

Thus,

\[ \varphi^{NT((p,q))}(S)\varphi^{(s,t)}(T) = \begin{cases} \varphi^{NT((p,q \lor s,q \lor t))} \left( i_{(p,q)}^{(q \lor s,q \lor t)}(S) i_{(s,t)}^{(q \lor s,q \lor t)}(T) \right) & \text{if } (p,q) \lor (s,t) < \infty \\ 0 & \text{otherwise} \end{cases} \]

and so \( \varphi \) is Nica covariant.

Next, we construct a \(*\)-homomorphism from \( NT_{Y^{NT}} \) to \( NT_{Z} \) by exhibiting a Nica covariant representation of \( Y^{NT} \) in \( NT_{Z} \).

**Proposition 3.3.16.** Suppose \( G \) is an amenable group so that the compactly aligned product system \( Y^{NT} \) from Proposition 3.3.7 is defined. For each \( q \in Q \), let \( \varphi^{NT}_{q} \) be the inclusion of \( Y^{NT}_{q} \) in \( NT_{Z} \). Then \( \varphi^{NT}_{q} \) is a Nica covariant representation of \( Y^{NT} \). Hence, there exists a \(*\)-homomorphism \( \Omega^{NT} : NT_{Y^{NT}} \rightarrow NT_{Z} \) such that \( \Omega^{NT} \circ i_{Y^{NT}}^{q} = \varphi^{NT}_{q} \) for each \( q \in Q \).

**Proof.** It is trivial to check that \( \varphi^{NT}_{q} \) is a representation. We show that \( \varphi^{NT}_{q} \) is Nica covariant. If \( q \in Q \) and \( b \in NT^{q}_{Z} \), then Lemma 3.3.10 tells us that \( M_{b} \in K_{NT_{x}}(Y^{NT}_{q}) \). We claim that \( \varphi^{NT(q)}(M_{b}) = b \). To see this, observe that if \( p, r \in P \)
and $z \in \mathbb{Z}_{(p,q)}$, $w \in \mathbb{Z}_{(r,q)}$, then

$$
\varphi'^{\mathcal{N}T(q)}(M_{i\mathbb{Z}_{(p,q)}(z)\mathbb{Z}_{(r,q)}(w)^*}) = \varphi'^{\mathcal{N}T(q)}(\Theta_{i\mathbb{Z}_{(p,q)}(z)\mathbb{Z}_{(r,q)}(w)})
= \varphi'^{\mathcal{N}T(q)}(i\mathbb{Z}_{(p,q)}(z)) \varphi'^{\mathcal{N}T(q)}(i\mathbb{Z}_{(r,q)}(w)^*)
= i\mathbb{Z}_{(p,q)}(z)i\mathbb{Z}_{(r,q)}(w)^*.
$$

Now if $S := \Theta_{y,w} \in \mathcal{K}_{\mathcal{N}T_X}(\mathbb{Y}^{\mathcal{N}T}_q)$ and $T := \Theta_{u,v} \in \mathcal{K}_{\mathcal{N}T_X}(\mathbb{Y}^{\mathcal{N}T}_t)$, making use of Lemma 3.3.11 and Proposition 3.3.12, we see that

$$
\varphi'^{\mathcal{N}T(q)}(S) \varphi'^{\mathcal{N}T(t)}(T) = \varphi'^{\mathcal{N}T(q)}(\Theta_{y,w}) \varphi'^{\mathcal{N}T(t)}(\Theta_{u,v})
= \varphi'^{\mathcal{N}T(q)}(M_{yw^*}) \varphi'^{\mathcal{N}T(t)}(M_{uv^*})
= yw^*uv^*
= \begin{cases} 
\varphi'^{\mathcal{N}T(q\lor t)}(M_{yw^*uv^*}) & \text{if } q \lor t < \infty \\
0 & \text{otherwise}
\end{cases}
= \begin{cases} 
\varphi'^{\mathcal{N}T(q\lor t)}(M_{yw^*M_{uv^*}}) & \text{if } q \lor t < \infty \\
0 & \text{otherwise}
\end{cases}
= \begin{cases} 
\varphi'^{\mathcal{N}T(q\lor t)}(i_{\mathbb{Y}^{\mathcal{N}T}_q}(S)i_{\mathbb{Y}^{\mathcal{N}T}_t}(T)) & \text{if } q \lor t < \infty \\
0 & \text{otherwise}
\end{cases}.
$$

Hence, $\varphi'$ is Nica covariant.

Putting all of this together, we get the following.

**Theorem 3.3.17.** Suppose $G$ is an amenable group so that the compactly aligned product system $\mathbb{Y}^{\mathcal{N}T}$ from Proposition 3.3.7 is defined. The $*$-homomorphisms $\Omega^{\mathcal{N}T} : \mathcal{N}T_{\mathbb{Z}} \to \mathcal{N}T_{\mathbb{Y}^{\mathcal{N}T}}$ and $\Omega^{\mathcal{N}T} : \mathcal{N}T_{\mathbb{Y}^{\mathcal{N}T}} \to \mathcal{N}T_{\mathbb{Z}}$ are mutually inverse isomorphisms.

**Proof.** We begin by showing that $\Omega^{\mathcal{N}T} \circ \Omega^{\mathcal{N}T} = \text{id}_{\mathcal{N}T_{\mathbb{Z}}}$. For any $(p,q) \in P \rtimes_\alpha Q$, we have

$$
(\Omega^{\mathcal{N}T} \circ \Omega^{\mathcal{N}T}) \circ i_{\mathbb{Z}_{(p,q)}} = \Omega^{\mathcal{N}T} \circ i_{\mathbb{Y}^{\mathcal{N}T}_q} \circ i_{\mathbb{Z}_{(p,q)}} = i_{\mathbb{Z}_{(p,q)}}.
$$

Since $\mathcal{N}T_{\mathbb{Z}}$ is generated by the image of $i_{\mathbb{Z}}$ and $\Omega^{\mathcal{N}T} \circ \Omega^{\mathcal{N}T}$ is a $*$-homomorphism, we conclude that $\Omega^{\mathcal{N}T} \circ \Omega^{\mathcal{N}T} = \text{id}_{\mathcal{N}T_{\mathbb{Z}}}$. Next we check that $\Omega^{\mathcal{N}T} \circ \Omega^{\mathcal{N}T} = \text{id}_{\mathcal{N}T_{\mathbb{Y}^{\mathcal{N}T}}}$. For any $p \in P$,

$$
(\Omega^{\mathcal{N}T} \circ \Omega^{\mathcal{N}T}) \circ (i_{\mathbb{Y}^{\mathcal{N}T}_H} \circ i_{\mathbb{Z}_{(p,e_H)}}) = \Omega^{\mathcal{N}T} \circ i_{\mathbb{Z}_{(p,e_H)}} = i_{\mathbb{Y}^{\mathcal{N}T}_H} \circ i_{\mathbb{Z}_{(p,e_H)}}.
$$
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Since $Y_{e_H}^{NT} = NT_X \cong \phi_X^{NT}(NT_X)$ is generated by the images of each $iZ_{(p,e_H)}$ we conclude that $(\Omega^{NT} \circ \Omega^{NT}) \circ iY_{e_H}^{NT} = iY_{e_H}^{NT}$. Now fix $q \in Q \setminus \{e_H\}$ and $m,n \in P$. For any $z \in Z_{(m,q)}$, $w \in Z_{(n,e_H)}$ we see that

$$(\Omega^{NT} \circ \Omega^{NT}) \circ iY_{e_H}^{NT}((iZ_{(m,q)}(z))iZ_{(n,e_H)}(w)^*) = \Omega^{NT}((iZ_{(m,q)}(z))iZ_{(n,e_H)}(w)^*) = \Omega^{NT}(iZ_{(m,q)}(z)) \Omega^{NT}(iZ_{(n,e_H)}(w)^*) = iY_{e_H}^{NT}(iZ_{(m,q)}(z))iY_{e_H}^{NT}(iZ_{(n,e_H)}(w)^*) = iY_{e_H}^{NT}(iZ_{(m,q)}(z)iZ_{(n,e_H)}(w)^*).$$

Since $Y_{e_H}^{NT} = \overline{\text{span}} \left\{ iZ_{(m,q)}(Z_{(m,q)})iZ_{(n,e_H)}(Z_{(n,e_H)})^* : m,n \in P \right\}$, and since $iY_{e_H}^{NT}$ is linear and isometric, we see that $(\Omega^{NT} \circ \Omega^{NT}) \circ iY_{e_H}^{NT} = iY_{e_H}^{NT}$ for each $q \in Q \setminus \{e_H\}$. Since $NT_{Y_{e_H}^{NT}}$ is generated by the image of $iY_{e_H}^{NT}$ and $\Omega^{NT} \circ \Omega^{NT}$ is a $*$-homomorphism, we conclude that $\Omega^{NT} \circ \Omega^{NT} = \text{id}_{NT_{Y_{e_H}^{NT}}}$. \hfill $\square$

### 3.4 Factorising Cuntz–Nica–Pimsner algebras

We would like to be able to replicate our work from Section 3.3 using Cuntz–Nica–Pimsner algebras in place of Nica–Toeplitz algebras. Specifically, can we construct a product system $Y^{NO}$ over $(H,Q)$ with coefficient algebra $NO_X$ such that the Cuntz–Nica–Pimsner algebras of $Z$ and $Y^{NO}$ are isomorphic?

The basic idea is to extend the commuting diagrams of Figures 3.1 and 3.2 by applying the canonical quotient homomorphisms $q_X : NO_X \to NO_X$ and $q_Z : NO_Z \to NO_Z$ at the relevant places (see Figures 3.3 and 3.4). Since every Cuntz–Nica–Pimsner covariant representation is by definition a Nica covariant representation, many of the results from Section 3.3 that we require work exactly as before. Sometimes we will impose additional hypotheses to get things to work, and unsurprisingly, our proofs often become more complicated. We will add in hypotheses as and when needed, since we are not sure if they are necessary, in the hope that future work may be able to relax/remove them.

We now list the results that we will prove in Section 3.4, and summarise the various spaces and maps that we will be working with in a pair of commuting diagrams. As before, we hope that if the reader loses track of what is going on, they will be able to return to this point for assistance.

1. In Proposition 3.4.3 we show that the inclusion of $X$ in $Z$ induces a homomorphism $\phi_X^{NO}$ from $NO_X$ to $NO_Z$ such that $\phi_X^{NO} \circ q_X = q_Z \circ \phi_X^{NT}$.

2. In Proposition 3.4.6 we show that the homomorphism $\phi_X^{NO}$ is injective.
(3) In Proposition 3.4.7, we use $\phi^{NT}_X$ to construct a collection of Hilbert $\mathcal{N}\mathcal{O}_X$-bimodules $\{Y^{NO}_q : q \in Q\}$ inside $\mathcal{N}\mathcal{O}_Z$.

(4) In Proposition 3.4.8, we show that $\mathcal{Y}^{NO} := \bigsqcup_{q \in Q} Y^{NO}_q$ is a compactly aligned product system over $(H, Q)$ with coefficient algebra $\mathcal{N}\mathcal{O}_X$.

(5) In Proposition 3.4.12, we find sufficient conditions for $\mathcal{N}\mathcal{O}_X$ to act faithfully on each fibre of $\mathcal{Y}^{NO}$.

Once we have the product system $\mathcal{Y}^{NO}$, we prove that $\mathcal{N}\mathcal{O}_{\mathcal{Y}^{NO}} \cong \mathcal{N}\mathcal{O}_Z$.

(6) In Proposition 3.4.16 we use the universal Cuntz–Nica–Pimsner covariant representations of $\mathcal{Z}$ and $\mathcal{Y}^{NO}$ to construct a Cuntz–Nica–Pimsner representation $\varphi^{NO}$ of $\mathcal{Z}$ in $\mathcal{N}\mathcal{O}_{\mathcal{Y}^{NO}}$. Using the universal property of $\mathcal{N}\mathcal{O}_Z$ we get a $\ast$-homomorphism $\Omega^{NO} : \mathcal{N}\mathcal{O}_Z \to \mathcal{N}\mathcal{O}_{\mathcal{Y}^{NO}}$ such that $\Omega^{NO} \circ j_{Z(p,q)} = j_{Y^{NO}_q} \circ j_{Z(p,q)}$.

(7) In Proposition 3.4.19 we find sufficient conditions for the inclusion of $\mathcal{Y}^{NO}$ in $\mathcal{N}\mathcal{O}_Z$ to be a Cuntz–Nica–Pimsner covariant representation. The universal property of $\mathcal{N}\mathcal{O}_{\mathcal{Y}^{NO}}$ then gives a $\ast$-homomorphism $\Omega^{NO} : \mathcal{N}\mathcal{O}_{\mathcal{Y}^{NO}} \to \mathcal{N}\mathcal{O}_Z$ such that $\Omega^{NO} \circ j_{Y^{NO}_q}$ is the inclusion of $\mathcal{Y}^{NO}_q$ in $\mathcal{N}\mathcal{O}_Z$.

(8) In Theorem 3.4.21 we prove that $\Omega^{NO}$ and $\Omega^{Y^{NO}}$ are mutually inverse isomorphisms.

In summary, we will show that for every $p \in P$ and $q \in Q$, the maps in the following two diagrams exist and make the diagrams commute.

**Figure 3.3:** The homomorphism $\phi^{NO}_X$
3.4.1 Constructing the product system

The first step is to check that the inclusion of $X$ in $Z$ induces a $*$-homomorphism from $\mathcal{N}O_X$ to $\mathcal{N}O_Z$, analogous to Proposition 3.3.1. To do this we need a couple of lemmas.

Recall Definition 3.1.1 and part (ii) of Remark 3.1.2 — since $Z$ is a product system, for every $(p,q), (m,n) \in P \rtimes_\alpha Q$ with $(p,q) \neq e_G \rtimes_\alpha H$ there exists a Hilbert $A$-bimodule isomorphism $M_{Z(p,q),Z(m,n)}^Z: Z(p,q) \otimes_A Z(m,n) \rightarrow Z(p\alpha q(m),q n)$.

**Lemma 3.4.1.** Let $p \in P$, $(s,t) \in P \rtimes_\alpha Q$, and suppose $T \in K_A(X_p)$. If $s \neq e_G$, then

$$
\iota_{(p,c)}^{(s,t)}(T) = M_{Z(s,c_H),(e_G,t)}^Z \circ (\iota_p^s(T) \otimes_A \text{id}_{Z(e_G,c)}) \circ (M_{Z(s,c_H),(e_G,t)}^Z)^{-1}.
$$

(3.6)
Proof. If \( y \in Z_{(s,e_H)} \) and \( z \in Z_{(e_G,t)} \), then
\[
\left( M^{Z\downarrow}_{(s,e_H),(e_G,t)} \circ (\iota_{e_G}^*(T) \otimes_A \text{id}_{Z_{(e_G,t)}}) \circ \left( M^{Z\downarrow}_{(s,e_H),(e_G,t)} \right)^{-1} \right) (yz) = \iota_{e_G}^*(T)(yz) = \phi_s(T)(yz) = \iota_{(e_G,t)}^*(T)(yz)
\]
Since \( M^{Z\downarrow}_{(s,e_H),(e_G,t)} \circ (\iota_{e_G}^*(T) \otimes_A \text{id}_{Z_{(e_G,t)}}) \circ \left( M^{Z\downarrow}_{(s,e_H),(e_G,t)} \right)^{-1} = (s,t) \) and \( (s,t) \) are linear and continuous, and since \( Z_{(s,t)} = \text{span} \{ yz : y \in Z_{(s,e_H)}, z \in Z_{(e_G,t)} \} \) (as \( s \neq e_G \)), we see that Equation 3.6 holds when \( p = e_G \).

Now suppose that \( p \neq e_G \). If \( p \leq s \), then \( (p,e_H) \leq (s,t) \) and so both sides of Equation 3.6 are zero. So suppose that \( p \leq s \). For any \( x \in Z_{(p,e_H)}, y \in Z_{(p^{-1}s,e_H)} \), and \( z \in Z_{(e_G,t)} \), we see that
\[
\left( M^{Z\downarrow}_{(s,e_H),(e_G,t)} \circ (\iota_{e_G}^*P) \otimes_A \text{id}_{Z_{(e_G,t)}} \circ \left( M^{Z\downarrow}_{(s,e_H),(e_G,t)} \right)^{-1} \right) (xyz) = \iota_{e_G}^*P(T)(xyz) = ((Tx)y)z = (T(xyz)) = \iota_{(e_G,t)}^*P(T)(xyz).
\]
Since \( Z_{(s,t)} = \text{span} \{ xyz : x \in Z_{(p,e_H)}, y \in Z_{(p^{-1}s,e_H)}, z \in Z_{(e_G,t)} \} \), we conclude that (3.6) holds when \( p \neq e_G \) as well.

Lemma 3.4.2. Suppose \( A \) acts faithfully on each fibre of \( X \) and each \( \tilde{\phi}_{(p,q)} \) is injective. If \( F \subseteq P \) is finite and \( \{ T_p \in K_A(X_p) : p \in F \} \) is a collection of compact operators such that
\[
\sum_{p \in F} \iota_p^*T_p = 0 \in L_A(X_s) \quad \text{for large } s \in P,
\]
then
\[
\sum_{p \in F} \iota_{(p,e_H)}^*T_p = 0 \in L_A \left( \tilde{Z}_{(s,t)} \right) \quad \text{for large } (s,t) \in P \rtimes \alpha Q.
\]

Proof. Let \((u,v) \in P \rtimes \alpha Q\). By (3.7) we can choose \( r \geq u \) such that whenever \( s \geq r \), we have
\[
\sum_{p \in F} \iota_p^*T_p = 0 \in L_A(X_s).
\]
Since \((G \rtimes \alpha H, P \rtimes \alpha Q)\) has the product order, we know that \((r,v) \geq (u,v)\). We
Claim that for any \((s, t) \geq (r, v)\), we have

\[
\sum_{p \in F} \gamma_{(p, e_H)}^{(s, t)}(T_p) = 0 \in \mathcal{L}_A \left( \tilde{Z}_{(s, t)} \right).
\]

Fix \((s, t) \geq (r, v)\) and let \(z \in \tilde{Z}_{(s, t)}\). We need to show that

\[
\left( \sum_{p \in F} \gamma_{(p, e_H)}^{(s, t)}(T_p) \right)(z) = 0 \in \tilde{Z}_{(s, t)},
\]

which is equivalent to showing that

\[
\left( \sum_{p \in F} \gamma_{(p, e_H)}^{(s, t)}(T_p) \right)(z) = 0 \in Z_{(m, n)}
\]

for every \((m, n) \leq (s, t)\). With this in mind, let \((m, n) \leq (s, t)\). If \(m \neq s\), then \(\alpha_{n-1}(m^{-1}s) \neq e_G\), and so

\[
(e_G, e_H) < (\alpha_{n-1}(m^{-1}s), e_H) \leq (\alpha_{n-1}(m^{-1}s), n^{-1}t).
\]

Since \(A\) acts faithfully on each fibre of \(X\), it acts faithfully on \(X_{\alpha_{n-1}(m^{-1}s)} = Z_{(\alpha_{n-1}(m^{-1}s), e_H)}\), and so

\[
I_{(m, n)-1(s, t)} = I_{(\alpha_{n-1}(m^{-1}s), n^{-1}t)} = \bigcap_{(e_G, e_H) < (x, y) \leq (\alpha_{n-1}(m^{-1}s), n^{-1}t)} \ker (\phi_{(x, y)}) = \{0\}.
\]

Thus, if \(m \neq s\), then \(z_{(m, n)} \in Z_{(m, n)} \cdot I_{(m, n)-1(s, t)} = \{0\}\). Hence, for any \((m, n) \leq (s, t)\), we have

\[
\left( \sum_{p \in F} \gamma_{(p, e_H)}^{(s, t)}(T_p) \right)(z) = \left( \sum_{p \in F} \gamma_{(p, e_H)}^{(m, n)}(T_p) \right)(z_{(m, n)}) = \begin{cases} (\sum_{p \in F} \gamma_{(p, e_H)}^{(s, n)}(T_p))(z_{(s, n)}) & \text{if } m = s \\ 0 & \text{otherwise.} \end{cases}
\]

Thus it remains to show that \((\sum_{p \in F} \gamma_{(p, e_H)}^{(s, n)}(T_p))(z_{(s, n)}) = 0\). We deal with the cases where \(s = e_G\) and \(s \neq e_G\) separately.

If \(s = e_G\), then

\[
0 = \sum_{p \in F} \gamma_{p}^{e_G}(T_p) = \begin{cases} \gamma_{e_G}^{e_G}(T_{e_G}) & \text{if } e_G \in F \\ 0 & \text{otherwise} \end{cases} = \begin{cases} T_{e_G} & \text{if } e_G \in F \\ 0 & \text{otherwise,} \end{cases}
\]
and so either $e \not\in F$ or $T_{eg} = 0$. Thus,

\[
\left(\sum_{p \in F} t^{(s,n)}_{(p,e_H)}(T_p)\right)(z_{(s,n)}) = \left(\sum_{p \in F} t^{(e_G,n)}_{(p,e_H)}(T_p)\right)(z_{(e_G,n)}) = 0,
\]

\[
= \begin{cases} 
  (\phi_{(e_G,n)}(T_{e_G})) (z_{(e_G,n)}) & \text{if } e_G \in F \\
  0 & \text{otherwise}
\end{cases}
\]

\[
= 0.
\]

It remains to deal with the case where $s \neq e_G$. Since $\sum_{p \in F} t^s_p(T_p) = 0$, an application of Lemma 3.4.1 shows that

\[
\left(\sum_{p \in F} t^{(s,n)}_{(p,e_H)}(T_p)\right)(z_{(s,n)}) = \left( M^Z_{(s,e_H),(e_G,n)} \circ \left(\sum_{p \in F} t^s_p(T_p) \otimes A \text{id}_{Z_{(e_G,n)}}\right) \circ M^Z_{(s,e_H),(e_G,n)^{-1}}\right)(z_{(s,n)}) = 0,
\]

We conclude that $\sum_{p \in F} \tilde{t}^{(s,t)}_{(p,e_H)}(T_p) = 0$ as claimed. \hfill \QED

**Proposition 3.4.3.** Suppose $A$ acts faithfully on each fibre of $X$ and each $\tilde{\phi}_{(p,q)}$ is injective. Then the inclusion of $X$ in $Z$ induces a $\ast$-homomorphism $\phi_X^{\mathcal{N}O} : \mathcal{N}O_X \to \mathcal{N}O_Z$ such that $\phi_X^{\mathcal{N}O}(j_X(x)) = j_Z(x)$ for each $x \in X$. Thus, $\phi_X^{\mathcal{N}O} \circ q_X = q_Z \circ \phi_X^{\mathcal{N}T}$.

**Proof.** We need to exhibit a Cuntz–Nica–Pimsner covariant representation of $X$ in $\mathcal{N}O_Z$. For each $p \in P$, define $\psi^\mathcal{N}O_p : X_p \to \mathcal{N}O_Z$ by $\psi^\mathcal{N}O_p := j_{Z_{(p,e_H)}}$. Because $\psi^\mathcal{N}O_p = q_Z \circ \psi^{\mathcal{N}T}_p$, where $\psi^{\mathcal{N}T}$ is the Nica covariant representation of $X$ from Proposition 3.3.1, we see that $\psi^\mathcal{N}O$ is a Nica covariant representation of $X$.

We now check that $\psi^\mathcal{N}O$ is Cuntz–Pimsner covariant. Suppose $F \subseteq P$ is finite and $\{T_p \in K_A(X_p) : p \in F\}$ is a collection of compact operators such that

\[
\sum_{p \in F} t^s_p(T_p) = 0 \in L_A(X_s) \quad \text{for large } s \in P.
\]

We need to show that $\sum_{p \in F} \psi^\mathcal{N}O_{(p)}(T_p) = 0$. Since $\psi^\mathcal{N}O_{(p)} = j_{\mathcal{N}O_Z((p,e_H))}$ for each $p \in P$ and $j_{\mathcal{N}O_Z}$ is Cuntz–Pimsner covariant, it suffices to prove that

\[
\sum_{p \in F} \tilde{t}^{(s,t)}_{(p,e_H)}(T_p) = 0 \in L_A\left(\mathcal{N}O_{(s,t)}\right) \quad \text{for large } (s, t) \in P \times Q.
\]

This follows from Lemma 3.4.2. Thus, $\psi^\mathcal{N}O$ is Cuntz–Pimsner covariant, and so the universal property of $\mathcal{N}O_X$ induces a $\ast$-homomorphism $\phi_X^{\mathcal{N}O} : \mathcal{N}O_X \to \mathcal{N}O_Z$ such
that $\phi_X^{NO}(j_X(x)) = \psi^{NO}(x) = j_Z(x)$ for each $x \in X$. Finally, since the image of $i_X$ generates $\mathcal{N}\mathcal{T}_X$, and

$$
\phi_X^{NO} \circ q_X \circ i_X = \phi_X^{NO} \circ j_X = j_Z|_X = q_Z \circ i_Z|_X = q_Z \circ \phi_X^{NT} \circ i_X,
$$

we conclude that

$$
\phi_X^{NO} \circ q_X = q_Z \circ \phi_X^{NT}.
$$

It is not clear whether the hypotheses on Proposition 3.4.3 can be relaxed. The hypothesis that each $\tilde{\phi}_{(p,q)}$ is injective is used only to ensure that the Cuntz–Nica–Pimsner algebra $\mathcal{N}\mathcal{O}_Z$ exists. The assumption that $A$ acts faithfully on each fibre of $X$ allows us to make use of Lemma 3.4.2. If were to try to prove Lemma 3.4.2 without assuming that $A$ acts faithfully on each fibre of $X$, we would need to prove that whenever $F \subseteq P$ is finite and \{ $T_p \in \mathcal{K}(X_p) : p \in F$ \} is a collection of compact operators such that

$$
\sum_{p \in F} i_p^s(T_p) = 0 \in \mathcal{L}_A \left( \tilde{X}_s \right) \text{ for large } s \in P,
$$

then

$$
\sum_{p \in F} i_{(p,e_H)}^{(s,t)}(T_p) = 0 \in \mathcal{L}_A \left( \tilde{Z}_{(s,t)} \right) \text{ for large } (s, t) \in P \rtimes_\alpha Q.
$$

The next example shows that this is generally not true.

**Example 3.4.4.** Let $\Gamma$ be the 2-graph consisting of three vertices $u, v, w$, one edge $\lambda$ of degree $(1, 0)$, one edge $\mu$ of degree $(0, 1)$, satisfying $s(\lambda) = u$, $s(\mu) = w$, and $r(\lambda) = r(\mu) = v$. The associated compactly aligned product system $Z \subseteq C^*(\Gamma)$ over $(\mathbb{Z}^2, \mathbb{N}^2)$ has fibres given by

$$
A := Z_{(0,0)} := \text{span}\{ t_u, t_v, t_w \} = C^*(\{ t_u, t_v, t_w \}),
$$

$$
Z_{(1,0)} := \text{span}\{ t_\lambda \},
$$

$$
Z_{(0,1)} := \text{span}\{ t_\mu \},
$$

$$
Z_{(m,n)} := \{ 0 \} \text{ for all other } (m, n) \in \mathbb{N}^2.
$$

From this, we define another compactly aligned product system $X$ over $(\mathbb{Z}, \mathbb{N})$, whose fibres are given by $X_n := Z_{(n,0)}$ for each $n \in \mathbb{N}$. For any $n \geq 1$, using Example 3.1.38, we see that

$$
\tilde{X}_n = \text{span} \{ t_\nu : \nu \in \Gamma^{(n,0)} \} = \text{span} \{ t_u, t_w, t_\lambda \},
$$

and for any $(n, m) \geq (1, 1)$ we have

$$
\tilde{Z}_{(n,m)} = \text{span} \{ t_\nu : \nu \in \Gamma^{(n,m)} \} = \text{span} \{ t_u, t_v, t_\lambda, t_\mu \}.
$$
Consider \( t_v \in A \cong K_A(X_0) \) and \( \Theta_{t_\lambda, t_\lambda} \in K_A(X_1) \). Then

\[
\tilde{i}^n_0(t_v) - \tilde{i}^n_1(\Theta_{t_\lambda, t_\lambda}) = 0 \in \mathcal{L}_A(\tilde{X}_n) \quad \text{for any } n \geq 1.
\]

For any \((n, m) \geq (1, 1)\), since \( r(\mu) = v \),

\[
\tilde{i}^{(n,m)}_0(t_v)(t_\mu) = t_\mu.
\]

Also, since \( \text{MCE}(\lambda, \mu) = \emptyset \),

\[
\tilde{i}^{(n,m)}_1(\Theta_{t_\lambda, t_\lambda})(t_\mu) = t_\lambda t_\lambda^* t_\mu = 0.
\]

Thus,

\[
\left( \tilde{i}^{(n,m)}_0(t_v) - \tilde{i}^{(n,m)}_1(\Theta_{t_\lambda, t_\lambda}) \right)(t_\mu) = t_\mu,
\]

and so

\[
\tilde{i}^{(n,m)}_0(t_v) - \tilde{i}^{(n,m)}_1(\Theta_{t_\lambda, t_\lambda}) \neq 0 \in \mathcal{L}_A(\tilde{Z}_{(n,m)}).
\]

We should not be particularly surprised by the previous example — the product system \( Z \) associated to \( \Gamma \) contains no two dimensional information (since the graph \( \Gamma \) contains no paths of degree \((1, 1)\)). It would be interesting to see whether this issue persists if we require the graph to be locally convex. If so, it would be interesting to see if there is a notion of local convexity for arbitrary product systems over \( \mathbb{N}^k \) (or even more general quasi-lattice ordered groups).

Similar to Subsection 3.3.1, in order to construct a product system with coefficient algebra \( NT_X \) sitting inside \( NT_Z \), we need to check that the \(*\)-homomorphism \( \phi_X^{\mathcal{N}O} : \mathcal{N}O_X \to \mathcal{N}O_Z \) is injective. The idea is to show that by restricting the canonical coaction of \( G \rtimes_\alpha H \) on \( \mathcal{N}O_Z \) to the image of \( \phi_X^{\mathcal{N}O} \), we get a coaction of \( G \) on \( \phi_X^{\mathcal{N}O}(\mathcal{N}O_X) \). As such, we need to know when the group \( C^*\)-algebra of \( G \rtimes_\alpha H \) contains a faithful copy of the group \( C^*\)-algebra of \( G \).

**Lemma 3.4.5.** Suppose \( G \) and \( H \) are groups with \( G \) amenable. Let \( \alpha : H \to \text{Aut}(G) \) be a group homomorphism. Then there exists an injective \(*\)-homomorphism \( \iota : C^*(G) \to C^*(G \rtimes_\alpha H) \) such that \( \iota(i_G(g)) = i_{G \rtimes_\alpha H}((g, e_H)) \) for each \( g \in G \) (where \( i_G \) and \( i_{G \rtimes_\alpha H} \) are the universal unitary representations of \( G \) and \( G \rtimes_\alpha H \) respectively). Thus, \( C^*(G) \cong \text{span}\{i_{G \rtimes_\alpha H}(G \rtimes_\alpha \{e_H\})\} \subseteq C^*(G \rtimes_\alpha H) \).

**Proof.** It is straightforward to check that \( g \mapsto i_{G \rtimes_\alpha H}((g, e_H)) \) is a unitary representation of \( G \) in \( C^*(G \rtimes_\alpha H) \). The universal property of \( C^*(G) \) then provides us with the \(*\)-homomorphism \( \iota \). We just need to check that \( \iota \) is injective. Since \( G \) is amenable, we know that

\[
C^*(G) \cong C^*(\{T_g : g \in G\}) \subseteq B(\ell^2(G)),
\]
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where for each \( g \in G \), \( T_g \in \mathcal{B}(\ell^2(G)) \) is defined by

\[
T_g(f)(h) := f(g^{-1}h)
\]

for each \( f \in \ell^2(G) \) and \( h \in G \). Similarly, the map \((g, h) \mapsto S_{(g, h)} \in \mathcal{B}(\ell^2(G \rtimes_\alpha H))\) where

\[
S_{(g, h)}(f)(k, l) := f((g, h)^{-1}(k, l)) = f(\alpha_{hl^{-1}}(g^{-1}k, h^{-1}l))
\]

for each \( f \in \ell^2(G \rtimes_\alpha H) \) and each \((k, l) \in G \rtimes_\alpha H\), is a unitary representation of \( G \rtimes_\alpha H \). Clearly, if \( f \in \ell^2(G) \), then the map \( \tilde{f} : G \rtimes_\alpha H \to \mathbb{C} \) defined by

\[
\tilde{f}(g, h) := \begin{cases} 
  f(g) & \text{if } h = e_H \\
  0 & \text{otherwise}
\end{cases}
\]

belongs to \( \ell^2(G \rtimes_\alpha H) \) and \( \|f\|_{\ell^2(G)} = \|\tilde{f}\|_{\ell^2(G \rtimes_\alpha H)} \). Now let \( F \subseteq G \) be a finite set. For any \( f \in \ell^2(G) \) we have

\[
\left\| \sum_{g \in F} S_{(g, e_H)}(f) \right\|^2_{\ell^2(G \rtimes_\alpha H)} = \sum_{(k, h) \in G \rtimes_\alpha H} \left| \sum_{g \in F} S_{(g, e_H)}(f)(k, h) \right|^2 = \sum_{(k, h) \in G \rtimes_\alpha H} \left| \sum_{g \in F} \tilde{f}(g^{-1}k, h) \right|^2
\]

\[
= \sum_{k \in G} \left| \sum_{g \in F} f(g^{-1}k) \right|^2
\]

\[
= \sum_{k \in G} \left| \sum_{g \in F} T_g(f)(k) \right|^2
\]

\[
= \left\| \sum_{g \in F} T_g(f) \right\|^2_{\ell^2(G)}.
\]

Thus,

\[
\left\| \sum_{g \in F} T_g \right\|_{\mathcal{B}(\ell^2(G))} \leq \left\| \sum_{g \in F} S_{(g, e_H)} \right\|_{\mathcal{B}(\ell^2(G \rtimes_\alpha H))}.
\]

As \( \iota \) is norm-decreasing, we have

\[
\left\| \sum_{g \in F} i_G(g) \right\|_{C^*(G)} \geq \left\| \iota \left( \sum_{g \in F} i_G(g) \right) \right\|_{C^*(G \rtimes_\alpha H)} = \left\| \sum_{g \in F} i_{G \rtimes_\alpha H}(g, e_H) \right\|_{C^*(G \rtimes_\alpha H)}.
\]

Since \( i_{G \rtimes_\alpha H} \) is the universal unitary representation of \( G \rtimes_\alpha H \), we deduce that

\[
\left\| \sum_{g \in F} i_G(g) \right\|_{C^*(G)} \geq \left\| \sum_{g \in F} S_{(g, e_H)} \right\|_{\mathcal{B}(\ell^2(G \rtimes_\alpha H))} \geq \left\| \sum_{g \in F} T_g \right\|_{\mathcal{B}(\ell^2(G))} = \left\| \sum_{g \in F} i_G(g) \right\|_{C^*(G)},
\]
Firstly, we need to check that

\[ \phi \text{ is injective, so that the } \ast \text{-homomorphism } \phi_{X}^{\mathcal{N}O} : \mathcal{N}O_{X} \to \mathcal{N}O_{Z} \text{ from Proposition 3.4.3 exists. If } G \text{ is an amenable group, then } \phi_{X}^{\mathcal{N}O} \text{ is injective.} \]

**Proof.** Since \( G \) is amenable, we can use Theorem 3.1.48 to show that \( \phi_{X}^{\mathcal{N}O} \) is injective. Firstly, we need to check that \( \phi_{X}^{\mathcal{N}O} \mid_{X_{G}(A)} \) is injective. Suppose \( a \in A \) is such that \( \phi_{X}^{\mathcal{N}O}(j_{X_{G}}(a)) = 0 \). Then \( j_{Z_{G}(p,eH)}(a) = 0 \), which forces \( a = 0 \) since \( j_{Z} \) is isometric by Theorem 3.1.46. Thus, \( \phi_{X}^{\mathcal{N}O} \mid_{X_{G}(A)} \) is injective.

