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Abstract

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Keywords

dependent, compressibility, permeability, drain, flow, vertical, darcian, non, consolidation, void, ratio

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Vertical drain consolidation with non-Darcian flow and void-ratio-dependent compressibility and permeability

R. WALKER*, B. INDRARATNA* and C. RUJIKIATKAMJORN*

Vertical drains increase the rate of consolidation in soft soils by facilitating faster dissipation of excess pore water pressure through short, horizontal drainage paths. This paper presents an analytical solution for non-linear radial consolidation under equal-strain conditions incorporating smear but ignoring well resistance. Three aspects of non-linearity are considered: (a) non-Darcian flow, (b) a log-linear void-ratio–stress relationship; and (b) a log-linear void-ratio–permeability relationship. The analytical solution to non-linear radial consolidation can explicitly capture the behaviour of both overconsolidated and normally consolidated soils. For non-linear material properties, consolidation may be faster or slower when compared with the cases with constant material properties. The difference depends on the compressibility/permeability ratios (C_c/C_k and C_r/C_k), the preconsolidation pressure and the stress increase. If $C_c/C_k < 1$ or $C_r/C_k < 1$ then the coefficient of consolidation increases as excess pore pressures dissipate, and the corresponding rate of consolidation is greater.

KEYWORDS: case history; consolidation; pore pressures; settlement

Les drains verticaux augmentent la vitesse de consolidation dans les sols meubles, en facilitant une dissipation plus rapide de la pression d'eau interstitielle à travers des canaux de drainage horizontaux courts. Cette communication présente une solution analytique pour la consolidation radiale non linéaire dans des conditions de contrainte égales, comprenant l'effet *smear*, mais ne tenant pas compte de la résistance de puits. On examine trois aspects de non linéarité suivants: (i) écoulement non Darcien, (ii) rapports ratio de vide/contrainte log-linéaire; et (iii) rapport ratio de vide/perméabilité log-linéaire. La solution analytique pour la consolidation radiale non linéaire est en mesure de capturer de façon explicite le comportement de sols surconsolidés et à consolidation normale. Pour les propriétés de matières non linéaires, la consolidation peut être plus rapide ou plus lente par rapport aux cas à propriétés matérielles constantes. La différence est tributaire des ratios de compressibilité/perméabilité (C_c/C_k et C_r/C_k), de la pression de pré-consolidation, et de l'augmentation des contraintes. Si $C_c/C_k < 1$ ou $C_r/C_k < 1$, le coefficient de consolidation augmente, les pressions interstitielles excessives se dissipant et les ratios de consolidation correspondants étant supérieurs.

INTRODUCTION

Vertical drains increase the rate of consolidation in soft soils by facilitating faster dissipation of excess pore water pressure through short, horizontal drainage paths. Usually, the simplest analytical methods used to analyse vertical drain consolidation problems simulate a single soil cylinder with constant soil properties over a given stress range (Barron, 1948; Hansbo, 1981). When material behaviour is non-linear, the simple models are often inadequate. This paper presents analytical solutions for non-linear radial consolidation under equal-strain conditions incorporating smear but ignoring well resistance. For most commonly installed drain lengths in Australia and Southeast Asia (less than 18 m), the well resistance is not significant compared with the smear effects (Indraratna & Redana, 2000). Three aspects of non-linearity are considered: (a) non-Darcian flow, (b) log-linear void-ratio–stress relationship, and (c) log-linear void-ratio–permeability relationship. In non-Darcian flow, the velocity of flow, v , is related to the hydraulic gradient, i , by the power law

$$v = \tilde{k} i^n \quad (1)$$

where $\tilde{k}$ is the coefficient of permeability under non-Darcian conditions, and n is the non-Darcian flow exponent. Void

ratio is related to effective stress and permeability by the following relationships.

$$e = e_0 - C_c \log \left(\frac{\sigma'}{\sigma'_0} \right) \quad (2)$$

$$e = e_0 + C_k \log \left(\frac{\tilde{k}}{\tilde{k}_0} \right) \quad (3)$$

where e is the void ratio; σ' is the effective stress; C_c is the compressibility index; C_k is the permeability index; and e_0 , σ'_0 and $\tilde{k}_0$ are the initial values of void ratio, effective stress and permeability respectively before the application of pre-loading.

A review of the current literature reveals various attempts to model the corresponding problem with Darcian flow (e.g. Basak & Madhav, 1978; Lekha *et al.*, 1998; Indraratna *et al.*, 2005). Lekha *et al.* (1998) proposed the average excess pore pressure, $\bar{u}$, for radial consolidation considering log-linear void-ratio–stress and log-linear void-ratio–permeability relationships as

$$\bar{u} = \sigma'_0 \left\{ \frac{\Delta\sigma}{\sigma'_0} - \frac{T_h}{\mu} \frac{1 - \beta}{2 - C_c/C_k} \times \left[\left(\beta + \frac{\Delta\sigma}{\sigma'_0} \right)^{2 - C_c/C_k} - \beta^{2 - C_c/C_k} \right] \right\} \quad (4)$$

$$\beta = 1 - \frac{1}{2} \left(1 + \frac{\Delta\sigma}{\sigma'_0} \right) \quad (5)$$

where $\Delta\sigma$ is the change in total stress, T_h is the time factor

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for horizontal consolidation, and μ is a smear zone parameter for Darcian flow.

Equation (4) is a linear function of the time factor T_h , whereby at large values of T_h unrealistic negative values of excess pore water pressure are predicted. The excess pore water pressure should decay to zero: thus equation (4) is unsuitable for estimating $\bar{u}$ as primary consolidation nears conclusion. Also, equation (4) is undefined if $C_c/C_k = 2$. Basak & Madhav (1978) presented a more useful solution with equation (2), but using a linear relationship between permeability and effective stress. Indraratna *et al.* (2005) expressed the excess pore pressure under instantaneous loading as

$$\bar{u} = \exp\left(-P_{av} \frac{8T_{h0}}{\mu}\right) \quad (6a)$$

where

$$P_{av} = \frac{1}{2} \left[1 + \left(1 + \frac{\Delta\sigma}{\sigma'_0} \right)^{1-C_c/C_k} \right] \quad (6b)$$

Equation (6) is very similar to the linear solution given by Hansbo (1981); the main difference is in the parameter P_{av} . P_{av} represents the average between the beginning and end values of the consolidation coefficient. This averaging of consolidation coefficient over the applied stress increment is implied in Hansbo's (1981) solution. Thus equation (6) simply provides the best choice of consolidation coefficient for use with Hansbo's (1981) equations. Although Indraratna *et al.* (2005) recommend using the log-linear void-ratio-stress relationship (equation (2)) for settlement calculations, the solution expressed by equation (6) does not reflect the non-linear processes involved with pore pressure dissipation.

The proposed model presented herein removes these simplifying assumptions. Hansbo's (2001) equal-strain solution for radial drainage with non-Darcian flow is extended to include the non-linear material properties expressed in equations (2) and (3). A series solution to the resulting non-linear partial differential equation is found, explicitly capturing the variation of permeability and compressibility in the consolidation of normally and overconsolidated soil. These solutions can also be used for benchmarking numerical analysis.

CONSTITUTIVE RELATIONSHIP

The horizontal coefficient of consolidation under non-Darcian flow, $\tilde{c}_h$, can be written as

$$\tilde{c}_h = \frac{\tilde{k}_h}{m_v \gamma'_w} \quad (7)$$

where m_v is the volume compressibility, and γ'_w is the unit weight of water. When considering material non-linearity, the coefficient of consolidation becomes dependent on the effective stress (under equal strain, the effective stress does not vary radially). The effective stress can be written as

$$\sigma' = \sigma'_0 + \Delta\sigma - W\Delta\sigma \quad (8)$$

where $\Delta\sigma$ is the instantaneous change in total stress, and W is a normalised pore pressure given by

$$W = \frac{\bar{u}}{\Delta\sigma} \quad (9)$$

The compressibility of soils previously subjected to higher effective stresses (overconsolidated) may increase markedly when the preconsolidation pressure, σ'_p , is exceeded. With reference to Fig. 1, the compressibility relationships in the recompression zone ($\sigma' \leq \sigma'_p$) and the compression zone ($\sigma' > \sigma'_p$) are described, respectively, by

$$e = e_0 - C_r \log\left(\frac{\sigma'}{\sigma'_0}\right) \quad (10a)$$

and

$$e = e_0 - C_r \log\left(\frac{\sigma'_p}{\sigma'_0}\right) - C_c \log\left(\frac{\sigma'}{\sigma'_p}\right) \quad (10b)$$

The consolidation coefficient involves permeability and volume compressibility. Volume compressibility is defined by the relationship

$$m_v = \frac{1}{1 + e_0} \frac{\partial e}{\partial \sigma'} \quad (11)$$

Differentiating equation (10) to find $\partial e / \partial \sigma'$ and then substituting into equation (11) yields

$$m_v = \frac{-0.434 C_r}{\sigma' (1 + e_0)}; \quad \sigma' \leq \sigma'_p \quad (12a)$$

and

$$m_v = \frac{-0.434 C_c}{\sigma' (1 + e_0)}; \quad \sigma' > \sigma'_p \quad (12b)$$

The relative change in volume compressibility with effective stress is thus expressed as

$$\frac{m_{v0}}{m_v} = \frac{\sigma'}{\sigma'_0}; \quad \sigma' \leq \sigma'_p \quad (13a)$$

and

$$\frac{m_{v0}}{m_v} = \frac{C_r}{C_c} \left(\frac{\sigma'}{\sigma'_0} \right); \quad \sigma' > \sigma'_p \quad (13b)$$

Fig. 1. Void-ratio-dependent compressibility and permeability

Substituting equation (10) into equation (3) gives the relative change in permeability with effective stress as

$$\frac{\tilde{k}}{\tilde{k}_0} = \left(\frac{\sigma'}{\sigma'_0} \right)^{-C_r/C_k} ; \quad \sigma' \leq \sigma'_p \quad (14a)$$

and

$$\frac{\tilde{k}}{\tilde{k}_0} = \left(\frac{\sigma'_p}{\sigma'_0} \right)^{(C_c - C_r)/C_k} \left(\frac{\sigma'}{\sigma'_0} \right)^{-C_c/C_k} ; \quad \sigma' > \sigma'_p \quad (14b)$$

Combining equations (7), (8), (13) and (14) results in the following expressions for the stress dependence of $\tilde{c}_h$ in the recompression and compression zones

$$\frac{\tilde{c}_h}{\tilde{c}_{h0}} = \left(1 + \frac{\Delta\sigma}{\sigma'_0} - \frac{W\Delta\sigma}{\sigma'_0} \right)^{1 - C_r/C_k} ; \quad \sigma' \leq \sigma'_p \quad (15a)$$

$$\frac{\tilde{c}_h}{\tilde{c}_{h0}} = \frac{C_r}{C_c} \left(\frac{\sigma'_p}{\sigma'_0} \right)^{(C_c - C_r)/C_k} \times \left(1 + \frac{\Delta\sigma}{\sigma'_0} - \frac{W\Delta\sigma}{\sigma'_0} \right)^{1 - C_c/C_k} ; \quad \sigma' > \sigma'_p \quad (15b)$$

The value of consolidation coefficient at the start of analysis is denoted $\tilde{c}_{h0}$.

DEVELOPMENT OF GOVERNING EQUATION FOR NON-DARCIAN FLOW

The consolidation coefficient is a function of the coefficient of volume compressibility, m_v , and of the soil permeability, which vary according to the effective stress. The stress dependence of the consolidation coefficient is treated implicitly and incorporated explicitly in the following.

Vertical drains, installed in a square or triangular pattern, are usually modelled analytically by considering an equivalent axisymmetric system. Pore water flows from a soil cylinder to a single central vertical drain with simplified boundary conditions. Moreover, this mathematical analysis considers only the lateral flow, because, for long vertical drains the contribution from the vertical consolidation coefficient (c_v) is insignificant. Therefore the question of anisotropy does not affect the mathematical derivations directly. For simplicity, the authors have assumed that the compressibility parameter m_v is the same for both smear and the undisturbed zone. Fig. 2 shows a unit cell with external radius r_e , drain radius r_w and smear zone radius r_s . Depending on the soil stiffness, the size and shape of the mandrel, and the installation method, the extent of the smear zone (r_s) can usually be estimated as $r_s = (2 \text{ to } 3)r_m$, where r_m is the equivalent radius of the mandrel (Hansbo, 1981). Ghandeharion *et al.* (2010) showed that the driving of the mandrel does follow the theory of lateral cavity expansion in a non-linear manner. Weber *et al.* (2010) showed that a very clearly structured smear zone was observed from environmental scanning electron microscope photographs. The soil permeability and compressibility parameters decrease non-uniformly with the radius within and just outside the smear zone, as vividly discussed elsewhere by Indraratna & Redana (1997, 1998a) and Walker (2006), based on measurements from large-scale consolidation tests. However, in order to simplify the mathematical formulation of already complex equations, reduced lateral soil permeability was assumed as an equivalent average (constant) across the smear zone annulus. The lateral coefficient of consolidation (c_h) is therefore changed proportionately too. This assumption may lead to some overestimation of the settlement. The velocity of

Fig. 2. Axisymmetric unit cell

pore water flow in the smear and undisturbed zones are respectively given by (Hansbo, 2001)

$$v = \tilde{k}_s \left(\frac{1}{\gamma_w} \frac{\partial u_s}{\partial r} \right)^n ; \quad n \geq 1 \quad (16a)$$

and

$$v = \tilde{k}_h \left(\frac{1}{\gamma_w} \frac{\partial u}{\partial r} \right)^n ; \quad n \geq 1 \quad (16b)$$

where $\tilde{k}_h$ is the undisturbed horizontal permeability for non-Darcian flow; $\tilde{k}_s$ is the horizontal permeability in the smear zone; u is the excess pore water pressure in the undisturbed zone; and u_s is the excess pore water pressure in the smear zone.

The rate of fluid flow through the internal face of the hollow cylindrical slice with internal radius r is

$$2\pi r v \quad (17)$$

The rate of volume change in the hollow cylindrical slice with internal radius r and outer radius r_e is

$$\pi(r_e^2 - r^2) \frac{\partial \varepsilon}{\partial t} \quad (18)$$

where $\partial\epsilon/\partial t$ is the one-dimensional strain rate. For instantaneous loading the strain rate can be expressed as

$$\frac{\partial\epsilon}{\partial t} = -m_v \frac{\partial\bar{u}}{\partial t} \quad (19)$$

where $\bar{u}$ is the average excess pore pressure, and m_v is the volume compressibility of the soil. For continuity, the volume changes in equations (17) and (18) can be equated; rearranging the resulting expression using equations (16) and (19), the pore pressure gradient in the smear and undisturbed zones is represented by

$$\frac{\partial u_s}{\partial y} - (r_w \gamma_w)^{1-1/n} \left[-\frac{r_e^2}{2\tilde{c}_h} \frac{\partial\bar{u}}{\partial t} \frac{\tilde{k}_h}{\tilde{k}_s} \left(1 - \frac{y^2}{N^2} \right) \right]^{1/n} y^{-1/n} \quad (20a)$$

and

$$\frac{\partial u}{\partial y} - (r_w \gamma_w)^{1-1/n} \left(-\frac{r_e^2}{2\tilde{c}_h} \frac{\partial\bar{u}}{\partial t} \right)^{1/n} \left(1 - \frac{y^2}{N^2} \right)^{1/n} y^{-1/n} \quad (20b)$$

where $N = r_e/r_w$, and a change of variable has been made such that $y = r/r_w$. (The derivation of equation (20) is given in the Appendix.)