Next, let \( \nu_{Z} : \mathcal{N}O_{Z} \to \mathcal{N}O_{Z} \otimes C^{\ast}(G \rtimes_{\alpha} H) \) denote the canonical gauge coaction of \( G \rtimes_{\alpha} H \) on \( \mathcal{N}O_{Z} \). Since

\[
\nu_{Z}(\phi_{X}^{\mathcal{N}O}(j_{X_{p}}(x))) = \phi_{X}^{\mathcal{N}O}(j_{X_{p}}(x)) = j_{Z_{p,eH}}(x) \otimes i_{G}(p, e_{H}) = \phi_{X}^{\mathcal{N}O}(j_{X_{p}}(x)) \otimes i_{G}(p),
\]

for any \( p \in P \) and \( x \in X_{p} \), we can define \( \beta : \phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X}) \to \phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X}) \otimes C^{\ast}(G) \) by

\[
\beta := (\text{id} \otimes i_{C^{\ast}(G)}) \circ \nu_{Z} \mid_{\phi(\mathcal{N}O_{X})}.
\]

We claim that \( \beta \) is a coaction of \( G \) on \( \phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X}) \). Since \( \nu_{Z} \) and \( \text{id} \otimes i_{C^{\ast}(G)} \otimes i_{G} \) are injective \( \ast \)-homomorphisms, so is \( \beta \). If \( p \in P \) and \( x \in X_{p} \), then

\[
((\beta \otimes \text{id}_{C^{\ast}(G)}) \circ \beta)(\phi_{X}^{\mathcal{N}O}(j_{X_{p}}(x))) = (\beta \otimes \text{id}_{C^{\ast}(G)})(j_{Z_{p,eH}}(x) \otimes i_{G}(p)) = j_{Z_{p,eH}}(x) \otimes i_{G}(p) \otimes i_{G}(p) = (\text{id}_{\phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X})} \otimes \delta_{G})(j_{Z_{p,eH}}(x) \otimes i_{G}(p)) = ((\text{id}_{\phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X})} \otimes \delta_{G}) \circ \beta)(\phi_{X}^{\mathcal{N}O}(j_{X_{p}}(x))).
\]

Since \( \phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X}) \) is generated by the image of \( \phi_{X}^{\mathcal{N}O} \circ j_{X} \), and both \( (\beta \otimes \text{id}_{C^{\ast}(G)}) \circ \beta \) and \( (\text{id}_{\phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X})} \otimes \delta_{G}) \circ \beta \) are \( \ast \)-homomorphisms, we conclude that \( \beta \) satisfies the coaction identity. For \( p, r \in P, x \in X_{p}, w \in X_{r} \), and \( g \in G \), we have

\[
\phi_{X}^{\mathcal{N}O}(j_{X_{p}}(x)j_{X_{r}}(w)^{*}) \otimes i_{G}(g) = (\phi_{X}^{\mathcal{N}O}(j_{X_{p}}(x)j_{X_{r}}(w)^{*}) \otimes i_{G}(pr^{-1}))(1_{M(\phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X}))} \otimes i_{G}(rp^{-1}g)) = \beta(\phi_{X}^{\mathcal{N}O}(j_{X_{p}}(x)j_{X_{r}}(w)^{*}))(1_{M(\phi_{X}^{\mathcal{N}O}(\mathcal{N}O_{X}))} \otimes i_{G}(rp^{-1}g)).
\]
Thus,
\[
\phi^N_X (\mathcal{N} \mathcal{O}_X) \otimes C^* (G) = \text{span} \left\{ \phi^N_X (j_{X_p} (x) j_{X_r} (w) ) \otimes i_G (g) : p, r \in P, x \in X_p, w \in X_r, g \in G \right\} = \text{span} \left\{ \beta \left( \phi^N_X (\mathcal{N} \mathcal{O}_X) \right) \left( 1_M (\phi^N_X (\mathcal{N} \mathcal{O}_X)) \otimes C^* (G) \right) \right\},
\]
and so \( \beta \) is coaction nondegenerate. Finally, for any \( p \in P \) and \( x \in X_p \) we have
\[
(\beta \circ \phi^N_X) (j_{X_p} (x)) = \beta \left( j_{z_{(p,e H)}} (x) \right)
= (\text{id}_{\phi^N_X (\mathcal{N} \mathcal{O}_X)} \otimes i^{-1}) \left( \nu_\mathcal{Z} (j_{z_{(p,e H)}} (x)) \right)
= (\text{id}_{\phi^N_X (\mathcal{N} \mathcal{O}_X)} \otimes i^{-1}) \left( j_{z_{(p,e H)}} (x) \otimes i_{G \times \alpha H} (p, e_H) \right)
= j_{z_{(p,e H)}} (x) \otimes i_G (p)
= (\phi^N_X \otimes \text{id}_{C^* (G)}) (j_{X_p} (x) \otimes i_G (p))
= ((\phi^N_X \otimes \text{id}_{C^* (G)}) \circ \nu_X) (j_{X_p} (x)).
\]
Since \( \beta \circ \phi^N_X \) and \( (\phi^N_X \otimes \text{id}_{C^* (G)}) \circ \nu_X \) are \(*\)-homomorphisms, and the image of \( j_X \) generates \( \mathcal{N} \mathcal{O}_X \), we see that \( \beta \circ \phi^N_X = (\phi^N_X \otimes \text{id}_{C^* (G)}) \circ \nu_X \). By Theorem 3.1.48, we conclude that \( \phi^N_X \) is injective. \( \square \)

Almost identically to Subsection 3.3.1, we can use the injective \(*\)-homomorphism \( \phi^N_X \) to construct a product system. Firstly, we construct a collection of Hilbert \( \mathcal{N} \mathcal{O}_X \)-bimodules \( \{ Y^N_q : q \in Q \setminus \{ e_H \} \} \). The idea is to make use of the collection of Hilbert \( \mathcal{N} \mathcal{T}_X \)-bimodules \( \{ Y^N_{e_H} : q \in Q \setminus \{ e_H \} \} \) defined in Propositions 3.3.3 and 3.3.6 and apply the quotient maps \( q_X \) and \( q_Z \) at the appropriate places.

For each \( q \in Q \setminus \{ e_H \} \), we let \( Y^N_q = q_Z (Y^N_{e_H}) \). We then show that there exists a right action of \( \mathcal{N} \mathcal{O}_X \) on \( Y^N_q \), and a \( \mathcal{N} \mathcal{O}_X \)-valued inner product on \( Y^N_q \), making the following pair of diagrams commute.

**Figure 3.5:** The Hilbert \( \mathcal{N} \mathcal{O}_X \)-module \( Y^N_q \)

\[
\begin{array}{ccc}
Y^N_{e_H} \times \mathcal{N} \mathcal{O}_X & \overset{(y,a) \mapsto y \cdot a}{\longrightarrow} & Y^N_{e_H} \\
\downarrow q_Z \times q_X & & \downarrow q_Z \\
Y^N_q \times \mathcal{N} \mathcal{O}_X & \overset{(\cdot,y)_q}{\longrightarrow} & \mathcal{N} \mathcal{T}_X \\
\downarrow q_Z \times q_Z & & \downarrow q_X \\
Y^N_q \times \mathcal{N} \mathcal{O}_X & \overset{(y,a) \mapsto y \cdot a}{\longrightarrow} & Y^N_q \\
\end{array}
\]

We also show that there exists a left action of \( \mathcal{N} \mathcal{O}_X \) on \( Y^N_q \), implemented by
a $*$-homomorphism $\Phi_{N^O}^q : NO_X \rightarrow L_{NO_X} (Y_q^{N^O})$, such that the following diagram commutes.

![Diagram](image)

**Figure 3.6:** The homomorphism $\Phi_{N^O}^q$

Proposition 3.4.7. Suppose $G$ is an amenable group, $A$ acts faithfully on each fibre of $X$, and each $\tilde{\phi}_{(p,q)}$ is injective, so that the $*$-homomorphism $\phi_X^{N^O}$ of Proposition 3.4.3 exists and is injective. For each $q \in Q \setminus \{e_H\}$, define

$$Y_q^{N^O} := \text{span} \left\{ j_{Z_{(e_G,q)}}(x)\phi_X^{N^O}(b) : x \in Z_{(e_G,q)}, \ b \in NO_X \right\} \subseteq NO_X.$$

Then $Y_q^{N^O}$ carries a right action of $NO_X$ such that

$$y \cdot b = y\phi_X^{N^O}(b) \quad \text{for each } y \in Y_q^{N^O} \text{ and } b \in NO_X.$$

For each $y, w \in Y_q^{N^O}$, we have $y^*w \in \phi_X^{N^O} (NO_X)$, and there is an $NO_X$ valued inner-product $\langle \cdot, \cdot \rangle_{NO_X} : Y_q^{N^O} \times Y_q^{N^O} \rightarrow NO_X$ such that

$$\langle y, w \rangle_{NO_X} = (\phi_X^{N^O})^{-1} (y^*w) \quad \text{for each } y, w \in Y_q^{N^O}.$$

With this structure, $Y_q^{N^O}$ becomes a right Hilbert $NO_X$-module. Furthermore, there exists a $*$-homomorphism $\Phi_{N^O}^q : NO_X \rightarrow L_{NO_X} (Y_q^{N^O})$ such that

$$\Phi_{N^O}^q(b)(y) = \phi_X^{N^O}(b)y \quad \text{for each } b \in NO_X \text{ and } y \in Y_q^{N^O}.$$

With this additional structure, $Y_q^{N^O}$ becomes a Hilbert $NO_X$-bimodule.

**Proof.** We have effectively already completed all of the necessary calculations in Subsection 3.3.1 to prove the result. Rather than just rerunning the arguments of Propositions 3.3.3 and 3.3.6 with $Y_q^{N^O}$ in place of $Y_q^{NT}$, we will show how these two spaces are related via the quotient maps on $NT_X$ and $NT_Z$ and use this to prove the result. The key observation is that the quotient homomorphisms $q_X$ and $q_Z$ intertwine the $*$-homomorphisms $\phi_X^{NT}$ and $\phi_X^{N^O}$.
Since $q_z \circ i_z = j_z$ and $q_z \circ \phi_X^{NT} = \phi_X^{NO} \circ q_x$, we have
\[
Y_q^{NO} = \text{span} \left\{ j_z((e,g)) (x) \phi_X^{NO} (b) : x \in Z_{(e,g)}, \ b \in \mathcal{NO}_X \right\} \\
= q_z \left( \text{span} \left\{ i_z((e,g)) (x) \phi_X^{NT} (b) : x \in Z_{(e,g)}, \ b \in \mathcal{N}_X \right\} \right) \\
= q_z (Y_q^{NT}).
\]

We now show how the right actions of $\mathcal{NO}_X$ on $Y_q^{NO}$ can be obtained from the right action of $\mathcal{N}_X$ on $Y_q^{NT}$. For any $a \in \mathcal{N}_X$ and $y \in Y_q^{NT}$, we have
\[
q_z (y \cdot a) = q_z \left( y \phi_X^{NT} (a) \right) = q_z(y) q_z \left( \phi_X^{NT} (a) \right) = q_z(y) \phi_X^{NO} \left( q_x(a) \right).
\] (3.8)

If $a' \in \mathcal{N}_X$ and $y' \in Y_q^{NT}$ with $q_x(a) = q_x(a')$ and $q_z(y) = q_z(y')$, then (3.8) is equal to
\[
q_z(y') \phi_X^{NO} (q_x(a')) = q_z(y') q_z \left( \phi_X^{NT} (a') \right) = q_z \left( y' \phi_X^{NT} (a') \right) = q_z (y' \cdot a')
\]

Thus, $Y_q^{NO}$ carries a right action of $\mathcal{NO}_X$ defined by the formula
\[
q_z(y) \cdot q_x(a) := q_z(y \cdot a) \quad \text{for each } a \in \mathcal{N}_X \text{ and } y \in Y_q^{NT}.
\]

Moreover, (3.8) shows that
\[
y \cdot a = y \phi_X^{NO} (a) \quad \text{for any } a \in \mathcal{NO}_X \text{ and } y \in Y_q^{NO}.
\]

Next we show how the $\mathcal{NO}_X$ valued inner-product on $Y_q^{NO}$ can be obtained from the $\mathcal{N}_X$ valued inner-product on $Y_q^{NT}$. For any $y, w \in Y_q^{NT}$, we have
\[
q_x \left( \langle y, w \rangle_{\mathcal{N}_X} \right) = q_x \left( \langle \phi_X^{NT}^{-1} (y^* w) \rangle \right) \\
= \langle \phi_X^{NO}^{-1} (q_x(y^* w)) \rangle \\
= \langle \phi_X^{NO}^{-1} (q_z(y^* w)) \rangle. \tag{3.9}
\]

If $y', w' \in Y_q^{NT}$ with $q_z(y) = q_z(y')$ and $q_z(w) = q_z(w')$, then this is equal to
\[
\langle \phi_X^{NO}^{-1} (q_z(y') \cdot q_z(w')) \rangle = \langle \phi_X^{NO}^{-1} (q_z(y'^* w')) \rangle \\
= q_x \left( \langle \phi_X^{NT}^{-1} (y'^* w') \rangle \right) \\
= q_x \left( \langle y', w' \rangle_{\mathcal{N}_X} \right).
\]

Thus, we can define $\langle \cdot, \cdot \rangle_{\mathcal{NO}_X} : Y_q^{NO} \times Y_q^{NO} \to \mathcal{NO}_X$ by
\[
\langle q_z(y), q_z(w) \rangle_{\mathcal{NO}_X} := q_x \left( \langle y, w \rangle_{\mathcal{N}_X} \right) \quad \text{for any } y, w \in Y_q^{NT}.
\]
Moreover, (3.9) shows that
\[
\langle y, w \rangle_{\mathcal{NO}} = (\phi \mathcal{NO})^{-1}(y^*w) \quad \text{for any } y, w \in \mathcal{Y} \mathcal{NO}.
\] (3.10)
Using (3.10), it is elementary to show that \((\mathcal{Y} \mathcal{NO}, \langle \cdot, \cdot \rangle_{\mathcal{NO}})\) is a Hilbert \(\mathcal{NO}X\)-module.

It remains to check that \(\mathcal{Y} \mathcal{NO}\) carries a left action of \(\mathcal{NO}X\) by adjointable operators. We now show how the \(*\)-homomorphism \(\Phi \mathcal{NO} : \mathcal{NO}X \to \mathcal{L}_{\mathcal{NO}X}(\mathcal{Y} \mathcal{NO})\) can be obtained from the \(*\)-homomorphism \(\Phi \mathcal{NT} : \mathcal{NT}X \to \mathcal{L}_{\mathcal{NT}X}(\mathcal{Y} \mathcal{NT})\). For any \(a \in \mathcal{NT}X\) and \(y \in \mathcal{Y} \mathcal{NT}\), we have
\[
\varphi \mathcal{NT}(a)(y) = \phi \mathcal{NT}(a)y = \phi \mathcal{NO}(\varphi \mathcal{X}(a))qz(y).
\] (3.11)
If \(a' \in \mathcal{NT}X\) and \(y' \in \mathcal{Y} \mathcal{NT}\) with \(\varphi \mathcal{X}(a) = \varphi \mathcal{X}(a')\) and \(qz(y) = qz(y')\), then (3.11) is equal to
\[
\phi \mathcal{NO}(\varphi \mathcal{X}(a'))qz(y') = \phi \mathcal{NT}(a')y' = \phi \mathcal{NO}(\varphi \mathcal{X}(a'))qz(y') = \phi \mathcal{NO}(\varphi \mathcal{X}(a'))qz(y').
\]
Thus, for each \(a \in \mathcal{NT}X\), there exists a well defined map \(\Phi \mathcal{NO}(\varphi \mathcal{X}(a)) : \mathcal{Y} \mathcal{NO} \to \mathcal{Y} \mathcal{NO}\) given by
\[
\Phi \mathcal{NO}(\varphi \mathcal{X}(a))(qz(y)) := qz(\Phi \mathcal{NT}(a)(y)) \quad \text{for each } y \in \mathcal{Y} \mathcal{NT}.
\]
Moreover, (3.11) shows that
\[
\Phi \mathcal{NO}(a)(y) = \phi \mathcal{NO}(a)y \quad \text{for any } a \in \mathcal{NO}X \text{ and } y \in \mathcal{Y} \mathcal{NO} \quad \text{(3.12)}
\]
Using (3.10) and (3.12), it is routine to check that \(\Phi \mathcal{NO}(\varphi \mathcal{X}(a))\) is an adjointable map on \(\mathcal{Y} \mathcal{NO}\) and the map \(\varphi \mathcal{X}(a) \mapsto \Phi \mathcal{NO}(\varphi \mathcal{X}(a))\) is a \(*\)-homomorphism from \(\mathcal{NO}X\) to \(\mathcal{L}_{\mathcal{NO}X}(\mathcal{Y} \mathcal{NO})\), which we denote by \(\Phi \mathcal{NO}\).

The next result shows that if \(\mathcal{Y} \mathcal{NO} := \mathcal{NO}X(\mathcal{NO}X)_{\mathcal{NO}X}\), then \(\mathcal{Y} \mathcal{NO} := \bigsqcup_{q \in \mathbb{Q}} \mathcal{Y} \mathcal{NO}\) has the structure of a compactly aligned product system.

The idea is to use the Hilbert \(\mathcal{NT}X\)-bimodule isomorphism \(M_{\mathcal{NT}Y} : \mathcal{Y} \mathcal{NT} \otimes_{\mathcal{NT}X} \mathcal{Y} \mathcal{NT} \to \mathcal{Y} \mathcal{NT}\) from Proposition 3.3.7 to define a Hilbert \(\mathcal{NO}X\)-bimodule isomorphism \(M_{\mathcal{NO}Y} : \mathcal{Y} \mathcal{NO} \otimes_{\mathcal{NO}X} \mathcal{Y} \mathcal{NO} \to \mathcal{Y} \mathcal{NO}\), such that the following diagram commutes.
Figure 3.7: The Hilbert $N\mathcal{O}_X$-bimodule isomorphism $M_{q,t}$

$$
\begin{array}{ccc}
Y_q^{NT} \otimes_{N\mathcal{T}_X} Y_t^{NT} & \xrightarrow{M_{q,t}^{Y^{NT}}} & Y_{qt}^{NT} \\
qz \otimes_{N\mathcal{T}_X} qz & \downarrow \downarrow & qz \\
Y_q^{N\mathcal{O}} \otimes_{N\mathcal{O}_X} Y_t^{N\mathcal{O}} & \xrightarrow{M_{q,t}^{Y^{N\mathcal{O}}}} & Y_{qt}^{N\mathcal{O}}
\end{array}
$$

**Proposition 3.4.8.** Suppose $G$ is an amenable group, $A$ acts faithfully on each fibre of $X$, and each $\tilde{\phi}_{(p,q)}$ is injective, so that the collection of Hilbert $N\mathcal{O}_X$-bimodules $\{Y_q^{N\mathcal{O}} : q \in Q \setminus \{e_H\}\}$ from Proposition 3.4.7 exists.

Let $Y_{e_H}^{N\mathcal{O}} := N\mathcal{O}_X (N\mathcal{O}_X)_{N\mathcal{O}_X}$, and for each $q \in Q \setminus \{e_H\}$, define $Y_q^{N\mathcal{O}}$ as in Proposition 3.4.7. Then $Y^{N\mathcal{O}} := \bigsqcup_{q \in Q} Y_q^{N\mathcal{O}}$ is a compactly aligned product system over $(H, Q)$ with coefficient algebra $N\mathcal{O}_X$, with multiplication in $Y^{N\mathcal{O}}$ given by multiplication in $N\mathcal{O}_Z$.

**Proof.** We have already shown that each $Y_q^{N\mathcal{O}}$ is a Hilbert $N\mathcal{O}_X$-bimodule. To show that $Y^{N\mathcal{O}} := \bigsqcup_{q \in Q} Y_q^{N\mathcal{O}}$ is a product system over $(H, Q)$ with coefficient algebra $N\mathcal{O}_X$ and multiplication inherited from $N\mathcal{O}_Z$, we need only check that there exists a Hilbert $N\mathcal{O}_X$-bimodule isomorphism $M_{q,t}^{Y^{N\mathcal{O}}} : Y_q^{N\mathcal{O}} \otimes_{N\mathcal{O}_X} Y_t^{N\mathcal{O}} \to Y_{qt}^{N\mathcal{O}}$ for each $q, t \in Q \setminus \{e_H\}$ such that

$$
M_{q,t}^{Y^{N\mathcal{O}}} (y \otimes_{N\mathcal{O}_X} w) = yw \quad \text{for each } y \in Y_q^{N\mathcal{O}}, \ w \in Y_t^{N\mathcal{O}}. \quad (3.13)
$$

Rather than just rerun our argument from Proposition 3.3.7 with $Y_q^{N\mathcal{O}}$ in place of $Y_q^{NT}$, we will show how the these isomorphisms can be obtained from the Hilbert $N\mathcal{T}_X$-bimodule isomorphisms $M_{q,t}^{Y^{NT}} : Y_q^{NT} \otimes_{N\mathcal{T}_X} Y_t^{NT} \to Y_{qt}^{NT}$ using the quotient map $qz$. If $y, y' \in Y_q^{NT}$ with $qz(y) = qz(y')$ and $w, w' \in Y_t^{NT}$ with $qz(w) = qz(w')$, then

$$
qz \left( M_{q,t}^{Y^{NT}} (y \otimes_{N\mathcal{T}_X} w) \right) = qz \left( yw \right) = qz \left( y'w' \right) = qz \left( M_{q,t}^{Y^{NT}} (y' \otimes_{N\mathcal{T}_X} w') \right).
$$

Hence, there is a well-defined map $M_{q,t}^{Y^{N\mathcal{O}}} : Y_q^{N\mathcal{O}} \otimes_{N\mathcal{O}_X} Y_t^{N\mathcal{O}} \to Y_{qt}^{N\mathcal{O}}$ given by

$$
M_{q,t}^{Y^{N\mathcal{O}}} ((qz \otimes_{N\mathcal{O}_X} qz)(z)) := qz \left( M_{q,t}^{Y^{NT}} (z) \right) \quad \text{for each } z \in Y_q^{NT} \otimes_{N\mathcal{T}_X} Y_t^{NT},
$$

which satisfies (3.13). Since $M_{q,t}^{Y^{NT}}$ is surjective and $qz \left( Y_q^{NT} \right) = Y_q^{N\mathcal{O}}$, the map $M_{q,t}^{Y^{N\mathcal{O}}}$ is also surjective. Routine calculations using (3.13) show that $M_{q,t}^{Y^{N\mathcal{O}}}$ is inner-
product preserving and left $\mathcal{N}\mathcal{O}_X$-linear. Hence, $M_{q,t}^{\mathcal{N}\mathcal{O}}$ is a Hilbert $\mathcal{N}\mathcal{O}_X$-bimodule isomorphism. We conclude that $Y^{\mathcal{N}\mathcal{O}}$ has the structure of a product system.

It remains to show that $Y^{\mathcal{N}\mathcal{O}}$ is compactly aligned. Let $q, t \in Q$ with $q \vee t < \infty$ and $S \in \mathcal{K}_{\mathcal{N}\mathcal{O}_X}(Y_q^{\mathcal{N}\mathcal{O}})$, $T \in \mathcal{K}_{\mathcal{N}\mathcal{O}_X}(Y_t^{\mathcal{N}\mathcal{O}})$. We need to show that

$$\iota_q^{q,t}(S)\iota_t^{q,t}(T) \in \mathcal{K}_{\mathcal{N}\mathcal{O}_X}(Y_q^{\mathcal{N}\mathcal{O}}).$$

If $q = e_H$ or $t = e_H$, the result is trivial, so we may as well suppose that $q, t \neq e_H$. Since $\iota_q^{q,t}$ and $\iota_t^{q,t}$ are linear and continuous, as is multiplication in $L_{\mathcal{N}\mathcal{O}_X}(Y_q^{\mathcal{N}\mathcal{O}})$, we may as well assume that $S$ and $T$ are rank one operators. Hence, $S = \Theta_{qz(x),qz(y)}$ and $T = \Theta_{qz(u),qz(v)}$ for some $x, y \in Y_q^{NT}$ and $u, v \in Y_t^{NT}$. Then for any $z \in Y_q^{NT}$, we have

$$(\iota_q^{q,t}(\Theta_{qz(x),qz(y)})) \iota_t^{q,t}(\Theta_{qz(u),qz(v)})(qz(z)) = (\Theta_{qz(x),qz(y)})(qz(z)) = qz(M_{xyuv^*}(z)).$$

Lemma 3.3.11 and Proposition 3.3.12 show that $M_{xyuv^*} \in \mathcal{K}_{\mathcal{N}\mathcal{O}_X}(Y_q^{\mathcal{N}\mathcal{O}})$. Thus, the previous line can be approximated by sums of the form

$$qz(\Theta_{\alpha,\beta}(z)) = \Theta_{qz(\alpha),qz(\beta)}(qz(z)),$$

where $\alpha, \beta \in Y_q^{NT}$. Since $z \in Y_q^{NT}$ was arbitrary and $Y_q^{\mathcal{N}\mathcal{O}} = qz(Y_r^{NT})$ for each $r \in Q \setminus \{e_H\}$, we conclude that $\iota_q^{q,t}(\Theta_{qz(x),qz(y)}) \iota_t^{q,t}(\Theta_{qz(u),qz(v)})$ can be approximated by sums of operators in $\mathcal{K}_{\mathcal{N}\mathcal{O}_X}(Y_q^{\mathcal{N}\mathcal{O}})$. Thus,

$$\iota_q^{q,t}(\Theta_{qz(x),qz(y)}) \iota_t^{q,t}(\Theta_{qz(u),qz(v)}) \in \mathcal{K}_{\mathcal{N}\mathcal{O}_X}(Y_q^{\mathcal{N}\mathcal{O}})$$

as required.

Shortly, we will examine the Cuntz–Nica–Pimsner algebra of the compactly aligned product system $Y^{\mathcal{N}\mathcal{O}}$. To make the calculations tractable, we seek sufficient conditions for the $*$-homomorphisms $\Phi_q^{\mathcal{N}\mathcal{O}} : \mathcal{N}\mathcal{O}_X \to L_{\mathcal{N}\mathcal{O}_X}(Y_q^{\mathcal{N}\mathcal{O}})$ that implement the left action of $\mathcal{N}\mathcal{O}_X$ on the fibres of $Y^{\mathcal{N}\mathcal{O}}$ to be injective. Again, we will make use of Theorem 3.1.48. The main step is showing that $\Phi_q^{\mathcal{N}\mathcal{O}}(\mathcal{N}\mathcal{O}_X) \subseteq L_{\mathcal{N}\mathcal{O}_X}(Y_q^{\mathcal{N}\mathcal{O}})$ carries a coaction of $G$ that intertwines $\Phi_q^{\mathcal{N}\mathcal{O}}$ with the canonical coaction of $G$ on $\mathcal{N}\mathcal{O}_X$. We first need some preliminary results.

**Lemma 3.4.9.** Suppose $G$ is an amenable group, $A$ acts faithfully on each fibre of $X$, and each $\phi_{(p,q)}$ is injective, so that the product system $Y^{\mathcal{N}\mathcal{O}}$ from Proposition 3.4.8 exists. Then

$$\nu_Z \circ \phi_X^{\mathcal{N}\mathcal{O}} = (\phi_X^{\mathcal{N}\mathcal{O}} \otimes \iota) \circ \nu_X.$$
If $a \in \mathcal{N}\mathcal{O}_X$ and $y \in Y_q^{\mathcal{N}\mathcal{O}}$, then

$$\nu_Z \left( \Phi_q^{\mathcal{N}\mathcal{O}}(a)(y) \right) = \left[ (\Phi_q^{\mathcal{N}\mathcal{O}} \otimes i) (\nu_X(a)) \right] (\nu_Z(y)).$$

Proof. Since both $\nu_Z \circ \phi_X^{\mathcal{N}\mathcal{O}}$ and $(\phi_X^{\mathcal{N}\mathcal{O}} \otimes i) \circ \nu_X$ are $*$-homomorphisms, it suffices to check that they agree on the generators of $\mathcal{N}\mathcal{O}_X$. Fix $p \in P$ and $x \in X_p$. Then

$$\nu_Z \left( \phi_X^{\mathcal{N}\mathcal{O}}(jx_p(x)) \right) = \nu_Z \left( jz_{(p,e_H)}(x) \right) = \nu_Z \left( jz_{(p,e_H)}(x) \otimes i_{G \times H} (p,e_H) \right) = (\phi_X^{\mathcal{N}\mathcal{O}} \otimes i) (jx_p(x) \otimes i_G(p))$$

as required. Since $\nu_Z$ is a $*$-homomorphism, we see that for any $a \in \mathcal{N}\mathcal{O}_X$ and $y \in Y_q^{\mathcal{N}\mathcal{O}}$,

$$\nu_Z \left( \Phi_q^{\mathcal{N}\mathcal{O}}(a)(y) \right) = \nu_Z \left( \phi_X^{\mathcal{N}\mathcal{O}}(a)(y) \right) = \nu_Z \left( \phi_X^{\mathcal{N}\mathcal{O}}(a) \right) \nu_Z(y)$$

Lemma 3.4.10. Suppose $G$ is an amenable group, $A$ acts faithfully on each fibre of $X$, and each $\tilde{\phi}_{(p,q)}$ is injective, so that the product system $Y_n^{\mathcal{N}\mathcal{O}}$ from Proposition 3.4.8 exists. Then for each $q \in Q$,

$$Y_q^{\mathcal{N}\mathcal{O}} \otimes C^*(G)$$

$$= \overline{\text{span}} \left\{ (\text{id}_{Y_q^{\mathcal{N}\mathcal{O}}} \otimes \iota^{-1}) \left[ \nu_Z \left( Y_q^{\mathcal{N}\mathcal{O}} \right) \left( 1_M(\mathcal{N}\mathcal{O}_Z) \otimes i_{G \times_H (G \times \{q^{-1}\})} \right) \right] \right\}.$$  \hspace{1cm} (3.14)

Proof. We begin by showing $\subseteq$. Fix $p,r \in P$, $g \in G$, $z \in Z_{(p,q)}$, $w \in Z_{(r,e_H)}$. Since

$$(r,e_H)(p,q)^{-1}(g,e_H) = (r \alpha_{q^{-1}}(p^{-1}g), q^{-1}),$$

we see that

$$\nu_Z \left( jz_{(p,q)}(z) jz_{(r,e_H)}(w)^* \right) \left( 1_M(\mathcal{N}\mathcal{O}_Z) \otimes i_{G \times_H (r \alpha_{q^{-1}}(p^{-1}g), q^{-1})) } \right)$$

$$= \left( jz_{(p,q)}(z) jz_{(r,e_H)}(w)^* \otimes i_{G \times_H (r \alpha_{q^{-1}}(p^{-1}g), q^{-1})} \right) \left( 1_M(\mathcal{N}\mathcal{O}_Z) \otimes i_{G \times_H (r \alpha_{q^{-1}}(p,q)^{-1}(g,e_H))} \right)$$

$$= jz_{(p,q)}(z) jz_{(r,e_H)}(w)^* \otimes i_{G \times_H (g,e_H)}$$

$$= \left( \text{id}_{Y_q^{\mathcal{N}\mathcal{O}}} \otimes i \right) \left( jz_{(p,q)}(z) jz_{(r,e_H)}(w)^* \otimes i_G(g) \right).$$
Thus,
\[
j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^* \otimes i_G(g)
= \left(\text{id}_{X^{\otimes} \otimes L^{-1}}\right)\left[\nu_Z\left(j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^*\right)\left(1, M(\mathcal{N} \mathcal{O} Z) \otimes i_{G_{r,e_H}}(r \alpha_{p,q}^{-1}(p^{-1}g), q^{-1})\right)\right]
\in \text{span} \left\{\left(\text{id}_{X^{\otimes} \otimes L^{-1}}\right)\left[\nu_Z\left(Y^{NO}_q\right)\left(1, M(\mathcal{N} \mathcal{O} Z) \otimes i_{G_{r,e_H}}(G \times \{q^{-1}\})\right)\right]\right\}.
\]

Since \(Y^{NO}_q = \text{span} \left\{j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^* : p, r \in P, z \in Z_{(p,q)}, w \in Z_{(r,e_H)}\right\}\) and \(C^*(G) = \text{span} \{i_G(g) : g \in G\}\), we conclude that \(\subseteq\) holds.

We now prove \(\supseteq\). If \(p, r \in P, g \in G, z \in Z_{(p,q)}, w \in Z_{(r,e_H)}\), then

\[
\nu_Z \left(j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^*\right) \left(1, M(\mathcal{N} \mathcal{O} Z) \otimes i_{G_{r,e_H}}(g^{-1})\right)
= \left(j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^* \otimes i_{G_{r,e_H}}((p,q)(r,e_H)^{-1})\right) \left(1, M(\mathcal{N} \mathcal{O} Z) \otimes i_{G_{r,e_H}}(g^{-1})\right)
= j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^* \otimes i_{G_{r,e_H}}((p,q)(r,e_H)^{-1}(g^{-1}))
= j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^* \otimes i_{G_{r,e_H}}(p \alpha_q(r^{-1}g), e_H)
= \left(1, Y^{NO}_q \otimes \text{id}_{C^*(G)}\right) \left(j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^* \otimes i_G(p \alpha_q(r^{-1}g))\right).
\]

Thus,

\[
\left(1, Y^{NO}_q \otimes \text{id}_{C^*(G)}\right) \left[\nu_Z \left(j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^*\right)\left(1, M(\mathcal{N} \mathcal{O} Z) \otimes i_{G_{r,e_H}}(g^{-1})\right)\right]
= j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^* \otimes i_G(p \alpha_q(r^{-1}g))
\in Y^{NO}_q \otimes C^*(G)
\]

Since \(Y^{NO}_q = \text{span} \left\{j_{z_{(p,q)}}(z)j_{z_{(r,e_H)}}(w)^* : p, r \in P, z \in Z_{(p,q)}, w \in Z_{(r,e_H)}\right\}\) and \(g \in G\) was arbitrary, we conclude that \(\supseteq\) holds. \(\Box\)

**Lemma 3.4.11.** Suppose \(G\) is an amenable group, \(A\) acts faithfully on each fibre of \(X\), and each \(\tilde{\phi}_{(p,q)}\) is injective, so that the product system \(Y^{NO}_q\) from Proposition 3.4.8 exists. If \(\Phi_q^{NO}(a) = 0 \in \mathcal{L}_{\mathcal{N} \mathcal{O}_X} (Y^{NO}_q)\) for some \(a \in \mathcal{N} \mathcal{O}_X\), then

\[
(\Phi_q^{NO} \otimes \text{id}_{C^*(G)})(\nu_X(a)) = 0 \in \mathcal{L}_{\mathcal{N} \mathcal{O}_X} (Y^{NO}_q) \otimes C^*(G).
\]

**Proof.** Suppose \(a \in \mathcal{N} \mathcal{O}_X\) is such that \(\Phi_q^{NO}(a) = 0 \in \mathcal{L}_{\mathcal{N} \mathcal{O}_X} (Y^{NO}_q)\). We want to show that \((\Phi_q^{NO} \otimes \text{id}_{C^*(G)})(\nu_X(a)) = 0 \in \mathcal{L}_{\mathcal{N} \mathcal{O}_X} (Y^{NO}_q) \otimes C^*(G)\). Since the tensor product \(\mathcal{L}_{\mathcal{N} \mathcal{O}_X} (Y^{NO}_q) \otimes C^*(G)\) is isomorphic to \(\mathcal{L}_{\mathcal{N} \mathcal{O}_X} (Y^{NO}_q) \otimes \mathcal{L}_{C^*(G)}(C^*(G))\), which embeds isometrically in \(\mathcal{L}_{\mathcal{N} \mathcal{O}_X \otimes C^*(G)} (Y^{NO}_q \otimes C^*(G))\) (see Chapter 4 of [40] for the details regarding exterior tensor products of Hilbert modules), it suffices to show that \((\Phi_q^{NO} \otimes \text{id}_{C^*(G)})(\nu_X(a))\) acts as the zero operator on \(Y^{NO}_q \otimes C^*(G)\). By Lemma 3.4.10, it suffices to show that \((\Phi_q^{NO} \otimes \text{id}_{C^*(G)})(\nu_X(a))\) is the zero operator.
For any $z$ by checking that $\Phi$

To do this we will use the fact that

$$ (\Phi_q^{NO} \otimes \text{id}_{C^*(G)}) (\nu_X(a)) \circ (\text{id}_{Y_q^{NO}} \otimes \iota^{-1}) = (\text{id}_{Y_q^{NO}} \otimes \iota^{-1}) \circ (\Phi_q^{NO} \otimes \iota)(\nu_X(a)), $$

and $(\Phi_q^{NO} \otimes \iota)(\nu_X(a))$ is left multiplication by $(\phi_X^{NO} \otimes \iota)(\nu_X(a))$ on

$$ Y_q^{NO} \otimes C^*(G \rtimes_{\alpha} H) \subseteq NOZ \otimes C^*(G \rtimes_{\alpha} H) \subseteq \mathcal{M}(NOZ) \otimes C^*(G \rtimes_{\alpha} H). $$

For any $y \in Y_q^{NO}$ and $g \in G$, since multiplication in $\mathcal{M}(NOZ) \otimes C^*(G \rtimes_{\alpha} H)$ is associative, we see that

$$ (\Phi_q^{NO} \otimes \text{id}_{C^*(G)}) (\nu_X(a)) \left( (\text{id}_{Y_q^{NO}} \otimes \iota^{-1}) \left[ \nu_Y \left( 1_{\mathcal{M}(NOZ)} \otimes i_{G \rtimes_{\alpha} H} \left( g, q^{-1} \right) \right) \right] \right) $$

$$ = (\text{id}_{Y_q^{NO}} \otimes \iota^{-1}) \left( (\Phi_q^{NO} \otimes \iota)(\nu_X(a)) \left[ \nu_Y \left( 1_{\mathcal{M}(NOZ)} \otimes i_{G \rtimes_{\alpha} H} \left( g, q^{-1} \right) \right) \right] \right) $$

$$ = (\text{id}_{Y_q^{NO}} \otimes \iota^{-1})(\nu_X(a))(\nu_Y(y)) \left( 1_{\mathcal{M}(NOZ)} \otimes i_{G \rtimes_{\alpha} H} \left( g, q^{-1} \right) \right). $$

(3.15)

By Lemma 3.4.9, we have

$$ (\Phi_q^{NO} \otimes \iota)(\nu_X(a))(\nu_Y(y)) = \nu_Y(\Phi_q^{NO}(a)(y)) = 0. $$

Thus, (3.15) is zero, and we conclude that $(\Phi_q^{NO} \otimes \text{id}_{C^*(G)}) (\nu_X(a))$ is the zero operator on $Y_q^{NO} \otimes C^*(G)$ as required. $\square$

We are now ready to show that the homomorphism $\Phi_q^{NO} : NO_X \rightarrow L_{NO_X}(Y_q^{NO})$ is injective.

**Proposition 3.4.12.** Suppose $G$ is an amenable group, $A$ acts faithfully on each fibre of $X$, and each $\phi_{(p,q)}$ is injective, so that the product system $Y^{NO}$ from Proposition 3.4.8 exists. If $A$ acts faithfully on $Z_{(e_G,q)}$, then $\Phi_q^{NO} : NO_X \rightarrow L_{NO_X}(Y_q^{NO})$ is injective.

**Proof.** Since $G$ is amenable we can use Theorem 3.1.48 to prove the result. We begin by checking that $\Phi_q^{NO}|_{j_{X_{e_G}}} \circ j_{X_{e_G}}$ is injective. Suppose $a \in A$ is such that $\Phi_q^{NO}(j_{X_{e_G}}(a)) = 0$. For any $z \in Z_{(e_G,q)}$, we have

$$ 0 = \Phi_q^{NO}(j_{X_{e_G}}(a)) \left( j_{Z_{(e_G,q)}}(z) \right) = j_{Z_{(e_G,q)}}(a)(j_{Z_{(e_G,q)}}(z)) = j_{Z_{(e_G,q)}}(a \cdot z). $$

Since $j_{Z_{(e_G,q)}}$ is isometric and $A$ acts faithfully on $Z_{(e_G,q)}$, we deduce that $a = 0$, which implies $j_{X_{e_G}}(a) = 0$. 


We now need to show that there is a coaction $\beta$ of $G$ on $\Phi^X_q (\mathcal{NO}_X)$ that intertwines $\Phi^X_q$ with the canonical gauge coaction of $G$ on $\mathcal{NO}_X$. Lemma 3.4.11 shows that there is a map $\beta : \Phi^X_q (\mathcal{NO}_X) \rightarrow \Phi^X_q (\mathcal{NO}_X) \otimes C^*(G)$ such that
\[
\beta \left( \Phi^X_q (a) \right) = \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \right) (\nu_X (a)) \quad \text{for each } a \in \mathcal{NO}_X.
\]
Since $\Phi^X_q$, $\Phi^X_q \otimes \text{id}_{C^*(G)}$, and $\nu_X$ are all $*$-homomorphisms, $\beta$ is a $*$-homomorphism.

It is elementary to check that $\beta$ intertwines $\Phi^X_q$ with the canonical gauge coaction $\nu_X$. For any $a \in \mathcal{NO}_X$, we have
\[
(\beta \circ \Phi^X_q) (a) = \beta \left( \Phi^X_q (a) \right) = \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \right) (\nu_X (a)) = \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \circ \nu_X \right) (a),
\]
which proves that
\[
\beta \circ \Phi^X_q = \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \right) \circ \nu_X. \tag{3.16}
\]

Next we check that $\beta$ satisfies the coaction identity. Making use of (3.16), we see that
\[
(\beta \otimes \text{id}_{C^*(G)}) \circ \beta \circ \Phi^X_q = (\beta \otimes \text{id}_{C^*(G)}) \circ \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \right) \circ \nu_X
\]
\[
= \left( \left( \beta \otimes \text{id}_{C^*(G)} \right) \otimes \text{id}_{C^*(G)} \right) \circ \nu_X
\]
\[
= \left( \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \right) \circ \nu_X \right) \otimes \text{id}_{C^*(G)} \circ \nu_X
\]
\[
= \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \otimes \text{id}_{C^*(G)} \right) \circ \left( \nu_X \otimes \text{id}_{C^*(G)} \right) \circ \nu_X.
\]
Since $\nu_X$ satisfies the coaction identity $\left( \nu_X \otimes \text{id}_{C^*(G)} \right) \circ \nu_X = (\text{id}_{\mathcal{NO}_X} \otimes \delta_G) \circ \nu_X$, we have
\[
(\beta \otimes \text{id}_{C^*(G)}) \circ \beta \circ \Phi^X_q = \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \otimes \text{id}_{C^*(G)} \right) \circ (\text{id}_{\mathcal{NO}_X} \otimes \delta_G) \circ \nu_X
\]
\[
= \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \right) \circ \nu_X
\]
\[
= \left( \text{id}_{\Phi^X_q (\mathcal{NO}_X) \otimes \delta_G} \right) \circ \left( \Phi^X_q \otimes \text{id}_{C^*(G)} \right) \circ \nu_X
\]
\[
= \left( \text{id}_{\Phi^X_q (\mathcal{NO}_X) \otimes \delta_G} \right) \circ \beta \circ \Phi^X_q.
\]
Thus, $(\beta \otimes \text{id}_{C^*(G)}) \circ \beta = (\text{id}_{\Phi^X_q (\mathcal{NO}_X) \otimes \delta_G}) \circ \beta$, and so $\beta$ satisfies the coaction identity.

We also need to show that $\beta$ is coaction nondegenerate. Again making use of
(3.16), we have

\[
\text{span} \left\{ \beta \left( \Phi_q^{N \O} (N \O_X) \right) \left( 1_{M(\Phi_q^{N \O}(N \O_X))} \otimes C^*(G) \right) \right\}
\]

\[
= \text{span} \left\{ \left( \Phi_q^{N \O} \otimes \text{id}_{C^*(G)} \right) (\nu_X (N \O_X)) \left( 1_{M(\Phi_q^{N \O}(N \O_X))} \otimes C^*(G) \right) \right\}
\]

\[
= \text{span} \left\{ \left( \Phi_q^{N \O} \otimes \text{id}_{C^*(G)} \right) (\nu_X (N \O_X) \left( 1_{M(N \O_X)} \otimes C^*(G) \right)) \right\}
\]

\[
= \left( \Phi_q^{N \O} \otimes \text{id}_{C^*(G)} \right) \text{span} \left\{ \nu_X (N \O_X) \left( 1_{M(N \O_X)} \otimes C^*(G) \right) \right\}.
\]

Since \( \nu_X \) is coaction nondegenerate, this is equal to

\[
\left( \Phi_q^{N \O} \otimes \text{id}_{C^*(G)} \right) (N \O_X \otimes C^*(G)) = \Phi_q^{N \O} (N \O_X) \otimes C^*(G),
\]

and we see that \( \beta \) is coaction nondegenerate.

Finally, we check that \( \beta \) is injective. Suppose \( \beta \left( \Phi_q^{N \O}(a) \right) = 0 \) for some \( a \in N \O_X \). We must show that \( \Phi_q^{N \O}(a) = 0 \). We will make use of the fact that \( \left( \Phi_q^{N \O} \otimes \iota \right) (\nu_X(a)) \) is left multiplication by \( \left( \Phi_q^{N \O} \otimes \iota \right) (\nu_X(a)) \) on

\[
Y_q^{N \O} \otimes C^*(G \rtimes_{\alpha} H) \subseteq N \O_Z \otimes C^*(G \rtimes_{\alpha} H) \subseteq M(N \O_Z) \otimes C^*(G \rtimes_{\alpha} H),
\]

and multiplication in \( M(N \O_Z) \otimes C^*(G \rtimes_{\alpha} H) \) is associative. For any \( y \in Y_q^{N \O} \), Lemma 3.4.9 shows that

\[
\nu_Z \left( \Phi_q^{N \O}(a)(y) \right)
\]

\[
= \left( \Phi_q^{N \O} \otimes \iota \right) (\nu_X(a))(\nu_Z(y))
\]

\[
= \left( \Phi_q^{N \O} \otimes \iota \right) (\nu_X(a)) \left[ \nu_Z(y) \left( 1_{M(N \O_Z)} \otimes i_{G \rtimes_{\alpha} H} \left( e_G ; q^{-1} \right) \right) \left( 1_{M(N \O_Z)} \otimes i_{G \rtimes_{\alpha} H} \left( e_G ; q \right) \right) \right]
\]

\[
= \left( \Phi_q^{N \O} \otimes \iota \right) (\nu_X(a)) \left[ \nu_Z(y) \left( 1_{M(N \O_Z)} \otimes i_{G \rtimes_{\alpha} H} \left( e_G ; q^{-1} \right) \right) \left( 1_{M(N \O_Z)} \otimes i_{G \rtimes_{\alpha} H} \left( e_G ; q \right) \right) \right].
\]

Lemma 3.4.10 tells us that

\[
\nu_Z(y) \left( 1_{M(N \O_Z)} \otimes i_{G \rtimes_{\alpha} H} \left( e_G ; q^{-1} \right) \right) \in Y_q^{N \O} \otimes \iota \left( C^*(G) \right).
\]

Since

\[
\left( \Phi_q^{N \O} \otimes \iota \right) (\nu_X(a)) = \left( \text{id}_{Y_q^{N \O}} \otimes \iota \right) \circ \left( \Phi_q^{N \O} \otimes \text{id}_{C^*(G)} \right) (\nu_X(a)) \circ \left( \text{id}_{Y_q^{N \O}} \otimes \iota^{-1} \right)
\]

\[
= \left( \text{id}_{Y_q^{N \O}} \otimes \iota \right) \circ \beta \left( \Phi_q^{N \O}(a) \right) \circ \left( \text{id}_{Y_q^{N \O}} \otimes \iota^{-1} \right)
\]

\[
= 0,
\]

we conclude that (3.17) is zero. As \( \nu_Z \) is injective, we must have \( \Phi_q^{N \O}(a)(y) = 0 \). As \( y \in Y_q^{N \O} \) was arbitrary, it follows that \( \Phi_q^{N \O}(a) = 0 \). Hence, \( \beta \) is injective.

Putting all of this together and applying Theorem 3.1.48, we conclude that \( \Phi_q^{N \O} \)
is injective.

In the next subsection we will investigate the Cuntz–Nica–Pimsner algebra of $Y^{N\mathcal{O}}$. To make our calculations tractable we will assume that $A$ acts faithfully on each fibre $Z$ — by Proposition 3.4.12 this implies that $N\mathcal{O}_X$ acts faithfully on each fibre of $Y^{N\mathcal{O}}$. Moreover, if $A$ acts faithfully on each fibre of $Z$, then $A$ acts faithfully on each fibre of $X$ and each $\tilde{\phi}_{(p,q)}$ is injective, ensuring that all of the results from Subsection 3.4.1 work.

### 3.4.2 Isomorphisms of Cuntz–Nica–Pimsner algebras

With the product system $Y^{N\mathcal{O}}$ defined in the previous subsection, we will show that $N\mathcal{O}_Y \cong N\mathcal{O}_Z$. To do this we will use the universal property of each $C^*$-algebra to induce a $*$-homomorphism from one to the other, and then check that these $*$-homomorphisms are the inverses of each other. To make our arguments easier to write down, we will identify the coefficient algebra $N\mathcal{O}_X$ of $Y^{N\mathcal{O}}$ with $\varphi_{N\mathcal{O}}(N\mathcal{O}_X) \subseteq N\mathcal{O}_Z$. Thus, every fibre of $Y^{N\mathcal{O}}$ can be viewed as sitting inside $N\mathcal{O}_Z$, and the left and right actions of $N\mathcal{O}_X \cong \phi_X^{N\mathcal{O}}(N\mathcal{O}_X)$ on each $Y_q^{N\mathcal{O}}$ are multiplication in $N\mathcal{O}_Z$.

To begin, we get a $*$-homomorphism from $N\mathcal{O}_Z$ to $N\mathcal{O}_Y^{N\mathcal{O}}$ by exhibiting a Cuntz–Nica–Pimsner covariant representation of $Z$ in $N\mathcal{O}_Y^{N\mathcal{O}}$.

**Definition 3.4.13.** Suppose $G$ is an amenable group, and $A$ acts faithfully on each fibre of $Z$, so that the product system $Y^{N\mathcal{O}}$ from Proposition 3.4.8 exists. Define $\varphi^{N\mathcal{O}}: Z \to N\mathcal{O}_Y^{N\mathcal{O}}$ by

$$\varphi^{N\mathcal{O}}_{(p,q)} := j_{Y_q^{N\mathcal{O}}} \circ j_{Z_{(p,q)}}$$

for each $(p,q) \in P \rtimes_\alpha Q$.

Since $j_{Y^{N\mathcal{O}}}$ and $j_Z$ are Nica covariant representations, the same reasoning as in the proofs of Proposition 3.3.13, Lemma 3.3.14, and Proposition 3.3.15 shows that $\varphi^{N\mathcal{O}}$ is a Nica covariant representation of $Z$. We have some work to do to show that it is Cuntz–Pimsner covariant.

For the next lemma recall Definition 3.1.39 — given a quasi-lattice ordered group $(G,P)$, we say that a predicate statement $\mathcal{P}(m)$ (where $m \in P$) is true for large $m$ if, given any $p \in P$, there exists $q \geq p$, such that $\mathcal{P}(m)$ is true whenever $m \geq q$.

**Lemma 3.4.14.** Suppose $(K,R)$ is a quasi-lattice ordered group. Let $F \subseteq R$ be finite. Then

$$F = \{ r \in F : r \leq m \} \sqcup \{ r \in F : r \vee m = \infty \}$$

for large $m \in R$. 

Thus, \( F \) for particular for large \( m \). Now fix \( p \in R \). Let \( F' \) be a maximal element of the collection

\[
\{ F' \subseteq F \cup \{ p \} : \text{\( F' \) contains \( p \) and is bounded above} \}
\]

(partially ordered by set inclusion), which exists since \( F \) is finite whilst \( \{ p \} \subseteq F \cup \{ p \} \) is bounded above (by \( p \)) and contains \( p \). Let \( n \) be an upper bound for \( F' \). Since \( p \in F' \), we have that \( p \leq n \). Suppose \( m \in R \) with \( n \leq m \); Let \( r \in F \). If \( r \in F' \) then \( r \leq n \leq m \). Alternatively, if \( r \in F \setminus F' \), then \( r \vee m = \infty \) (if \( r \vee m < \infty \), then \( r \vee m \) is an upper bound for \( F' \cup \{ r \} \), which contradicts the maximality of \( F' \)).

Thus, \( F \subseteq \{ r \in F : r \leq m \} \cup \{ r \in F : r \vee m = \infty \} \). Since \( p \in R \) was arbitrary, we conclude that that \( F \subseteq \{ r \in F : r \leq m \} \cup \{ r \in F : r \vee m = \infty \} \) for large \( m \). \( \square \)

**Lemma 3.4.15.** Suppose \( G \) is an amenable group, and \( A \) acts faithfully on each fibre of \( Z \), so that the product system \( Y^{NO} \) from Proposition 3.4.8 exists. Let \( (p, q) \in P \rtimes \alpha Q \) and \( T \in K_A(Z_{(p,q)}) \). If \( m \in Q \) and \( q \vee m = \infty \), then

\[
M_{\tilde{j}_Z^{(p,q)}(T)} = 0 \in K_{NO_X}(Y_m^{NO}) .
\]

**Proof.** Fix \( T \in K_A(Z_{(p,q)}) \) and \( m \in Q \) with \( m \vee q = \infty \). Let \( r, n \in P \) and fix \( z \in Z_{(r,m)} \) and \( w \in Z_{(n,e_H)} \). Since \( (p, q) \vee (r, m) = \infty \) as \( q \vee m = \infty \), an application of Lemma 3.1.22 shows that

\[
M_{\tilde{j}_Z^{(p,q)}(T)} \left( j_{Z_{(r,m)}}(z)j_{Z_{(n,e_H)}}(w)^* \right) = \tilde{j}_Z^{(p,q)}(T)j_{Z_{(r,m)}}(z)j_{Z_{(n,e_H)}}(w)^* \\
\in \text{span} \left\{ j_{Z_{(p,q)}}(Z_{(p,q)})j_{Z_{(p,q)}}(Z_{(p,q)})^* j_{Z_{(r,m)}}(z)j_{Z_{(n,e_H)}}(w)^* \right\} \\
= \{0\} .
\]

As \( Y_m^{NO} = \text{span} \left\{ j_{Z_{(r,m)}}(z)j_{Z_{(n,e_H)}}(w)^*: r, n \in P, z \in Z_{(r,m)}, w \in Z_{(n,e_H)} \right\} \), we conclude that \( M_{\tilde{j}_Z^{(p,q)}(T)} = 0 \in K_{NO_X}(Y_m^{NO}) \). \( \square \)

**Proposition 3.4.16.** Suppose \( G \) is an amenable group, and \( A \) acts faithfully on each fibre of \( Z \), so that the product system \( Y^{NO} \) from Proposition 3.4.8 exists, and \( NO_X \) acts faithfully on each fibre of \( Y^{NO} \) by Proposition 3.4.12.

Then the Nica covariant representation \( \varphi^{NO} : Z \to NO_{Y^{NO}} \) from Definition 3.4.13 is Cuntz–Pimsner covariant. Hence, there exists a \(*\)-homomorphism \( \Omega^{NO} : NO_Z \to NO_{Y^{NO}} \) such that \( \Omega^{NO} \circ j_{Z_{(p,q)}} = \varphi^{NO}_{(p,q)} = j_{Y^{NO}} \circ j_{Z_{(p,q)}} \) for each \((p, q) \in P \rtimes \alpha Q \).

**Proof.** Let \( F \) be a finite subset of \( P \rtimes \alpha Q \) and suppose we have a set of compact
operators \( \{ T_{(p,q)} \in \mathcal{K}_A (Z_{(p,q)}) : (p,q) \in F \} \) such that
\[
\sum_{(p,q) \in F} t^{(s,t)}_{(p,q)} (T_{(p,q)}) = 0 \in \mathcal{L}_A (Z_{(s,t)})
\]
for large \( (s,t) \in P \rtimes Q \). Since \( j_Z \) is Cuntz–Pimsner covariant, we have
\[
\sum_{(p,q) \in F} j^{((p,q))}_{Z} (T_{(p,q)}) = 0.
\]

To show that \( \varphi^{\mathcal{N}\mathcal{O}} \) is Cuntz–Pimsner covariant, we need to show that
\[
\sum_{(p,q) \in F} \varphi^{\mathcal{N}\mathcal{O}} ((p,q)) (T_{(p,q)}) = 0 \in \mathcal{N}\mathcal{O}_{\mathcal{Y}^{\mathcal{N}\mathcal{O}}}.
\]

By Lemma 3.3.14, this is equivalent to showing that
\[
\sum_{(p,q) \in F} j^{(q)}_{\mathcal{Y}^{\mathcal{N}\mathcal{O}}} \left( M_{j^{((p,q))}_{Z} (T_{(p,q)})} \right) = 0 \in \mathcal{N}\mathcal{O}_{\mathcal{Y}^{\mathcal{N}\mathcal{O}}}.
\]

Since \( j_{\mathcal{Y}^{\mathcal{N}\mathcal{O}}} \) is a Cuntz–Pimsner covariant representation of \( \mathcal{Y}^{\mathcal{N}\mathcal{O}} \), it suffices to show that
\[
\sum_{(p,q) \in F} t^{m}_{q} \left( M_{j^{((p,q))}_{Z} (T_{(p,q)})} \right) = 0 \in \mathcal{L}_{\mathcal{N}\mathcal{O}_{\mathcal{X}}} (\mathcal{Y}^{\mathcal{N}\mathcal{O}}_{m})
\]
for large \( m \in Q \), which is equivalent to showing that
\[
\sum_{\{ (p,q) \in F : q \leq m \}} M_{j^{((p,q))}_{Z} (T_{(p,q)})} = 0 \in \mathcal{L}_{\mathcal{N}\mathcal{O}_{\mathcal{X}}} (\mathcal{Y}^{\mathcal{N}\mathcal{O}}_{m})
\]
for large \( m \in Q \). Making use of Lemma 3.4.14, we see that for large \( m \in Q \),
\[
\sum_{\{ (p,q) \in F : q \leq m \}} M_{j^{((p,q))}_{Z} (T_{(p,q)})} = \sum_{(p,q) \in F} M_{j^{((p,q))}_{Z} (T_{(p,q)})} - \sum_{\{ (p,q) \in F : q \geq m = \infty \}} M_{j^{((p,q))}_{Z} (T_{(p,q)})}
\]
\[
= \sum_{(p,q) \in F} M_{j^{((p,q))}_{Z} (T_{(p,q)})} - \sum_{(p,q) \in F : q \geq m = \infty} M_{j^{((p,q))}_{Z} (T_{(p,q)})}
\]
\[
= - \sum_{\{ (p,q) \in F : q \geq m = \infty \}} M_{j^{((p,q))}_{Z} (T_{(p,q)})},
\]
It then follows from Lemma 3.4.15, that the last line is zero, as required. Hence, \( \varphi^{\mathcal{N}\mathcal{O}} \) is Cuntz–Pimsner covariant. The final statement follows from the universal property of \( \mathcal{N}\mathcal{O}_{Z} \).

We now work towards getting a \(*\)-homomorphism from \( \mathcal{N}\mathcal{O}_{\mathcal{Y}^{\mathcal{N}\mathcal{O}}} \) to \( \mathcal{N}\mathcal{O}_{Z} \) inverse to the one from \( \mathcal{N}\mathcal{O}_{Z} \) to \( \mathcal{N}\mathcal{O}_{\mathcal{Y}^{\mathcal{N}\mathcal{O}}} \) just constructed. Our plan is to exhibit a Nica covariant representation of \( \mathcal{Y}^{\mathcal{N}\mathcal{O}} \) in \( \mathcal{N}\mathcal{O}_{Z} \) and then use the universal property of
\(\mathcal{NO}_X\) to induce a \(*\)-homomorphism.

In Proposition 3.4.12 we found conditions on the group \(G\) and the product system \(Z\) to ensure that the left actions of \(\mathcal{NO}_X\) on each \(Y^\mathcal{NO}_q\) are faithful. In the next result, we exhibit sufficient conditions for these actions to be by compact operators.

**Lemma 3.4.17.** Suppose \(G\) is an amenable group, and \(A\) acts faithfully on each fibre of \(Z\), so that the product system \(Y^\mathcal{NO}\) from Proposition 3.4.8 exists. If \(q \in \mathbb{Q}\) and \(a \in \phi^{-1}_{(eG,q)}(\mathcal{K}_A(Z_{(eG,q)}))\), then

\[
\Phi^\mathcal{NO}_q(jx_{eG}(a)) = M_{jz_{(eG,q)}}(\phi_{(eG,q)}(a)) \in \mathcal{K}\mathcal{NO}_X(Y^\mathcal{NO}_q). \tag{3.18}
\]

In particular, if \(A\) acts compactly on \(Z_{(eG,q)}\), then \(\mathcal{NO}_X\) acts compactly on \(Y^\mathcal{NO}_q\).

**Proof.** Fix \(q \in \mathbb{Q}\) and \(a \in \phi^{-1}_{(eG,q)}(\mathcal{K}_A(Z_{(eG,q)}))\). Hence, we can write

\[
\phi_{(eG,q)}(a) = \lim_{i \to \infty} \sum_{k_i=1}^{n_i} \Theta_{\mu_{k_i},\nu_{k_i}} \in \mathcal{K}_A(Z_{(eG,q)}).
\]

For any \(z \in Z_{(eG,q)}\) and \(b \in \mathcal{NO}_X\), we have

\[
\Phi^\mathcal{NO}_q(jx_{eG}(a)) \left( jz_{(eG,q)}(z) \phi^\mathcal{NO}_X(b) \right) = jz_{(eG,q)} \left( \phi_{(eG,q)}(a)(z) \right) \phi^\mathcal{NO}_X(b)
\]

\[
= jz_{(eG,q)} \left( \lim_{i \to \infty} \sum_{k_i=1}^{n_i} \Theta_{\mu_{k_i},\nu_{k_i}}(z) \right) \phi^\mathcal{NO}_X(b)
\]

\[
= jz_{(eG,q)} \left( \lim_{i \to \infty} \sum_{k_i=1}^{n_i} \mu_{k_i} \cdot \langle \nu_{k_i}, z \rangle_A \right) \phi^\mathcal{NO}_X(b).
\]

Since \(jz_{(eG,q)}\) is linear and norm-decreasing, this is equal to

\[
\lim_{i \to \infty} \sum_{k_i=1}^{n_i} jz_{(eG,q)} \left( \mu_{k_i} \cdot \langle \nu_{k_i}, z \rangle_A \right) \phi^\mathcal{NO}_X(b)
\]

\[
= \lim_{i \to \infty} \sum_{k_i=1}^{n_i} jz_{(eG,q)}(\mu_{k_i}) jz_{(eG,q)}(\nu_{k_i})^* jz_{(eG,q)}(z) \phi^\mathcal{NO}_X(b)
\]

\[
= \lim_{i \to \infty} \sum_{k_i=1}^{n_i} M_{jz_{(eG,q)}}(\Theta_{\mu_{k_i},\nu_{k_i}}) \left( jz_{(eG,q)}(z) \phi^\mathcal{NO}_X(b) \right).
\]

Since the map \(b \mapsto M_b\) is linear, and \(\|M_b\|_{\mathcal{NO}_X(Y^\mathcal{NO}_q)} \leq \|b\|_{\mathcal{NO}_Z}\) whenever \(b \in \mathcal{NO}_Z\)
is such that \( M_b \in \mathcal{L}_{\mathcal{NO}_X} (Y_q^{\mathcal{NO}}) \) (see Lemma 3.3.9), this is equal to

\[
M_{\lim_{i \to \infty} \sum_{k_i=1}^{n_i} j_{\mathcal{Z}(e_G,q)} (\theta_{p_{k_i}}, q_{k_i})} (j_{\mathcal{Z}(e_G,q)}(z)\phi_X^{\mathcal{NO}}(b))
\]

\[
= M_{j_{\mathcal{Z}(e_G,q)} (\lim_{i \to \infty} \sum_{k_i=1}^{n_i} \theta_{p_{k_i}}, q_{k_i})} (j_{\mathcal{Z}(e_G,q)}(z)\phi_X^{\mathcal{NO}}(b))
\]

\[
= M_{j_{\mathcal{Z}(e_G,q)} (\phi_{e_G,q}(a))} (j_{\mathcal{Z}(e_G,q)}(z)\phi_X^{\mathcal{NO}}(b))
\]

where the first equality follows from the fact that \( j_{\mathcal{Z}(e_G,q)} \) is a *-homomorphism and so is continuous. Since \( \Phi_q^{\mathcal{NO}} \) and \( M_{j_{\mathcal{Z}(e_G,q)} (\phi_{e_G,q}(a))} \) are linear and continuous, and \( Y_q^{\mathcal{NO}} = \text{span} \{ j_{\mathcal{Z}(e_G,q)}(z)\phi_X^{\mathcal{NO}}(b) : z \in \mathcal{Z}(e_G,q), b \in \mathcal{NO}_X \} \), we conclude that

\[
\Phi_q^{\mathcal{NO}} (j_{\mathcal{X}(q)}(a)) = M_{j_{\mathcal{Z}(e_G,q)} (\phi_{e_G,q}(a))}.
\]

To establish (3.18), it remains to show that \( M_{j_{\mathcal{Z}(e_G,q)} (\phi_{e_G,q}(a))} \in \mathcal{K}_{\mathcal{NO}_X} (Y_q^{\mathcal{NO}}) \). This follows from Lemma 3.3.10 since \( j_{\mathcal{Z}(e_G,q)} (\phi_{e_G,q}(a)) \in \mathcal{NO}_Z \).

Now suppose that \( A \) acts compactly on \( \mathcal{Z}(e_G,q) \). Fix \( p \in P \) and \( x \in \mathcal{X}_p \). Choose \( x' \in \mathcal{X}_p \) so that \( x = x' \cdot \langle x', x' \rangle_A \) by the Hewitt–Cohen–Blanchard factorisation theorem. Since \( \langle x', x' \rangle_A \in \phi_{e_G,q}^{-1} (\mathcal{K}_A (\mathcal{Z}(e_G,q))) \), we can apply the first part of the result to see that

\[
\Phi_q^{\mathcal{NO}} (j_{\mathcal{X}_p}(x)) = \Phi_q^{\mathcal{NO}} (j_{\mathcal{X}_p}(x')) \Phi_q^{\mathcal{NO}} (j_{\mathcal{X}_p}(\langle x', x' \rangle_A)) \in \mathcal{K}_{\mathcal{NO}_X} (Y_q^{\mathcal{NO}}).
\]

Since \( \Phi_q^{\mathcal{NO}} \) is a *-homomorphism and \( \mathcal{NO}_X \) is generated by the image of \( j_{\mathcal{X}} \), we have that \( \Phi_q^{\mathcal{NO}} (\mathcal{NO}_X) \subseteq \mathcal{K}_{\mathcal{NO}_X} (Y_q^{\mathcal{NO}}) \), and so \( \mathcal{NO}_X \) acts compactly on \( Y_q^{\mathcal{NO}} \). \( \square \)

We are almost ready to show that the inclusion of \( Y^{\mathcal{NO}} \) in \( \mathcal{NO}_Z \) is a Cuntz–Nica–Pimsner covariant representation. Before we do so, we need one last lemma.

**Lemma 3.4.18.** Let \( q \in Q \) and \( a \in \phi_{e_G,q}^{-1} (\mathcal{K}_A (\mathcal{Z}(e_G,q))) \). If \( (H, Q) \) is directed, then

\[
j_{\mathcal{Z}(e_G, e_H)} (a) = j_{\mathcal{Z}(e_G,q)} (\phi_{e_G,q}(a)).
\]

**Proof.** Since \( j_{\mathcal{Z}} \) is Cuntz–Pimsner covariant, it suffices to show that

\[
(3.19) \quad \lim_{i \to \infty} \mathcal{L}_A (Z_{s_i, t_i}) (a) = 0
\]

for large \( (s, t) \in P \times_a Q \).

Fix \( (m, n) \in P \times_a Q \). Since \( (H, Q) \) is directed, \( (m, n) \leq (m, n \lor q) \). Suppose that \( (s, t) \in P \times_a Q \) with \( (m, n \lor q) \leq (s, t) \). For any \( z \in \mathcal{Z}(e_G,q) \) and \( w \in \mathcal{Z}(\alpha_{q-1}(s), q^{-1} t) \),
we have
\[
\left( \iota_{(e_G,e_H)}^{(s,t)} (a) - \iota_{(e_G,q)}^{(s,t)} (\phi_{(e_G,q)} (a)) \right) (zw) = \phi_{(s,t)} (ap) (zw) - \left( \phi_{(e_G,q)} (a) (z) \right) w
\]
\[
= \phi_{(e_G,q)} (a) (z) w - \left( \phi_{(e_G,q)} (a) (z) \right) w
\]
\[
= \left[ \left( \phi_{(e_G,q)} (a) - \left( \phi_{(e_G,q)} (a) \right) \right) (z) \right] w
\]
\[
= 0.
\]

Since \( \iota_{(e_G,e_H)}^{(s,t)} (a) - \iota_{(e_G,q)}^{(s,t)} (\phi_{(e_G,q)} (a)) \in \mathcal{L}_A (Z_{(s,t)}) \) is linear and continuous, and since \( Z_{(s,t)} = \text{span} \left\{ zw : z \in Z_{(e_G,q)}, w \in Z_{(\alpha_{q-1}(s),q^{-1}t)} \right\} \), we conclude that (3.19) holds.

\[ \square \]

**Proposition 3.4.19.** Suppose \( G \) is an amenable group, \( A \) acts faithfully on each fibre of \( Z \), so that the product system \( Y^{\mathcal{N}O} \) from Proposition 3.4.8 exists, and \( N\mathcal{O}_X \) acts faithfully on each fibre of \( Y^{\mathcal{N}O} \) by Proposition 3.4.12. Moreover, suppose that \( A \) acts compactly on each \( Z_{(e_G,q)} \), so that \( N\mathcal{O}_X \) acts compactly on each fibre of \( Y^{\mathcal{N}O} \) by Lemma 3.4.17.

Suppose that the quasi-lattice ordered group \((H,Q)\) is directed. For each \( q \in Q \), let \( \varphi_q^{\mathcal{N}O} \) be the inclusion of \( Y_q^{\mathcal{N}O} \) in \( N\mathcal{O}_Z \). Then \( \varphi_q^{\mathcal{N}O} \) is a Cuntz–Nica–Pimsner covariant representation of \( Y^{\mathcal{N}O} \). Hence, there exists a \( * \)-homomorphism \( \Omega^{\mathcal{N}O} : N\mathcal{O}_{Y^{\mathcal{N}O}} \to N\mathcal{O}_Z \) such that \( \Omega^{\mathcal{N}O} \circ j_{Y_q^{\mathcal{N}O}} = \varphi_q' \) for each \( q \in Q \).

**Proof.** The same reasoning as in Proposition 3.3.16 shows that \( \varphi_q^{\mathcal{N}O} \) is a Nica covariant representation of \( Y_q^{\mathcal{N}O} \) in \( N\mathcal{O}_Z \). It remains to show that \( \varphi_q^{\mathcal{N}O} \) is Cuntz–Pimsner covariant. By Proposition 3.4.12 and Lemma 3.4.17, \( N\mathcal{O}_X \) acts faithfully and compactly on each \( Y_q^{\mathcal{N}O} \). Hence, by Proposition 3.1.45, it suffices to check that \( (\varphi_q^{\mathcal{N}O})^{(q)} \circ \Phi_q^{\mathcal{N}O} = \varphi_q^{\mathcal{N}O} \) for each \( q \in Q \). This is clear when \( q = e_H \), so we just need to worry about when \( q \neq e_H \). As \( Y_{e_H}^{\mathcal{N}O} = N\mathcal{O}_X \cong \phi_X^{\mathcal{N}O} (N\mathcal{O}_X) \) is generated by the image of \( j_L |_X \) it suffices to show that

\[ \left( (\varphi_q^{\mathcal{N}O})^{(q)} \circ \Phi_q^{\mathcal{N}O} \right) (j_L (x)) = \varphi_{e_H}^{\mathcal{N}O} (j_L (x)) \quad \text{for all } x \in X. \] (3.20)

Let \( p \in P \) and \( x \in X_p \). Choosing \( x' \in X_p \) so that \( x = x' \cdot (x',x')_A^p \) by the Hewitt–Cohen–Blanchard factorisation theorem, we have

\[ \varphi_{e_H}^{\mathcal{N}O} (j_L (p,e_H)) (x) = j_L (p,e_H) (x) = j_L (p,e_H) (x') j_L (e_G,e_H) ((x',x')_A^p). \]

By the first part of Lemma 3.4.17, we see that

\[ \Phi_q^{\mathcal{N}O} (j_L (p,e_H)) (x) = M_{j_L (p,e_H)} (x') j_L ((e_G,q)) (\phi_{(e_G,q)} ((x',x')_A^p)) \in K_{N\mathcal{O}_X} (Y_q^{\mathcal{N}O}). \]
Since \( j_{(e_G, q)} ((e_G, q)) (\phi_{(e_G, q)} ((\langle x', x' \rangle_A)^p )) \in \mathcal{N}O_Z \), it follows from the argument in Proposition 3.3.16 that

\[
\left( (\varphi \circ \Phi_N) (x) \right) = \left( j_{(e_G, q)} ((e_G, q)) (\phi_{(e_G, q)} ((\langle x', x' \rangle_A)^p )) \right).
\]

Thus, for (3.20) to hold, it suffices to show that

\[
j_{(e_G, q)} ((e_G, q)) (\phi_{(e_G, q)} ((\langle x', x' \rangle_A)^p )) = j_{(e_G, q)} ((e_G, q)) (\phi_{(e_G, q)} ((\langle x', x' \rangle_A)^p )) \]

which follows from Lemma 3.4.18.

\[\square\]

Remark 3.4.20. It is not clear if all of the hypotheses in the previous result are necessary — in particular, we would like to know whether the assumptions that \( A \) acts compactly on each \( Z(e_G, q) \) and that \((H, Q)\) is directed are necessary.

Putting all of this together, we can show that \( NO_Z \) and \( NO_Y NO \) are isomorphic.

**Theorem 3.4.21.** Suppose \( G \) is an amenable group, \( A \) acts faithfully on each fibre of \( Z \), so that the product system \( Y NO \) from Proposition 3.4.8 exists, and \( NO_X \) acts faithfully on each fibre of \( Y NO \) by Proposition 3.4.12. Moreover, suppose that \( A \) acts compactly on each \( Z(e_G, q) \), so that \( NO_X \) acts compactly on each fibre of \( Y NO \) by Lemma 3.4.17.

Suppose that \((H, Q)\) is directed, so that the \(*\)-homomorphism \( \Omega^{NO} \) from Proposition 3.4.19 exists. Then the \(*\)-homomorphisms \( \Omega^{NO} : NO_Z \to NO_Y NO \) and \( \Omega^{NO} : NO_Y NO \to NO_Z \) are mutually inverse. Thus, \( NO_Z \cong NO_Y NO \).

**Proof.** The same reasoning as in the proof of Theorem 3.3.17 shows that \( \Omega^{NO} \) and \( \Omega^{NO} \) are mutually inverse. \[\square\]

### 3.5 Relative Cuntz–Nica–Pimsner algebras

In Section 3.3 we showed that there exists a product system \( X \) over \((G, P)\) with coefficient algebra \( A \), and a product system \( Y NT \) with coefficient algebra \( NT_X \) over \((H, Q)\), such that \( NT_Z \cong NT_Y NT \). In this section we consider the Cuntz–Nica–Pimsner algebra of \( Y NT \).

We show that when the action \( \alpha \) of \( H \) on \( G \) is trivial, there exists a product system \( V \) over \((H, Q)\), and a product system \( W NO \) over \((G, P)\) with coefficient algebra \( NO_V \), such that the Cuntz–Nica–Pimsner algebra of \( Y NT \) is isomorphic to...
the Nica–Toeplitz algebra of $\mathcal{W}^{NO}$. In a sense, we have shown that the Cuntz–Pimsner covariance in the $C^*$-algebra $\mathcal{N}O_{YNT}$ can be moved into the coefficient algebra of the product system $\mathcal{W}^{NO}$.

Since in the $C^*$-algebras $\mathcal{N}O_{YNT}$ and $\mathcal{N}T_{\mathcal{W}^{NO}}$ we are, in effect, only asking for Cuntz–Pimsner covariance in some of the fibres of $\mathcal{Z}$, we like to think of these algebras as relative Cuntz–Nica–Pimsner algebras. This nomenclature is motivated by the relative Cuntz–Krieger algebras of higher-rank graphs introduced by Sims [66], and the relative Cuntz–Pimsner algebras introduced by Muhly and Solel [45] (which were studied further by Fowler, Muhly, and Raeburn [25]).

**Standing Hypotheses.** Throughout Section 3.5 we will assume that the action $\alpha : H \to \text{Aut}(G)$ is trivial. Hence, $\mathcal{Z}$ is a compactly aligned product system over $(G \times H, P \times Q)$. Since $G \rtimes_{\alpha} H \cong G \times H \cong H \times G$ and $P \rtimes_{\alpha} Q \cong P \times Q \cong Q \times P$, we can swap the roles of $G$ and $H$, and $P$ and $Q$ in our results from Sections 3.3 and 3.4. Moreover, to ensure that the various $C^*$-algebras and product systems that we want to work with actually exist, we assume that both $G$ and $H$ are amenable, $A$ acts faithfully on each $\mathcal{Z}(e_G,q)$, and each $\tilde{\phi}(p,q)$ is injective.

We now summarise the setup, as well as fixing some notation.

(1) For each $q \in Q$, we let $V_q := \mathcal{Z}(e_G,q)$. Then $V := \bigsqcup_{q \in Q} V_q$ is a compactly aligned product system over $(H, Q)$ with coefficient algebra $A$.

(2) By Proposition 3.4.3, since $A$ acts faithfully on each fibre of $V$ and each homomorphism $\tilde{\phi}(p,q) : A \to \mathbb{L}(\tilde{\mathcal{Z}}(p,q))$ is injective, there exists a homomorphism $\phi_V^{NO} : \mathcal{N}O_V \to \mathcal{N}O_Z$ such that that $\phi_V^{NO} \circ j_V = j_Z$. Furthermore, since $H$ is an amenable group, Proposition 3.4.6 says that $\phi_V^{NO}$ is injective.

(3) Since $H$ is amenable, $A$ acts faithfully on each fibre of $V$, and each $\tilde{\phi}(p,q)$ is injective, Proposition 3.3.7 and Proposition 3.3.12 give the existence of a compactly aligned product system $\mathcal{W}^{NO}$ over $(G, P)$ with coefficient algebra $\mathcal{N}O_V$, with fibres given by

$$\mathcal{W}_p^{NO} := \text{span}\{j_{z(p,e_H)}(x)\phi_V^{NO}(b) : x \in \mathcal{Z}(p,e_H), b \in \mathcal{N}O_V\}$$

$$\subseteq \mathcal{N}O_Z$$

for each $p \in P \setminus \{e_G\}$.