By using the binomial expansion, the terms involving y on the right-hand side of equation (20) can be represented by a series as

$$\left(1 - \frac{y^2}{N^2} \right)^{1/n} y^{-1/n} = y^{-1/n} \sum_{j=0}^{\infty} \frac{\{-1/n\}_j}{j!} \left(\frac{y}{N} \right)^{2j} \quad (21)$$

where $\{x\}_m$, sometimes called the Pochhammer symbol or rising factorial, is defined by

$$\{x\}_m = x(x+1)(x+2) \dots (x+m-1), \quad \{x\}_0 = 1 \quad (22)$$

The series expansion performed in equation (21) and others below are best obtained using computer algebra systems (CAS). For example, entering the text '(1-(x^2)/(N^2))^(1/n)*x^(1/n)' into the Wolfram Alpha LLC. (2011) website yields, after simplification, equation (21). Substituting equation (21) into equation (20) and integrating (with the boundary conditions $u_s = 0$ at $y = 1$, and $u = u_s$ at $y = s$, where $s = r_s/r_w$) yields the following pore pressure expressions

$$u_s(y) = (r_w \gamma_w)^{1-1/n} \left(-\frac{r_e^2}{2\tilde{c}_{h0}} \frac{\partial\bar{u}}{\partial t} \right)^{1/n} \left(\frac{\tilde{c}_{h0}}{\tilde{c}_h} \right)^{1/n} \times \left(\frac{\tilde{k}_h}{\tilde{k}_s} \right)^{1/n} [g(y) - g(1)]; \quad 1 \leq y < s \quad (23a)$$

and

$$u(y) = (r_w \gamma_w)^{1-1/n} \left(-\frac{r_e^2}{2\tilde{c}_{h0}} \frac{\partial\bar{u}}{\partial t} \right)^{1/n} \left(\frac{\tilde{c}_{h0}}{\tilde{c}_h} \right)^{1/n} \times \left\{ g(y) - g(s) + \left(\frac{\tilde{k}_h}{\tilde{k}_s} \right)^{1/n} [g(s) - g(1)] \right\}; \quad 1 \leq y < N \quad (23b)$$

where $g(y)$ is the integral of equation (21) with respect to y , given by

$$g(y) = n y^{1-1/n} \sum_{j=0}^{\infty} \frac{\{-1/n\}_j}{j![(2j+1)n-1]} \left(\frac{y}{N} \right)^{2j} \quad (24a)$$

(The derivation of $g(y)$ is given in the Appendix.)

Equation (24a) can be formulated using a recurrence relationship as

$$g(y) = \sum_{j=0}^{\infty} a_j(y) \quad (24b)$$

$$a_0 = \frac{n y^{1-1/n}}{n-1} \quad (24c)$$

$$a_j = a_{j-1} \left(\frac{y}{N} \right)^2 \frac{(jn-n-1)(2jn-n-1)}{jn(2jn+n-1)}$$

The average excess pore pressure satisfies the algebraic expression

$$\pi(r_e^2 - r_w^2)\bar{u} = 2\pi \int_{r_w}^{r_s} r u_s(r) dr + 2\pi \int_{r_s}^{r_e} r u(r) dr \quad (25a)$$

or, in the transformed coordinate system

$$\bar{u} = \frac{2}{N^2-1} \left[\int_1^s y u_s(y) dy + \int_s^N y u(y) dy \right] \quad (25b)$$

Substituting equation (23) into equation (25) and performing the appropriate integrations gives the excess pore pressure (and governing equation) as

$$\bar{u} = (r_w \gamma_w)^{1-1/n} \left(-\frac{r_e^2}{2\tilde{c}_{h0}} \frac{\partial\bar{u}}{\partial t} \right)^{1/n} \left(\frac{\tilde{c}_{h0}}{\tilde{c}_h} \right)^{1/n} \beta \quad (26)$$

with

$$\beta = \frac{1}{N^2-1} \left\{ 2\bar{g}(N) - \kappa^{1/n} [2\bar{g}(1) + g(1)(N^2-1)] \right. \\ \left. + (\kappa^{1/n}-1)[g(s)(N^2-s^2) + 2\bar{g}(s)] \right\} \quad (27a)$$

where $\kappa = \tilde{k}_h/\tilde{k}_s$ and $\bar{g}(y)$ is the product of y and equation (24a), integrated with respect to y to give

$$\bar{g}(y) = n^2 y^{3-1/n} \sum_{j=0}^{\infty} \frac{\{-1/n\}_j}{j![(2j+1)n-1][(2j+3)n-1]} \left(\frac{y}{N} \right)^{2j} \quad (27b)$$

Equation (27b) can be formulated using a recurrence relationship as

$$\bar{g}(y) = \sum_{j=0}^{\infty} \bar{a}_j(y) \quad (27c)$$

$$\bar{a}_0 = \frac{n^2 y^{3-1/n}}{(n-1)(3n-1)} \quad (27d)$$

$$\bar{a}_j = \bar{a}_{j-1} \left(\frac{y}{N} \right)^2 \frac{(jn-n-1)(2jn-n-1)}{jn(2jn+3n-1)}$$

The pore pressure at any point in the soil can now be related to the average excess pore water pressure by substituting equation (26) into equation (23). The resulting expressions for pore water pressure in the smear and undisturbed zones are

$$u_s(y) = \frac{\bar{u}}{\beta} \left(\frac{\tilde{k}_h}{\tilde{k}_s} \right)^{1/n} [g(y) - g(1)] \quad (28a)$$

$$u(y) = \frac{\bar{u}}{\beta} \left\{ g(y) - g(s) + \left(\frac{\tilde{k}_h}{\tilde{k}_s} \right)^{1/n} [g(s) - g(1)] \right\} \quad (28b)$$

Usually the expressions for excess pore water pressure would involve explicit functions of time (Hansbo, 1981, 2001); however, as shown below, the solution of equation (26) gives consolidation time as a non-invertible function of $\bar{u}$. The corresponding time is determined when a suitable value for $\bar{u}$ is specified.

ANALYTICAL SOLUTION OF THE GOVERNING EQUATION

Equation (26) can be rewritten as

$$-\partial \tilde{T} = \frac{\partial W}{W^n} \left(\frac{\tilde{c}_{h0}}{\tilde{c}_h} \right) \quad (29)$$

where $\tilde{T}$ is a modified time factor defined by

$$\tilde{T} = 8 \frac{T_{h0}}{\beta^n} \left(\frac{\Delta \sigma}{r_w \gamma_w} \right)^{n-1} \quad (30a)$$

$$T_{h0} = \frac{\tilde{c}_{h0} t}{4r_e^2} \quad (30b)$$

Using a Taylor power series, expansion of equation (15a) about the point $W = 0$ gives (in the recompression zone)

$$\frac{\tilde{c}_{h0}}{W^n \tilde{c}_h} = \left(1 + \frac{\Delta \sigma}{\sigma'_0} \right)^{-(1-C_r/C_k)} \times \sum_{j=0}^{\infty} \frac{\{1 - C_r/C_k\}_j}{j!} \left(1 + \frac{\sigma'_0}{\Delta \sigma} \right) (W)^{j-n} \quad (31)$$