Our aim is to show that $\mathcal{N}O_{YNT} \cong \mathcal{N}T_{\mathcal{W}^{NO}}$. To do this we will exhibit homomorphisms $\omega : \mathcal{N}O_{YNT} \to \mathcal{N}T_{\mathcal{W}^{NO}}$ and $\omega : \mathcal{N}T_{\mathcal{W}^{NO}} \to \mathcal{N}O_{YNT}$, and show
that they are inverses of each other. We now list the results that we will prove in
Section 3.5, and summarise the various spaces and maps that we will be working
with in a pair of commuting diagrams.

(4) In Lemma 3.5.1 we use the universal Cuntz–Nica–Pimsner covariant represen-
tation of $\mathbb{Z}$ and the universal Nica covariant representation of $\mathbb{W}_NO$ to define
a Nica covariant representation $\vartheta$ of $\mathbb{Z}$ in $\mathcal{N}T_{\mathbb{W}_NO}$. The universal property
of $\mathcal{N}T_{\mathbb{Z}}$ then gives us a $\ast$-homomorphism $\Xi : \mathcal{N}T_{\mathbb{Z}} \to \mathcal{N}T_{\mathbb{W}_NO}$ such that
$\Xi \circ i_{Z(p,q)} = \vartheta(p,q) = i_{\mathbb{W}_NO} \circ j_{Z(p,q)}$ for each $(p,q) \in P \times Q$.

(5) In Proposition 3.5.2 we show that the restriction of $\Xi$ to the product system
$Y_{\mathcal{N}T} \subseteq \mathcal{N}T_{\mathbb{Z}}$ gives a Nica covariant representation of $Y_{\mathcal{N}T}$, which we de-
note by $\Psi$. The idea is that $\Xi$ plays the same role as the inclusion map in
Proposition 3.3.16 and Proposition 3.4.19.

(6) In Proposition 3.5.3 we find sufficient conditions for $\Psi$ to be Cuntz–Pimsner
covariant, and make use of the universal property of $\mathcal{N}O_{Y_{\mathcal{N}T}}$ to induce a
$\ast$-homomorphism $\omega$ such that $\omega \circ j_{Y_{\mathcal{N}T}} = \Psi$.

(7) In Proposition 3.5.5 we use the universal Nica covariant representation of $\mathbb{Z}$ and
the universal Cuntz–Nica–Pimsner covariant representation of $Y_{\mathcal{N}T}$ to define a
Cuntz–Nica–Pimsner covariant representation $\vartheta'$ of $V$ in $\mathcal{N}O_{Y_{\mathcal{N}T}}$. The universal property
of $\mathcal{N}O_V$ then gives us a $\ast$-homomorphism $\Psi_{eG}' : \mathcal{N}O_V \to \mathcal{N}O_{Y_{\mathcal{N}T}}$
such that $\Psi_{eG}' \circ j_{V_q} = j_{Y_{\mathcal{N}T}} \circ i_{Z(eG,q)}$ for each $q \in Q$.

(8) In Proposition 3.5.6 we use the $\ast$-homomorphism $\Psi_{eG}'$ to construct a linear
map $\Psi_p' : \mathbb{W}_p^{NO} \to \mathcal{N}O_{Y_{\mathcal{N}T}}$ such that $\Psi_p' \circ j_{Z(p,q)} = j_{Y_{\mathcal{N}T}} \circ i_{Z(p,q)}$ for each
$(p,q) \in P \times Q$.

(9) In Proposition 3.5.8 and Proposition 3.5.9 we show that the collection of maps
$\{\Psi_p' : p \in P\}$ gives a Nica covariant representation of the product system
$\mathbb{W}_NO$. The universal property of $\mathcal{N}T_{\mathbb{W}_NO}$ then gives a $\ast$-homomorphism
$\omega' : \mathcal{N}T_{\mathbb{W}_NO} \to \mathcal{N}O_{Y_{\mathcal{N}T}}$ such that $\omega' \circ i_{\mathbb{W}_NO} = \Psi'$.

(10) In Theorem 3.5.10 we prove that $\omega$ and $\omega'$ are mutually inverse isomorphisms.

In summary, we will show that for every $(p,q) \in P \times Q$, the maps in the following
two diagrams exist and make the diagrams commute.
To begin, we will show that there exists a $\ast$-homomorphism $\omega$ from $\mathcal{N}\mathcal{O}_Y$ to $\mathcal{N}\mathcal{T}_W$ by exhibiting a Cuntz–Nica–Pimsner covariant representation $\Psi$ of $Y$ in $\mathcal{N}\mathcal{T}_W$. Recall from Proposition 3.3.8 that $Y$ acts faithfully on each fibre of $Y$, since $A$ acts faithfully on each $Z$.

To get this representation, we first define a linear map from $Y$ to $W$ for each $q \in Q$. To make our arguments easier to write down, we will identify the coefficient algebra $\mathcal{N}\mathcal{O}_Y$ of $Y$ with $\phi_\mathcal{N}\mathcal{O}_Y(\mathcal{N}\mathcal{O}_Y)$, and the coefficient algebra $\mathcal{N}\mathcal{O}_V$ of $W$ with $\phi_\mathcal{N}\mathcal{O}_V(\mathcal{N}\mathcal{O}_V)$. Since $Y$ acts faithfully on each fibre of $Y$, it suffices to produce a $\ast$-homomorphism from $\mathcal{N}\mathcal{T}_Z$ to $\mathcal{N}\mathcal{T}_W$, and then restrict this map to each $Y_q$. 

Figure 3.8: The homomorphisms $\phi_{\mathcal{N}\mathcal{O}}$, $\Psi'_{\mathcal{E}_G}$

Figure 3.9: The homomorphisms $\omega$, $\omega'$
Lemma 3.5.1. Define \( \vartheta : Z \rightarrow \mathcal{N} \mathcal{T}_{W^{\infty}} \) by

\[
\vartheta_{(p,q)} := i_{W^{\infty}} \circ j_{Z_{(p,q)}}.
\]

Then \( \vartheta \) is a Nica covariant representation of \( Z \). Hence, there exists a homomorphism \( \Xi : \mathcal{N} \mathcal{T}_{Z} \rightarrow \mathcal{N} \mathcal{T}_{W^{\infty}} \) such that \( \Xi \circ i_{Z_{(p,q)}} = \vartheta_{(p,q)} = i_{W^{\infty}} \circ j_{Z_{(p,q)}} \) for each \( (p,q) \in P \times Q \).

Proof. Since both \( i_{W^{\infty}} \) and \( j_{Z_{(p,q)}} \) are Nica covariant representations, the same reasoning as in Proposition 3.3.13 and Proposition 3.3.15 shows that \( \vartheta \) is a Nica covariant representation. The universal property of \( \mathcal{N} \mathcal{T}_{Z} \) then gives the existence of the homomorphism \( \Xi \).

Proposition 3.5.2. Define \( \Psi : Y^{\mathcal{N}} \rightarrow \mathcal{N} \mathcal{T}_{W^{\infty}} \) by \( \Psi_{q} := \Xi_{|Y^{\mathcal{N}}_{q}} \) for each \( q \in Q \). Then \( \Psi \) is a Nica covariant representation of \( Y^{\mathcal{N}} \).

Proof. Since \( \Xi \) is a \( \ast \)-homomorphism, it is trivial to check that \( \Psi \) is a representation. Before we show that \( \Psi \) is Nica covariant, we prove that

\[
\Psi^{(q)}(M_{b}) = \Xi(b) \quad \text{for any } b \in \mathcal{N} \mathcal{T}^{q}_{Z}.
\] (3.21)

If \( m, n \in P \) and \( z \in Z_{(m,q)} \), \( w \in Z_{(n,q)} \), then \( i_{Z_{(m,q)}}(z) i_{Z_{(n,q)}}(w)^* \in \mathcal{N} \mathcal{T}^{q}_{Z} \) and \( M_{i_{Z_{(m,q)}}(z)} i_{Z_{(n,q)}}(w)^* = \Theta_{i_{Z_{(m,q)}}(z), i_{Z_{(n,q)}}(w)} \in \mathcal{K}_{\mathcal{N} \mathcal{T}_{X}}(Y^{\mathcal{N}}_{q}) \). Hence,

\[
\Psi^{(q)}(M_{i_{Z_{(m,q)}}(z)} i_{Z_{(n,q)}}(w)^*) = \Psi^{(q)}(i_{Z_{(m,q)}}(z)) \Xi^{(q)}(i_{Z_{(n,q)}}(w))^* \\
= \Xi(z) \Xi(w)^* \\
= \Xi(z) \Xi(w)^*.
\]

Since \( \mathcal{N} \mathcal{T}^{q}_{Z} = \text{span} \left\{ i_{Z_{(m,q)}}(z) i_{Z_{(n,q)}}(w)^* ; m, n \in P, z \in Z_{(m,q)}, w \in Z_{(n,q)} \right\} \), we conclude that (3.21) holds by linearity and continuity and an application of Lemma 3.3.9. We now prove that \( \Psi \) is Nica covariant. Fix \( q, t \in Q \) and \( S \in \mathcal{K}_{\mathcal{N} \mathcal{T}_{X}}(Y^{\mathcal{N}}_{q}) \), \( T \in \mathcal{K}_{\mathcal{N} \mathcal{T}_{X}}(Y^{\mathcal{N}}_{t}) \). By Lemma 3.3.10, \( S = M_{b} \) and \( T = M_{c} \) for some \( b \in \mathcal{N} \mathcal{T}^{q}_{Z} \), \( c \in \mathcal{N} \mathcal{T}^{t}_{Z} \). Moreover, by Lemma 3.3.11 it follows that

\[
bc \in \begin{cases} 
\mathcal{N} \mathcal{T}^{(q \vee t)}_{Z} & \text{if } q \vee t < \infty \\
\{0\} & \text{otherwise.}
\end{cases}
\]
Hence,
\[
\Psi^{(q)}(S)\Psi^{(t)}(T) = \Psi^{(q)}(M_b)\Psi^{(t)}(M_c) = \Xi(b)\Xi(c)
\]
\[
= \Xi(bc)
\]
\[
= \begin{cases} 
\Psi^{(q\vee t)}(M_{bc}) & \text{if } q \vee t < \infty \\
0 & \text{otherwise}
\end{cases}
\]
\[
= \begin{cases} 
\Psi^{(q\vee t)}(t^{(q\vee t)}_b(M_b)t^{(q\vee t)}_t(M_c)) & \text{if } q \vee t < \infty \\
0 & \text{otherwise}
\end{cases}
\]
\[
= \begin{cases} 
\Psi^{(q\vee t)}(t^{(q\vee t)(S)}(T)) & \text{if } q \vee t < \infty \\
0 & \text{otherwise}.
\end{cases}
\]

**Proposition 3.5.3.** Suppose \(A\) acts compactly on each \(Z_{(eG,q)}\) and \((H,Q)\) is directed. Then \(\Psi\) is a Cuntz–Pimsner covariant representation of \(Y^{N_T}\). Hence, there exists a \(*\)-homomorphism \(\omega : NO_{Y^{N_T}} \to NT_{WNO}\) such that \(\omega \circ j_{Y^{N_T}} = \Psi_q = \Xi|_{Y^{N_T}}\) for each \(q \in Q\).

**Proof.** Since \(G\) is amenable and \(A\) acts faithfully on each \(Z_{(eG,q)}\), Proposition 3.3.8 tells us that \(N_T_X\) acts faithfully on each fibre of \(Y^{N_T}\). Additionally, since \(A\) acts compactly on each \(Z_{(eG,q)}\), the same reasoning as in the proof of Lemma 3.4.17 shows that \(N_T_X\) acts compactly on each fibre of \(Y^{N_T}\). Hence, to see that \(\Psi\) is Cuntz–Pimsner covariant, it suffices by Proposition 3.1.45 to check that \(\Psi^{(q)} \circ \Phi^{N_T}_{q} = \Psi_{eH}\) for each \(q \in Q\). As \(Y^{N_T}_{eH} = N_T_X \cong \phi^{N_T}_{X}(N_T_X)\) is generated by the image of \(\phi^{N_T}_{X} \circ i_X = i_Z|_X\) it suffices to show that
\[
(\Psi^{(q)} \circ \Phi^{N_T}_{q})(i_Z(x)) = \Psi_{eH}(i_Z(x)) \quad \text{for each } x \in X.
\] (3.22)

To this end, fix \(p \in P\) and \(x \in X_p\). Choosing \(x' \in X_p\) so that \(x = x' \cdot (x', x')^p_A\) by the Hewitt–Cohen–Blanchard factorisation theorem, we have
\[
\Psi_{eH}(i_{Z_{(p,eH)}}(x)) = \Xi(i_{Z_{(p,eH)}}(x))
\]
\[
= i_{W^{N_O}_p}(j_{Z_{(p,eH)}}(x))
\]
\[
= i_{W^{N_O}_p}(j_{Z_{(p,eH)}}(x')) i_{W^{N_O}_{(eG,eH)}}(j_{Z_{(eG,eH)}}((x', x')^p_A))
\]

By the first part of Lemma 3.4.17, we see that
\[
\phi^{N_T}_q(i_{Z_{(p,eH)}}(x)) = M_{i_{Z_{(p,eH)}}(x')} (\phi_{(eG,q)}((x', x')^p_A) ) \in K_{N_T_X}(Y^{N_T}_q)
\]
and \( i_{Z(p,eH)} \left( (x')^{i(eG,q)} \right) (\phi_{(eG,q)} (\langle x', x' \rangle^p_A)) \in \mathcal{N}^G \). Thus, by (3.21)

\[
\Psi(q) \left( \Phi^\mathcal{N}_q \left( i_{Z(p,eH)}(x) \right) \right) = \Xi \left( i_{Z(p,eH)}(x') \right) \phi^{i(eG,q)}(\langle x', x' \rangle^p_A)
\]

\[
= \Xi \left( i_{Z(p,eH)}(x') \right) \phi^{i(eG,q)}(\langle x', x' \rangle^p_A)
\]

\[
= i_{W^{NO}} \left( j_{Z(p,q)}(x') \right) i_{W_{eG}^{NO}} \left( j^e_G(\langle x', x' \rangle^p_A) \right).
\]

Hence, for (3.22) to hold, it suffices to show that

\[
j_{Z(p,q)}(\langle x', x' \rangle^p_A) = j^e_G(\langle x', x' \rangle^p_A),
\]

which follows from Lemma 3.4.18. \( \square \)

**Remark 3.5.4.** As in Remark 3.4.20, it is not clear if all of the hypotheses in the previous result are necessary. We would like to be able to rerun the argument used in the proof of Proposition 3.4.16 (where we did not need \( A \) to act compactly on each \( Z_{(eG,q)} \), nor for \( (H,Q) \) to be directed), but \( i_{W^{NO}} \) need not be Cuntz–Pimsner covariant in general.

To establish the isomorphism between \( \mathcal{N}^{\mathcal{Y}_{H}} \) and \( \mathcal{N}^T_{W^{NO}} \) it remains to show that there exists a \( * \)-homomorphism \( \omega' : \mathcal{N}^T_{W^{NO}} \to \mathcal{N}^{\mathcal{Y}_{H}} \) that is the inverse of \( \omega \). To get \( \omega' \), we will exhibit a Nica covariant representation of \( W^{NO} \) in \( \mathcal{N}^{\mathcal{Y}_{H}} \).

Unfortunately, defining this representation is somewhat more difficult than when we defined the representation \( \Psi \) of \( Y^T \) in \( \mathcal{N}^T_{W^{NO}} \). Whilst each fibre of \( W^{NO} \) sits inside the C*-algebra \( \mathcal{N}^{\mathcal{Z}} \), we cannot in general get a \( * \)-homomorphism from \( \mathcal{N}^{\mathcal{Z}} \) to \( \mathcal{N}^T_{W^{NO}} \) which we can then just restrict to \( W^{NO} \). We get around this difficulty as follows. Firstly, we will produce a \( * \)-homomorphism \( \Psi_{eG}^{nG} \) from \( W_{eG}^{NO} = \mathcal{N}^{\mathcal{V}} \) to \( \mathcal{N}^{\mathcal{Y}_{H}} \) by exhibiting a Cuntz–Nica–Pimsner covariant representation of \( V \) in \( \mathcal{N}^{\mathcal{Y}_{H}} \) and using the universal property of \( \mathcal{N}^{\mathcal{V}} \). Secondly, we will use the homomorphism \( \Psi_{eG}^{nG} \) to construct a linear map \( \Psi_p \) from \( W_{p}^{NO} \) to \( \mathcal{N}^{\mathcal{Y}_{H}} \) for each \( p \in \mathcal{P} \). We then argue that the collection of maps \( \{ \Psi_p : p \in \mathcal{P} \} \) forms a Nica covariant representation of \( W^{NO} \).

**Proposition 3.5.5.** Define \( \vartheta' : V \to \mathcal{N}^{\mathcal{Y}_{H}} \) by \( \vartheta'_q := j_{\mathcal{Y}_{H}^{T}} \circ i_{Z_{(eG,q)}} \) for each \( q \in \mathcal{Q} \). Then \( \vartheta' \) is a Cuntz–Nica–Pimsner covariant representation of \( V \). Hence, identifying \( \mathcal{N}^{\mathcal{V}} \) with \( \phi_{V}^{NO}(\mathcal{N}^{\mathcal{V}}) \), there exists a \( * \)-homomorphism \( \Psi_{eG}^{nG} : \phi_{V}^{NO}(\mathcal{N}^{\mathcal{V}}) \to \mathcal{N}^{\mathcal{Y}_{H}} \) such that \( \vartheta'_{eG} \circ j_{Z_{(eG,e)}} = \vartheta'_q = j_{\mathcal{Y}_{H}^{T}} \circ i_{Z_{(eG,q)}} \) for each \( q \in \mathcal{Q} \).

**Proof.** The same reasoning as used in the proof of Proposition 3.3.13 and Proposition 3.3.15 shows that \( \vartheta' \) is a Nica covariant representation of \( V \). We now show that \( \vartheta' \) is Cuntz–Pimsner covariant. Suppose \( F \subseteq \mathcal{Q} \) is finite and \( \{ T_q \in \mathcal{K}_{A}(V_q) : q \in F \} \) satisfies \( \sum_{q \in F} l_q^t(T_q) = 0 \in \mathcal{L}_A(V_t) \) for large \( t \in \mathcal{Q} \). We need to show that
\[ \sum_{q \in F} \vartheta(q)(T_q) = 0 \in \mathcal{N} \mathcal{O}_{Y^N T}. \] We claim that
\[
\sum_{q \in F} t_q^t \left( M_{i_{\mathcal{I}}^N(T_q)}(e_{G,q}) \right) = 0 \in \mathcal{L}_{N T X} \quad \text{for large } t. \tag{3.23}
\]
For any \( T \in \mathcal{K}_A(V_q) \), and \( z \in \mathcal{Z}(e_{G,q}) \), \( b \in \mathcal{N} T X \), we have
\[
t_q^t \left( M_{i_{\mathcal{I}}^N(T_q)}(e_{G,q}) \right) (i_{\mathcal{I}}(z)) \phi_{X^T}(b) = i_{\mathcal{I}}(i_{\mathcal{I}}(z)) \phi_{X^T}(b). \]
Given any \( r \in Q \), fix \( s \geq r \), such that for all \( t \geq s \), we have
\[
\sum_{q \in F} t_q^t \left( M_{i_{\mathcal{I}}^N(T_q)}(e_{G,q}) \right) = 0 \in \mathcal{L}_A(V_t) = \mathcal{L}_A(Z_{e_{G,t}}). \]
Then for any \( t \geq s \), we see that
\[
\left( \sum_{q \in F} t_q^t \left( M_{i_{\mathcal{I}}^N(T_q)}(e_{G,q}) \right) \right) (Y^N T) = \left\{ 0 \right\}.
\]
Thus, (3.23) holds. Since \( j_{Y^N} \) is Cuntz–Pimsner covariant, we conclude that
\[
\sum_{q \in F} \vartheta(q)(T_q) = \sum_{q \in F} j_{Y^N} \left( M_{i_{\mathcal{I}}^N(T_q)}(e_{G,q}) \right) = 0,
\]
and so \( \vartheta \) is Cuntz–Pimsner covariant.

**Proposition 3.5.6.** For each \( p \in P \setminus \{e_G\} \), there exists a norm-decreasing linear map \( \Psi_p^t : \mathcal{W}_p^{N^0} \to \mathcal{N} \mathcal{O}_{Y^N T} \) such that
\[
\Psi_p^t \left( j_{(p,e_H)}^t(x) \phi_{V^O}(b) \right) = j_{Y^N T} \left( i_{Z_{(p,e_H)}}^t(x) \right) \Psi_p^t \left( \phi_{V^O}(b) \right) \tag{3.24}
\]
for each \( x \in Z_{(p,e_H)} \) and \( b \in \mathcal{N} \mathcal{O}_V \). Thus,
\[
\Psi_p^t \left( j_{(p,q)}^t(z) j_{(e_t,t)}(w)^* \right) = j_{Y^N T} \left( i_{Z_{(p,q)}}^t(z) \right) j_{Y^N T} \left( i_{Z_{(e,t)}}^t(w) \right)^* \tag{3.25}
\]
for each \( q,t \in Q \) and \( z \in Z_{(p,q)} \), \( w \in Z_{(e,t)} \).
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Proof. We claim that for any finite set $F \subseteq \mathbb{Z}_{(p,e_H)} \times \mathcal{N} \mathcal{O}_V$, we have

$$\left\| \sum_{(x,b) \in F} j_{Y_{eH}^+} (i_{Z_{(p,e_H)}}(x)) \Psi'_{eG} (\phi_{NO}^V(b)) \right\|_{\mathcal{N} \mathcal{O}_{Y_{eH}^+}} \leq \left\| \sum_{(x,b) \in F} j_{Z_{(p,e_H)}} (x) \phi_{NO}^V(b) \right\|_{W_p^{NO}}. \quad (3.26)$$

Making use of the $C^*$-identity, we see that

$$\left\| \sum_{(x,b) \in F} j_{Y_{eH}^+} (i_{Z_{(p,e_H)}}(x)) \Psi'_{eG} (\phi_{NO}^V(b)) \right\|^2_{\mathcal{N} \mathcal{O}_{Y_{eH}^+}}$$

$$= \left\| \sum_{(x,b), (y,c) \in F} \Psi'_{eG} (\phi_{NO}^V(b^*)) j_{Y_{eH}^+} (i_{Z_{(p,e_H)}}(x))^* i_{Z_{(p,e_H)}}(y) \Psi'_{eG} (\phi_{NO}^V(c)) \right\|_{\mathcal{N} \mathcal{O}_{Y_{eH}^+}}$$

$$= \left\| \sum_{(x,b), (y,c) \in F} \Psi'_{eG} (\phi_{NO}^V(b^*)) j_{Y_{eH}^+} (i_{Z_{(p,e_H)}}(y)_{(p,e_H)}) \Psi'_{eG} (\phi_{NO}^V(c)) \right\|_{\mathcal{N} \mathcal{O}_{Y_{eH}^+}}.$$

Since $\Psi'_{eG} \circ j_{Z_{(p,e_H)}} = \phi_{NO}^V \circ j_{Y_{eH}^+} \circ i_{Z_{(p,e_H)}}$, this is equal to

$$\left\| \sum_{(x,b), (y,c) \in F} \Psi'_{eG} (\phi_{NO}^V(b^*)) j_{Z_{(p,e_H)}} (x) (y)_{(p,e_H)}) \Psi'_{eG} (\phi_{NO}^V(c)) \right\|_{\mathcal{N} \mathcal{O}_{Y_{eH}^+}}$$

$$= \left\| \sum_{(x,b), (y,c) \in F} \Psi'_{eG} (\phi_{NO}^V(b^*) j_{Z_{(p,e_H)}} (x) (y)_{(p,e_H)}) \phi_{NO}^V(c) \right\|_{\mathcal{N} \mathcal{O}_{Y_{eH}^+}}.$$

Since $\Psi'_{eG}$ is a $*$-homomorphism, this is no greater than

$$\left\| \sum_{(x,b), (y,c) \in F} \phi_{NO}^V(b^*) j_{Z_{(p,e_H)}} (x) (p,e_H) (y)_{(p,e_H)}) \phi_{NO}^V(c) \right\|_{\mathcal{N} \mathcal{O}_{Y_{eH}^+}}$$

$$= \left\| \sum_{(x,b), (y,c) \in F} \phi_{NO}^V(b^*) j_{Z_{(p,e_H)}} (x) (p,e_H) \phi_{NO}^V(c) \right\|_{\mathcal{N} \mathcal{O}_{Y_{eH}^+}}$$

$$= \left\| \left( \sum_{(x,b) \in F} \phi_{NO}^V(b) j_{Z_{(p,e_H)}} (x) \right)^* \left( \sum_{(y,c) \in F} \phi_{NO}^V(c) \right) \right\|_{\mathcal{N} \mathcal{O}_{Z}}$$

$$= \left\| \left( \sum_{(x,b) \in F} \phi_{NO}^V(b) \right)^* \left( \sum_{(y,c) \in F} \phi_{NO}^V(c) \right) \right\|_{\mathcal{N} \mathcal{O}_{Z}}$$

$$= \left\| \sum_{(x,b), (y,c) \in F} \phi_{NO}^V(b^*) j_{Z_{(p,e_H)}} (x) (p,e_H) \phi_{NO}^V(c) \right\|_{\mathcal{N} \mathcal{O}_{Z}}$$

$$= \left\| \sum_{(x,b) \in F} j_{Z_{(p,e_H)}} (x) \phi_{NO}^V(b) \right\|_{W_p^{NO}}^2.$$
Thus, (3.26) holds. It follows that the formula

\[ j_{z(p,e_H)}(x)φ_N^O(b) \mapsto j_{Y_{N^T}^H}(i_{z(p,e_H)}(x)Ψ_{e_G}(φ_N^O(b))) \]

determines a well-defined norm-decreasing linear map \( Ψ'_p \) on \( W_p^N^O \) as claimed. \( \square \)

Next we prove that the collection of maps \( \{ Ψ'_p : p \in P \} \) defined in Propositions 3.5.5 and 3.5.6 gives a representation of the product system \( W^N^O \). The proof that \( Ψ \) satisfies relation (T2) of Definition 3.1.7 is particularly onerous, so we present it first as a lemma.

**Lemma 3.5.7.** The map \( Ψ : W^N^O \to NO_{Y_{N^T}} \) satisfies relation (T2) of Definition 3.1.7.

**Proof.** We need to show that

\[ Ψ'_p(x)Ψ'_r(y) = Ψ'_p(xy) \text{ for any } p, r ∈ P \text{ and } x ∈ W_p^N^O, y ∈ W_r^N^O. \tag{3.27} \]

We begin by showing that (3.27) holds when \( r = e_G \). If \( z ∈ Z(p,e_H) \) and \( b, c ∈ NO_V \), using (3.24), we see that

\[
Ψ'_p(j_{z(p,e_H)}(z)φ_N^O(b))Ψ'_e_G(φ_N^O(c)) = j_{Y_{N^T}^e_G}(i_{z(p,e_H)}(x))Ψ'_e_G(φ_N^O(b))Ψ'_e_G(φ_N^O(c)) = j_{Y_{N^T}^e_G}(i_{z(p,e_H)}(x))Ψ'_e_G(φ_N^O(bc)) = Ψ'_p(j_{z(p,e_H)}(z)φ_N^O(bc)) = Ψ'_p(j_{z(p,e_H)}(z)φ_N^O(b)φ_N^O(c)).
\]

Since \( W_p^N^O = \text{span}\{ j_{z(p,e_H)}(z)φ_N^O(b) : z ∈ Z(p,e_H), b ∈ NO_V \} \) and \( W_{e_G}^N^O ≃ φ_N^O(NO_V) \), we conclude that (3.27) holds when \( r = e_G \).

We now move on to the case where \( r ≠ e_G \). Since \( Ψ'_p \) and \( Ψ'_r \) are linear and norm-decreasing, and multiplication in \( NO_{Y_{N^T}} \) is linear and continuous, it suffices to prove that (3.27) holds when \( x = j_{z(q,e_H)}(z)j_{z(e_G,t)}(w)^* \) and \( y = j_{z(t,m)}(u)j_{z(e_G,n)}(v)^* \) for some \( q, t, m, n ∈ Q \), and \( z ∈ Z(p,q), w ∈ Z(e_G,t), u ∈ Z(t,m), v ∈ Z(e_G,n) \). We deal with the cases where \( t = e_H \) and \( t ≠ e_H \) separately.

If \( t = e_H \), then \( w ∈ Z(e_G,e_H) = A \), and so an application of (3.25) gives

\[
Ψ'_{pr}(j_{z(q,e_H)}(z)j_{z(e_G,n)}(w)^*j_{z(t,m)}(u)j_{z(e_G,n)}(v)^*) = Ψ'_{pr}(j_{z(pr,qm)}(zw^*u)j_{z(e_G,n)}(v)^*) = j_{Y_{N^T}^q}(i_{z(pr,qm)}(zw^*u))j_{Y_{N^T}^n}(i_{z(e_G,n)}(v))^*.
\tag{3.28}
\]
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Another application of (3.25), shows that (3.28) is equal to

\[ j_{Y_N^N} \left( i_{Z_{(q,q)}}(z) i_{Z_{(r,m)}}(u) j_{Y_N^N} \left( i_{Z_{(e,g,n)}}(v) \right)^* \right) \]

\[ = j_{Y_N^N} \left( i_{Z_{(q,q)}}(z) j_{Y_N^N} \left( i_{Z_{(e,g,\eta)}}(w) \right)^* j_{Y_N^N} \left( i_{Z_{(e,g,\eta)}}(u) \right) j_{Y_N^N} \left( i_{Z_{(e,g,n)}}(v) \right)^* \right) \]

\[ = \Psi_p' \left( j_{Z_{(q,q)}}(z) j_{Z_{(e,g,\eta)}}(w)^* j_{Z_{(r,m)}}(u) j_{Z_{(e,g,n)}}(v)^* \right) \]

as required.

It remains to deal with the situation where \( t \neq e_H \). We make use of Remark 3.1.23 to rewrite the product \( j_{Z_{(q,q)}}(z) j_{Z_{(e,g,\eta)}}(w)^* j_{Z_{(r,m)}}(u) j_{Z_{(e,g,n)}}(v)^* \) in the form required to apply (3.25). If \( t \vee m = \infty \), then \( (e_g, t) \vee (r, m) = \infty \), and so applying Lemma 3.1.22 to the Nica covariant representations \( j_{Z} \) and \( j_{Y_N^N} \), we see that

\[ \Psi_{pr}' \left( j_{Z_{(q,q)}}(z) j_{Z_{(e,g,\eta)}}(w)^* j_{Z_{(r,m)}}(u) j_{Z_{(e,g,n)}}(v)^* \right) = \Psi_{pr}'(0) = 0 \]

and

\[ \Psi_p' \left( j_{Z_{(q,q)}}(z) j_{Z_{(e,g,\eta)}}(w)^* \right) \Psi_r' \left( j_{Z_{(r,m)}}(u) j_{Z_{(e,g,n)}}(v)^* \right) \]

\[ = j_{Y_N^N} \left( i_{Z_{(q,q)}}(z) j_{Y_N^N} \left( i_{Z_{(e,g,\eta)}}(w) \right)^* j_{Y_N^N} \left( i_{Z_{(e,g,\eta)}}(u) \right) j_{Y_N^N} \left( i_{Z_{(e,g,n)}}(v) \right)^* \right) \]

\[ = 0. \]

Thus, we may as well suppose that \( t \vee m < \infty \). Choose \( w' \in Z_{(e_g,t)} \) and \( u' \in Z_{(r,m)} \) so that \( w = w' \cdot (w', u')_{(e_g,t)} \) and \( u = u' \cdot (u', u')_{(r,m)} \) by the Hewitt–Cohen–Blanchard factorisation theorem. Since \( Z \) is compactly aligned, and

\[ Z_{(r,t \vee m)} = \text{span} \left\{ \sigma \tau, \sigma \in Z_{(e_g,t)}, \tau \in Z_{(r,t^{-1}(t \vee m))} \right\} = \text{span} \left\{ \eta \rho, \eta \in Z_{(r,m)}, \rho \in Z_{(e_g,m^{-1}(t \vee m))} \right\}; \]

we can write

\[ \ell'(r \vee m) (\Theta_{w', u'}) \ell'(r \vee m) (\Theta_{u', u'}) = \lim_{t \to \infty} \sum_{j=1}^{K_{i}} \Theta_{\sigma_{j}, \tau_{j}, \eta_{j}, \rho_{j}} \in K_A \left( Z_{(r,t \vee m)} \right), \quad (3.29) \]

where

\[ \sigma_{j}, \tau_{j}, \in Z_{(e_g,t)}, \tau_{j} \in Z_{(r,t^{-1}(t \vee m))}, \eta_{j} \in Z_{(r,m)}, \rho_{j} \in Z_{(e_g,m^{-1}(t \vee m))}. \]
Combining (3.25) and Remark 3.1.23, we have

\[
\Psi_{pr}' \left( jz_{(p,q)}(z)jz_{(eG,t)}(w)^*jz_{(r,m)}(u)jz_{(eG,n)}(v)^* \right)
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{k_i} \Psi_{pr}' \left( jz_{(pr,qt-1(t\vee m))} \left( z \langle w', \sigma_{j_i}\rangle_{A} \tau_{j_i} \right) jz_{(eG, nm-1(t\vee m))} \left( \langle u', \eta_{j_i}\rangle_{A} \rho_{j_i} \right) \right) \]

\[
= \lim_{i \to \infty} \sum_{j=1}^{k_i} j_{Y^{NT}} \left( \sum_{j=1}^{k_i} \left( i_{z_{(eG,t)}}(w) \cdot \left( i_{z_{(eG,t)}}(w') \cdot i_{z_{(eG,t)}}(w') \right) \right) \right)_{N^TX}^{\infty \vee m}
\]

(3.30)

We now need to calculate \( \Psi_p' \left( jz_{(p,q)}(z)jz_{(eG,t)}(w)^* \right) \Psi_r' \left( jz_{(r,m)}(u)jz_{(eG,n)}(v)^* \right) \), which requires applying Remark 3.1.23 to the product system \( Y^{NT} \) and the Nica covariant representation \( j_{Y^{NT}} \). Since \( w = w' \cdot \langle w', w' \rangle_{A} \) and \( u = u' \cdot \langle u', u' \rangle_{A} \),

\[
i_{z_{(eG,t)}}(w) = i_{z_{(eG,t)}}(w') \cdot \left( i_{z_{(eG,t)}}(w') \cdot i_{z_{(eG,t)}}(w') \right)_{N^TX}^{\infty \vee m}
\]

and

\[
i_{z_{(r,m)}}(u) = i_{z_{(r,m)}}(u') \cdot \left( i_{z_{(r,m)}}(u') \cdot i_{z_{(r,m)}}(u') \right)_{N^TX}^{\infty \vee m}
\]

are the Hewitt–Cohen–Blanchard factorisations of \( i_{z_{(eG,t)}}(w) \) in \( Y^{NT}_t \) and \( i_{z_{(r,m)}}(u) \) in \( Y^{NT}_m \) respectively. Furthermore, (3.29) implies that

\[
i_{t}^{\infty \vee m} \left( \Theta i_{z_{(eG,t)}}(w'), i_{z_{(eG,t)}}(w') \right) i_{t}^{\infty \vee m} \left( \Theta i_{z_{(r,m)}}(u'), i_{z_{(r,m)}}(u') \right)
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{k_i} \Theta i_{z_{(eG,t)}}(\sigma_{j_i}) i_{z_{(eG,t-1(t\vee m))}}(\tau_{j_i}) i_{z_{(r,m)}}(\eta_{j_i}) i_{z_{(eG,nm-1(t\vee m))}}(\rho_{j_i})
\]

\[
\in \mathcal{K}_{N^TX} (Y^{NT}_t^{\infty \vee m}),
\]

whilst

\[
i_{z_{(eG,t)}}(\sigma_{j_i}) \in Y^{NT}_t,
\]

\[
i_{z_{(r,m)}}(\eta_{j_i}) \in Y^{NT}_m,
\]

\[
i_{z_{(eG,t-1(t\vee m))}}(\tau_{j_i}) \in Y^{NT}_{t-1(t\vee m)},
\]

\[
i_{z_{(eG,nm-1(t\vee m))}}(\rho_{j_i}) \in Y^{NT}_{m-1(t\vee m)}.
\]
Thus, $(3.25)$ and Remark 3.1.23 imply that

\[
\Psi'_p \left( j_{(p,q)}(z) j_{(e_G,t)}(w)^* \right) \Psi'_r \left( j_{(r,m)}(u) j_{(e_G,n)}(v)^* \right) \\
= j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(p,q)}}(z) \right) j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(e_G,t)}}(w) \right)^* j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(r,m)}}(u) \right) j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(e_G,n)}}(v) \right)^* \\
= \lim_{i \to \infty} \sum_{j=1}^{k_i} j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(pr,qt-1)}}(z) \right) j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(e_G,t)}}(w') \right)^* j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(r,m)}}(\sigma_j) \right) \left( j_{\mathcal{Y}^{\mathcal{O}X}}(v) \right)^* * \\
= \lim_{i \to \infty} \sum_{j=1}^{k_i} j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(pr,qt-1)}}(z) \right) j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(e_G,t)}}(w') \right)^* j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(r,m)}}(\eta_j) \right) \left( j_{\mathcal{Y}^{\mathcal{O}X}}(v) \right)^* * \\
\]

which is $(3.30)$. Thus,

\[
\Psi'_p \left( j_{Z_{(p,q)}}(z) j_{Z_{(e_G,t)}}(w)^* j_{Z_{(r,m)}}(u) j_{Z_{(e_G,n)}}(v)^* \right) \\
= \Psi'_p \left( j_{Z_{(p,q)}}(z) j_{Z_{(e_G,t)}}(w)^* \right) \Psi'_r \left( j_{Z_{(r,m)}}(u) j_{Z_{(e_G,n)}}(v)^* \right) \\
\]

when \( t \neq e_H \). We conclude that the map \( \Psi' : \mathcal{W}^{\mathcal{O}X} \to \mathcal{N}\mathcal{O}_{\mathcal{Y}^{\mathcal{O}X}} \) satisfies relation (T2) of Definition 3.1.7 \( \square \)

**Proposition 3.5.8.** The map \( \Psi' : \mathcal{W}^{\mathcal{O}X} \to \mathcal{N}\mathcal{O}_{\mathcal{Y}^{\mathcal{O}X}} \) is a representation of \( \mathcal{W}^{\mathcal{O}X} \).

**Proof.** By construction, each \( \Psi'_p \) is linear and \( \Psi'_{e_G} \) is a \( * \)-homomorphism. Hence, \( \Psi' \) satisfies (T1). We already showed that \( \Psi' \) satisfies (T2) in Lemma 3.5.7. It remains to show that \( \Psi' \) satisfies (T3).

Fix \( p \in P \) and let \( z, w \in Z_{(p,e_H)} \), \( b, c \in \mathcal{N}\mathcal{O}_V \). Since \( i_Z \) is a representation of \( Z \), making use of $(3.24)$, we see that

\[
\Psi'_p \left( j_{Z_{(p,e_H)}}(z) \phi_{\mathcal{O}V}(b) \right)^* \Psi'_p \left( j_{(p,e_H)}(w) \phi_{\mathcal{O}V}(c) \right) \\
= \left( j_{\mathcal{Y}^{\mathcal{O}X}}(i_{Z_{(p,e_H)}}(z)) \right) \Psi'_p \left( \phi_{\mathcal{O}V}(b) \right)^* \left( j_{\mathcal{Y}^{\mathcal{O}X}}(i_{Z_{(p,e_H)}}(w)) \right) \Psi'_p \left( \phi_{\mathcal{O}V}(c) \right) \\
= \Psi'_p \left( \phi_{\mathcal{O}V}(b)^* \right) j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(p,e_H)}}(z)^* \right) j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(p,e_H)}}(w) \right) \Psi'_p \left( \phi_{\mathcal{O}V}(c) \right) \\
= \Psi'_p \left( \phi_{\mathcal{O}V}(b)^* \right) j_{\mathcal{Y}^{\mathcal{O}X}} \left( i_{Z_{(p,e_H)}}(z)^* i_{Z_{(p,e_H)}}(w) \right) j_{\mathcal{Y}^{\mathcal{O}X}} \left( \left( z, w \right)_{(p,e_H)} \right) \Psi'_p \left( \phi_{\mathcal{O}V}(c) \right). \\
\]

Since \( j_{\mathcal{Y}^{\mathcal{O}X}} \circ i_{Z_{(e_G,s)}} = \eta_{e_H} = \Psi'_p \circ j_{Z_{(e_G,s)}} \) and \( j_Z \) is a representation of \( Z \), this is
equal to

\[
\Psi'_{e_G} \left( \phi_V^{N\mathcal{O}}(b)^* \right) \Psi'_{e_G} \left( j_{Z_{(p,e_H)}}(z, w)^{(p,e_H)}_A \right) \Psi'_{e_G} \left( \phi_V^{N\mathcal{O}}(c) \right) \\
= \Psi'_{e_G} \left( \phi_V^{N\mathcal{O}}(b)^* j_{Z_{(p,e_H)}}(z, w)^{(p,e_H)}_A \phi_V^{N\mathcal{O}}(c) \right) \\
= \Psi'_{e_G} \left( (j_{Z_{(p,e_H)}}(z) \phi_V^{N\mathcal{O}}(b))^* (j_{Z_{(p,e_H)}}(w) \phi_V^{N\mathcal{O}}(c)) \right) \\
= \Psi'_{e_G} \left( (j_{Z_{(p,e_H)}}(z) \phi_V^{N\mathcal{O}}(b), j_{Z_{(p,e_H)}}(w) \phi_V^{N\mathcal{O}}(c))^p_{N\mathcal{O}V} \right).
\]

Since \( W_p^{N\mathcal{O}} = \text{span} \left\{ j_{Z_{(p,e_H)}}(Z_{(p,e_H)}) \phi_V^{N\mathcal{O}}(N\mathcal{O}_V) \right\} \) for each \( p \in P \), \( \Psi' \) is linear and norm-decreasing, and multiplication in \( N\mathcal{O}_{X_N} \) is linear and continuous, we conclude that \( \Psi'(x)^* \Psi'(y) = \Psi'_{e_G} \left( (x, y)^p_{N\mathcal{O}V} \right) \) for each \( x, y \in W_p^{N\mathcal{O}} \). Thus, \( \Psi' \) satisfies (T3), and we conclude that \( \Psi' \) is a representation of \( W^{N\mathcal{O}_V} \) \( \square \)

We can also show that the representation \( \Psi' \) is Nica covariant.

**Proposition 3.5.9.** The representation \( \Psi' : W^{N\mathcal{O}} \to N\mathcal{O}_{X_N} \) is Nica covariant. Hence, by the universal property of \( N\mathcal{T}_{W^{N\mathcal{O}}} \), there exists a \(*\)-homomorphism \( \omega' : N\mathcal{T}_{W^{N\mathcal{O}}} \to N\mathcal{O}_{X_N} \) such that \( \omega' \circ i_{W_p^{N\mathcal{O}}} = \Psi'_p \) for each \( p \in P \).

**Proof.** We need to show that

\[
\Psi'^{(p)}(S) \Psi'^{(r)}(T) = \begin{cases} \Psi'^{(p \lor r)} \left( i_{p \lor r}^{N\mathcal{O}}(S) i_{p \lor r}^{N\mathcal{O}}(T) \right) & \text{if } p \lor r < \infty \\ 0 & \text{otherwise} \end{cases}
\]

for any \( S \in K_{N\mathcal{O}_V} \left( W_p^{N\mathcal{O}} \right) \) and \( T \in K_{N\mathcal{O}_V} \left( W_r^{N\mathcal{O}} \right) \). If \( p = e_G \) or \( r = e_G \) the result is trivial, so we suppose that \( p, r \neq e_G \).

By Lemma 3.3.10, it suffices to show that

\[
\Psi'^{(p)} \left( \Theta_{j_{Z_{(p,e_H)}}(z), j_{Z_{(p,t)}}(w)} \right) \Psi'^{(r)} \left( \Theta_{j_{Z_{(r,m)}}(u), j_{Z_{(r,n)}}(v)} \right) = \begin{cases} \Psi'^{(p \lor r)} \left( M_{j_{Z_{(p,e_H)}}(z), j_{Z_{(p,t)}}(w)} \ast j_{Z_{(r,m)}}(u), j_{Z_{(r,n)}}(v)) \right) & \text{if } p \lor r < \infty \quad (3.32) \\ 0 & \text{otherwise} \end{cases}
\]

whenever \( z \in Z_{(p,e_H)} \), \( w \in Z_{(p,t)} \), \( u \in Z_{(r,m)} \), \( v \in Z_{(r,n)} \).

We begin by showing that (3.32) holds whenever \( p \lor r = \infty \) or \( t \lor m = \infty \). Since \( j_{Z_{(p,e_H)}}(z) j_{Z_{(p,t)}}(w)^* j_{Z_{(r,m)}}(u) j_{Z_{(r,n)}}(v) = 0 \) if \( t \lor m = \infty \) (as \( j_{Z} \) is Nica covariant), it suffices to show that

\[
\Psi'^{(p)} \left( \Theta_{j_{Z_{(p,e_H)}}(z), j_{Z_{(p,t)}}(w)} \right) \Psi'^{(r)} \left( \Theta_{j_{Z_{(r,m)}}(u), j_{Z_{(r,n)}}(v)} \right) = 0
\]

whenever \( p \lor r = \infty \) or \( t \lor m = \infty \).
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To see this, observe that

\[
\Psi^{(p)} \left( \Theta_{jz_{(p,q)}}(z), jz_{(p,t)}(w) \right) \Psi^{(r)} \left( \Theta_{jz_{(r,m)}}(u), jz_{(r,n)}(v) \right) \\
= \Psi'_p \left( jz_{(p,q)}(z) \right) \Psi'_p \left( jz_{(p,t)}(w) \right) * \Psi'_r \left( jz_{(r,m)}(u) \right) \Psi'_r \left( jz_{(r,n)}(v) \right) * \\
= jY^{(m)} \left( i_{z_{(p,t)}}(z) \right) jY_{m}^{(m)} \left( iz_{(p,t)}(w) \right) * jY_{m}^{(m)} \left( iz_{(r,m)}(u) \right) jY_{m}^{(m)} \left( iz_{(r,n)}(v) \right) *.
\]

(3.34)

If \( t \lor m = \infty \) then (3.34) is certainly zero since \( jY^{\lor} \) is Nica covariant. We now show that (3.34) is still zero when \( t \lor m < \infty \) — we will use the Nica covariance of \( iz \) and the fact that \( p \lor r = \infty \).

Choose \( w' \in Z_{(p,t)} \) and \( u' \in Z_{(r,m)} \) so that \( w = w' \cdot \langle w', w' \rangle_A \), \( u = u' \cdot \langle u', u' \rangle_A \) by the Hewitt–Cohen–Blanchard factorisation theorem. Then

\[
jY_{m}^{(m)} \left( iz_{(p,t)}(w) \right) jY_{m}^{(m)} \left( iz_{(r,m)}(u) \right) = jY_{m}^{(m)} \left( \Theta_{iz_{(p,t)}(w)} \cdot iz_{(r,m)}(u) \right) jY_{m}^{(m)} \left( iz_{(r,m)}(u) \right) *.
\]

(3.35)

Since \( jY^{\lor} \) is Nica covariant, we have

\[
jY_{m}^{(m)} \left( \Theta_{iz_{(p,t)}(w)} \cdot iz_{(r,m)}(u) \right) jY_{m}^{(m)} \left( iz_{(r,m)}(u) \right) = jY_{m}^{(m)} \left( \Theta_{iz_{(p,t)}(w)} \cdot iz_{(r,m)}(u) \right) jY_{m}^{(m)} \left( iz_{(r,m)}(u) \right) *.
\]

which is zero since \( iz \) is Nica covariant and \( (p, t) \lor (r, m) = \infty \) (as \( p \lor r = \infty \)). Thus, (3.35) is zero, and so (3.34) is zero as well. This completes the proof of (3.33).

It remains to prove that (3.32) holds whenever \( p \lor r < \infty \) and \( t \lor m < \infty \). As in Lemma 3.5.7, we will need to make use of Remark 3.1.23 to rewrite things in a form that allows us to apply the description of \( \Psi' \) given by (3.25).

Again, choose \( w' \in Z_{(p,t)} \) and \( u' \in Z_{(r,m)} \) so that \( w = w' \cdot \langle w', w' \rangle_A \) and \( u = u' \cdot \langle u', u' \rangle_A \) by the Hewitt–Cohen–Blanchard factorisation theorem. Writing

\[
l_{i}^{(p \lor r, t \lor m)} \left( \Theta_{u', u'} \right) l_{i}^{(p \lor r, t \lor m)} \left( \Theta_{u', u'} \right) = \lim_{i \rightarrow \infty} \sum_{j=1}^{k_i} \Theta_{\sigma_i, \tau_i, \eta_i, \rho_i} \in \mathcal{K}_{A} (Z_{(p \lor r, t \lor m)}),
\]

(3.36)

where

\[
\sigma_i, \tau_i \in Z_{(p, t)}, \eta_i \in Z_{(p^{-1}(p \lor r), t^{-1}(t \lor m))}, \rho_i \in Z_{(r, m)}, \rho_i \in Z_{(p^{-1}(p \lor r), m^{-1}(t \lor m))},
\]
CHAPTER 3. FACTORIZATION OF PRODUCT SYSTEMS

Remark 3.1.23 tells us that

\[
j_{\mathbf{Z}_{(p,q)}}(z) j_{\mathbf{Z}_{(p,t)}}(w)^* j_{\mathbf{Z}_{(r,m)}}(u) j_{\mathbf{Z}_{(r,n)}}(v)^* \\
= \lim_{i \to \infty} \sum_{j_i=1}^{k_i} j_{\mathbf{Z}_{(p\lor r,qt-1(t\lor m)}}(z(u', \sigma_{j_i} A^{(p,t)} \tau_{j_i})) j_{\mathbf{Z}_{(p\lor r,rm-1(t\lor m)}}(v(u', \eta_{j_i} A^{(r,m)} \rho_{j_i}))^*.
\]

Thus,

\[
\Psi^{(p\lor r)}(M_{j_{\mathbf{Z}_{(p,q)}}(z) j_{\mathbf{Z}_{(p,t)}}(w)^* j_{\mathbf{Z}_{(r,m)}}(u) j_{\mathbf{Z}_{(r,n)}}(v)^*)} \\
= \lim_{i \to \infty} \sum_{j_i=1}^{k_i} \Psi^{(p\lor r)}(\Theta j_{\mathbf{Z}_{(p\lor r,qt-1(t\lor m)}}(z(u', \sigma_{j_i} A^{(p,t)} \tau_{j_i})) j_{\mathbf{Z}_{(p\lor r,rm-1(t\lor m)}}(v(u', \eta_{j_i} A^{(r,m)} \rho_{j_i}))^*) \\
= \lim_{i \to \infty} \sum_{j_i=1}^{k_i} \Psi^{(p\lor r)}_j j_{\mathbf{Z}_{(p\lor r,qt-1(t\lor m)}}(z(u', \sigma_{j_i} A^{(p,t)} \tau_{j_i}))^* \\
= \lim_{i \to \infty} \sum_{j_i=1}^{k_i} \Psi^{(p\lor r)}_j j_{\mathbf{Z}_{(p\lor r,rm-1(t\lor m)}}(z(u', \sigma_{j_i} A^{(r,m)} \rho_{j_i}))^*.
\]

To complete the proof we just need to show that (3.37) is equal to the left hand side of (3.32). Since \( w = u' \cdot \langle u', u' \rangle^{(p,t)} A \) and \( u = u' \cdot \langle u', u' \rangle^{(r,m)} A \in \mathbf{Z}_{(r,m)} \),

\[
i_{\mathbf{Z}_{(p,t)}}(w) = i_{\mathbf{Z}_{(p,t)}}(w') \cdot \left( i_{\mathbf{Z}_{(p,t)}}(w'), i_{\mathbf{Z}_{(p,t)}}(w') \right)^{t^{N,T}_X}
\]

and

\[
i_{\mathbf{Z}_{(r,m)}}(u) = i_{\mathbf{Z}_{(r,m)}}(u') \cdot \left( i_{\mathbf{Z}_{(r,m)}}(u'), i_{\mathbf{Z}_{(r,m)}}(u') \right)^{m^{N,T}_X}
\]

are the Hewitt–Cohen–Blanchard factorisations of \( i_{\mathbf{Z}_{(r,m)}}(w) \in Y_{N,T} \) and \( i_{\mathbf{Z}_{(r,m)}}(w) \in Y_{m}^{N,T} \) respectively. Moreover, (3.36) implies that

\[
l_{\mathbf{Z}_{(p\lor r),qt-1(t\lor m)}}(\Theta i_{\mathbf{Z}_{(p,t)}}(w'), i_{\mathbf{Z}_{(p,t)}}(w')) l_{\mathbf{Z}_{(r,m)}}(\Theta i_{\mathbf{Z}_{(r,m)}}(w'), i_{\mathbf{Z}_{(r,m)}}(w')) \\
= \lim_{i \to \infty} \sum_{j_i=1}^{k_i} \Theta i_{\mathbf{Z}_{(p,t)}}(\sigma_{j_i}) i_{\mathbf{Z}_{(p\lor r,qt-1(t\lor m)}}(\tau_{j_i}) i_{\mathbf{Z}_{(r,m)}}(\eta_{j_i}) i_{\mathbf{Z}_{(r\lor r,rm-1(t\lor m)}}(\rho_{j_i}) \\
\in K_{N,T_X}(Y_{N,T}^{N,T})
\]
whilst

\[ i_{Z_{(p,t)}}(\sigma_{ji}) \in Y_{l}^{NT}, \quad i_{Z_{(p-1(pvr),t-1(tvm))}}(\tau_{ji}) \in Y_{l-1(tvm)}^{NT}, \]

\[ i_{Z_{(r,m)}}(\eta_{ji}) \in Y_{m}^{NT}, \quad i_{Z_{(r-1(pvr),m-1(tvm))}}(\rho_{ji}) \in Y_{m-1(tvm)}^{NT}. \]

Hence, by Remark 3.1.23, we have

\[ \Psi(\Theta j_{Z_{(p,q)}}(z) j_{Z_{(p,t)}}(w)) \Psi'(\Theta j_{Z_{(r,m)}}(u) j_{Z_{(r,n)}}(v)) \]

\[ = \Psi_p(j_{Z_{(p,q)}}(z)) \Psi'_p(j_{Z_{(p,t)}}(w))^* \Psi'_r(j_{Z_{(r,m)}}(u)) \Psi_r(j_{Z_{(r,n)}}(v))^* \]

\[ = j_{Y_{q}^{NT}}(i_{Z_{(p,q)}}(z)) j_{Y_{l}^{NT}}(i_{Z_{(p,t)}}(w))^* j_{Y_{n}^{NT}}(i_{Z_{(r,m)}}(u)) j_{Y_{m}^{NT}}(i_{Z_{(r,n)}}(v))^* \]

\[ = \lim_{t \to \infty} \sum_{j_{i}=1}^{k_{i}} j_{Y_{q}^{NT}}(i_{Z_{(p,q)}}(z)) (i_{Z_{(p,t)}}(w'), i_{Z_{(p,t)}}(\sigma_{ji})) \]

\[ j_{Y_{l}^{NT}}(i_{Z_{(r,m)}}(u)) (i_{Z_{(r,n)}}(\eta_{ji})) \]

\[ = \lim_{t \to \infty} \sum_{j_{i}=1}^{k_{i}} j_{Y_{q}^{NT}}(i_{Z_{(p,q,r-1(tvm))}}(z(w', \sigma_{ji},(p,t)^{t} \tau_{ji}))) \]

\[ j_{Y_{l}^{NT}}(i_{Z_{(r,m,n-1(tvm))}}(v(w', \eta_{ji},(r,m)^{r} \rho_{ji})))^*. \]

which is precisely (3.37). This completes the proof that \( \Psi' \) is Nica covariant. \( \square \)

Finally, we are ready to prove that \( NO_{Y_{NT}} \) and \( NT_{W_{NO}} \) are isomorphic.

**Theorem 3.5.10.** Suppose that \( A \) acts compactly on each \( Z_{(eg,q)} \) and \( (H, Q) \) is directed, so that the homomorphism \( \omega \) of Proposition 3.5.3 exists. Then the homomorphisms \( \omega : NO_{Y_{NT}} \to NT_{W_{NO}} \) and \( \omega' : NT_{W_{NO}} \to NO_{Y_{NT}} \) are mutually inverse isomorphisms. Thus, \( NO_{Y_{NT}} \cong NT_{W_{NO}} \).

**Proof.** Firstly, we show that \( \omega \circ \omega' = id_{NT_{W_{NO}}} \). As \( NT_{W_{NO}} \) is generated by \( i_{W_{NO}} \), it suffices to show that \( \omega \circ \omega' \circ i_{W_{NO}} = i_{W_{NO}} \). Let \( p \in P, q,t \in Q \) and fix \( z \in Z_{(p,q)} \), \( w \in Z_{(eg,t)} \). Then

\[ (\omega \circ \omega')(i_{W_{p}^{NO}}(j_{Z_{(p,q)}}(z) j_{Z_{(eg,t)}}(w)^*)) = \omega(\Psi_p(j_{Z_{(p,q)}}(z) j_{Z_{(eg,t)}}(w)^*)) \]

\[ = \omega(j_{Y_{q}^{NT}}(i_{Z_{(p,q)}}(z)) j_{Y_{l}^{NT}}(i_{Z_{(eg,t)}}(w)^*)) \]

\[ = \Xi(i_{Z_{(p,q)}}(z) j_{Z_{(eg,t)}}(w)^*)) \]

\[ = i_{W_{p}^{NO}}(j_{Z_{(p,q)}}(z)) i_{W_{eg}^{NO}}(j_{Z_{(eg,t)}}(w))^* \]

\[ = i_{W_{p}^{NO}}(j_{Z_{(p,q)}}(z) j_{Z_{(eg,t)}}(w)^*). \]
As $\mathcal{W}^\mathcal{N}_p = \text{span} \left\{ j_{(p,q)} \, (Z_{(p,q)}) \, j_{(e_G,t)} \, (Z_{(e_G,t)})^*: q,t \in Q \right\}$ for each $p \in P$, whilst both of the maps $\omega \circ \omega' \circ i_{\mathcal{W}^\mathcal{N}_p}$ and $i_{\mathcal{W}^\mathcal{N}_p}$ are linear and continuous, we conclude that $\omega \circ \omega' \circ i_{\mathcal{W}^\mathcal{N}_p} = i_{\mathcal{W}^\mathcal{N}_p}$ for each $p \in P$. Thus, $\omega \circ \omega' = \text{id}_{\mathcal{N}^\mathcal{N}_{\mathcal{W}^\mathcal{N}_p}}$.

Secondly, we check that $\omega' \circ \omega = \text{id}_{\mathcal{N}^\mathcal{O}_{\mathcal{Y}^\mathcal{N}_T}}$. As $\mathcal{N}^\mathcal{O}_{\mathcal{Y}^\mathcal{N}_T}$ is generated by $j_{\mathcal{Y}^\mathcal{N}_T}$, it suffices to check that $\omega' \circ \omega \circ j_{\mathcal{Y}^\mathcal{N}_T} = j_{\mathcal{Y}^\mathcal{N}_T}$. Let $q \in Q$, $p,r \in P$ and fix $z \in Z_{(p,q)}$, $w \in Z_{(r,e_H)}$. Then

\[
(\omega' \circ \omega) \left( j_{\mathcal{Y}^\mathcal{N}_T} \left( i_{Z_{(p,q)}}(z) i_{Z_{(r,e_H)}}(w)^* \right) \right) = (\omega' \circ \Xi') \left( i_{Z_{(p,q)}}(z) i_{Z_{(r,e_H)}}(w)^* \right) \\
= \omega' \left( i_{\mathcal{W}^\mathcal{N}_p} \left( j_{Z_{(p,q)}}(z) \right) i_{\mathcal{W}^\mathcal{N}_H} \left( j_{Z_{(r,e_H)}}(w) \right)^* \right) \\
= \Psi'_p \left( j_{Z_{(p,q)}}(z) \right) \Psi'_r \left( j_{Z_{(r,e_H)}}(w) \right)^* \\
= j_{\mathcal{Y}^\mathcal{N}_T} \left( i_{Z_{(p,q)}}(z) \right) j_{\mathcal{Y}^\mathcal{N}_H} \left( i_{Z_{(r,e_H)}}(w) \right)^* \\
= j_{\mathcal{Y}^\mathcal{N}_T} \left( i_{Z_{(p,q)}}(z) i_{Z_{(r,e_H)}}(w)^* \right).
\]

As $\mathcal{Y}^\mathcal{N}_q = \text{span} \left\{ i_{Z_{(p,q)}} \, (Z_{(p,q)}) \, i_{Z_{(r,e_H)}} \, (Z_{(r,e_H)})^*: p,q \in P \right\}$ for each $q \in Q$, whilst both of the maps $\omega' \circ \omega \circ j_{\mathcal{Y}^\mathcal{N}_T}$ and $j_{\mathcal{Y}^\mathcal{N}_T}$ are linear and continuous, we conclude that $\omega' \circ \omega \circ j_{\mathcal{Y}^\mathcal{N}_T} = j_{\mathcal{Y}^\mathcal{N}_T}$ for each $q \in Q$. Thus, $\omega' \circ \omega = \text{id}_{\mathcal{N}^\mathcal{O}_{\mathcal{Y}^\mathcal{N}_T}}$. \qed
Chapter 4

Iterating the Pimsner construction

4.1 Background and setup

In [13], Deaconu investigates a process that he calls iterating the Pimsner construction. Given two Hilbert $A$-bimodules $E_1$ and $E_2$ (subject to some hypotheses that we will detail shortly) and a Hilbert $A$-bimodule isomorphism $\chi : E_1 \otimes_A E_2 \rightarrow E_2 \otimes_A E_1$, he shows that the scalars of $E_1$ can be extended to $\mathcal{T}_{E_2}$ and $\mathcal{O}_{E_2}$, whilst the scalars of $E_2$ can be extended to $\mathcal{T}_{E_1}$ and $\mathcal{O}_{E_1}$. He then considers the Toeplitz and Cuntz–Pimsner algebras of these new Hilbert bimodules and establishes various isomorphisms between them.

Unfortunately, it is not readily apparent which of Deaconu’s hypotheses are actually necessary to make this procedure work. Furthermore, some of the proofs in [13] lack detail, which leads to difficulty in telling at which point in the argument each hypothesis is used. In this chapter, we aim to show which of these hypotheses are necessary for Deaconu’s construction and which can be relaxed, as well as showing how Deaconu’s iterative procedure fits onto our framework of factorising product systems developed in Chapter 3.

Deaconu’s setup is as follows. Let $A$ be a unital $C^*$-algebra with $E_1$ and $E_2$ full finitely generated Hilbert $A$-bimodules, such that the left action of $A$ on each of $E_1$ and $E_2$ is faithful and nondegenerate. Moreover, suppose that there exists an $A$-bimodule isomorphism $\chi : E_1 \otimes_A E_2 \rightarrow E_2 \otimes_A E_1$.

In Section 3 and 4 of [13], Deaconu discusses what he calls extending the scalars. Subject to the hypotheses in the previous paragraph, he shows that the balanced tensor product $E_2 \otimes_A \mathcal{T}_{E_1}$ can be equipped with a left action of $\mathcal{T}_{E_1}$ by adjointable operators, whilst $E_2 \otimes_A \mathcal{O}_{E_1}$ can be equipped with a left action of $\mathcal{O}_{E_1}$ by adjointable operators. Similarly, $E_1 \otimes_A \mathcal{T}_{E_2}$ and $E_1 \otimes_A \mathcal{O}_{E_2}$ carry left actions of $\mathcal{T}_{E_2}$ and $\mathcal{O}_{E_2}$ respectively. Deaconu then proves the following results concerning the Toeplitz and Cuntz–Pimsner algebras of these bimodules.
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Proposition 4.1.1 ([13], Lemma 4.1). With the setup as above,

$$\mathcal{T}_{E_2 \otimes_A T_{E_1}} \cong \mathcal{T}_{E_1 \otimes_A T_{E_2}}.$$ 

Proposition 4.1.2 ([13], Lemma 4.2). With the setup as above,

$$\mathcal{O}_{E_2 \otimes_A O_{E_1}} \cong \mathcal{O}_{E_1 \otimes_A O_{E_2}}.$$ 

Furthermore,

$$\mathcal{T}_{E_2 \otimes_A O_{E_1}} \cong \mathcal{O}_{E_1 \otimes_A T_{E_2}} \quad \text{and} \quad \mathcal{T}_{E_1 \otimes_A O_{E_2}} \cong \mathcal{O}_{E_2 \otimes_A T_{E_1}}.$$ 

We now discuss Deaconu’s procedure and attempt to explain exactly what is going on. In doing so, we will see that the assumptions that $A$ is unital, $E_1$ and $E_2$ are full, and that the left actions of $A$ are nondegenerate, are not necessary for extending the scalars, nor for establishing the isomorphisms in Propositions 4.1.1 and 4.1.2. We also show that the assumption that $E_1$ and $E_2$ are finitely generated can be relaxed. We will also see that the assumption that $A$ acts faithfully on $E_1$ and $E_2$ is not necessary for extending the scalars to $T_{E_1}$ and $T_{E_2}$, nor for establishing the isomorphism in Proposition 4.1.1.

4.2 Extending the scalars to $T_{E_1}$

Firstly, we recap how Deaconu extends the scalars of $E_2$ from $A$ to $T_{E_1}$ — see Sections 3 and 4 of [13] (the same arguments with $E_2$ replacing $E_1$ and $\chi^{-1}$ replacing $\chi$ show how the scalars of $E_1$ can be extended from $A$ to $T_{E_2}$). As in Definition 2.1.19, the balanced tensor product $E_2 \otimes_A T_{E_1}$ with respect to the injective $\ast$-homomorphism $i_A : A \to T_{E_1}$, gives a (right) Hilbert $T_{E_1}$-module. It remains to show that $E_2 \otimes_A T_{E_1}$ carries a left action of $T_{E_1}$ by adjointable operators, that is there exists a $\ast$-homomorphism $\Phi_{T_{E_1}} : T_{E_1} \to L(T_{E_1}) (E_2 \otimes_A T_{E_1})$. Since $T_{E_1}$ is generated by $E_1$ (because $E_1$ is assumed to be full), Deaconu claims that to define this left action of $T_{E_1}$, it suffices to show how elements of $E_1$ act on $E_2 \otimes_A T_{E_1}$. He describes the action of $E_1$ on $E_2 \otimes_A T_{E_1}$ as being given by the composition

$$E_1 \otimes_A E_2 \otimes_A T_{E_1} \to E_2 \otimes_A E_1 \otimes_A T_{E_1} \to E_2 \otimes_A T_{E_1},$$

where the first map is $\chi \otimes_A \text{id}_{T_{E_1}}$ and the second map is is given by absorbing $E_1$ into $T_{E_1}$. The next result makes this precise.

Proposition 4.2.1. Let $A$ be a $C^*$-algebra and $E_1$, $E_2$ be Hilbert $A$-bimodules. There exists a linear inner-product preserving map $\text{abs} : E_1 \otimes_A T_{E_1} \to T_{E_1}$ such that
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there remain issues with Deaconu’s approach. Firstly, it is not clear why each $\psi$ is adjointable, why this collection of maps defines a $\ast$-is adjointable. Secondly, it is not clear from Deaconu’s arguments, even if each $\psi$ is adjointable, why this collection of maps defines a $\ast$-homomorphism from $\mathcal{T}_{E_1}$ to $\mathcal{T}_{E_1}$.

Moreover, if there exists an $A$-bimodule isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$, then for each $x \in E_1$ there exists a continuous linear map $\psi(x) : E_2 \otimes_A \mathcal{T}_{E_1} \to E_2 \otimes_A \mathcal{T}_{E_1}$ such that

$$
\psi(x) (y \otimes_A t) = (\mathrm{id}_{E_2} \otimes_A \mathrm{abs}) (\chi \otimes_A \mathrm{id}_{\mathcal{T}_{E_1}}) (x \otimes_A y \otimes_A t) 
$$

for each $y \in E_2$, $t \in \mathcal{T}_{E_1}$.

**Proof.** Since $(i_{E_1}, i_A)$ is a Toeplitz representation, for any $x, x' \in E_1$, $t, t' \in \mathcal{T}_{E_1}$, we have

$$
\langle x \otimes_A t, x' \otimes_A t' \rangle = \langle t, (x, x')_A \cdot t' \rangle_{\mathcal{T}_{E_1}} = t^* i_A ((x, x')_A) t' 
$$

$$
= (i_{E_1}(x)t)^* i_{E_1}(x') t' 
$$

$$
= \langle i_{E_1}(x)t, i_{E_1}(x') t' \rangle_{\mathcal{T}_{E_1}}. 
$$

Thus, there exists a linear inner-product preserving map $\mathrm{abs} : E_1 \otimes_A \mathcal{T}_{E_1} \to \mathcal{T}_{E_1}$ satisfying (4.2). To see that the map $\mathrm{abs}$ satisfies (4.3), observe that for any $a \in A$, $x \in E_1$, and $t \in \mathcal{T}_{E_1}$, we have

$$
\mathrm{abs} (a \cdot (x \otimes_A t)) = \mathrm{abs} ((a \cdot x) \otimes_A t) = i_{E_1}(a \cdot x)t = i_A(a) i_{E_1}(x)t 
$$

$$
= i_A(a) \mathrm{abs} (x \otimes_A t). 
$$

Now suppose that there exists an $A$-bimodule isomorphism $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$ and fix $x \in E_1$. Since $\mathrm{abs}$ and $\chi$ are inner-product preserving, for any $y \in E_2$, $t \in \mathcal{T}_{E_1}$,

$$
\| (\mathrm{id}_{E_2} \otimes_A \mathrm{abs}) (\chi \otimes_A \mathrm{id}_{\mathcal{T}_{E_1}}) (x \otimes_A y \otimes_A t) \|_{E_2 \otimes_A \mathcal{T}_{E_1}}^2 = \| x \otimes_A y \otimes_A t \|_{E_1 \otimes_A E_2 \otimes_A \mathcal{T}_{E_1}}^2 
$$

$$
\leq \| x \|_{E_1}^2 \| y \otimes_A t \|_{E_2 \otimes_A \mathcal{T}_{E_1}}^2. 
$$

Thus, there exists a well-defined continuous linear map $\psi(x)$ on $E_2 \otimes_A \mathcal{T}_{E_1}$, satisfying (4.4). \qed

Proposition 4.2.1 describes a linear map $\psi(x)$ on $E_2 \otimes_A \mathcal{T}_{E_1}$ for each $x \in E_1$, but there remain issues with Deaconu’s approach. Firstly, it is not clear why each $\psi(x)$ is adjointable. Secondly, it is not clear from Deaconu’s arguments, even if each $\psi(x)$ is adjointable, why this collection of maps defines a $\ast$-homomorphism from $\mathcal{T}_{E_1}$ to $\mathcal{T}_{E_1}$.\]
The question of adjointability is discussed by Deaconu — we will shortly recap the argument presented in [13] and indicate where more argument is needed. The question of why the collection of maps \( \{ \psi(x) : x \in E_1 \} \) is sufficient to define a *-homomorphism on \( \mathcal{T}_{E_1} \) is not discussed at all in [13].

One possible approach, which we do not pursue here, is to show that for a suitable choice of *-homomorphism \( \pi : A \to \mathcal{L}_{\mathcal{T}_{E_1}}(E_2 \otimes_A \mathcal{T}_{E_1}) \) (\( \pi \) should just be the left action of \( A \) on the first factor \( E_2 \) of \( E_2 \otimes_A \mathcal{T}_{E_1} \)), the pair \((\psi, \pi)\) is a Toeplitz representation of \( E_1 \) in \( \mathcal{L}_{\mathcal{T}_{E_1}}(E_2 \otimes_A \mathcal{T}_{E_1}) \). The universal property of \( \mathcal{T}_{E_1} \) would then provide a *-homomorphism from \( \mathcal{T}_{E_1} \) to \( \mathcal{L}_{\mathcal{T}_{E_1}}(E_2 \otimes_A \mathcal{T}_{E_1}) \). If \((\psi, \pi)\) is a Toeplitz representation and \( E_1 \) is full (as in Deaconu’s situation), then \( \pi \) is fully determined by \( \psi \) and relation (T3).

Our approach to establishing that each \( \psi(x) \) is adjointable and \( \psi \) extends to a *-homomorphism on \( \mathcal{T}_{E_1} \) uses the machinery of product systems. Firstly, we show that having a pair of Hilbert \( A \)-bimodules \( E_1 \) and \( E_2 \) with a Hilbert \( A \)-bimodule isomorphism \( \chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1 \) is equivalent to having a product system \( \mathcal{Z} \) over \( \mathbb{N}^2 \). Using our results from Section 3.3, we will show that if the product system \( \mathcal{Z} \) is compactly aligned then the maps \( \psi(x) \) are all adjointable. We also show that \( \mathcal{Z} \) is always compactly aligned if \( E_1 \) and \( E_2 \) are finitely generated (as in the situation considered by Deaconu). We will also use our work from Section 3.3 to show that there exists a *-homomorphism \( \Phi_{\mathcal{T}_{E_1}} : \mathcal{T}_{E_1} \to \mathcal{L}_{\mathcal{T}_{E_1}}(E_2 \otimes_A \mathcal{T}_{E_1}) \) such that \( \Phi_{\mathcal{T}_{E_1}}(i_{E_1}(x)) = \psi(x) \) for each \( x \in E_1 \) (so the *-homomorphism \( \Phi_{\mathcal{T}_{E_1}} \) determines an action of \( \mathcal{T}_{E_1} \) on \( E_2 \otimes_A \mathcal{T}_{E_1} \) satisfying (4.1)). It is interesting that Fowler’s compact alignment condition also shows up as the key hypothesis for making Deaconu’s approach to \( C^* \)-algebras of product systems over \( \mathbb{N}^2 \) work.

We now give an overview of the argument presented in [13] as to why \( \psi(x) \) is adjointable. Deaconu argues that \( \psi(x) \) is an adjointable map on \( E_2 \otimes_A \mathcal{T}_{E_1} \) using the sequence of maps

\[
E_1^* \otimes_A E_2 \otimes_A E_1 \to E_1^* \otimes_A E_1 \otimes_A E_2 \to E_2
\]

where the first map is \( \text{id}_{E_1^*} \otimes_A \chi^{-1} \) and the second map is given by left multiplication with the inner-product in \( E_1 \) (i.e. identifying \( E_1^* \otimes_A E_1 \) with \( \langle E_1, E_1 \rangle_A \subseteq A \), which is all of \( A \) if \( E_1 \) is full, and then letting this act on the left of \( E_2 \)). It is not difficult to verify that (4.5) determines a map on the the subspace

\[
E_2 \otimes_A \text{span} \{ i_{E_1}(E_1) \mathcal{T}_{E_1} \} = E_2 \otimes_A \text{span} \{ i_{E_1}^m(E_1^m) i_{E_1}^n(E_1^n)^* : m \geq 1, n \geq 0 \}
\]

of \( E_2 \otimes_A \mathcal{T}_{E_1} \), that acts as the adjoint of \( \psi(x) \) should. However, no explanation is
given as to why (4.5) defines an adjoint for $\psi(x)$ on

$$E_2 \otimes_A \text{Span} \{i_{E_1}^0 (E_1^{\otimes 0}) i_{E_1}^0 (E_1^{\otimes n})^* : n \geq 0 \} = E_2 \otimes_A \text{Span} \{i_{E_1}^0 (E_1^{\otimes n})^* : n \geq 0 \}.$$ 

To demonstrate that there is some subtlety here, which we do not believe is addressed in [13], we present the following example. The modules $E_1$ and $E_2$ in the example are not finitely generated, so do not satisfy Deaconu’s hypotheses, but the example illustrates a genuine issue with Deaconu’s arguments.

**Example 4.2.2.** Let $E_1 = E_2 = \ell^2(\mathbb{N})$ regarded as a Hilbert $\mathbb{C}$-bimodule. Let $\chi : E_1 \otimes_\mathbb{C} E_2 \rightarrow E_2 \otimes_\mathbb{C} E_1$ be the Hilbert $\mathbb{C}$-bimodule isomorphism defined by $\chi(x \otimes y) := x \otimes_C y$. Then the linear map $\psi(e_1) : E_2 \otimes_A \mathcal{T}_{E_1} \rightarrow E_2 \otimes_A \mathcal{T}_{E_1}$ is not adjointable.