Substituting equation (31) into equation (29), and integrating with the initial condition $W = 1$ at $t = 0$, results in the following time factor-normalised pore pressure relationship in the recompression zone

$$\tilde{T} = - \left(1 + \frac{\Delta \sigma}{\sigma'_0} \right)^{-(1-C_r/C_k)} \times \left[\sum_{j=0}^{\infty} \frac{\{1 - C_r/C_k\}_j}{j!(j-n+1)} \left(1 + \frac{\sigma'_0}{\Delta \sigma} \right)^{-j} (W^{j-n+1} - 1) \right] \quad (32)$$

To avoid writing such large expressions as the right-hand side of equation (32), a shorthand notation is used, whereby a function F , depending on parameters α , θ and W , is described by

$$F[W, \alpha, \theta] = - \left(1 + \frac{\Delta \sigma}{\sigma'_0} \right)^{-(1-\alpha/C_k)} \times \left[\sum_{j=0}^{\infty} \frac{\{1 - \alpha/C_k\}_j}{j!(j-n+1)} \left(1 + \frac{\sigma'_0}{\Delta \sigma} \right)^{-j} (W^{j-n+1} - \theta^{j-n+1}) \right] \quad (33a)$$

Equation (33a) can be formulated using a recurrence relationship as

$$F[W, \alpha, \theta] = - \left(1 + \frac{\Delta \sigma}{\sigma'_0} \right)^{-(1-\alpha/C_k)} \sum_{l=0}^{\infty} \tilde{a}_l \quad (33b)$$

$$\tilde{a}_0 = \frac{W^{1-n} - \theta^{1-n}}{1-n}$$

$$\tilde{a}_j = \tilde{a}_{j-1} \frac{(j-n)(j-\alpha/C_k)}{j(1+j-n)} \left(1 + \frac{\sigma'_0}{\Delta \sigma} \right)^{-1} \frac{W^{1+j-n} - \theta^{1+j-n}}{W^{j-n} - \theta^{j-n}} \quad (33c)$$

For the special case of Darcian flow (i.e. derive equation (32) with $n = 1$) the recurrence relationships become

$$\begin{aligned} \tilde{a}_0 &= \ln \left(\frac{W}{\theta} \right) \\ \tilde{a}_1 &= \left(1 - \frac{\alpha}{C_k} \right) \left(1 + \frac{\sigma'_0}{\Delta \sigma} \right)^{-1} (W - \theta) \\ \tilde{a}_j &= \tilde{a}_{j-1} \frac{(j-1)(j-\alpha/C_k)}{j^2} \left(1 + \frac{\sigma'_0}{\Delta \sigma} \right)^{-1} \frac{W^j - \theta^j}{W^{j-1} - \theta^{j-1}} \end{aligned} \quad (33d)$$

Equation (32) can now be written in the concise notation as

$$\tilde{T} = F[W, C_r, 1] \quad (34)$$

Equation (34) is valid during recompression: that is, when $W \geq W_p$, where W_p corresponds to the preconsolidation pressure. W_p is calculated from

$$W_p = 1 - \frac{\sigma'_0}{\Delta \sigma} \left(\frac{\sigma'_p}{\sigma'_0} - 1 \right) \quad (35)$$

The time factor required to reach the preconsolidation pressure, $\tilde{T}_p$, is determined by substituting W_p into equation (34): hence

$$\tilde{T}_p = F[W_p, C_r, 1] \quad (36)$$

Now the power series representation of equation (15b) is substituted into equation (29) and solved with the initial condition $W = W_p$ at $\tilde{T} = \tilde{T}_p$. The resulting expression for consolidation in the compression phase is

$$\tilde{T} = F[W, C_c, W_p] \frac{C_c}{C_r} \left(\frac{\sigma'_p}{\sigma'_0} \right)^{-(C_c - C_r)/C_k} + F[W_p, C_r, 1] \quad (37)$$

Equation (37) can be used for normally consolidated soils by putting $C_r = C_c$ and $W_p = 1$. The m_{v0} value in $\tilde{T}$ should always be calculated using the recompression index, C_r .

Substituting a given normalised excess pore pressure into equations (34) and (37), the resulting value of $\tilde{T}$ can be found. The time required to reach the specified degree of consolidation is then found from equation (30). Equation (32) is undefined for integer values of n ; however, values very close to integer values give the appropriate consolidation times (e.g. use $n = 1.0001$, not $n = 1$).

When $C_r/C_k = C_c/C_k = 1$ (i.e. $\tilde{c}_h$ does not change during consolidation), equation (34) is numerically equivalent to Hansbo's (2001) non-Darcian radial consolidation equation. For the case of Darcian flow $n = 1$, the corresponding value of $\tilde{T}_{\text{Darcy}}$ is given by

$$\tilde{T}_{\text{Darcy}} = \frac{8T_{h0}}{\mu} \quad (38)$$

The μ parameter can then be any of those derived for Darcian flow conditions. The following expressions for μ were derived for

- (a) a smear zone of constant reduced permeability (Hansbo, 1981)

$$\mu = \ln\left(\frac{N}{s}\right) + (\kappa)\ln(s) - 0.75 \quad (39a)$$

- (b) similar to (a), but plane-strain conditions (Indraratna & Redana, 1997)

$$\mu = \kappa \left[\frac{2}{3} \left(1 - \frac{1}{N^2} \right) - \frac{2(N-s)^3}{3(N-1)N^2} \right] + \frac{2(N-s)^3}{3(N-1)N^2} \quad (39b)$$

- (c) a smear zone with parabolic variation of permeability in the smear zone (Walker & Indraratna, 2006)

$$\mu = \ln\left(\frac{N}{s}\right) - \frac{3}{4} + \frac{\kappa(s-1)^2}{s^2 - 2\kappa s + \kappa} \ln\left(\frac{s}{\sqrt{\kappa}}\right) - \frac{s(s-1)\sqrt{\kappa(\kappa-1)}}{2(s^2 - 2\kappa s + \kappa)} \ln\left(\frac{\sqrt{\kappa} + \sqrt{\kappa-1}}{\sqrt{\kappa} - \sqrt{\kappa-1}}\right) \quad (39c)$$

where κ is the ratio of undisturbed permeability to the permeability at the drain/soil interface. Equation (39c) was simplified by ignoring insignificant terms, and can be used when $n > 10$; otherwise the full expression presented by Walker & Indraratna (2006) is more appropriate, usually for smaller drain spacing. When $n = 1$ and $C_r/C_k = C_c/C_k = 1$, equation (34) is numerically equivalent to Hansbo's (1981) radial consolidation equation.

EFFECT OF C_c/C_k ON NORMALLY CONSOLIDATED SOIL

For normally consolidated soils, if $C_c/C_k < 1$, then c_h increases over time, and the consolidation rate is faster. If $C_c/C_k > 1$ then c_h decreases over time, and the consolidation rate is slower. The degree to which consolidation is slowed or hastened is controlled by the normalised factor $\Delta\sigma/\sigma'_0$. Taken together, C_c/C_k and $\Delta\sigma/\sigma'_0$ give an indication of the total change in c_h over the consolidation period, in accordance with equation (15). In practice, C_c/C_k ranges between 0.5 and 2.0 (Berry & Wilkinson, 1969; Mesri & Rokhsar, 1974), with C_k taken from the empirical relation $C_k = 0.5e_0$.

Figure 3 gives consolidation curves based on the ratio between the final and initial consolidation coefficients (c_{hf} and c_{h0} respectively) for Darcian flow. Although the same change in c_h can be obtained with different C_c/C_k and $\Delta\sigma/\sigma'_0$ values, the resulting difference in degree of consolidation is minimal. The actual values of C_c/C_k and $\Delta\sigma/\sigma'_0$ used in producing Fig. 3 are given in Table 1. Using different parameter values from those in Table 1 will result in slightly faster or slower consolidation. Faster consolidation would occur if c_{hf}/c_{h0} was more than unity; otherwise, slower

Fig. 3. Consolidation curves for normally consolidated soil under Darcian flow conditions

Table 1. Parameters used to produce consolidation curves shown in Fig. 3

C_c/C_k	$\Delta\sigma/\sigma'_0$	c_{hf}/c_{h0}
0.500	5.758	2.600
0.500	3.840	2.200
0.500	2.240	1.800
0.500	0.960	1.400
1.000	1.000	1.000
1.115	6.000	0.800
1.263	6.000	0.600
1.471	6.000	0.400

consolidation would occur. Analysis including the non-linear soil properties is particularly relevant for soils where relatively high values of $\Delta\sigma/\sigma'_0$ will lead to large changes in c_h , and thus potentially large deviations in degree of consolidation compared with calculations based on constant soil properties. For c_{hf}/c_{h0} values close to unity, each 10% increase in c_{hf}/c_{h0} decreases the time to reach 90% consolidation by approximately 5% (compared with the ideal case of $c_{hf}/c_{h0} = 1$). Each 10% decrease in c_{hf}/c_{h0} increases the time to reach 90% consolidation by approximately 10%.

SETTLEMENT

Primary consolidation settlements, ρ , result from changes in effective stress. Once excess pore water pressures are found (i.e. based on $W = \bar{u}/\Delta\sigma$), then the settlements can be calculated using the following expressions. If the initial and final effective stresses (σ'_0 and $\sigma'_0 + \Delta\sigma$) fall in the recompression zone then settlements are calculated with

$$\rho = \frac{HC_r}{1 + e_0} \log \left[1 + \frac{\Delta\sigma}{\sigma'_0} (1 - W) \right] \quad (40a)$$

where H is the depth of soil. If the initial and final effective stresses fall in the compression zone then settlements are given by

$$\rho = \frac{HC_c}{1 + e_0} \log \left[1 + \frac{\Delta\sigma}{\sigma'_0} (1 - W) \right] \quad (40b)$$

If the initial effective stress is less than the preconsolidation pressure, σ'_p , and the final effective stress is greater than σ'_p then settlements in the recompression zone are the same as in equation (40a); settlements in the compression zone are then expressed as

$$\rho = \frac{H}{1 + e_0} \left\{ (C_r - C_c) \log(\text{OCR}) + C_c \log \left[1 + \frac{\Delta\sigma}{\sigma'_0} (1 - W) \right] \right\} \quad (40c)$$

where OCR is the overconsolidation ratio, defined by

$$\text{OCR} = \frac{\sigma'_p}{\sigma'_0} \quad (41)$$

The total primary settlement, ρ_∞ , can be calculated by putting $W = 1$ in the above equations.

DEGREE OF CONSOLIDATION BASED ON SETTLEMENT AND PORE PRESSURE

The degree of consolidation, as determined by pore pressure dissipation, is simply given by

$$U_h = 1 - \frac{\bar{u}}{\Delta\sigma} = 1 - W \quad (42)$$

The degree of consolidation, based on settlement, is written as

$$U_{hs} = \frac{\rho}{\rho_\infty} \quad (43)$$

The relationship between U_{hs} and W depends on the stress history of the soil.

In equation (40), when the stress range is either completely in the recompression zone or in the compression zone, the term U_{hs} is then related to W by

$$U_{hs} = \frac{\log[1 + (\Delta\sigma/\sigma'_0)(1 - W)]}{\log[1 + (\Delta\sigma/\sigma'_0)]} \quad (44a)$$

For other cases, if $\sigma' \leq \sigma'_p$ then

$$U_{hs} = \frac{\log[1 + (\Delta\sigma/\sigma'_0)(1 - W)]}{(1 - C_c/C_r) \log(OCR) + C_c/C_r \log[1 + (\Delta\sigma/\sigma'_0)]} \quad (44b)$$

If $\sigma' > \sigma'_p$ then

$$U_{hs} = \frac{(1 - C_c/C_r) \log(OCR) + C_c/C_r \log[1 + (\Delta\sigma/\sigma'_0)(1 - W)]}{(1 - C_c/C_r) \log(OCR) + C_c/C_r \log[1 + (\Delta\sigma/\sigma'_0)]} \quad (44c)$$

The degrees of consolidation based on pore pressure and settlement are different (Leroueil *et al.*, 1990; Indraratna *et al.*, 2005). For equation (44a) U_h lags behind U_{hs} depending on $\Delta\sigma/\sigma'_0$, as shown in Fig. 4. When determining the degree of consolidation for normally consolidated soils by settlement data, it is important to note that the effective stress in the soil will be less than expected, particularly during the middle stages of consolidation, if U_h is assumed to be equal to U_{hs} . A large variety of behaviour can occur for overconsolidated soil, as shown in Fig. 5, depending on the value of C_c/C_r , OCR and $\Delta\sigma/\sigma'_0$. Dissipation of excess pore pressure is fast during the recompression stage, because C_r is low relative to C_c , while the resulting settlements are relatively small compared with the ultimate settlement. This means that U_h will be greater than U_{hs} during recompression, but U_{hs} may or may not become greater than U_h during the compression stage. The many parameters involved, and the subsequent large variety of behaviour relating U_{hs} to U_h for

Fig. 4. Comparison between degree of consolidation based on settlement and pore pressure for normally consolidated soil

Fig. 5. Comparison between degree of consolidation based on settlement and pore pressure for overconsolidated soil (OCR = 1.5): (a) $C_c/C_r = 2$; (b) $C_c/C_r = 10$

overconsolidated soil, emphasises the need for accurate determination of the soil stress history, and the care needed when specifying construction milestones based on degree of consolidation.

APPROXIMATION FOR ARBITRARY LOADING

The consolidation behaviour expressed by equation (37) is valid only for instantaneous loading. Unfortunately, including arbitrary time-dependent loading such as Conte & Troncone (2009), or even a simple ramp load (Walker & Indraratna, 2009), will make equation (29) non-separable and, to the authors' knowledge, analytically unsolvable. However, arbitrary loading can be simulated by subdividing a continuous loading function into a finite number of instantaneous step loads (see Fig. 6). The only restriction is that average excess pore pressure cannot become negative ($u > 0$). For example, slightly decreasing loads associated with submergence of fill can be modelled, but the swelling associated with preload removal (caused by dissipation of negative pore pressure) cannot be modelled with the consolidation equations presented.

For the m th loading stage, as shown in Fig. 6, at $\tilde{T}_m$ the load increases from $\Delta\sigma_{m-1}$ to $\Delta\sigma_m$. The loading stage ends at $\tilde{T}_{m+1}$. The excess pore water pressure at the end of the last increment, u_m^- , is known (e.g. in the first loading step $u_m^- = 0$). The pore pressure after the load application, u_m^+ , is given by

Fig. 6. Schematic diagram of piecewise loading in terms of loading rate combined with resulting excess pore water pressure

$$u_m^+ = u_m^- + \Delta\sigma_m - \Delta\sigma_{m-1} \quad (45)$$

The normalised pore pressure at the beginning of the load increment, W_m^+ , is then described by

$$W_m^+ = \frac{u_m^+}{\Delta\sigma_m} \quad (46)$$

The initial conditions for solution of equation (29) have now been found: $W = W_m^+$ at $\tilde{T} = \tilde{T}_m$. There are three cases to consider in the solution of equation (29). If the preconsolidation pressure has been exceeded in previous loading steps, then the normalised pore pressure in the m th loading step is governed by

$$\tilde{T} = \tilde{T}_m + F[W, C_c, W_m^+] \frac{C_c}{C_r} \left(\frac{\sigma'_p}{\sigma'_0} \right)^{-(C_c - C_r)/C_k} \quad (47a)$$

If the preconsolidation pressure has not been exceeded in previous load steps, then the expressions for normalised pore pressure in the recompression and compression zones are

$$\tilde{T} = \tilde{T}_m + F[W, C_r, W_m^+] \quad (47b)$$

and

$$\begin{aligned} \tilde{T} &= \tilde{T}_m + F[W, C_c, W_p] \frac{C_c}{C_r} \left(\frac{\sigma'_p}{\sigma'_0} \right)^{-(C_c - C_r)/C_k} + \tilde{T} \\ &= \tilde{T}_m + F[W_p, C_r, W_m^+] \end{aligned} \quad (47c)$$

The consolidation behaviour can now be described at any point. The following list of steps describes the process of constructing a graph of pore pressure against time. The process is best automated in a computer program.