**Proof.** It is elementary to check that the Toeplitz algebra of $E_1$ is the universal $C^*$-algebra generated by countably-infinitely many isometries with mutually orthogonal ranges, i.e. $\mathcal{T}_{E_1} = C^*(\{S_i : i \in \mathbb{N}\})$ where $S_i^* S_j = \delta_{i,j}1$. Specifically, if the maps $i_C : \mathbb{C} \rightarrow C^*(\{S_i : i \in \mathbb{N}\})$ and $i_{E_1} : E_1 \rightarrow C^*(\{S_i : i \in \mathbb{N}\})$ are defined by $\lambda \mapsto \lambda 1$ and $e_1 \mapsto S_1$ respectively, then $(i_C, i_{E_1})$ is the universal Toeplitz representation of $E_1$. For any $j \in \mathbb{N}$, we have

$$\psi(e_1) (e_j \otimes_\mathbb{C} 1) = (\text{id}_{E_2} \otimes_A \text{abs}) (\chi \otimes_A \text{id}_{E_1}) (e_j \otimes_A e_j \otimes_A 1)
= (\text{id}_{E_2} \otimes_A \text{abs}) (e_1 \otimes_A e_j \otimes_A 1)
= e_1 \otimes_A S_j.$$ 

We suppose that $\psi(e_1)$ is adjointable and derive a contradiction. Write

$$\psi(e_1)^* (e_1 \otimes_A 1) = \sum_{i=1}^\infty e_i \otimes_A t_i$$

for some $t_i \in \mathcal{T}_{E_1}$. For each $j \in \mathbb{N}$ we have

$$S_j^* = S_j^* i_C ((e_1, e_1)_C) 1 = (e_1 \otimes_A S_j, e_1 \otimes_A 1)_{\mathcal{T}_{E_1}} = (\phi(e_1)(e_j \otimes_A 1), e_1 \otimes_A 1)_{\mathcal{T}_{E_1}}
= \langle e_j \otimes_A 1, \phi(e_1)^*(e_1 \otimes_A 1) \rangle_{\mathcal{T}_{E_1}}
= \left\langle e_j \otimes_A 1, \sum_{i=1}^\infty e_i \otimes_A t_i \right\rangle_{\mathcal{T}_{E_1}}
= \sum_{i=1}^\infty 1^* i_C ((e_j, e_i)_C) t_i
= t_j.$$

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Thus, \( t_j = S_j^* \) for each \( j \in \mathbb{N} \), forcing \( \psi(e_1)^*(e_1 \otimes_A 1) = \sum_{i=1}^{\infty} e_i \otimes_A S_i^* \), which does not converge. Hence, \( \psi(e_1) \) is not adjointable on \( E_2 \otimes_A T_{E_1} \). \( \square \)

The next result and remark show that having a pair of Hilbert \( A \)-bimodules \( E_1 \) and \( E_2 \), and a Hilbert \( A \)-bimodule isomorphism \( \chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1 \), is equivalent to having a product system over \( \mathbb{N}^2 \).

**Proposition 4.2.3.** Let \( A \) be a \( C^* \)-algebra and let \( E_1, E_2 \) be Hilbert \( A \)-bimodules. Suppose that \( \chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1 \) is a Hilbert \( A \)-bimodule isomorphism. Then there exists a product system \( Z := \bigcup_{(m,n) \in \mathbb{N}^2} Z_{(m,n)} \) over \( \mathbb{N}^2 \) with coefficient algebra \( A \) such that

\[
Z_{(m,n)} = \begin{cases} 
E_1 \otimes_A E_2 & \text{if } m \geq 1 \\
E_2 \otimes_A E_1 & \text{if } m = 0.
\end{cases}
\]

(4.6)

for each \((m,n) \in \mathbb{N}^2\). Furthermore, the Hilbert \( A \)-bimodule isomorphisms \( M^Z_{(1,0),(0,1)} : Z_{(1,0)} \otimes_A Z_{(0,1)} \to Z_{(1,1)} \) and \( M^Z_{(0,1),(1,0)} : Z_{(0,1)} \otimes_A Z_{(1,0)} \to Z_{(1,1)} \) are given by

\[
M^Z_{(1,0),(0,1)} = \id\otimes\chi^{-1} \quad \text{and} \quad M^Z_{(0,1),(1,0)} = \chi^{-1}.
\]

**Proof.** The result follows from an application of ([28], Theorem 2.1). \( \square \)

**Remark 4.2.4.** Given a product system \( Z \) over \( \mathbb{N}^2 \) with coefficient algebra \( A \), if we let \( E_1 := Z_{(1,0)} \) and \( E_2 := Z_{(0,1)} \), then \( \chi := \left( M^Z_{(0,1),(1,0)} \right)^{-1} \circ M^Z_{(1,0),(0,1)} \) is an \( A \)-bimodule isomorphism from \( E_1 \otimes_A E_2 \) to \( E_2 \otimes_A E_1 \). It is straightforward to see that the product system produced by applying Proposition 4.2.3 to \( E_1, E_2 \), and \( \chi \), recovers the product system \( Z \) (up to isomorphism).

Our goal is to show that when the product system \( Z \) constructed from \( E_1, E_2 \), and \( \chi \) (as in Proposition 4.2.3) is compactly aligned, the Hilbert \( T_{E_1} \)-module \( E_2 \otimes_A T_{E_1} \) can be identified with a subspace of \( NT_Z \). To see that this assumption is consistent with Deaconu’s setup, observe that when \( E_1 \) and \( E_2 \) are finitely generated, Proposition 2.1.22 implies that \( A \) acts compactly on each fibre of \( Z \), and hence \( Z \) is compactly aligned by Proposition 3.1.24.

Since the product system \( Z \) is determined by the data \((E_1, E_2, \chi)\), it is natural to ask whether \( Z \) being compactly aligned is equivalent to having

\[
(S \otimes_A \id_{E_2}) \left( \chi^{-1} \circ (T \otimes_A \id_{E_1}) \circ \chi \right) \in \mathcal{K}_A \left( E_1 \otimes_A E_2 \right)
\]

for all \( S \in \mathcal{K}_A \left( E_1 \right) \), \( T \in \mathcal{K}_A \left( E_2 \right) \).

We now work towards showing that this is the case. Firstly, we need a couple of lemmas.
Lemma 4.2.5. Let \((G, P)\) be a quasi-lattice ordered group, and \(X\) a product system over \(P\) with coefficient algebra \(A\). Let \(p, q \in P\) with \(p \neq e\) and \(x, w \in X_p, y, z \in X_q\). Choosing \(w' \in X_p\) so that \(w = w' \cdot (w', w')_A^p\) by the Hewitt–Cohen–Blanchard factorisation theorem, we have

\[
\Theta_{xy,wz} = \Theta_{xy,w'z} \circ i_p^{pq}(\Theta_{w', w'}) .
\]

Proof. For any \(u \in X_p\) and \(v \in X_q\),

\[
(\Theta_{xy,w'z} \circ i_p^{pq}(\Theta_{w', w'}))(uv) = \Theta_{xy,w'z}(\Theta_{w', w'}(u)v) = \Theta_{xy,w'z}(w'(w', u)_A^p v) = xy(w', w')_A^p = xy(w'z, w'(w', u)_A^p) .
\]

Since \(M_{p,q} : X_p \otimes_A X_q \rightarrow X_{pq}\) is inner product preserving, this must equal

\[
xy(w' \otimes_A z, (\Theta_{w', w'} \otimes_A \text{id}_{X_p}) (u \otimes_A v))_A = xy((\Theta_{w', w'} \otimes_A \text{id}_{X_p}) (u \otimes_A v))_A = x(y\langle \Theta_{w', w'} \otimes_A (u \otimes_A v) \rangle_A = xy(wz, uv)_A^pq = \Theta_{xy,wuz}(uv).
\]

Since \(X_{pq} = \bigwedge \{uv : u \in X_p, v \in X_q\}\), whilst both \(\Theta_{xy,wuz}\) and \(\Theta_{xy,w'z} \circ i_p^{pq}(\Theta_{w', w'})\) are linear and continuous, we conclude that \(\Theta_{xy,wuz} = \Theta_{xy,w'z} \circ i_p^{pq}(\Theta_{w', w'})\). \(\square\)

Recall Proposition 3.1.6: for any \(p \in P \setminus \{e\}\), \(x \in X_p\), there exists an adjointable operator \(\Theta_x \in \mathcal{L}_A(X_q, X_{pq})\) defined by \(\Theta_x(y) = xy\), whose adjoint is determined by the formula \(\Theta_x^*(uv) = \langle x, u \rangle_A^p v\) for each \(u \in X_p, v \in X_q\).

Lemma 4.2.6. Let \((G, P)\) be a quasi-lattice ordered group, and \(X\) a product system over \(P\) with coefficient algebra \(A\). Let \(p, q, r \in P\) with \(r \geq p \neq e\). For any \(x, w, y, z \in X_p, y, z \in X_q\),

\[
i_p^{r}(\Theta_{xy,wz}) = (\Theta_{xy,wz})^r .
\]

Proof. Since \(r \geq pq\) if and only if \(p^{-1}r \geq q\), both sides of the above expression are zero if \(r \not\geq pq\). Hence, we may as well suppose that \(r \geq pq\). For any \(a \in X_p, b \in X_q, c \in X_{(pq)^{-1}}\), we have

\[
(\Theta_x \circ i_q^{pq}(\Theta_{y,z}) \circ \Theta^*_z)(abc) = x(y\langle w, a \rangle_A^p b)_A^q c
\]

\[
= xy(z, \langle w, a \rangle_A^p b)_A^q c
\]

\[
= xy(w \otimes_A z, a \otimes_A b)_A c.
\]
Because $M_{p,q} : X_p \otimes_A X_q \to X_{pq}$ is inner-product preserving, this must equal

$$xy(wz, ab)^{pq}c = \Theta_{xy,wz}(ab)c = \iota^\ast_{pq}(\Theta_{xy,wz})(abc).$$

Since $X_r = \text{span}\{abc : a \in X_p, b \in X_q, c \in X_{(pq)-1}\}$, whilst both $\iota^\ast_{pq}(\Theta_{xy,wz})$ and $\Theta_x \circ \iota^\ast_{pq} \circ (\Theta_y,z) \circ \Theta^\ast_w$ are linear and continuous, we conclude that $\iota^\ast_{pq}(\Theta_{xy,wz}) = \Theta_x \circ \iota^\ast_{pq} \circ (\Theta_y,z) \circ \Theta^\ast_w$. \qed

The next result shows that a product system over $\mathbb{N}^k$ is compactly aligned provided the product of any two compact operators on distinct generating fibres is compact.

**Proposition 4.2.7.** Let $X$ be a product system over $\mathbb{N}^k$. If

$$\iota^\ast_{e_i+e_j}(S)\iota^\ast_{e_j+e_j}(T) \in \mathcal{K}_A \left(X_{e_i+e_j}\right) \text{ for each } S \in \mathcal{K}_A \left(X_{e_i}\right), T \in \mathcal{K}_A \left(X_{e_j}\right) \text{ with } i \neq j,$$

then $X$ is compactly aligned.

**Proof.** We need to show that for any $m, n \in \mathbb{N}^k$,

$$\iota^\ast_{m\vee n}(S)\iota^\ast_{n\vee n}(T) \in \mathcal{K}_A \left(X_{m\vee n}\right) \text{ for each } S \in \mathcal{K}_A \left(X_m\right), T \in \mathcal{K}_A \left(X_n\right). \quad (4.7)$$

To do this we will use induction on $\sum_{j=1}^k (m+n)j$.

If $\sum_{j=1}^k (m+n)j \leq 2$ then either $m = 0$ or $n = 0$, or $m = e_i$ and $n = e_j$ for some $i \neq j$. In the first situation $m \vee n \in \{m, n\}$, so either $\iota^\ast_{m\vee n}(S) = S$ or $\iota^\ast_{n\vee n}(T) = T$, and (4.7) holds. In the second situation, our hypothesis says that (4.7) holds.

Now fix $N \geq 2$ and suppose that (4.7) holds whenever $\sum_{j=1}^k (m+n)j \leq N$. Fix $m, n \in \mathbb{N}^k$ with $\sum_{j=1}^k (m+n)j = N + 1$ and let $S \in \mathcal{K}_A \left(X_m\right), T \in \mathcal{K}_A \left(X_n\right)$.

We begin by showing that we can always reduce to the situation where the coordinate-wise minimum of $m$ and $n$ (denoted by $m \land n$) is zero. If $m \land n \neq 0$, Lemma 4.2.6 says we can approximate $\iota^\ast_{m\vee n}(S)\iota^\ast_{n\vee n}(T)$ by linear combinations of operators of the form

$$\left(\Theta_x \circ \iota^\ast_{m-m \land n}(S') \circ \Theta^\ast_w\right) \left(\Theta_y \circ \iota^\ast_{n-m \land n}(T') \circ \Theta^\ast_z\right) = \Theta_x \circ \iota^\ast_{m-m \land n}(S') \iota^\ast_{n-m \land n}(\phi_{m-m \land n}(w, y_{A}^m T') \circ \Theta^\ast_z$$

where $x, w, y, z \in X_{m \land n}$ and $S' \in \mathcal{K}_A \left(X_{m-m \land n}\right), T' \in \mathcal{K}_A \left(X_{n-m \land n}\right)$. Since

$$(m - m \land n) \lor (n - m \land n) = m \lor n - m \land n$$

...
and

\[ \sum_{j=1}^{k} ((m - m \wedge n) + (n - m \wedge n))_j = \sum_{j=1}^{k} (m+n)_j - 2(m \wedge n)_j \]

or

\[ \sum_{j=1}^{k} (m+n)_j \]

the inductive hypothesis tells us that

\[ \iota_{m\wedge n}^{m\wedge n}(S') \iota_{n\wedge m}^{m\wedge n}(\phi_{n\wedge m\wedge n} ((w, y)_A^{m\wedge n}) T') \in K_A (X_{m\wedge n\wedge m\wedge n}). \]

Hence,

\[ \Theta_x \circ \iota_{m\wedge n}^{m\wedge n}(S') \iota_{n\wedge m}^{m\wedge n}(\phi_{n\wedge m\wedge n} ((w, y)_A^{m\wedge n}) T') \circ \Theta^*_x \in K_A (X_{m\wedge n}), \]

and we conclude that

\[ \iota_{m\wedge n}(S) \iota_{n\wedge n\wedge n}(T) \in K_A (X_{m\wedge n}). \]

Now suppose that \( m \wedge n = 0 \). Thus, \( m \vee n = m + n \). Since \( \sum_{j=1}^{k} (m+n)_j = N + 1 \geq 3 \), either \( m > e_i \) or \( n > e_i \) for some \( i \in \{1, \ldots, l\} \). By taking the adjoint of (4.7), we may assume without loss of generality that \( m > e_i \). By Lemma 3.1.5 and Lemma 4.2.5, \( \iota_{m\wedge n}^{m\wedge n}(S) = \iota_{m\wedge n}(S) \) can be approximated by linear combinations of operators of the form

\[ \iota_{m\wedge n}(S) \iota_{m\wedge n}(T) \in K_A (X_{m\wedge n}). \]

Thus, the left hand side of (4.7) can be approximated by linear combinations of operators of the form

\[ \iota_{m\wedge n}(S') \iota_{m\wedge n}(T') \]

where \( S' \in K_A (X_m) \) and \( S'' \in K_A (X_{m-e_i}) \). By Lemma 3.1.5, we can also write

\[ \iota_{m\wedge n}(T) = \iota_{m\wedge n}(T'). \]

Thus, the left hand side of (4.7) can be approximated by linear combinations of operators of the form

\[ \iota_{m\wedge n}(S') \iota_{m\wedge n}(T') \]

where \( S' \in K_A (X_m) \) and \( S'' \in K_A (X_{m-e_i}) \). Since

\[ (m - e_i) \vee n = m + n - e_i \]
and
\[ \sum_{j=1}^{k} (m - e_i + n)_j = \left( \sum_{j=1}^{k} (m + n)_j \right) - 1 = N, \]
we can apply the inductive hypothesis to \( i_{m-e_i}^{m+n-e_i} (S') i_{m+n-e_i}^{m+n} T \). Hence, (4.8) can itself be approximated by linear combinations of operators of the form
\[ i_{m}^{m+n} (S') i_{m+n-e_i}^{m+n} (R) \] (4.9)
where \( R \in K_A (X_{m+n-e_i}) \). By Lemma 4.2.6, (4.9) can then be approximated by linear combinations of operators of the form
\[
(\Theta_x \circ i_{e_i}^{e_i+n} (S') \circ \Theta_z^*) (\Theta_y \circ i_{n}^{e_i+n} (R') \circ \Theta_z^*)
= \Theta_x \circ i_{e_i}^{e_i+n} (S') i_{n}^{e_i+n} (\phi_n (\langle w, y \rangle_A^{m-e_i}) R') \circ \Theta_z^*
\]
where \( x, w, y, z \in X_{m-e_i} \) and \( S' \in K_A (X_{e_i}) \), \( R' \in K_A (X_n) \). As \( m > e_i \) and \( m \wedge n = 0, \)
\[ e_i \vee n = e_i + n \]
and
\[ \sum_{j=1}^{k} (e_i + n)_j < \sum_{j=1}^{k} (m + n)_j = N + 1, \]
and so we can apply the inductive hypothesis one more time to see that
\[ i_{e_i}^{e_i+n} (S') i_{n}^{e_i+n} (\phi_n (\langle w, y \rangle_A^{m-e_i}) R') \in K_A (X_{e_i+n}) \].
Hence,
\[ \Theta_x \circ i_{e_i}^{e_i+n} (S') i_{n}^{e_i+n} (\phi_n (\langle w, y \rangle_A^{m-e_i}) R') \circ \Theta_z^* \in K_A (X_{e_i+n+m+e_i}) = K_A (X_{m\vee n}), \]
and we conclude that
\[ i_{m}^{m\vee n} (S) i_{n}^{m\vee n} (T) \in K_A (X_{m\vee n}) \]
as required. \( \square \)

In particular, we are interested in the product system \( Z \) determined by the data \((E_1, E_2, \chi)\). In this situation the previous result shows that \( Z \) is compactly aligned provided the product of a compact operator on \( E_1 \) and a compact operator on \( E_2 \) is compact on \( E_1 \otimes_A E_2 \).

**Corollary 4.2.8.** Let \( Z \) be the product system determined by \( E_1, E_2, \) and \( \chi \) (as in
Proposition 4.2.3). If

\[(S \otimes A \text{id}_{E_2})\left(\chi^{-1} \circ (T \otimes A \text{id}_{E_1}) \circ \chi\right) \in \mathcal{K}_A(E_1 \otimes A E_2)\]

for all \(S \in \mathcal{K}_A(E_1),\ T \in \mathcal{K}_A(E_2),\) then \(Z\) is compactly aligned.

Proof. Since \(M^Z\)_{(0,1),(1,0)} = \(\chi^{-1}\),

\[\iota^{(1,1)}_p(S)\iota^{(1,1)}_q(T) = (S \otimes A \text{id}_{E_2})\left(\chi^{-1} \circ (T \otimes A \text{id}_{E_1}) \circ \chi\right) \in \mathcal{K}_A(E_1 \otimes A E_2) = \mathcal{K}_A(Z)_{(1,1)}\]

for any \(S \in \mathcal{K}_A(E_1) = \mathcal{K}_A(Z)_{(0,0)}\) and \(T \in \mathcal{K}_A(E_2) = \mathcal{K}_A(Z)_{(0,1)}\). The result now follows from Proposition 4.2.7. \(\Box\)

Remark 4.2.9. It would be interesting to see if, given a product system \(X\) over a quasi-lattice ordered group \((G, P)\) with \(P\) generated by some set \(F\), whether \(X\) being compactly aligned is equivalent to having \(\iota^{p, q} p (S) \iota^{p, q} q (T) \in \mathcal{K}_A(X)_{p, q}\) whenever \(p, q \in F\) with \(p \vee q < \infty\) and \(S \in \mathcal{K}_A(X)_{p}\), \(T \in \mathcal{K}_A(X)_{q}\). We believe that the technique used in proving Proposition 4.2.7 should still work when the quasi-lattice ordered group \((\mathbb{Z}^k, \mathbb{N}^k)\) is replaced by an arbitrary right-angled Artin group. We are not sure if it is possible to extend this result to arbitrary quasi-lattice ordered groups.

Proposition 4.2.10. Let \(A\) be a \(C^*\)-algebra. Suppose \(E_1\) and \(E_2\) are Hilbert \(A\)-bimodules, and \(\chi : E_1 \otimes A E_2 \to E_2 \otimes A E_1\) is an \(A\)-bimodule isomorphism such that (4.10) holds. Let \(X := \bigsqcup \mathbb{N} E_1^\otimes n\) be the product system over \(\mathbb{N}\) associated to the Hilbert \(A\)-bimodule \(E_1\). Then there exists an injective \(*\)-homomorphism \(\phi^{NT}_X : \mathcal{N}T_X \to \mathcal{N}T_Z\) such that \(\phi^{NT}_X \circ \iota_X = \iota_Z\). Furthermore, there exists a Hilbert \(\mathcal{N}T_X\)-module \(Y^{NT}_1\) such that

\[Y^{NT}_1 = \overline{\text{span}} \left\{ \iota_{Z(0,1)}(Z(0,1)) \phi^{NT}_X(N X) \right\}\]

and the \(\mathcal{N}T_X\)-valued inner-product on \(Y^{NT}_1\) satisfies

\[\phi^{NT}_X((y, w)_{\mathcal{N}T_X}) = y^* w \text{ for all } y, w \in Y^{NT}_1.\]

After identifying \(\mathcal{N}T_X\) with \(\mathcal{T}_{E_1}\), there exists a Hilbert \(\mathcal{T}_{E_1}\)-module isomorphism...
\[ \xi : E_2 \otimes_A T_{E_1} \to Y_1^{NT} \text{ such that} \]
\[ \xi(y \otimes_A t) = i_{z_{(0,1)}}(y)\phi_X^{NT}(t) \quad \text{for each } y \in E_2 \text{ and } t \in T_{E_1}. \]  

(4.13)

**Proof.** Since (4.10) holds, the product system determined by the data \((E_1, E_2, \chi)\) is compactly aligned by Corollary 4.2.8. Thus, we can make use of the machinery developed in Chapter 3. Since \(Z\) is amenable, Propositions 3.3.1 and 3.3.2 show that there exists an injective \(*\)-homomorphism \(\phi_X^{NT} : \mathcal{N}T_X \to \mathcal{N}T_Z\) such that \(\phi_X^{NT} \circ i_Z = i_Z\). Proposition 3.3.3 shows that there exists a Hilbert \(\mathcal{N}T_X\)-module \(Y_1^{NT}\) satisfying (4.11) and (4.12). Hence, for any \(y, y' \in Z_{(0,1)} = E_2\) and \(t, t' \in \mathcal{N}T_X \cong T_{E_1}\),

\[
\left\langle i_{z_{(0,1)}}(y)\phi_X^{NT}(t), i_{z_{(0,1)}}(y')\phi_X^{NT}(t') \right\rangle_{\mathcal{N}T_X} \\
= \phi_X^{NT-1}\left(\phi_X^{NT}(t)^* i_{z_{(0,1)}}(y) i_{z_{(0,1)}}(y') \phi_X^{NT}(t')\right) \\
= \phi_X^{NT-1}\left(\phi_X^{NT}(t)^* i_{z_{(0,0)}}(y, y')_{A_1} \phi_X^{NT}(t')\right) \\
= \phi_X^{NT-1}\left(\phi_X^{NT}(t^* i_{x_{0}}(y, y')_{A_1}) t'\right) \\
= t^* i_{x_{0}}(y, y')_{A_1} t'.
\]

Thus, there exists a well-defined inner-product preserving map \(\xi : E_2 \otimes_A T_{E_1} \to Y_1^{NT}\) satisfying (4.13). Since \(Z_{(0,1)} = E_2\), it is clear that \(\xi\) is surjective, and so \(\xi\) is a Hilbert \(T_{E_1}\)-module isomorphism. \(\square\)

After identifying \(E_2 \otimes_A T_{E_1}\) with \(Y_1^{NT} \subseteq \mathcal{N}T_Z\), we can apply our results from Section 3.3 to show that each \(\psi(x)\) is adjointable and \(E_2 \otimes_A T_{E_1}\) carries an action of \(T_{E_1}\) by adjointable operators.

**Proposition 4.2.11.** Let \(A\) be a \(C^*\)-algebra. Suppose \(E_1\) and \(E_2\) are Hilbert \(A\)-bimodules, and \(\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1\) is an \(A\)-bimodule isomorphism such that (4.10) holds. Then there exists a \(*\)-homomorphism \(\Phi_1^{NT} : \mathcal{N}T_X \to \mathcal{L}_{\mathcal{N}T_X}(Y_1^{NT})\) such that

\[ \Phi_1^{NT}(b)(y) = \phi_X^{NT}(b)y \quad \text{for each } b \in \mathcal{N}T_X \text{ and } y \in Y_1^{NT}. \]  

(4.14)

If \(\Phi_{E_1} : T_{E_1} \to \mathcal{L}_{T_{E_1}}(E_2 \otimes_A T_{E_1})\) is the \(*\)-homomorphism defined by

\[ \Phi_{E_1}(b) := \xi^{-1} \circ \Phi_1^{NT}(b) \circ \xi \quad \text{for each } b \in T_{E_1} \cong \mathcal{N}T_X, \]

then

\[ \Phi_{E_1}(i_{E_1}(x)) = \psi(x) \]
for each $x \in X_1 = E_1$. In particular, each $\psi(x)$ is an adjointable map on $E_2 \otimes_A \mathcal{T}_{E_1}$ with

$$\psi(x)^* = \Phi_{\mathcal{T}_{E_1}}(i_{E_1}(x))^* = \xi^{-1} \circ \Phi_1^{N_T}(i_{X_1}(x)) \circ \xi.$$ 

\textbf{Proof.} Proposition 3.3.6 yields a $\ast$-homomorphism $\Phi_1^{N_T} : N \mathcal{T}_X \to \mathcal{L}_{N \mathcal{T}_X}(Y_1^{N_T})$ satisfying (4.14). Fix $x \in E_1$. For $y \in E_2$ and $t \in \mathcal{T}_{E_1}$,

$$(\xi^{-1} \circ \Phi_1^{N_T}(i_{X_1}(x)) \circ \xi)(y \otimes_A t) = (\xi^{-1} \circ \Phi_1^{N_T}(i_{X_1}(x))) (i_{Z_{(0,1)}}(y) \phi_{X}^{N_T}(t)) = \xi^{-1} \begin{pmatrix} i_{Z_{(1,0)}}(x) i_{Z_{(0,1)}}(y) \phi_{X}^{N_T}(t) \end{pmatrix} = \xi^{-1} \begin{pmatrix} i_{Z_{(1,1)}}(x) y \phi_{X}^{N_T}(t) \end{pmatrix}. (4.15)$$

Let

$$M^Z_{(0,1),(1,0)}(xy) = \chi(x \otimes_A y) = \lim_{j \to \infty} \sum_{i_j=1}^{n_j} u_{i_j} \otimes_A v_{i_j} \in E_2 \otimes_A E_1 = Z_{(0,1)} \otimes_A Z_{(1,0)}.$$ 

Then (4.15) is equal to

$$\xi^{-1} \begin{pmatrix} \lim_{j \to \infty} \sum_{i_j=1}^{n_j} i_{Z_{(0,1)}}(u_{i_j}) i_{Z_{(1,0)}}(v_{i_j}) \phi_{X}^{N_T}(t) \end{pmatrix} = \xi^{-1} \begin{pmatrix} \lim_{j \to \infty} \sum_{i_j=1}^{n_j} i_{Z_{(0,1)}}(u_{i_j}) \phi_{X}^{N_T}(i_{X_1}(v_{i_j})t) \end{pmatrix} = \lim_{j \to \infty} \sum_{i_j=1}^{n_j} u_{i_j} \otimes_A i_{E_1}(v_{i_j})t = (id_{E_2} \otimes A \text{ abs})(\chi \otimes_A id_{\mathcal{T}_{E_1}})(x \otimes_A y \otimes_A t) = \psi(x)(y \otimes_A t).$$

Since $\xi^{-1} \circ \Phi_1^{N_T}(i_{X_1}(x)) \circ \xi$ and $\psi(x)$ are linear and continuous, we conclude that $\xi^{-1} \circ \Phi_1^{N_T}(i_{X_1}(x)) \circ \xi = \psi(x).$ \hfill $\square$

We are now ready to prove ([13], Lemma 4.1) with relaxed hypotheses — we will see that it is a special case of Theorem 3.3.17 for product systems over $\mathbb{N}^2$. We do not assume that $A$ is unital, that $E_1$, $E_2$ are full, or that the left actions of $A$ on $E_1$ and $E_2$ are faithful and nondegenerate. Our only assumption is that the product system $Z$ determined by $E_1$, $E_2$, and $\chi$ is compactly aligned (or equivalently that (4.10) holds). This certainly occurs when $E_1$ and $E_2$ are finitely generated, so our result generalises Deaconu’s. Note that by swapping $E_1$ with $E_2$ and using $\chi^{-1}$ in place of $\chi$, the arguments from above show that $E_1 \otimes_A \mathcal{T}_{E_2}$ has the structure of a Hilbert $\mathcal{T}_{E_2}$-bimodule.

\textbf{Theorem 4.2.12.} Let $A$ be a $C^*$-algebra. Suppose $E_1$ and $E_2$ are Hilbert $A$-bimodules, and $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$ is an $A$-bimodule isomorphism such
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that (4.10) holds. Then

$$T_{E_2 \otimes_A T_{E_1}} \cong T_{E_1 \otimes_A T_{E_2}}.$$  

Proof. By Propositions 4.2.10 and 4.2.11, \(E_2 \otimes A T_{E_1}\) and \(Y^{NT}_1\) are isomorphic as Hilbert \(T_{E_1} = \mathcal{N}T_X\)-bimodules. If \(Y^{NT} := \bigsqcup_{n \in \mathbb{N}} \left( Y^{NT}_1 \right)^{\otimes n}\), then Theorem 3.3.17 gives

$$T_{E_2 \otimes_A T_{E_1}} \cong T_{Y^{NT}1} \cong \mathcal{N}T_{Y^{NT}} \cong \mathcal{N}T_Z.$$  

By symmetry, \(T_{E_1 \otimes_A T_{E_2}}\) is also isomorphic to \(\mathcal{N}T_Z\), which establishes the result. \(\square\)

Remark 4.2.13. In ([13], Remark 4.3), Deaconu considers the product system determined by \(E_1, E_2\), and \(\chi\) that we have denoted by \(Z\). He claims that \(T_{E_2 \otimes_A T_{E_1}}\) and \(T_{E_1 \otimes_A T_{E_2}}\) are isomorphic to the Toeplitz algebra of \(Z\) as defined by Fowler. Unfortunately, it is not clear whether Deaconu is referring to what Fowler calls the Toeplitz algebra of \(Z\), i.e. the \(C^*\)-algebra generated by a universal representation of \(Z\) (as in Definition 3.1.7), or the Nica–Toeplitz algebra of \(Z\), i.e. the \(C^*\)-algebra generated by a universal Nica covariant representation of \(Z\) — both are discussed by Fowler in [24]. Deaconu’s choice of notation and the discussion in ([13], Remark 4.3) suggest that he means the former; but Theorem 4.2.12 shows that the latter is in general correct. Note that even under Deaconu’s hypotheses, the two \(C^*\)-algebras do not coincide. For example, if \(E_1 = E_2 = \mathbb{C}\), then Fowler’s Toeplitz algebra is the universal \(C^*\)-algebra generated by two commuting, but not \(\ast\)-commuting isometries, which is much larger than \(\mathcal{N}T_Z = T \otimes T\) (where \(T\) is the classical Teoplitz algebra).

4.3 Extending the scalars to \(\mathcal{O}_{E_1}\)

We now discuss how the scalars of \(E_2\) can be extended from \(A\) to \(\mathcal{O}_{E_1}\). As before, the balanced tensor product \(E_2 \otimes_A \mathcal{O}_{E_1}\) with respect to the injective \(\ast\)-homomorphism \(j_A : A \to \mathcal{O}_{E_1}\), gives a (right) Hilbert \(\mathcal{O}_{E_1}\)-module. We must show that \(E_2 \otimes_A \mathcal{O}_{E_1}\) carries a left action of \(\mathcal{O}_{E_1}\) by adjointable operators. That is, there exists a homomorphism \(\Phi^{\mathcal{O}_{E_1}} : \mathcal{O}_{E_1} \to \mathcal{L}_{E_1}\) \((E_2 \otimes_A \mathcal{O}_{E_1})\). Deaconu does not provide any details on how to do this. One possible approach would be to exhibit a Cuntz–Pimsner covariant Toeplitz representation of \(E_1\) in \(\mathcal{L}_{E_1}\) \((E_2 \otimes_A \mathcal{O}_{E_1})\) and then invoke the universal property of \(\mathcal{O}_{E_1}\). Our approach, which makes use of our results from Section 3.4, is to identify \(E_2 \otimes_A \mathcal{O}_{E_1}\) with a subspace of \(\mathcal{N}O_Z\) that we already know has the structure of a Hilbert \(\mathcal{O}_{E_1}\)-bimodule.

Proposition 4.3.1. Let \(A\) be a \(C^*\)-algebra. Suppose \(E_1\) and \(E_2\) are Hilbert \(A\)-bimodules, and \(\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1\) is an \(A\)-bimodule isomorphism such that (4.10) holds. Suppose that \(A\) acts faithfully on \(E_1\). Let \(X := \bigsqcup_{n \in \mathbb{N}} E_1^{\otimes n}\) be the product system over \(\mathbb{N}\) associated to the Hilbert \(A\)-bimodule \(E_1\). Then there
exists an injective \( * \)-homomorphism \( \phi_{X}^{N} : NO_{X} \to NO_{Z} \) such that \( \phi_{X}^{N} \circ j_{X} = j_{Z} \). Furthermore, there exists a Hilbert \( NO_{X} \)-bimodule \( Y_{1}^{NO} \) such that
\[
Y_{1}^{NO} = \text{span} \{ j_{(0,1)} (Z_{(0,1)}) \phi_{X}^{N} (NO_{X}) \} \tag{4.16}
\]
and the \( NO_{X} \)-valued inner-product on \( Y_{1}^{NO} \) satisfies
\[
\phi_{X}^{N} (\langle y, w \rangle_{NO_{X}}) = y^{*} w \quad \text{for all } y, w \in Y_{1}^{NO}. \tag{4.17}
\]
The left action of \( NO_{X} \) on \( Y_{1}^{NO} \) by adjointable operators is implemented by a \( * \)-homomorphism \( \Phi_{1}^{NO} : NO_{X} \to L_{NO_{X}} (Y_{1}^{NO}) \) satisfying
\[
\Phi_{q}^{N} (b) (y) = \phi_{X}^{N} (b) y \quad \text{for each } b \in NO_{X} \text{ and } y \in Y_{1}^{NO}. \tag{4.18}
\]
After identifying \( NO_{X} \) with \( OE_{1} \), there exists a Hilbert \( OE_{1} \)-module isomorphism \( \zeta : E_{2} \otimes_{A} OE_{1} \to Y_{1}^{NO} \) such that
\[
\zeta (y \otimes_{A} t) = i_{Z_{(0,1)}} (y) \phi_{X}^{N} (t) \quad \text{for each } y \in E_{2} \text{ and } t \in OE_{1}. \tag{4.19}
\]
Thus \( E_{2} \otimes_{A} OE_{1} \) can be equipped with a left action of \( OE_{1} \cong NO_{X} \) by defining a \( * \)-homomorphism \( \Phi_{OE_{1}}^{E_{1}} : OE_{1} \to L_{OE_{1}} (E_{2} \otimes_{A} OE_{1}) \) by
\[
\Phi_{OE_{1}}^{E_{1}} (b) := \zeta^{-1} \circ \phi_{X}^{N} (b) \circ \zeta \quad \text{for each } b \in OE_{1} \cong NO_{X}. \]

Proof. We begin by checking that the hypotheses of Proposition 3.4.3 are satisfied. Since (4.10) holds, the product system determined by the data \((E_{1}, E_{2}, \chi)\) is compactly aligned by Corollary 4.2.8. Since \( A \) acts faithfully on \( E_{1} \), \( A \) acts faithfully on each tensor power of \( E_{1} \), and so on every fibre of \( \mathbf{X} \). Since the semigroup \( \mathbb{N}^{2} \) satisfies (3.3) (as any bounded subset of \( \mathbb{N}^{2} \) is finite), Proposition 3.1.43 implies that \( \tilde{\phi}_{(m,n)} : A \to L_{A} (\tilde{Z}_{(m,n)}) \) is injective for each \((m, n) \in \mathbb{N}^{2}\). Thus, Proposition 3.4.3 gives a \( * \)-homomorphism \( \phi_{X}^{N} : NO_{X} \to NO_{Z} \) such that \( \phi_{X}^{N} \circ j_{X} = j_{Z} \). Since \( Z \) is amenable, Proposition 3.4.6 ensures that \( \phi_{X}^{N} \) is injective.

The existence of a Hilbert \( NO_{X} \)-bimodule \( Y_{1}^{NO} \) satisfying (4.16), (4.17), and (4.18) is given by Proposition 3.4.8. Finally, almost exactly the same inner-product calculation as in the proof of Proposition 4.2.10 shows that there exists a well-defined Hilbert \( OE_{1} \)-module isomorphism \( \zeta : E_{2} \otimes_{A} OE_{1} \to Y_{1}^{NO} \) satisfying (4.19). \( \square \)

We are now ready to prove the first part of ([13], Lemma 4.1) with relaxed hypotheses — it is a special case of Theorem 3.4.21 for product systems over \( \mathbb{N}^{2} \). Again, we significantly weaken Deaconu’s hypotheses. Our only assumptions are that \( A \) acts faithfully and compactly on each of \( E_{1} \) and \( E_{2} \) (the latter is automatic.
when $E_1$ and $E_2$ are finitely generated).

**Theorem 4.3.2.** Let $A$ be a $C^*$-algebra, $E_1$ and $E_2$ Hilbert $A$-bimodules, and $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$ an $A$-bimodule isomorphism. Suppose that $A$ acts faithfully and compactly on $E_1$ and $E_2$. Then

$$O_{E_2 \otimes_A O_{E_1}} \cong O_{E_1 \otimes_A O_{E_2}}.$$  

*Proof.* Since $A$ acts compactly on $E_1$ and $E_2$, it follows from Proposition 2.1.22 that $A$ acts compactly on each fibre of $Z$. Thus, by Proposition 3.1.24, $Z$ is compactly aligned (equivalently (4.10) holds). Since $A$ acts faithfully on $E_1$, Proposition 4.3.1 shows that $E_2 \otimes_A T_{E_1}$ has the structure of a Hilbert $O_{E_1}$-bimodule, and is isomorphic to $\mathcal{Y}_1^{NO}$. Let $\mathcal{Y}_1^{NO} := \bigoplus_{n \in \mathbb{N}} (\mathcal{Y}_1^{NO})^\otimes n$. Since $(Z, N)$ is directed, an application of Theorem 3.4.21 shows that $O_{E_2 \otimes_A O_{E_1}} \cong O_{\mathcal{Y}_1^{NO}} \cong NO_{\mathcal{Y}_1^{NO}} \cong NO_{Z}$.