- Step 1: approximate the arbitrary loading by a finite number of instantaneous step loads.
- Step 2: convert the loading times to time factor values, $\tilde{T}$ using equation (30).
- Step 3: calculate W_m^+ using equations (45) and (46).

- Step 4: calculate $\tilde{T}$ in small increments of W , with equation (47), until $\tilde{T} > \tilde{T}_{m+1}$.
- Step 5: calculate u_{m+1}^- . Either use the last W value obtained in step 4, or interpolate between the last two values of W found in step 4.
- Step 6: repeat steps 3 to 5 for each loading stage.
- Step 7: convert the time factor values obtained in above steps to time values using equation (30).

In the steps just described, $\Delta\sigma$ changes, and in the m th loading stage is equal to $\Delta\sigma_m$. For cases where soil properties vary with depth, the soil profile can be divided into sublayers and the above process performed for each sublayer. The above steps have been implemented in a VBA/spreadsheet program that can be downloaded from <http://www.uow.edu.au/eng/research/geotechnical/software/index.html>.

ILLUSTRATIVE EXAMPLE

A pore pressure and settlement analysis has been performed on a soil/drain system with the following properties: $r_w = r_s = 0.07$ m; $r_e = 1.4$ m; $C_c = 0.7$; $C_r = 0.175$; $C_k = 0.875$; $\sigma'_0 = 10$ kPa; $\sigma'_p = 10, 20, 30$ kPa; $n = 1.001, 1.3$; $\gamma_w = 10$ kN/m³; $e_0 = 1$; $H = 1$ m. No smear zone is modelled. The resulting average pore pressure and settlement plots are shown in Fig. 7. There are a few salient points to note from this figure. For the two overconsolidated soils ($\sigma'_p = 20, 30$ kPa), the change from recompression to compression can be observed during the first ramp loading stage. At the preconsolidation pressure, the slope of the excess pore pressure plot sharply increases, reflecting the slower rate of consolidation during compression. Just prior to the preconsolidation pressure being reached, the dissipation of pore pressure in the highly overconsolidated soil ($\sigma'_p = 30$ kPa) actually exceeds that generated by the load application; the pore pressure reduces despite load still being applied. This occurs because $C_r/C_k < 1$, resulting in an increasingly rapid consolidation reaching a maximum just prior to the preconsolidation pressure. The difference between the Darcian flow of Fig. 7(a) and the non-Darcian flow of Fig. 7(b) is small for the highly overconsolidated soils. For the normally consolidated soil ($\sigma'_p = 10$ kPa), the difference is illustrated during the first ramp loading stage. For non-Darcian flow the pore pressure curve is slightly flatter, because, owing to the power law flow relationship, the higher pore pressures result in faster flow. The settlement plots illustrate the importance of minimising the disturbance caused by vertical drain installation, which can to some extent destroy any existing structure in the soil. Hence the preconsolidation pressure may be lowered, which, as shown in Fig. 7, causes greater settlements for the same pressure increase. This example shows that, with the equations presented above, almost any primary consolidation problem involving radial drainage can be modelled (provided the effective stress increases with time). When more than one soil layer is present, the analysis can simply be repeated with different soil properties (Walker & Indraratna, 2009).

LABORATORY VERIFICATION

The large-scale consolidometer at the University of Wollongong is a steel cylinder 450 mm in diameter and 950 mm high, with the ability to monitor the pore pressure and settlement response of soils under load. Since 1995, the consolidometer has been used by University of Wollongong researchers to study various aspects of vertical drain consolidation (Indraratna & Redana, 1995, 1998a, 1998b; Indraratna *et al.*, 2002, 2004). Fig. 8 shows a schematic diagram of the large-scale consolidometer. The cylindrical

Fig. 7. Non-linear radial consolidation for non-Darcian flow exponent: (a) $n = 1.001$; (b) $n = 1.3$

cell consists of two stainless steel sections bolted together along the two joining flanges. Top and bottom drainage can be facilitated by placing a geotextile on the cell base and clay surface. If only radial drainage is required, the permeable geotextile is replaced by an impervious plastic sheet. The clay is thoroughly mixed with water to ensure full saturation, and the resulting slurry is placed in the consolidation cell in layers approximately 20 cm thick. It is not necessary to fill the entire consolidometer, since an internal 'riser' can be used to transfer loads from the loading piston to the shortened sample. The ring friction expected with such a large height/diameter ratio (1.5–2) is almost eliminated by using an ultra-smooth Teflon membrane around the cell boundary (friction coefficient less than 0.03). Surcharge loading with a maximum capacity of 1200 kN can be applied by an air jack compressor system by way of a rigid piston 50 mm thick. Vacuum loading with a maximum capacity of 100 kPa can be applied through the central hole of the rigid piston. An LVDT (linear variable differential

Fig. 8. Schematic diagram of large-scale consolidation apparatus (Indraratna & Redana, 1998a, with permission from ASCE)

transformer) transducer is placed on top of the piston to monitor surface settlement. Pore water pressures are monitored by strain gauge type pore pressure transducers installed through small holes in the steel cell at various positions in the soil. Transducers are easily located on the cell periphery, or can be placed within the clay by using small-diameter stainless steel tubes. The LVDT and pore pressure transducers are connected to a PC-based data logger.

The soil is subjected to an initial preconsolidation pressure (usually $\sigma'_p = 20$ kPa), until the settlement rate becomes negligible (strain rate < 1 mm/day). The load is then removed, and a single vertical drain is installed using a rectangular steel mandrel that acts as a protector for the drain and a guide during penetration. Sand compaction piles may be installed with a circular pipe mandrel. Depending on the purpose of the test, vertical and horizontal samples may be cored to investigate the effect of drain installation on soil properties. Once the drain has been installed, the required loading sequence is applied to the sample.

The consolidation test described below was conducted by Rujikiatkamjorn (2006) to verify the radial consolidation model of Indraratna *et al.* (2005) expressed by equation (6). The model of Indraratna *et al.* (2005) determines the best value of c_h to use in Hansbo's (1981) consolidation equation with reference to C_c , C_k and the stress range. The soil consisted of reconstituted alluvial clay from the Moruya floodplain. The clay size particles ($< 2 \mu\text{m}$) accounted for about 40–50% of the soil specimen. For this Moruya clay, the water content and plastic limits were 45% and 17% respectively. The saturated unit weight was 17 kN/m^3 . In two separate tests, the soil was subjected to an initial preconsolidation pressure, $\sigma'_p = 20$ kPa and 50 kPa for five days. The load was removed, a $100 \text{ mm} \times 4 \text{ mm}$ band drain was centrally installed, and the preconsolidation loads were re-applied. Drainage in the vertical direction was prevented. Once settlements became negligible, the load was increased by 30 kPa and 50 kPa respectively (i.e. $\Delta\sigma = 30, 50 \text{ kPa}$). Settlements were monitored after this point, with the soil

consolidating in the compression range. The properties of the soil–drain system are given in Table 2, based on the smear zone with constant reduced permeability. Only for the purpose of analysis, Darcian flow was considered (i.e. $n = 1.001$). The degree of consolidation based on settlement was calculated and compared with laboratory data, the proposed model, and Indraratna *et al.* (2005). The comparison of the three approaches is shown in Fig. 9.

Figure 9 shows good agreement between the measured and predicted values of degree of consolidation for both the proposed non-linear equations and Indraratna *et al.* (2005). As mentioned above, in this case the method of Indraratna *et al.* (2005) will give the same results as Hansbo (1981). The proposed equations give a slightly better match in the early stages of consolidation; the difference is less for Test 2 than for Test 1. Both methods are expected to converge to 100% consolidation at longer times, as both are based on the same value of ultimate settlement. As the consolidation coefficient is increasing during consolidation ($C_c/C_k < 1$), by using an average value of c_h as in Indraratna *et al.* (2005), settlement is expected to be overpredicted in the initial stages of consolidation, as in Fig. 9. The combination of $C_c/C_k = 0.64$ and

$\Delta\sigma/\sigma'_0 = 1.5$ and 1 produces ratios of final to initial consolidation coefficient of 1.39 and 1.28 respectively, for Test 1 and Test 2. Fig. 3 showed that for such values of c_{hf}/c_{h0} the difference between the non-linear equations and the Hansbo (1981) equation are small. This explains the small difference between the analysis based on the proposed relationship and that of Indraratna *et al.* (2005). Larger and more significant discrepancies between the two approaches would be expected for combinations of C_c/C_k and $\Delta\sigma/\sigma'_0$ producing c_{hf}/c_{h0} values much greater or much less than one.

APPLICATION TO A CASE STUDY AND THE IMPLICATIONS OF PARAMETER n

At the site of the Second Bangkok International Airport (SBIA), several embankments were constructed over soft clay stabilised by vertical drains and vacuum pressure, combined with surcharge loading. The surface settlement and excess pore water pressure at the middle of two embankments, TV2, incorporating vacuum loading and vertical drains are analysed here. The basic soil parameters were reported by Indraratna *et al.* (2005). The initial in situ soil properties are assumed to be the same for both Darcian and non-Darcian flow analysis. A summary of the properties used for analysis is given in Table 3. Soil layers are divided into 1 m thick sublayers (0.5 m for when required), with stresses and pore pressure calculated at sublayer midpoints. Surcharge load reduces with depth according to a Boussinesq stress distribution under the centre of a trapezoidal embankment with 40 m base width and 36 m crest width. Vacuum pressure load factor reduces from unity at surface to zero at 15 m depth. Linear load graphs in Fig. 10 are discretised into small step loads every 1.744 days. Geometry/smear properties for all layers are: $k_h/k_s = 2$, $r_e = 0.5$ m, $r_w = 0.025$ m, $r_s = 0.1$ m. The analysis in detail can be found in the VBA/spreadsheet mentioned above. Fig. 10 presents the loading history with both measured surface settlements at the embankment centreline and pore water pressures 3 m below the embankment centreline.

To study the effect of non-Darcian flow, the predicted settlements and pore pressures are calculated based on n values varying between 1 and 1.5 (vacuum pressure is treated as an equivalent surcharge). For settlement predictions (Fig. 10(b)), when $n = 1$, a good agreement between the predicted and measured results can be obtained. When n increases, the predicted settlements overestimate the measured data. The predicted and measured pore pressures are shown in Fig. 10(c). As expected, the pore pressure dissipates rapidly when n increases. Within 60 days, the agreement between field observations and predictions is better according to Darcian flow. However, beyond 60 days, the prediction with non-Darcian flow ($n = 1.5$) is more realistic.

MODEL LIMITATIONS AND FUTURE RECOMMENDATIONS

Although the proposed analytical model is capable of determining degree of consolidation based on both pore

Table 2. Parameters used in analysis (Indraratna *et al.*, 2005)

Parameter	Test 1	Test 2
C_c	0.29	0.29
C_k	0.45	0.45
Influence radius, r_e : m	0.45	0.45
Equivalent drain radius, r_w : m	0.066	0.066
Smear zone radius, r_s : m	0.2	0.2
$N = r_e/r_w$	6.79	6.79
$s = r_s/r_w$	3.02	3.02
Initial horizontal permeability, k_{h0} : $\times 10^{-10}$ m/s	4.4	4
k_h/k_s	1.5	1.5
Initial void ratio, e_0	1	0.95
c_{h0} : $\times 10^{-3}$ m ² /day	1.20	2.68
Initial height, H : m	0.925	0.87
Preconsolidation pressure, σ'_p : kPa	20	50
Load, $\Delta\sigma$: kPa	30	50

Fig. 9. Comparison between non-linear model and Indraratna *et al.* (2005)

Table 3. Parameters used in analysis for embankment TV2

Layer	Unit weight: kN/m ³	σ'_p :* kPa	C_r	C_c	C_k	e_0	k_{h0} : $\times 10^{-10}$ m/s
0.0–2.0 m	16	58	0.06	0.37	0.9	1.8	90.3
2.0–8.5 m	15	45	0.08	1.6	0.4	2.8	38.1
8.5–10.5 m	15	70	0.05	1.7	1.2	2.4	18.1
10.5–12.0 m	16	80	0.03	0.95	0.9	1.8	7.68

* Constant within a layer.

Fig. 10. Embankment TV2: (a) loading history; (b) surface settlements at centreline; and (c) pore pressure at 3 m depth below centreline

pressure dissipation and strain, it has some inherent limitations, either through the assumptions made, or through simplifications applied to facilitate the mathematical formulations and solutions in this analytical approach.

Some of these limitations are highlighted below.

- (a) The equations do not capture the strain-rate effects directly, as the model is a stress-based approach, and does not incorporate creep effects. For a given time-settlement curve, the rate of dissipation of excess pore pressure tends to be slower when considering creep that constitutes part of the total time-dependent settlement. However, extension of this work is currently being carried out at the University of Wollongong adopting a large-strain radial consolidation theory.
- (b) The proposed model does not include the effects of vertical consolidation through c_v , because, for long vertical drains (say exceeding 12 m or so) with relatively close spacing (1.0–1.5 m), the radial consolidation is predominantly governed by c_h . Therefore the question of anisotropy (c_h/c_v or k_h/k_v ratio) is not a major concern in this analysis. However, for short vertical drains with larger drain spacing, the contribution from c_v will become increasingly relevant, and this is to be captured as future work where short prefabricated vertical drains

can still be used effectively, for example under rail tracks, as demonstrated by Indraratna *et al.* (2010).