By symmetry, $E_1 \otimes_A O_{E_2}$ has the structure of a Hilbert $O_{E_2}$-bimodule and $O_{E_1 \otimes_A O_{E_2}}$ is also isomorphic to $NO_{Z}$, which establishes the result. 

We now prove the second part of ([13], Lemma 4.1) with relaxed hypotheses — it is a special case of Theorem 3.5.10 for product systems over $\mathbb{N}^2$.

**Theorem 4.3.3.** Let $A$ be a $C^*$-algebra, $E_1$ and $E_2$ Hilbert $A$-bimodules, and $\chi : E_1 \otimes_A E_2 \to E_2 \otimes_A E_1$ an $A$-bimodule isomorphism. If $A$ acts faithfully and compactly on $E_2$, then

$$O_{E_2 \otimes_A O_{E_1}} \cong O_{\mathcal{Y}_1^{NO}} \cong NO_{\mathcal{Y}_1^{NO}} \cong NO_{Z}.$$  

*Proof.* We begin by showing that the product system $Z$ from Proposition 4.2.3 is compactly aligned. Since $A$ acts compactly on $E_2$, Proposition 2.1.22 says that $S \otimes_A id_{E_2} \in \mathcal{K}_A(E_1 \otimes_A E_2)$ for each $S \in \mathcal{K}_A(E_1)$. Hence, for any $S \in \mathcal{K}_A(E_1)$ and $T \in \mathcal{K}_A(E_2)$,

$$(S \otimes_A id_{E_2}) (\chi^{-1} \circ (T \otimes_A id_{E_1}) \circ \chi) \in \mathcal{K}_A(E_1 \otimes_A E_2),$$

and so Corollary 4.2.8 shows that $Z$ is compactly aligned.

Since $Z$ is compactly aligned, Propositions 4.2.10 and 4.2.11 show that $E_2 \otimes_A T_{E_1}$ and $\mathcal{Y}_1^{NO}$ are isomorphic as Hilbert $T_{E_1}$-bimodules. If we define $V := \bigsqcup_{n \in \mathbb{N}} E_2^\otimes n$, since $A$ acts faithfully on $E_2$, we can apply Proposition 4.3.1 (with the roles of $E_1$ and $E_2$ interchanged) to see that $E_1 \otimes_A O_{E_2}$ is isomorphic as a Hilbert $O_{E_2}$-bimodule to

$$W_1^{NO} = \operatorname{span} \left\{ j_{Z,(1,0)}(Z_{(1,0)}) \phi_{V}^{NO} (NO_{V}) \right\} \subseteq NO_{Z},$$
where \( \phi^O : N\mathcal{O}_V \to N\mathcal{O}_Z \) is the injective \( \ast \)-homomorphism satisfying \( \phi^O \circ j_V = j_Z \) (see the introduction to Section 3.5 for the details).

We again use the fact that \( A \) acts compactly on \( E_2 \), to see that \( A \) acts compactly on \( Z_{(0,n)} = E_2^\otimes_n \) for each \( n \in \mathbb{N} \). By Theorem 3.5.10, we see that

\[
\mathcal{O}_{E_2 \otimes A \mathcal{T}_{E_1}} \cong \mathcal{O}_{Y^T} \cong N\mathcal{O}_{Y^T} \cong N\mathcal{T}_{W^O} \cong \mathcal{T}_{W^O} \cong \mathcal{T}_{E_1 \otimes A \mathcal{T}_{E_2}},
\]

where \( Y^T := \bigsqcup_{n \in \mathbb{N}} (Y^T)^\otimes_n \) and \( W^O := \bigsqcup_{n \in \mathbb{N}} (Y^O)^\otimes_n \).

\[\square\]
Appendix A

A uniqueness theorem for Nica–Toeplitz algebras

In this appendix we provide a proof of the uniqueness theorem for Nica–Toeplitz algebras associated to compactly aligned product systems over quasi-lattice ordered groups. The result gives a sufficient condition for the induced representation $\psi^*$ of a Nica covariant representation $\psi$ (on a Hilbert space $\mathcal{H}$) to be faithful. This condition basically says that the ranges of all the operators $\{\psi_p(x) : x \in X_p, \ p \in P \setminus \{e\}\}$ should leave enough room in $\mathcal{H}$ for $A$ to act faithfully. When $A$ acts by compacts on each fibre of $X$, this condition is also necessary. The proof is strongly based on that of ([24], Theorem 7.2). However, unlike in Fowler’s original paper on product systems of Hilbert bimodules, we choose not to view the Nica–Toeplitz algebra as a subalgebra of a twisted semigroup crossed product. Moreover, we also remove the hypothesis that each fibre of the product system is essential (i.e. we do not require that $X_p = \Sigma \phi_p(A) X_p$ for each $p \in P$).

The first part of the proof shows that averaging over the canonical gauge coaction $\delta_X$ gives an expectation $E_{\delta_X}$ of $\mathcal{N}\mathcal{T}_X$ onto the generalised fixed-point algebra $\mathcal{N}\mathcal{T}^{\delta_X}_X$.

**Proposition A.1** ([55], Lemma 1.3). Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$ with coefficient algebra $A$. Then there exists a positive linear map $E_{\delta_X} : \mathcal{N}\mathcal{T}_X \to \mathcal{N}\mathcal{T}^{\delta_X}_X$ defined by $E_{\delta_X} := (\mathrm{id}_{\mathcal{N}\mathcal{T}_X} \otimes \rho) \circ \delta_X$, where $\rho : C^*(G) \to \mathbb{C}$ is the canonical trace.

**Remark A.2.** For any $p, q \in P$ and $x \in X_p$, $y \in X_q$, we have

$$E_{\delta_X} (i_{X_p}(x)i_{X_q}(y)^*) = (\mathrm{id}_{\mathcal{N}\mathcal{T}_X} \otimes \rho) (\delta_X (i_{X_p}(x)i_{X_q}(y)^*))$$

$$= (\mathrm{id}_{\mathcal{N}\mathcal{T}_X} \otimes \rho) (i_{X_p}(x)i_{X_q}(y)^* \otimes i_G (pq^{-1}))$$

$$= \delta_{pq^{-1},e} i_{X_p}(x)i_{X_q}(y)^*$$

$$= \delta_{p,q} i_{X_p}(x)i_{X_q}(y)^*.$$
Remark A.3. We are particularly interested in the situation where the expectation $E_{\delta_X}$ of Proposition A.1 is faithful on positive elements, i.e. $E_{\delta_X}(b^*b) = 0 \Rightarrow b = 0$ for any $b \in \mathcal{N}T_X$. Inspired by Definition 7.1 of [24], we say that a compactly aligned product system $X$ is amenable if $E_{\delta_X}$ is faithful on positive elements. The argument of ([39], Lemma 6.5) shows that if $G$ is an amenable group, then $X$ is automatically amenable.

The main step in the proof of the uniqueness theorem is to show that this expectation is also implemented spatially, i.e. there is a compatible expectation $E_{\psi}$ of $\psi^*(\mathcal{N}T_X)$ onto $\psi^*(\mathcal{N}T_X^\delta)$. To get this compatible expectation we need to be able to calculate the norms of elements of $\psi^*(\mathcal{N}T_X^\delta)$. To do this we will make use of the following well-known fact about operators on Hilbert spaces.

Proposition A.4. Let $\mathcal{H}$ be a Hilbert space. Suppose $T \in \mathcal{B}(\mathcal{H})$ and $P_1, \ldots, P_n \in \mathcal{B}(\mathcal{H})$ are mutually orthogonal projections that commute with $T$ and satisfy $\sum_{i=1}^n P_i = \text{id}_\mathcal{H}$. Then

$$\|T\|_{\mathcal{B}(\mathcal{H})} = \max \left\{ \|P_i T\|_{\mathcal{B}(\mathcal{H})} : 1 \leq i \leq n \right\}.$$ 

We now work towards showing that there exists a collection of mutually orthogonal projections in $\mathcal{B}(\mathcal{H})$ that decompose the identity and commute with everything in $\psi^*(\mathcal{N}T_X^\delta)$. Firstly, we fix some notation.

Definition A.5. Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$ with coefficient algebra $A$. Let $\psi : X \to \mathcal{B}(\mathcal{H})$ be a Nica covariant representation of $X$ on a Hilbert space $\mathcal{H}$ (see Definition 3.1.21). We define a collection $\{P^\psi_p : p \in P\}$ of projections in $\mathcal{B}(\mathcal{H})$ by $P^\psi_e := \text{id}_\mathcal{H}$, $P^\psi_\infty := 0$, and $P^\psi_p := \text{proj}_{\psi_p(X_p)\mathcal{H}}$ for each $p \in P \setminus \{e\}$.

We now show that the $*$-homomorphism $\psi^{(p)} : \mathcal{K}_A(X_p) \to \mathcal{B}(\mathcal{H})$ (see Remark 3.1.10) has a canonical extension to all of $\mathcal{L}_A(X_p)$ (for each $p \in P$), and establish some properties of this extension.

Proposition A.6. Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$ with coefficient algebra $A$. Let $\psi : X \to \mathcal{B}(\mathcal{H})$ be a Nica covariant representation of $X$ on a Hilbert space $\mathcal{H}$. Then

(i) For each $p \in P$, there exists a representation $\rho^\psi_p : \mathcal{L}_A(X_p) \to \mathcal{B}(\mathcal{H})$ such that for each $S \in \mathcal{L}_A(X_p)$

$$\rho^\psi_p(S)(\psi_p(x)h) = \psi_p(Sx)h \quad \text{for each } x \in X_p, \ h \in \mathcal{H}$$

and $\rho^\psi_p(S)$ is zero on $(\psi_p(X_p)\mathcal{H})^\perp$. 

(ii) \( \rho_p^\psi|_{\mathcal{K}_A(\mathbf{x}_p)} = \psi^{(p)} \).

(iii) For any \( q \in P \),
\[
\rho_q^\psi (\iota_e(a)) = \rho_e^\psi (a) P_q^\psi
\]
for any \( a \in A \cong \mathcal{K}_A(\mathbf{x}_e) \). Furthermore, if \( p \in P \setminus \{ e \} \), then
\[
\rho_p^\psi (\iota_p^q(S)) = \rho_p^\psi (S) P_p^\psi
\]
for any and \( S \in \mathcal{L}_A(\mathbf{x}_p) \).

(iv) If \( \mathcal{K} \subseteq \mathcal{H} \) is a \( \psi_e \)-invariant subspace of \( \mathcal{H} \), then the subspace \( M := \psi_p(\mathbf{x}_p) \mathcal{K} \) is \( \rho_p^\psi \)-invariant. Furthermore, if \( \psi_e|_{\mathcal{K}} \) is faithful, then \( \rho_p^\psi|_{\mathcal{M}} \) is also faithful.

Proof. Observe that for any \( p \in P \) and \( x, y \in \mathbf{x}_p \) and \( h, k \in \mathcal{H} \), we have
\[
\langle \psi_p(x)h, \psi_p(y)k \rangle_C = \langle h, \psi_p(x)^* \psi_p(y)k \rangle_C = \langle h, \psi_e(\langle x, y \rangle_A)k \rangle_C = \langle x \otimes_A h, y \otimes_A k \rangle_C.
\]
(A.1)

Thus, there exists a linear isometry \( U : \mathbf{x}_p \otimes_A \mathcal{H} \to \mathcal{H} \) such that
\[
U (x \otimes_A h) = \psi_p(x)h
\]
for each \( x \in \mathbf{x}_p, h \in \mathcal{H} \). Equation (A.1) shows that \( U^* (\psi_p(x)h) = x \otimes_A h \) for each \( x \in \mathbf{x}_p \) and \( h \in \mathcal{H} \). We claim that \( U^*|_{(\psi_p(\mathbf{x}_p) \mathcal{H})^\perp} = 0 \). To see this, observe that for any \( f \in (\psi_p(\mathbf{x}_p) \mathcal{H})^\perp \) and \( y \in \mathbf{x}_p, h \in \mathcal{H} \) we have
\[
\langle U(y \otimes_A k), f \rangle_C = \langle \psi_p(y)k, f \rangle_C = 0,
\]
and hence \( U^*(f) = 0 \). Since
\[
\mathbf{x}_p \otimes_A \mathcal{H} = (\mathbf{x}_p \cdot A) \otimes_A \mathcal{H} = \mathbf{x}_p \otimes_A \overline{\psi_e(A)} \mathcal{H},
\]
we may assume that the representation \( \psi_e \) is nondegenerate without loss of generality. With this in mind, define \( \rho_p^\psi : \mathcal{L}_A(\mathbf{x}_p) \to \mathcal{B}(\mathcal{H}) \) by
\[
\rho_p^\psi(S) := U \circ \left( \mathbf{x}_p \text{Ind}_A^\mathcal{L}_A(\mathbf{x}_p) \psi_e(S) \right) \circ U^*
\]
for each \( S \in \mathcal{L}_A(\mathbf{x}_p) \). Thus, for each \( S \in \mathcal{L}_A(\mathbf{x}_p) \), \( \rho_p^\psi(S)|_{(\psi_p(\mathbf{x}_p) \mathcal{H})^\perp} = 0 \), whilst for
any \( x \in X_p \) and \( h \in \mathcal{H} \) we have
\[
\rho_p^\psi(S)(\psi_p(x)h) = \left( U \circ \left( X_p \text{-Ind}_{A}^{C}(X_p)\psi_c(S) \right) \circ U^* \right)(\psi_p(x)h)
\]
\[
= \left( U \circ \left( X_p \text{-Ind}_{A}^{C}(X_p)\psi_c(S) \right) \right)(x \otimes_A h)
\]
\[
= U(Sx \otimes_A h)
\]
\[
= \psi_p(Sx)h.
\]

This completes the proof of part (i).

Since both \( \psi^{(p)} \) and \( \rho_p^\psi \) are *-homomorphisms, to prove (ii) it suffices to show that \( \psi^{(p)} \) and \( \rho_p^\psi \) agree on rank one operators. Fix \( x, y \in X_p \). Firstly, we check that \( \psi^{(p)}(\Theta_{x,y}) \) and \( \rho_p^\psi(\Theta_{x,y}) \) agree on \( \psi_p(X_p)\mathcal{H} \). For any \( z \in X_p \) and \( h \in \mathcal{H} \), we have
\[
\rho_p^\psi(\Theta_{x,y})(\psi_p(z)h) = \psi_p(\Theta_{x,y}(z))h = \psi_p(x \cdot (y,z)_A)h
\]
\[
= \psi_p(x)\psi_p(y)\psi_p(z)h = \psi^{(p)}(\Theta_{x,y})(\psi_p(z)h).
\]

Since both \( \psi^{(p)}(\Theta_{x,y}) \) and \( \rho_p^\psi(\Theta_{x,y}) \) are linear and continuous, we conclude that they agree on \( \psi_p(X_p)\mathcal{H} \). It remains to check that \( \psi^{(p)}(\Theta_{x,y}) \) and \( \rho_p^\psi(\Theta_{x,y}) \) agree on \( (\psi_p(X_p)\mathcal{H})^\perp \). By part (i), we know that \( \psi^{(p)}(\Theta_{x,y})|_{(\psi_p(X_p)\mathcal{H})^\perp} = 0 \). Since
\[
\langle \psi^{(p)}(\Theta_{x,y})h,k \rangle_C = \langle \psi_p(x)\psi_p(y)\psi_p(z)h,\psi_p(x)k \rangle_C = \langle h, \psi_p(y)\psi_p(x)^*k \rangle_C = 0
\]
for any \( h \in (\psi_p(X_p)\mathcal{H})^\perp \), \( k \in \mathcal{H} \), we conclude that \( \psi^{(p)}(\Theta_{x,y})|_{(\psi_p(X_p)\mathcal{H})^\perp} = 0 \) as well. This completes the proof of part (ii).

We now prove part (iii). Let \( q \in P \) and \( a \in A \). If \( q = e \), then \( P_q^\psi = \text{id}_\mathcal{H} \), and so
\[
\rho_c^\psi(a)P_q = \rho_c^\psi(a) = \rho_c^\psi(i_c^q(a)) = \rho_c^\psi(i_c^q(a)).
\]

On the other hand, if \( q \neq e \), then \( P_q^\psi = \text{proj}_{\psi_q(X_q)\mathcal{H}} \). Hence, both \( \rho_c^\psi(a)P_q \) and \( \rho_c^\psi(i_c^q(a)) \) are zero on \( (\psi_q(X_q)\mathcal{H})^\perp \). Since \( \rho_c^\psi(i_c^q(a)) \) and \( \rho_c^\psi(a)P_q^\psi \) are linear and continuous, whilst
\[
\rho_q^\psi(i_c^q(a)) = \psi_q(i_c^q(a)x)h = \psi_q(a \cdot x)h = \rho_q^\psi(a)P_q^\psi \psi_q(x)h
\]
for any \( x \in X_q, h \in \mathcal{H} \), we conclude that \( \rho_q^\psi(i_c^q(a)) \) and \( \rho_c^\psi(a)P_q^\psi \) also agree on \( \psi_q(X_q)\mathcal{H} \). Thus, \( \rho_q^\psi(i_c^q(a)) = \rho_c^\psi(a)P_q^\psi \).

Now fix \( p \in P \setminus \{e\} \) and \( S \in \mathcal{L}_A(X_p) \). Since \( pq \neq e \), both \( \rho_{pq}^\psi(P_{pq}^\psi(S)) \) and \( \rho_p^\psi(S)P_{pq}^\psi \) are zero on the orthogonal complement \( (\psi_{pq}(X_{pq})\mathcal{H})^\perp \). Observe that for
any \( x \in X_p, \, y \in X_q, \) and \( h \in \mathcal{H}, \) we have

\[
\rho^\psi_{pq} (\iota^{pq}_p(S)) \psi_{pq}(xy) h = \psi_{pq} (\iota^{pq}_p(S)(xy)) h = \psi_{pq} ((Sx)y) h
\]

= \rho^\psi_p(S) \psi_p(y) h

= \rho^\psi_p(S) \rho^\psi_q(S) P_{pq}^\psi (\psi_{pq}(xy)) h.

Since \( \psi_{pq}(X_{pq}) \mathcal{H} = \psi_p(X_p) \psi_q(X_q) \mathcal{H}, \) whilst \( \rho^\psi_{pq}(\iota^{pq}_p(S)) \) and \( \rho^\psi_p(S) P_{pq}^\psi \) are linear and continuous, we conclude that \( \rho^\psi_{pq}(\iota^{pq}_p(S)) \) and \( \rho^\psi_p(S) P_{pq}^\psi \) are also equal on \( \psi_{pq}(X_{pq}) \mathcal{H}. \)

Finally, we prove part (iv). Firstly, observe that \( \mathcal{M} \) is \( \rho^\psi_p \) invariant, since \( \rho^\psi_p(S) (\psi_p(x)k) = \psi_p(Sx)k \in \mathcal{M} \) for any \( S \in \mathcal{L}_A(X_p), \, x \in X_p, \) and \( k \in K. \) Now suppose that \( \psi_e|_K \) is faithful. Since \( \mathcal{L}_A(X_p) \) acts faithfully on \( X_p, \) the induced representation \( X_p \text{Ind}_{A}^{\mathcal{L}_A(X_p)}(\psi_e|_K) : \mathcal{L}_A(X_p) \to \mathcal{B}(X_p \otimes_A K) \) is faithful by ([51], Corollary 2.74). Since \( U \) implements a unitary equivalence between \( X_p \text{Ind}_{A}^{\mathcal{L}_A(X_p)}(\psi_e|_K) \) and \( \rho^\psi_p|_\mathcal{M}, \) and unitary equivalence preserves the faithfulness of representations, we conclude that \( \rho^\psi_e|_\mathcal{M} \) is faithful.

We will use the projections defined in Definition A.5 to construct a collection of mutually orthogonal projections in \( \mathcal{B}(\mathcal{H}) \) that decompose the identity and commute with everything in \( \psi_*(\mathcal{N}^T_{X} \mathcal{K}). \) Before we do this, we need to know what the product of projections in \( \{ P^\psi_p : p \in P \} \) look like.

**Proposition A.7.** Let \( (G, P) \) be a quasi-lattice ordered group and \( X \) a compactly aligned product system over \( P \) with coefficient algebra \( A. \) Let \( \psi : X \to \mathcal{B}(\mathcal{H}) \) be a Nica covariant representation of \( X \) on a Hilbert space \( \mathcal{H}. \) Then for each \( p, q \in P, \)

\[
P^\psi_p P^\psi_q = P^\psi_{p \wedge q}.
\]

In particular, the projections \( \{ P^\psi_p : p \in P \} \) commute.

**Proof.** Firstly, observe that part (i) of Proposition A.6 implies that \( P^\psi_p = \rho^\psi_p(\text{id}_{X_p}) \) for any \( p \in P \setminus \{ e \}. \)

Next, we show that if \( p \in P \) and \( (e_i)_{i \in I} \) is the canonical approximate identity for the \( C^* \)-algebra \( K_A(X_p), \) then

(i) \( \lim_{i \in I} (e_i x) = x \) for each \( x \in X_p; \)

(ii) \( \lim_{i \in I} (\rho^\psi_p(e_i)) = \rho^\psi_p(\text{id}_{X_p}) \) (converging in the strong operator topology).
To see (i), fix \(x \in X_p\) and \(\varepsilon > 0\). Choose \(x' \in X_p\) so that \(x = x' \cdot \langle x', x' \rangle A\) by the Hewitt–Cohen–Blanchard factorisation theorem. Choose \(i \in I\) such that for all \(j \geq i\),
\[
\left\| e_j\Theta_{x', x'} - \Theta_{x', x'} \right\|_{K_A(X_p)} < \frac{\varepsilon}{\|x\|_{X_p} + 1}.
\]
Thus, for all \(j \geq i\), we have
\[
\left\| e_j x - x \right\|_{X_p} = \left\| e_j x' \cdot \langle x', x' \rangle A - x' \cdot \langle x', x' \rangle A \right\|_{X_p} \\
\leq \left\| e_j\Theta_{x', x'} - \Theta_{x', x'} \right\|_{K_A(X_p)} \left\| x' \right\|_{X_p} < \varepsilon.
\]
Since \(\varepsilon > 0\) was arbitrary, we conclude that \(\lim_{i \in I} (e_i x) = x\) for each \(x \in X_p\).

We now move on to proving (ii). Fix \(h \in H\) and \(\varepsilon > 0\). If \(h \in \psi_p(X_p)H\perp\), then
\[
\rho_p^\psi(e_i)h = 0 = \rho_p^\psi(id_{X_p})h
\]
for each \(i \in I\). Thus, \(\lim_{i \in I} \left\| \rho_p^\psi(e_i)h - \rho_p^\psi(id_{X_p})h \right\|_H = 0\). On the other hand, if \(h \in \psi_p(X_p)H\perp\), then we can choose \(x_1, \ldots, x_n \in X_p\) and \(h_1, \ldots, h_n \in H\) such that
\[
\left\| h - \sum_{i=1}^n \psi_p(x_i)h_i \right\|_H < \frac{\varepsilon}{4}.
\]
Since \(\left\| e_i \right\|_{K_A(X_p)} \leq 1\) for each \(i \in I\) and \(\rho_p^\psi\) is norm-decreasing, we see that
\[
\left\| \rho_p^\psi(e_i - id_{X_p}) \left( h - \sum_{i=1}^n \psi_p(x_i)h_i \right) \right\|_H \\
\leq \left\| \rho_p^\psi(e_i - id_{X_p}) \right\|_{B(H)} \left\| h - \sum_{i=1}^n \psi_p(x_i)h_i \right\|_H \\
\leq \left\| e_i - id_{X_p} \right\|_{K_A(X_p)} \left\| h - \sum_{i=1}^n \psi_p(x_i)h_i \right\|_H \\
\leq 2 \left\| h - \sum_{i=1}^n \psi_p(x_i)h_i \right\|_H \\
< \frac{\varepsilon}{2}.
\]
By (i), for each \(1 \leq i \leq n\), we can choose \(j_i \in I\) such that whenever \(k \geq j_i\),
\[
\left\| e_k x_i - x_i \right\|_{X_p} < \frac{\varepsilon}{2n \max_{1 \leq i \leq n} \|h_i\|_H + 1}.
\]
As \(I\) is directed, we can choose \(m \in I\) such that \(m \geq j_i\) for each \(1 \leq i \leq n\). Since
\( \psi_p \) is norm-decreasing by Proposition 3.1.9, we see that for any \( k \geq m \),

\[
\left\| \rho_p^\psi (e_k - \text{id}_{X_p}) \left( \sum_{i=1}^n \psi_p(x_i) h_i \right) \right\|_\mathcal{H} = \left\| \sum_{i=1}^n \psi_p \left( (e_k - \text{id}_{X_p}) x \right) h_i \right\|_\mathcal{H} \\
\leq \sum_{i=1}^n \left\| \psi_p \left( (e_k - \text{id}_{X_p}) x \right) \right\|_{\mathcal{B}(\mathcal{H})} \| h_i \|_\mathcal{H} \\
\leq \sum_{i=1}^n \left\| (e_k - \text{id}_{X_p}) x \right\|_{X_p} \| h_i \|_\mathcal{H} \\
< \sum_{i=1}^n \frac{\varepsilon}{2n} (\max_{1 \leq i \leq n} \| h_i \|_\mathcal{H} + 1) \\
< \varepsilon.
\]

Thus, for each \( k \geq m \),

\[
\left\| \rho_p^\psi (e_k) h - \rho_p^\psi \left( \text{id}_{X_p} \right) h \right\|_\mathcal{H} = \left\| \rho_p^\psi (e_k - \text{id}_{X_p}) h \right\|_\mathcal{H} \\
\leq \left\| \rho_p^\psi (e_k - \text{id}_{X_p}) \left( h - \sum_{i=1}^n \psi_p(x_i) h_i \right) \right\|_\mathcal{H} + \left\| \rho_p^\psi (e_k - \text{id}_{X_p}) \left( \sum_{i=1}^n \psi_p(x_i) h_i \right) \right\|_\mathcal{H} \\
< \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, we conclude that \( \lim_{i \in I} \left\| \rho_p^\psi (e_i) h - \rho_p^\psi \left( \text{id}_{X_p} \right) h \right\|_\mathcal{H} = 0 \) for each \( h \in \mathcal{H} \). Thus, \( \lim_{i \in I} \rho_p^\psi (e_i) = \rho_p^\psi \left( \text{id}_{X_p} \right) \) in the strong operator topology.

Finally, we are ready to prove that \( P_p^\psi P_q^\psi = P_{p,q}^\psi \) for every \( p, q \in P \). Since \( P_e^\psi = \text{id}_{\mathcal{H}} \), the result is trivial when \( p = e \) or \( q = e \). Thus, we may as well suppose that \( p, q \neq e \). Let \( (e_i)_{i \in I} \) and \( (f_j)_{j \in J} \) be the canonical approximate identities for \( \mathcal{K}_A(X_p) \) and \( \mathcal{K}_A(X_q) \) respectively. Then for any \( i \in I \) and \( j \in J \), Proposition A.6 and the Nica covariance of \( \psi \) tells us that

\[
\rho_p^\psi (e_i) \rho_q^\psi (f_j) = \psi^{(p)} (e_i) \psi^{(q)} (f_j) = \begin{cases} 
\psi^{(p\lor q)} (i_p^{p\lor q}(e_i) i_q^{p\lor q}(f_j)) & \text{if } p \lor q < \infty \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
\rho_{p\lor q}^\psi (i_p^{p\lor q}(e_i) i_q^{p\lor q}(f_j)) & \text{if } p \lor q < \infty \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
\rho_{p\lor q}^\psi (i_p^{p\lor q}(e_i) i_q^{p\lor q}(f_j)) \rho_{p\lor q}^\psi (f_j) & \text{if } p \lor q < \infty \\
0 & \text{otherwise}
\end{cases}
\]

\[
= \begin{cases} 
\rho_p^\psi (e_i) P_{p\lor q}^\psi (f_j) P_{p\lor q}^\psi & \text{if } p \lor q < \infty \\
0 & \text{otherwise}
\end{cases}
\]
Hence, by (ii), we have

\[
P_p^\psi P_q^\psi = \rho_p^\psi (\text{id}_{X_p}) \rho_q^\psi (\text{id}_{X_q}) = \begin{cases} 
\rho_p^\psi (\text{id}_{X_p}) P_{p \vee q}^\psi \rho_q^\psi (\text{id}_{X_q}) & \text{if } p \vee q < \infty \\
0 & \text{otherwise} 
\end{cases}
\]

\[
= \begin{cases} 
\rho_{p \vee q}^\psi (\text{id}_{X_p}) \rho_{p \vee q}^\psi (\text{id}_{X_q}) & \text{if } p \vee q < \infty \\
0 & \text{otherwise} 
\end{cases}
\]

\[
= \begin{cases} 
\rho_{p \vee q}^\psi (\text{id}_{X_{p \vee q}}) & \text{if } p \vee q < \infty \\
0 & \text{otherwise} 
\end{cases}
\]

\[
= \begin{cases} 
\rho_{p \vee q}^\psi & \text{if } p \vee q < \infty \\
0 & \text{otherwise.} 
\end{cases}
\]

Thus, \(P_p^\psi P_q^\psi = P_{p \vee q}^\psi\) for each \(p, q \in P\). \(\Box\)

For the next definition, recall Remark 3.1.15 — given a quasi-lattice ordered group \((G, P)\), and a finite set \(C \subseteq P\), we define \(\bigvee C\) to be the least upper bound of \(C\) (with the convention that \(\bigvee \emptyset = e\)).

**Definition A.8.** Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\) with coefficient algebra \(A\). Let \(\psi : X \to \mathcal{B}(\mathcal{H})\) be a Nica covariant representation of \(X\) on a Hilbert space \(\mathcal{H}\). Let \(F \subseteq P\) be finite. For each \(C \subseteq F\), define

\[
Q_{C,F}^\psi := P_{\bigvee C}^\psi \prod_{p \in F \setminus C} (\text{id}_{\mathcal{H}} - P_p^\psi),
\]

where, by convention, the product over the empty set is \(\text{id}_{\mathcal{H}}\).

Whilst we have defined the projections \(Q_{C,F}^\psi\), for every subset \(C\) of \(F\), we are particularly interested in the projections corresponding to so called initial segments of \(F\).

**Definition A.9.** Let \((G, P)\) be a quasi-lattice ordered group. Fix a finite set \(F \subseteq P\). We say that a subset \(C \subseteq F\) is an initial segment of \(F\) if \(\bigvee C < \infty\) and \(C = \{t \in F : t \leq \bigvee C\}\).

The next result shows how the projections \(Q_{C,F}^\psi\) and \(\{P_p^\psi : p \in P\}\) interact with the operators \(\{\psi_p(x) : p \in P, x \in X_p\}\).

**Lemma A.10.** Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\) with coefficient algebra \(A\). Let \(\psi : X \to \mathcal{B}(\mathcal{H})\) be a Nica covariant representation of \(X\) on a Hilbert space \(\mathcal{H}\).
(i) Let \( p, q \in P \) and \( x \in X_p \). Then

\[
P^\psi_{p,q} \psi_p(x) = \begin{cases} 
\psi_p(x)P^\psi_{p-1(p\lor q)} & \text{if } p \lor q < \infty \\
0 & \text{otherwise.}
\end{cases}
\]

(ii) If \( F \subseteq P \) is finite and \( p \in F \), then

\[
Q^\psi_{C,F} \psi_p(x) = \begin{cases} 
Q^\psi_{C,F} \psi_p(x)P^\psi_{p-1(\forall C)} & \text{if } p \leq \forall C \\
0 & \text{otherwise}
\end{cases}
\]

for any initial segment \( C \) of \( F \).

Proof. Fix \( p, q \in P \) and \( x \in X_p \). If \( p \lor q = \infty \), then Lemma 3.1.22 tells us that for any \( y \in X_q \), we have \( \psi_q(y)^* \psi_p(x) = 0 \). Hence, for any \( h, g \in \mathcal{H} \), it follows that

\[
\langle \psi_q(y)h, \psi_p(x)g \rangle_C = (h, \psi_q(y)^* \psi_p(x)g)_C = 0.
\]

Thus, \( \psi_p(X_p)\mathcal{H} \subseteq (\psi_q(X_q)\mathcal{H})^\perp \), and so \( P^\psi_{p,q} \psi_p(x) = 0 \).

Now suppose that \( p \lor q < \infty \). If \( p = p \lor q \), then \( p \geq q \), and so

\[
P^\psi_{p,q} \psi_p(x) = \psi_p(x) = \psi_p(x)P^\psi_e = \psi_p(x)P^\psi_{p-1(p \lor q)}.
\]

If \( p \neq p \lor q \), then for any \( y \in X_{p-1(p \lor q)} \), \( h \in \mathcal{H} \), we have

\[
\psi_p(x)P^\psi_{p-1(p \lor q)} \psi_{p-1(p \lor q)}(y)h = \psi_p(x)\psi_{p-1(p \lor q)}(y)h = \psi_{p \lor q}(xy)h
\]

\[= P^\psi_{p,q} \psi_{p \lor q}(xy)h
\]

\[= P^\psi_{p,q} \psi_p(x)\psi_{p-1(p \lor q)}(y)h.
\]

Consequently, \( \psi_p(x)P^\psi_{p-1(p \lor q)} \) and \( P^\psi_{p,q} \psi_p(x) \) agree on \( \overline{\psi_{p-1(p \lor q)}(X_{p-1(p \lor q)})\mathcal{H}} \). It remains to check that they agree on the complement \( (\psi_{p-1(p \lor q)}(X_{p-1(p \lor q)})\mathcal{H})^\perp \). Let \( f \in (\psi_{p-1(p \lor q)}(X_{p-1(p \lor q)})\mathcal{H})^\perp \). We need to show that \( P^\psi_{p,q} \psi_p(x)f = 0 \). It suffices to show that \( \psi_p(x)f \in (\psi_q(X_q)\mathcal{H})^\perp : \) for any \( y \in X_q \) and \( h \in \mathcal{H} \), we have

\[
\langle \psi_p(x)f, \psi_q(y)h \rangle_C = \langle f, \psi_p(x)^* \psi_q(y)h \rangle_C
\]

\[
\in \langle f, \text{span}\{\psi_{p-1(p \lor q)}(X_{p-1(p \lor q)})\psi_{q-1(p \lor q)}(X_{q-1(p \lor q)})^*\} \rangle_C
\]

\[
\subseteq \langle f, \psi_{p-1(p \lor q)}(X_{p-1(p \lor q)})\mathcal{H} \rangle_C
\]

\[= \{0\}.
\]

This completes the proof of (i).

We now prove part (ii). Fix a finite set \( F \subseteq P \) with \( p \in F \). Let \( C \) be an initial
segment of $F$. If $p \leq \bigvee C$, then $p \vee (\bigvee C) = \bigvee C < \infty$. By part (i) it follows that
\[
Q_{C,F}^{\psi} \psi_p(x) = Q_{C,F}^{\psi} P_{C,F}^{\psi} \psi_p(x) = Q_{C,F}^{\psi} \psi_p(x) P_{P^{-1}(\bigvee C)}^{\psi} = Q_{C,F}^{\psi} \psi_p(x) P_{P^{-1}(\bigvee C)}^{\psi}.
\]
On the other hand, suppose that $p \not\leq \bigvee C$. Since $p \in F$, this implies that $C \not= F$. Moreover, since $C$ is an initial segment of $F$, this forces $p \in F \setminus C$. Therefore,
\[
Q_{C,F}^{\psi} \psi_p(x) = Q_{C,F}^{\psi} (iden_H - P_p^{\psi}) \psi_p(x) = 0.
\]
This completes the proof of part (ii).

Using the previous result we can show that the projections $\{P_p^{\psi} : p \in P\}$ and $\{Q_{C,F}^{\psi} : C \subseteq F\}$ commute with every element of $\psi_*(\mathcal{N}\mathcal{T}_{X}^{\delta_X})$.

**Proposition A.11.** For any $q \in P$, the projection $P_q^{\psi}$ commutes with every element of $\psi_*(\mathcal{N}\mathcal{T}_{X}^{\delta_X})$. In particular, if $F \subseteq P$ is finite and $C \subseteq F$, then $Q_{C,F}^{\psi}$ commutes with every element of $\psi_*(\mathcal{N}\mathcal{T}_{X}^{\delta_X})$.

**Proof.** Since $\psi_*(\mathcal{N}\mathcal{T}_{X}^{\delta_X}) = \text{span}\{\psi_p(X_p) \psi_p(X_p)^*\}$, it suffices to show that
\[
P_q^{\psi} \psi_p(x) \psi_p(y)^* = \psi_p(x) \psi_p(y)^* P_q^{\psi}
\]
for each $x, y \in X_p$. Via two applications of Lemma A.10, we see that
\[
P_q^{\psi} \psi_p(x) \psi_p(y)^* = \begin{cases} 
\psi_p(x) P_{P^{-1}(p \vee q)}^{\psi} \psi_p(y)^* & \text{if } p \vee q < \infty \\
0 & \text{otherwise}
\end{cases}
\]
\[
= \begin{cases} 
\psi_p(x) \left( \psi_p(y) P_{P^{-1}(p \vee q)}^{\psi} \right)^* & \text{if } p \vee q < \infty \\
0 & \text{otherwise}
\end{cases}
\]
\[
= \psi_p(x) \left( P_q^{\psi} \psi_p(y) \right)^* \\
= \psi_p(x) \psi_p(y)^* P_q^{\psi},
\]
as required. \qed

We now show that the projections $\{Q_{C,F}^{\psi} : C \text{ is an initial segment of } F\}$ are mutually orthogonal and decompose the identity operator on $\mathcal{H}$.

**Proposition A.12.** Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$ with coefficient algebra $A$. Let $\psi : X \to \mathcal{B}(\mathcal{H})$ be a Nica covariant representation of $X$. Let $F \subseteq P$ be finite. Then

(i) if $C \subseteq F$ is not an initial segment of $F$, then $Q_{C,F}^{\psi} = 0$;
(ii) \( \{ Q^\psi_{C,F} : C \subseteq F \} \) is a decomposition of the identity on \( \mathcal{H} \) into mutually orthogonal projections.

**Proof.** Suppose \( C \subseteq F \) is not an initial segment of \( F \). If \( \bigvee C = \infty \), then \( Q^\psi_{C,F} = Q^\psi_{C,F} P^\psi_{\bigvee C} = Q^\psi_{C,F} P^\psi_{\infty} = 0 \).

Alternatively, \( \bigvee C < \infty \) and \( C \neq \{ t \in F : t \leq \bigvee C \} \). Thus, \( C \neq F \). Choose \( t \in F \setminus C \) with \( t \leq \bigvee C \). Since \( t \lor (\bigvee C) = \bigvee C \), we see that

\[
Q^\psi_{C,F} = Q^\psi_{C,F} P^\psi_{\bigvee C} (\text{id}_\mathcal{H} - P^\psi_t) = Q^\psi_{C,F} \left( P^\psi_{\bigvee C} - P^\psi_{\bigvee C \lor (\bigvee C)} \right)
\]

Thus, \( Q^\psi_{C,F} = 0 \), which proves part (i).

We now prove part (ii). Since \( Q^\psi_{C,F} = 0 \) whenever \( C \) is not an initial segment of \( F \), it suffices to show that \( \{ Q^\psi_{C,F} : C \subseteq F \} \) is a decomposition of the identity into mutually orthogonal projections. Firstly, we show orthogonality. Suppose \( C,D \subseteq F \) are distinct. Without loss of generality, we may assume that \( D \setminus C \neq \emptyset \). Thus, \( C \neq F \) and we can choose \( t \in D \setminus C \). Since \( t \lor (\bigvee D) = \bigvee D \), we have

\[
Q^\psi_{C,D} Q^\psi_{D,F} = Q^\psi_{C,F} P^\psi_{\bigvee C} (\text{id}_\mathcal{H} - P^\psi_t) P^\psi_{\bigvee D} Q^\psi_{D,F}
\]

It remains to check that \( \sum_{C \subseteq F} Q^\psi_{C,F} = \text{id}_\mathcal{H} \). To prove this, we will use induction on \( |F| \). When \( |F| = 0 \) we have

\[
\sum_{C \subseteq F} Q^\psi_{C,F} = Q^\psi_{\emptyset,\emptyset} = P^\psi_{\emptyset} = P^\psi_e = \text{id}_\mathcal{H}.
\]

Now let \( n \geq 0 \) and suppose that \( \sum_{C \subseteq F} Q^\psi_{C,F} = \text{id}_\mathcal{H} \) whenever \( F \subseteq P \) and \( |F| = n \).
Fix $F' \subseteq P$ with $|F'| = n + 1$. Then, for any $y \in F'$, we have
\[
\sum_{C \subseteq F'} Q^\psi_{C,F'} = \sum_{C \subseteq F', y \in C} Q^\psi_{C,F'} + \sum_{C \subseteq F', y \notin C} Q^\psi_{C,F'} = \sum_{C \subseteq F', y \in C} P^\psi_C \prod_{p \in F' \setminus C} (\text{id}_H - P^\psi_p) + \sum_{C \subseteq F', y \notin C} P^\psi_C \prod_{p \in F' \setminus C} (\text{id}_H - P^\psi_p)
\]
\[
= \sum_{C \subseteq F' \setminus \{y\}} P^\psi_C \prod_{p \in (F' \setminus \{y\}) \setminus C} (\text{id}_H - P^\psi_p)
\]
\[
\sum_{C \subseteq F' \setminus \{y\}} P^\psi_C \prod_{p \in (F' \setminus \{y\}) \setminus C} (\text{id}_H - P^\psi_p)
\]
\[
= (P^\psi_y + (\text{id}_H - P^\psi_y)) \sum_{C \subseteq F' \setminus \{y\}} P^\psi_C \prod_{p \in (F' \setminus \{y\}) \setminus C} (\text{id}_H - P^\psi_p)
\]
\[
= \sum_{C \subseteq F' \setminus \{y\}} P^\psi_C \prod_{p \in (F' \setminus \{y\}) \setminus C} (\text{id}_H - P^\psi_p)
\]
\[
= \sum_{C \subseteq F' \setminus \{y\}} Q^\psi_{C,F' \setminus \{y\}} = \text{id}_H,
\]
where the last equality follows from applying the inductive hypothesis to $F' \setminus \{y\}$. □

Putting these results together we get an expression for the norm of an element
in $\psi_*(\mathcal{N}T^\psi_X)$ that doesn’t depend on the representation $\psi$.

**Lemma A.13.** Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned
product system over $P$ with coefficient algebra $A$. Let $\psi : X \to B(H)$ be a Nica
covariant representation of $X$ on a Hilbert space $H$. Then for any finite sum $Z := \sum_k \psi_{p_k}(x_k)\psi_{p_k}(y_k)^* \in \psi_*(\mathcal{N}T^\psi_X)$, we have
\[
\|Z\|_{B(H)} = \max \left\{ \left\| Q^\psi_{C,F \setminus C} \left( \sum_k \lambda^\psi_{p_k}(\Theta_{x_k,y_k}) \right) \right\|_{B(H)} : C \text{ is an initial segment of } F \right\}
\]
for any finite set $F \subseteq P$ containing each $p_k$. Furthermore, if for any finite set
$K \subseteq P \setminus \{e\}$, the representation
\[
A \ni a \mapsto \psi_e(a) \prod_{t \in K} (\text{id}_H - P^\psi_t) \in B(H)
\]
is faithful, then
\[ \|Z\|_{B(H)} = \max \left\{ \left\| \sum_k l_{p_k}^C (\Theta_{x_k,y_k}) \right\|_{\mathcal{L}_A(X\vee C)} : C \text{ is an initial segment of } F \right\}. \]

Proof. Let \( F \) be a finite subset of \( P \) containing each \( p_k \). Since \( Q_{C,F}^\psi \) commutes with \( Z \) for each \( C \subseteq F \) (by Proposition A.11) and \( \{Q_{C,F}^\psi : C \text{ is an initial segment of } F\} \) is an orthogonal decomposition of the identity, Proposition A.4 tells us that
\[ \|Z\|_{B(H)} = \max \left\{ \left\| Q_{C,F}^\psi Z \right\|_{B(H)} : C \text{ is an initial segment of } F \right\}. \]

However, for any initial segment \( C \) of \( F \), we see that
\[
Q_{C,F}^\psi Z = Q_{C,F}^\psi \sum_k \psi_{p_k}(x_k)\psi_{p_k}(y_k)^* \\
= Q_{C,F}^\psi \sum_{k:p_k \leq \vee C} \psi_{p_k}(x_k)\psi_{p_k}(y_k)^* P_{\vee C}^{\psi} \\
\text{by Lemma A.10 and Proposition A.11} \\
= Q_{C,F}^\psi \sum_{k:p_k \leq \vee C} \rho_{p_k}^\psi (\Theta_{x_k,y_k}) P_{\vee C}^{\psi} \\
= Q_{C,F}^\psi \sum_{k:p_k \leq \vee C} \rho_{\vee C}^\psi (l_{p_k}^C (\Theta_{x_k,y_k})) \\
= Q_{C,F}^\psi \rho_{\vee C}^\psi \left( \sum_k l_{p_k}^C (\Theta_{x_k,y_k}) \right).
\]

Now suppose that for any finite set \( K \subseteq P \setminus \{e\} \), the representation
\[ A \ni a \mapsto \psi_e(a) \prod_{t \in K} \left( \text{id}_H - P_t^{\psi} \right) \in B(H) \]
is faithful. To complete the proof we will show that the representation
\[ \mathcal{L}_A(X\vee C) \ni T \mapsto Q_{C,F}^\psi \rho_{\vee C}^\psi (T) \in B(H) \]
is faithful, and hence
\[
\|Z\|_{B(H)} = \max \left\{ \left\| Q_{C,F}^\psi Z \right\|_{B(H)} : C \text{ is an initial segment of } F \right\} = \max \left\{ \left\| Q_{C,F}^\psi \rho_{\vee C}^\psi \left( \sum_k l_{p_k}^C (\Theta_{x_k,y_k}) \right) \right\|_{B(H)} : C \text{ is an initial segment of } F \right\} = \max \left\{ \left\| \sum_k l_{p_k}^C (\Theta_{x_k,y_k}) \right\|_{\mathcal{L}_A(X\vee C)} : C \text{ is an initial segment of } F \right\}.
\]
Let \( \mathcal{K} := \prod_{t \in F \setminus C, t \vee (\mathcal{V} \setminus C) < \infty} \left( \text{id}_\mathcal{H} - P^\psi_\mathcal{V}(\mathcal{V})^{-1}(t \vee (\mathcal{V} \setminus C)) \right) \mathcal{H} \). Since \( \psi_e(a) P^\psi_p = P^\psi_p \psi_e(a) \), for each \( a \in A \) and \( p \in P \), by Lemma A.10, we see that \( \mathcal{K} \) is a \( \psi_e \)-invariant subspace of \( \mathcal{H} \). As \( C \) is an initial segment of \( F \), if \( t \in F \setminus C \) with \( t \vee (\mathcal{V} \setminus C) < \infty \), then \( t \not\leq \mathcal{V} \setminus C \), and so \( (\mathcal{V} \setminus C)^{-1}(t \vee (\mathcal{V} \setminus C)) \neq e \). Thus, \( \psi_e|_\mathcal{K} \) is faithful. Therefore, by Proposition A.6 it follows that \( \mathcal{M} := \psi^\psi_\mathcal{V}(\mathcal{X} \setminus \mathcal{V} \setminus C)\mathcal{K} \) is a \( \rho^\psi_\mathcal{V} \)-invariant subspace and \( \rho^\psi_\mathcal{V}|_{\mathcal{M}} \) is faithful. To show that \( \mathcal{L}_A(\mathcal{X} \setminus \mathcal{V} \setminus C) \ni T \mapsto Q^\psi_{C,F} \rho^\psi_\mathcal{V}(T) \in \mathcal{B}(\mathcal{H}) \) is faithful, it remains to check that \( \mathcal{M} \subseteq Q^\psi_{C,F} \mathcal{H} \). Lemma A.10 tells us that for any \( x \in \mathcal{X} \setminus \mathcal{V} \setminus C \), we have

\[
Q^\psi_{C,F} \psi_\mathcal{V}(x) = P^\psi_{\mathcal{V} \setminus C} \prod_{t \in F \setminus C, t \vee (\mathcal{V} \setminus C) < \infty} \left( \text{id}_\mathcal{H} - P^\psi_{\mathcal{V}(\mathcal{V})^{-1}(t \vee (\mathcal{V} \setminus C))} \right) \psi_\mathcal{V}(x)
\]

Therefore, \( Q^\psi_{C,F} \) is the identity on

\[
\psi_\mathcal{V}(\mathcal{X} \setminus \mathcal{V} \setminus C) \prod_{t \in F \setminus C, t \vee (\mathcal{V} \setminus C) < \infty} \left( \text{id}_\mathcal{H} - P^\psi_{\mathcal{V}(\mathcal{V})^{-1}(t \vee (\mathcal{V} \setminus C))} \right) \mathcal{H} = \psi_\mathcal{V}(\mathcal{X} \setminus \mathcal{V} \setminus C) \mathcal{K} = \mathcal{M},
\]

and so \( \mathcal{M} \subseteq Q^\psi_{C,F} \mathcal{H} \).

Now that we have an expression for the norm of elements in \( \psi_*(\mathcal{N}^{-e}_A^X) \), we are ready to show that the expectation \( E^e_{\mathcal{X}} \) can be implemented spatially.

**Proposition A.14.** Let \((G, P)\) be a quasi-lattice ordered group and \(X\) a compactly aligned product system over \(P\) with coefficient algebra \(A\). Let \(\psi : X \to \mathcal{B}(\mathcal{H})\) be a Nica covariant representation of \(X\) on a Hilbert space \(\mathcal{H}\). Suppose that for any finite set \(K \subseteq P \setminus \{e\}\), the representation

\[
A \ni a \mapsto \psi_e(a) \prod_{t \in K} \left( \text{id}_\mathcal{H} - P^\psi_t \right) \in \mathcal{B}(\mathcal{H})
\]

is faithful. Then

(i) \(\psi_*(\mathcal{N}^{-e}_A^X)\) is faithful; and

(ii) there exists a linear map \(E_\psi : \psi_*(\mathcal{N}^{-e}_A^X) \to \psi_*(\mathcal{N}^{-e}_A^X)\) such that

\[
E_\psi \circ \psi_* = \psi_* \circ E^e_{\mathcal{X}}.
\]

**Proof.** Firstly, we prove that \(\psi_*(\mathcal{N}^{-e}_A^X)\) is faithful. Fix a finite sum \(Z := \sum_k i_{x_k^e} (x_k) i_{x_k^e} (y_k)^* \in \mathcal{N}^{-e}_A^X\). Let \(\sigma : \mathcal{N}_X \to \mathcal{B}(\mathcal{H}')\) be a faithful representa-
tion. Thus, $\sigma \circ i_X$ is a Nica covariant representation of $X$ on $\mathcal{H}'$. For any finite set $F \subseteq P$ containing each $p_k$, two applications of Lemma A.13 tell us that

$$\|Z\|_{\mathcal{N}'(X)} = \|\sigma(Z)\|_{B(\mathcal{H}')} = \left\| \sum_k (\sigma \circ i_X)_p(x_k)(\sigma \circ i_X)_p(y_k)^* \right\|_{B(\mathcal{H}')}$$

$$= \max \left\{ \left\| Q^{\sigma \circ i_X}_{C,F} \rho_{\mathcal{N}'(X)}^C \left( \sum_k \psi_{p_k}(C;_{x_k,y_k})^* \right) \right\|_{B(\mathcal{H}')} : C \text{ is an initial segment of } F \right\}$$

$$\leq \max \left\{ \left\| \sum_k \psi_{p_k}(C;_{x_k,y_k})^* \right\|_{\mathcal{L}_A(X_{\mathcal{V}})} : C \text{ is an initial segment of } F \right\}$$

$$= \left\| \sum_k \psi_{p_k}(x_k)\psi_{p_k}(y_k)^* \right\|_{B(\mathcal{H})}$$

$$= \|\psi_s(Z)\|_{B(\mathcal{H})},$$

where we used the fact that each $Q^{\sigma \circ i_X}_{C,F}$ is a projection and each $*$-homomorphism $\rho_{\mathcal{N}'(X)}^C$ is norm-decreasing. Thus, $\psi_s|_{\mathcal{N}'(X)}$ is faithful.

Next we prove part (ii). We first show that for any finite sum $\sum_k \psi_{p_k}(x_k)\psi_{q_k}(y_k)^*$ we have

$$\left\| \sum_{k: p_k = q_k} \psi_{p_k}(x_k)\psi_{p_k}(y_k)^* \right\|_{B(\mathcal{H})} \leq \left\| \sum_k \psi_{p_k}(x_k)\psi_{q_k}(y_k)^* \right\|_{B(\mathcal{H})}. \quad (A.2)$$

Let $F \subseteq P$ be the finite set consisting of each $p_k$, $q_k$. By Lemma A.13, there exists an initial segment $C$ of $F$ such that

$$\left\| \sum_{k: p_k = q_k} \psi_{p_k}(x_k)\psi_{p_k}(y_k)^* \right\|_{B(\mathcal{H})} = \left\| \sum_{k: p_k = q_k} \psi_{p}(C;_{x_k,y_k})^* \right\|_{\mathcal{L}_A(X_{\mathcal{V}})}.$$

For each $s, t \in C$ with $s \neq t$ and $(s^{-1}(\mathcal{V}C)) \lor (t^{-1}(\mathcal{V}C)) < \infty$ define

$$\beta_{s,t} := \begin{cases} s ((s^{-1}(\mathcal{V}C)) \lor (t^{-1}(\mathcal{V}C))) & \text{if } s^{-1}(\mathcal{V}C) < (s^{-1}(\mathcal{V}C)) \lor (t^{-1}(\mathcal{V}C)) \\ t ((s^{-1}(\mathcal{V}C)) \lor (t^{-1}(\mathcal{V}C))) & \text{otherwise.} \end{cases}$$

Observe that for any $s, t \in C$ with $s \neq t$ and $(s^{-1}(\mathcal{V}C)) \lor (t^{-1}(\mathcal{V}C)) < \infty$ we have

$$s \left( (s^{-1}(\mathcal{V}C)) \lor (t^{-1}(\mathcal{V}C)) \right) \geq s \left( s^{-1}(\mathcal{V}C) \right) = \mathcal{V}C$$

and

$$t \left( (s^{-1}(\mathcal{V}C)) \lor (t^{-1}(\mathcal{V}C)) \right) \geq t \left( t^{-1}(\mathcal{V}C) \right) = \mathcal{V}C.$$
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projection
\[ R^\psi_{C,F} := Q^\psi_{C,F} \prod_{s,t \in C, s \neq t, (s^{-1}(\vee C)) \cup (t^{-1}(\vee C)) < \infty} \left( P^\psi_{\vee C} - P^\psi_{\beta_{s,t}} \right). \]

We claim that
\[ R^\psi_{C,F} \left( \sum_k \psi_{p_k}(x_k)\psi_{q_k}(y_k)^* \right) R^\psi_{C,F} = R^\psi_{C,F} \sum_{k:p_k=q_k} \psi_{p_k}(x_k)\psi_{p_k}(y_k)^*. \]

Since \( R^\psi_{C,F} \) commutes with \( \sum_{(k:p_k=q_k)} \psi_{p_k}(x_k)\psi_{p_k}(y_k)^* \) by Proposition A.11, it suffices to show that \( R^\psi_{C,F} \psi_p(x)\psi_q(y)^* R^\psi_{C,F} = 0 \) whenever \( x \in X_p, y \in X_q \), with \( p, q \in F \) and \( p \neq q \). Firstly, if \( p \not\in C \) or \( q \not\in C \), then \( p \not\in \vee C \) or \( q \not\in \vee C \) (since \( C \) is an initial segment of \( F \)), and so by Lemma A.10 we have \( Q^\psi_{C,F} \psi_p(x) = 0 \) or \( Q^\psi_{C,F} \psi_q(y) = 0 \). Consequently,
\[ R^\psi_{C,F} \psi_p(x)\psi_q(y)^* R^\psi_{C,F} = R^\psi_{C,F} Q^\psi_{C,F} \psi_p(x)\psi_q(y)^* Q^\psi_{C,F} R^\psi_{C,F} = R^\psi_{C,F} Q^\psi_{C,F} \psi_p(x)(Q^\psi_{C,F} \psi_q(y))^* R^\psi_{C,F} = 0. \]

Alternatively, if \( p, q \leq \vee C \) and \( (p^{-1}(\vee C)) \cup (q^{-1}(\vee C)) = \infty \), then
\[ P^\psi_{p^{-1}(\vee C)} P^\psi_{q^{-1}(\vee C)} = 0. \]

Hence, Lemma A.10 tells us that
\[ R^\psi_{C,F} \psi_p(x)\psi_q(y)^* R^\psi_{C,F} = R^\psi_{C,F} Q^\psi_{C,F} \psi_p(x)\psi_q(y)^* Q^\psi_{C,F} R^\psi_{C,F} = R^\psi_{C,F} Q^\psi_{C,F} \psi_p(x) P^\psi_{p^{-1}(\vee C)} P^\psi_{q^{-1}(\vee C)} \psi_q(y)^* Q^\psi_{C,F} R^\psi_{C,F} = 0. \]

With this in mind, suppose that \( p, q \in C \) and \( (p^{-1}(\vee C)) \cup (q^{-1}(\vee C)) < \infty \). Since \( p \) and \( q \) are distinct, it follows that either \( p^{-1}(\vee C) \subset (p^{-1}(\vee C)) \cup (q^{-1}(\vee C)) \) or \( q^{-1}(\vee C) \subset (p^{-1}(\vee C)) \cup (q^{-1}(\vee C)) \). By taking adjoints, we may assume, without loss of generality, that \( p^{-1} < (p^{-1}(\vee C)) \cup (q^{-1}(\vee C)) \). Therefore,
\[ \beta_{p,q} = p\left( (p^{-1}(\vee C)) \cup (q^{-1}(\vee C)) \right). \]
Consequently, an application of Lemma A.10 shows that

\[
R_{C,F}^\psi \psi_p(x) \psi_q(y)^* R_{C,F}^\psi
= R_{C,F}^\psi \left( P_{\psi}^C - P_{\beta_p,q}^\psi \right) \psi_p(x) \psi_q(y)^* P_{\psi}^C R_{C,F}^\psi
= R_{C,F}^\psi \psi_p(x) \left( P_{\psi}^{p^{-1}(V_C)} - P_{\beta_p,q}^{p^{-1}(V_C)} \right) P_{\psi}^{q^{-1}(V_C)} \psi_q(y)^* R_{C,F}^\psi
= R_{C,F}^\psi \psi_p(x) \left( P_{\psi}^{p^{-1}(V_C)} - P_{\psi}^{(p^{-1}(V_C) \vee (q^{-1}(V_C)))} \right) P_{\psi}^{q^{-1}(V_C)} \psi_q(y)^* R_{C,F}^\psi
= 0.
\]

Additionally, we claim that the representation

\[
\mathcal{L}_A(X_C) \ni T \mapsto R_{C,F}^\psi \rho_{\psi}^C(T) \in \mathcal{B}(\mathcal{H})
\]

is faithful. Let

\[
\mathcal{K} := \prod_{p \in F, \, C \ni s, \, t \leq C, \, p \vee (V_C) < \infty, \, s \neq t, \, (s^{-1}(V_C)) \vee (t^{-1}(V_C)) < \infty} (\text{id}_\mathcal{H} - P_{\psi}^{(V_C)^{-1}(p \vee (V_C))} - P_{\psi}^{(V_C)^{-1}(p \vee (V_C))}) \mathcal{H}.
\]

As \( \psi_e(a) P_q^\psi = P_q^\psi \psi_e(a) \), for each \( a \in A \) and \( q \in P \), by Lemma A.10, we see that \( \mathcal{K} \) is a \( \psi_e \)-invariant subspace of \( \mathcal{H} \). Since \( C \) is an initial segment of \( F \), if \( p \in F \setminus C \) with \( p \vee (V_C) < \infty \), then \( p \not\in V_C \), and so \( (V_C)^{-1}(p \vee (V_C)) \neq e \). We also claim that for any \( s, t \in C \) with \( s \neq t \) and \( (s^{-1}(V_C)) \vee (t^{-1}(V_C)) < \infty \), we have \( (V_C)^{-1}(s,t) \neq e \). Firstly, if \( s^{-1}(V_C) = (s^{-1}(V_C)) \vee (t^{-1}(V_C)) \), then

\[
(V_C)^{-1}(s,t) = (V_C)^{-1} \left( s^{-1}(V_C) \vee (t^{-1}(V_C)) \right)
= (s^{-1}(V_C)) \vee (t^{-1}(V_C))
\neq e.
\]

On the other hand, if \( s^{-1}(V_C) = (s^{-1}(V_C)) \vee (t^{-1}(V_C)) \), then

\[
(V_C)^{-1}(s,t) = (V_C)^{-1} ts^{-1}(V_C) \neq e
\]

since \( s \neq t \). Thus, \( \psi_e|_\mathcal{K} \) is faithful. Hence, by Proposition A.6 it follows that \( \mathcal{M} := \overline{\mathcal{L}_A(X_C) \mathcal{K}} \) is a \( \rho_{\psi}^C \)-invariant subspace and \( \rho_{\psi}^C|_{\mathcal{M}} \) is faithful. To see that \( \mathcal{L}_A(X_C) \ni T \mapsto R_{C,F}^\psi \rho_{\psi}^C(T) \in \mathcal{B}(\mathcal{H}) \) is faithful, it remains to show that \( \mathcal{M} \subseteq R_{C,F}^\psi \mathcal{H} \). Suppose \( s, t \in C \) with \( s \neq t \) and \( (s^{-1}(V_C)) \vee (t^{-1}(V_C)) < \infty \).
APPENDIX A. A C*-ALGEBRAIC UNIQUENESS THEOREM

Since $\forall C \leq \beta_{s,t}$, Lemma A.10 tells us that for any $x \in X_{\lor C}$ we have

$$P_{\beta_{s,t}} \psi_{\lor C}(x) = \psi_{\lor C}(x) P_{(\lor C)^{-1} \beta_{s,t}}.$$ 

Thus,

$$R_{C,F}^\psi \psi_{\lor C}(x) = Q_{C,F}^\psi \prod_{s,t \in C, s \neq t, (s^{-1}(\lor C)) \lor (t^{-1}(\lor C)) < \infty} \left( P_{\lor C}^\psi - P_{\beta_{s,t}}^\psi \right) \psi_{\lor C}(x)$$

$$= Q_{C,F}^\psi \psi_{\lor C}(x) \prod_{s,t \in C, s \neq t, (s^{-1}(\lor C)) \lor (t^{-1}(\lor C)) < \infty} \left( \text{id}_H - P_{(\lor C)^{-1} \beta_{s,t}}^\psi \right)$$

$$= \psi_{\lor C}(x) \prod_{p \in F \setminus C, s,t \in C, p \lor (\lor C) < \infty, s \neq t, (s^{-1}(\lor C)) \lor (t^{-1}(\lor C)) < \infty} \left( \text{id}_H - P_{(\lor C)^{-1}(p \lor (\lor C))}^\psi \right) \left( \text{id}_H - P_{(\lor C)^{-1} \beta_{s,t}}^\psi \right).$$

Hence, $R_{C,F}^\psi$ is the identity on

$$\mathcal{M} = \psi_{\lor C}(X_{\lor C}) \mathcal{K}$$

$$= \psi_{\lor C}(X_{\lor C}) \prod_{p \in F \setminus C, s,t \in C, p \lor (\lor C) < \infty, s \neq t, (s^{-1}(\lor C)) \lor (t^{-1}(\lor C)) < \infty} \left( \text{id}_H - P_{(\lor C)^{-1}(p \lor (\lor C))}^\psi \right) \left( \text{id}_H - P_{(\lor C)^{-1} \beta_{s,t}}^\psi \right) \mathcal{H},$$

and so $\mathcal{M} \subseteq R_{C,F}^\psi \mathcal{H}$.

Putting all of this together, we see that

$$\left\| \sum_{k : p_k = q_k} \psi_{p_k}(x_k) \psi_{p_k}(y_k)^* \right\|_{B(\mathcal{H})} = \left\| \sum_{k : p_k = q_k} \psi_{p_k}(x_k) \Theta_{x_k,y_k} \right\|_{\mathcal{L}_A(X_{\lor C})}$$

$$= \left\| R_{C,F}^\psi \left( \sum_{k : p_k = q_k} \psi_{p_k}(x_k) \Theta_{x_k,y_k} \right) \right\|_{B(\mathcal{H})}$$

$$= \left\| R_{C,F}^\psi \sum_{k : p_k = q_k} \psi_{p_k}(x_k) \psi_{p_k}(y_k)^* \right\|_{B(\mathcal{H})}$$

$$= \left\| R_{C,F}^\psi \left( \sum_k \psi_{p_k}(x_k) \psi_{q_k}(y_k)^* \right) R_{C,F}^\psi \right\|_{B(H)}$$

$$\leq \left\| \sum_k \psi_{p_k}(x_k) \psi_{q_k}(y_k)^* \right\|_{B(\mathcal{H})}.$$

Since the norm estimate (A.2) holds, the formula $\psi_p(x) \psi_q(y)^* \mapsto \delta_{p,q} \psi_p(x) \psi_q(y)^*$ extends to a map on $\psi_*(\mathcal{N}T_X) = \text{span} \left\{ \psi_p(x) \psi_q(y)^* : p, q \in P, x \in X_p, y \in X_q \right\}$ by
linearity and continuity, which we denote by $E_{\psi}$. Furthermore, for any $p, q \in P$, $x \in X_p$, $y \in X_q$, we have

\[
(E_{\psi} \circ \psi^*) \left( (i_{X_p}(x)i_{X_q}(y)^*) \right) = E_{\psi} (\psi_p(x)\psi_q(y)^*) = \delta_{p, q}\psi_p(x)\psi_q(y)^* \\
= \psi_* \left( \delta_{p, q}(i_{X_p}(x)i_{X_q}(y)^*) \right) \\
= (\psi_*e \circ E_{\delta_X}) \left( (i_{X_p}(x)i_{X_q}(y)^*) \right).
\]

Since $N_T X = \text{span} \{ i_{X_p}(x)i_{X_q}(y)^* : p, q \in P, x \in X_p, y \in X_q \}$, whilst $E_{\psi} \circ \psi^*$ and $\psi_* \circ E_{\delta_X}$ are linear and norm-decreasing, we conclude that $E_{\psi} \circ \psi^* = \psi_* \circ E_{\delta_X}$. This completes the proof of part (ii).

Finally, we prove the uniqueness theorem for Nica–Toeplitz algebras. We remind the reader of Remark A.3 — if $(G, P)$ is a quasi-lattice ordered group with $G$ amenable, then any compactly aligned product system $X$ over $P$ is automatically amenable.

**Theorem A.15.** Let $(G, P)$ be a quasi-lattice ordered group and $X$ a compactly aligned product system over $P$ with coefficient algebra $A$. Suppose $\psi : X \to \mathcal{B}(\mathcal{H})$ is a Nica covariant representation of $X$ on a Hilbert space $\mathcal{H}$.

(i) If the product system $X$ is amenable, and, for any finite set $K \subseteq P \setminus \{e\}$, the representation

\[
A \ni a \mapsto \psi_e(a) \prod_{t \in K} \left( \text{id}_\mathcal{H} - P_t^\psi \right) \in \mathcal{B}(\mathcal{H})
\]

is faithful, then the induced $*$-homomorphism $\psi_* : N_T X \to \mathcal{B}(\mathcal{H})$ is faithful.

(ii) If $\psi_*$ is faithful and $\phi_p(A) \subseteq K_A(X_p)$ for each $p \in P$, then the representation

\[
A \ni a \mapsto \psi_e(a) \prod_{t \in K} \left( \text{id}_\mathcal{H} - P_t^\psi \right) \in \mathcal{B}(\mathcal{H})
\]

is faithful.

**Proof.** Suppose that the product system $X$ is amenable and the representation

\[
A \ni a \mapsto \psi_e(a) \prod_{t \in K} \left( \text{id}_\mathcal{H} - P_t^\psi \right) \in \mathcal{B}(\mathcal{H})
\]

is faithful for any finite set $K \subseteq P \setminus \{e\}$. Let $b \in N_T X$ be such that $\psi_*(b) = 0$. Thus,

\[
\psi_*(E_{\delta_X}(b^*b)) = E_{\psi}(\psi_*(b^*b)) = E_{\psi}(\psi_*(b)^*\psi_*(b)) = 0.
\]

Since $\psi_*$ is faithful on $N_T X = E_{\delta_X}(N_T X)$, we must have $E_{\delta_X}(b^*b) = 0$. As $E_{\delta_X}$ is faithful on positive elements, we conclude that $b = 0$. Hence, $\psi_*$ is faithful.
We now prove (ii). Suppose that $\psi_*$ is faithful and $\phi_p(A) \subseteq \mathcal{K}_A(X_p)$ for each $p \in P$. Fix a finite set $K \subseteq P \setminus \{e\}$. For each $a \in A$, define

$$T_a := \sum_{J \subseteq K, \vee J < \infty} (-1)^{|J|} (\phi_{\vee J}(a)) \in \mathcal{N}\mathcal{T}_X.$$ 

We claim that for any Nica covariant representation $\varphi : X \to B(H')$ we have

$$\varphi_*(T_a) = \varphi_e(a) \prod_{t \in K} (\text{id}_{H'} - P_t^\varepsilon). \tag{A.3}$$

To see this, firstly observe that for any $t \in P$ and $a \in A$, we have

$$\varphi_* \left( i_{X}^t(\phi_t(a)) \right) = P_t^\varepsilon(\phi_p(a)) = \varphi_e(a) P_t^\varepsilon.$$

Therefore,

$$\varphi_*(T_a) = \sum_{J \subseteq K, \vee J < \infty} (-1)^{|J|} \varphi_* \left( i_{X}^\vee J(\phi_{\vee J}(a)) \right)$$

$$= \varphi_e(a) \sum_{J \subseteq K, \vee J < \infty} (-1)^{|J|} P_{\vee J}^\varepsilon$$

$$= \varphi_e(a) \sum_{J \subseteq K} (-1)^{|J|} P_{\vee J}^\varepsilon.$$

Hence, to prove that Equation (A.3) holds, it suffices to show that

$$\sum_{J \subseteq K} (-1)^{|J|} P_{\vee J}^\varepsilon = \prod_{t \in K} (\text{id}_{H'} - P_t^\varepsilon). \tag{A.4}$$

To prove this we use induction on $|K|$. When $|K| = 0$ we have

$$\sum_{J \subseteq K} (-1)^{|J|} P_{\vee J}^\varepsilon = \sum_{J \subseteq \emptyset} (-1)^{|J|} P_{\vee J}^\varepsilon = (-1)^{|\emptyset|} P_{\vee \emptyset}^\varepsilon = P_e^\varepsilon$$

$$= \text{id}_{H'}$$

$$= \prod_{t \in \emptyset} (\text{id}_{H'} - P_t^\varepsilon)$$

$$= \prod_{t \in K} (\text{id}_{H'} - P_t^\varepsilon).$$

Now let $n \in \mathbb{N}$ and suppose we have equality whenever $K \subseteq P$ and $|K| = n$. Fix
$K' \subseteq P$ with $|K'| = n + 1$. Let $s \in K'$. Then

$$
\sum_{J \subseteq K'} (-1)^{|J|} P^\varphi_{VJ} = \sum_{J \subseteq K' \setminus \{s\}} (-1)^{|J|} P^\varphi_{VJ} + \sum_{\{J \subseteq K' : s \in J\}} (-1)^{|J|} P^\varphi_{VJ}
$$

$$= \sum_{J \subseteq K' \setminus \{s\}} (-1)^{|J|} P^\varphi_{VJ} + \sum_{J \subseteq K' \setminus \{s\}} (-1)^{|\cup J\{s\}|} P^\varphi_{V(J\{s\})}
$$

$$= \sum_{J \subseteq K' \setminus \{s\}} (-1)^{|J|} P^\varphi_{VJ} - \sum_{J \subseteq K' \setminus \{s\}} (-1)^{|J|} P^\varphi_{VJ} P^\varphi_s
$$

$$= (\text{id}_{H'} - P^\varphi_s) \sum_{J \subseteq K' \setminus \{s\}} (-1)^{|J|} P^\varphi_{VJ}
$$

$$= (\text{id}_{H'} - P^\varphi_s) \prod_{t \in K' \setminus \{s\}} (\text{id}_{H'} - P^\varphi_t)
$$

$$= \prod_{t \in K'} (\text{id}_{H'} - P^\varphi_t).
$$

This proves that Equation (A.4) holds, and so Equation (A.3) follows.

Now let $\pi : A \rightarrow \mathcal{B}(\mathcal{H}')$ be a faithful nondegenerate representation of $A$. Define $\Psi : X \rightarrow \mathcal{B}(\mathcal{F}_X \otimes_A \mathcal{H}')$ by

$$\Psi := \left( \mathcal{F}_X - \text{Ind}_{A}^{\mathcal{L}_A(\mathcal{F}_X)} \pi \right) \circ \iota$$

where $\iota : X \rightarrow \mathcal{L}_A(\mathcal{F}_X)$ is the Fock representation of $X$. Since $\iota$ is a Nica covariant representation of $X$ and $\mathcal{F}_X - \text{Ind}_{A}^{\mathcal{L}_A(\mathcal{F}_X)} \pi$ is a $\ast$-homomorphism, $\Psi$ is a Nica covariant representation of $X$. We claim that the representation

$$A \ni a \mapsto \Psi_e(a) \prod_{i \in K} (\text{id}_{\mathcal{H}} - P^\varphi_i) \in \mathcal{B}(\mathcal{F}_X \otimes_A \mathcal{H}')$$

is faithful. To see this, suppose that $a \in A \setminus \{0\}$, so that $aa^* \neq 0$. As $\pi$ is faithful, we can find $h \in \mathcal{H}'$ such that $\pi(aa^*)h \neq 0$. For any $t \in P \setminus \{e\}$ we have

$$
\Psi_t(X_t) (\mathcal{F}_X \otimes_A \mathcal{H}') = \left( \mathcal{F}_X - \text{Ind}_{A}^{\mathcal{L}_A(\mathcal{F}_X)} \pi \left( l_t(X_t) \right) \right) (\mathcal{F}_X \otimes_A \mathcal{H}')
$$

$$= l_t(X_t) (\mathcal{F}_X) \otimes_A \mathcal{H}'
$$

$$= \bigoplus_{s \geq t} X_s \otimes_A \mathcal{H}'.
$$

Hence, it follows that $P^\varphi_t = \text{proj}_{\Psi_t(X_t)(\mathcal{F}_X \otimes_A \mathcal{H}')} \pi$ is zero on $A \otimes_A \mathcal{H}' \subseteq \mathcal{F}_X \otimes_A \mathcal{H}'$. 
Thus, as \( e \not\in K \) we see that
\[
\Psi_e (a) \prod_{t \in K} (\text{id}_H - P_t^\psi) (a^* \otimes_A h) = \Psi_e (a^* \otimes_A h)
\]
\[
= l_e (a^* \otimes_A h)
\]
\[
= aa^* \otimes_A h,
\]
which is nonzero since
\[
\|aa^* \otimes_A h\|_{\mathcal{F}_X \otimes_A \mathcal{H}'}^2 = \langle aa^* \otimes_A h, aa^* \otimes_A h \rangle_C
\]
\[
= \langle h, \pi((aa^*)^*aa^*)h \rangle_C
\]
\[
= \langle \pi(aa^*)h, \pi(aa^*)h \rangle_C
\]
\[
= \|\pi(aa^*)h\|_{\mathcal{H}'}^2
\]
\[
\neq 0.
\]
Therefore, \( A \ni a \mapsto \Psi_e (a) \prod_{t \in K} (\text{id}_H - P_t^\psi) \in \mathcal{B} (\mathcal{F}_X \otimes_A \mathcal{H'}) \) is faithful.

Putting all of this together, and using that \( \psi_* \) is faithful at the penultimate equality, we see that for any \( a \in A \),
\[
\|a\|_A = \left\| \Psi_e (a) \prod_{t \in K} (\text{id}_H - P_t^\psi) \right\|_{\mathcal{B}(\mathcal{F}_X \otimes_A \mathcal{H}')} = \left\| \Psi_* (T_a) \right\|_{\mathcal{B}(\mathcal{F}_X \otimes_A \mathcal{H}')} \leq \|T_a\|_{\mathcal{N}^{\mathcal{T}_X}} = \|\psi_* (T_a)\|_{\mathcal{B}(\mathcal{H})} = \left\| \psi_e (a) \prod_{t \in K} (\text{id}_H - P_t^\psi) \right\|_{\mathcal{B}(\mathcal{H})}.
\]
Hence, \( A \ni a \mapsto \psi_e (a) \prod_{t \in K} (\text{id}_H - P_t^\psi) \in \mathcal{B} (\mathcal{H}) \) is faithful. \( \square \)
Bibliography