- (c) Lateral soil displacements based on elliptical cavity expansion theory (ECET) have been recently analysed (Ghandeharioon *et al.*, 2010); however, for simplicity, the mathematical formulations related to ECET have not been incorporated in the current analysis.
- (d) The non-Darcian flow equation adopted in this study is similar to Hansbo (2001) and Sathanathan & Indraratna (2006). However, the scope of possible flow is limited by the value of n and the type of algebraic expression.
- (e) Well resistance is ignored in the equations, as the authors have shown elsewhere that unless the vertical drains exceed 20 m or so in length, the well resistance is insignificant compared with the smear effects (Indraratna & Redana, 2000).
- (f) Indraratna & Redana (1997) proved, using large-scale consolidation tests, that even within the smear zone, the lateral permeability k_h and hence the lateral coefficient of consolidation c_h can vary. Subsequently, Walker & Indraratna (2006) introduced a parabolic fit to this variation. However, for simplicity of the mathematical formulations, the authors have used a constant k_h within the smear zone, but it is a reduced value (an equivalent average) compared with the higher k_h in the outer undisturbed zone. Use of a parabolic fit may provide a more comprehensive model, but the associated cumbersome mathematical solutions may not necessarily give a much greater accuracy.
- (g) The model assumes full saturation and steady-state flow (both Darcian and non-Darcian). The unsaturated conditions at the time of installation of relatively dry drains and the occurrence of air gaps (occluded air) at the drain/soil interfaces (Indraratna *et al.*, 2004) have not been considered in this analysis. The current solutions will be further enhanced if the effects of soil/drain interfaces for partially saturated conditions are incorporated.
- (h) In situ (undisturbed) soils usually possess a characteristic structure, whose behaviour can be significantly different from that of the same material in a reconstituted state, as explained below.
 - (i) Soil structure creates a material that is initially quite stiff at relatively low stress levels.
 - (ii) When a soil with structure is remoulded/reconstituted, usually there is a reduction in the voids ratio.
 - (iii) During virgin yielding, structured soil is generally more compressible than the reconstituted soil (Liu & Carter, 2002).

A short preconsolidation period in the laboratory may cause minimal effect on soil settlements and excess pore pressure dissipation. The use of reconstituted soil parameters to predict field behaviour may result in the underestimation of soil settlement and excess pore pressure. The selection of soil parameters (reconstituted or in situ soil) should be based on soil conditions to obtain accurate predictions of soil settlement and excess pore pressure.

CONCLUSION

An analytical solution to non-linear radial consolidation problems has been presented and verified against a large-scale laboratory test. The evolution of pore pressure with time described by equation (37) is valid for both Darcian and non-Darcian flow, and can capture the behaviour of overconsolidated and normally consolidated soils. Consolidation may be faster or slower when compared with the cases with constant material properties for non-linear material properties. The difference depends on the compressibility/permeability ratios (C_c/C_k and C_r/C_k), the preconsolidation

pressure (σ'_p) and the stress increase ($\Delta\sigma/\sigma'_0$). If $C_c/C_k < 1$ or $C_r/C_k < 1$, then the coefficient of consolidation increases as excess pore pressures dissipate, and the corresponding consolidation is faster. If $C_c/C_k > 1$ or $C_r/C_k > 1$, then the coefficient of consolidation decreases as excess pore pressures dissipate, and the corresponding consolidation becomes slower. For the case where $C_c/C_k = 1$, the solution is identical to Hansbo (2001). The equations presented give an analytical solution to non-linear radial consolidation that can be used to verify purely numerical methods. Many vertical drain problems where vertical drainage is negligible and effective stresses always increase can be analysed with the approximation for arbitrary loading.

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APPENDIX

Derivations of equations (20a) and (20b)

By equating equations (17) and (18), the radial flow rate is assumed to be equal to the rate of volume change of soil mass in the vertical direction. Therefore

$$2\pi r v = \pi(r_e^2 - r^2) \frac{\partial \varepsilon}{\partial t} \quad (48)$$

Substituting equations (16) and (19) gives

$$\frac{\partial u_s}{\partial r} = \left[\frac{-\pi(r_e^2 - r^2) m_v \gamma_w}{2\pi r \tilde{k}_s} \frac{\partial \bar{u}}{\partial t} \right]^{1/n}; \quad n \geq 1 \quad (49a)$$

$$\frac{\partial u}{\partial r} = \left[\frac{-\pi(r_e^2 - r^2) m_v \gamma_w}{2\pi r \tilde{k}_h} \frac{\partial \bar{u}}{\partial t} \right]^{1/n}; \quad n \geq 1 \quad (49b)$$

Substituting $N = r_e/r_w$ and $y = r/r_w$ in equations (49a) and (49b) yields equations (20a) and (20b).

Derivation of $g(y)$

From equation (20), $g(y)$ can be expressed as

$$g(y) = \int y^{-1/n} \sum_{j=0}^{\infty} \frac{\{-1/n\}_j}{j!} \left(\frac{y}{N}\right)^{2j} \quad (50)$$

$$g(y) = \sum_{j=0}^{\infty} \frac{\{-1/n\}_j}{j! N^{2j}} \int y^{2j-1/n} \quad (51)$$

$$g(y) = \sum_{j=0}^{\infty} \frac{\{-1/n\}_j}{j! N^{2j}} \frac{y^{2j-(1/n)+1}}{2j-(1/n)+1} \quad (52)$$

Let

$$a_j(y) = \sum_{j=0}^{\infty} \frac{\{-1/n\}_j}{j! N^{2j}} \frac{y^{2j-(1/n)+1}}{2j-(1/n)+1} \quad (53)$$

Then

$$a_0(y) = \frac{ny^{1-(1/n)}}{n-1} \quad (24b)$$

$$a_{j-1}(y) = \frac{\{-1/n\}_{j-1}}{(j-1)! N^{2(j-1)}} \frac{y^{2(j-1)-(1/n)+1}}{2(j-1)-(1/n)+1} \quad (54)$$

Dividing equation (53) by equation (54) gives

$$\frac{a_j(y)}{a_{j-1}(y)} = \frac{\frac{\{-1/n\}_j}{j! N^{2j}} \frac{y^{2j-(1/n)+1}}{2j-(1/n)+1}}{\frac{\{-1/n\}_{j-1}}{(j-1)! N^{2(j-1)}} \frac{y^{2(j-1)-(1/n)+1}}{2(j-1)-(1/n)+1}} \quad (55)$$

$$\frac{a_j(y)}{a_{j-1}(y)} = \frac{\{-1/n\}_j}{\{-1/n\}_{j-1}} \frac{(j-1)! N^{2(j-1)}}{j! N^{2j}} \times \frac{y^{2j-(1/n)+1}}{y^{2(j-1)-(1/n)+1}} \frac{2(j-1)-(1/n)+1}{2j-(1/n)+1} \quad (56)$$

$$\frac{a_j(y)}{a_{j-1}(y)} = \frac{-1/n + j - 1}{j} \frac{y^2}{N^2} \frac{2jn - n - 1}{2jn - 1 + n} \quad (57)$$

$$\frac{a_j(y)}{a_{j-1}(y)} = \frac{y^2}{N^2} \frac{(jn - n - 1)(2jn - n - 1)}{jn(2jn + n - 1)} \quad (58)$$

NOTATION

C_c	compression index
C_k	permeability change index
C_r	recompression index
$\tilde{c}_h$	horizontal coefficient of consolidation for non-Darcian flow
c_{hf}	final value of horizontal consolidation coefficient
$\tilde{c}_{h0}$	initial horizontal coefficient of consolidation for non-Darcian flow
d_e	diameter of influence area
e	void ratio
e_0	initial void ratio
e_f	final void ratio
F	function
g	function
H	height of soil; drainage length
i	hydraulic gradient; integer
j	integer
$\tilde{k}$	non-Darcian coefficient of consolidation
$\tilde{k}_h$	undisturbed non-Darcian horizontal permeability
$\tilde{k}_s$	smear zone non-Darcian horizontal permeability
m	integer
m_v	coefficient of volume compressibility
m_{v0}	initial value of volume compressibility
N	ratio of influence radius to drain radius
n	non-Darcian flow index
OCR	overconsolidation ratio
P_{av}	averaging factor to account for changing coefficient of consolidation
r	radial coordinate
r_e	radius of influence area
r_m	equivalent radius of mandrel
r_s	radius of smear zone
r_w	radius of drain
s	ratio of smear zone radius to drain radius
T_h	time factor for horizontal consolidation
T_{h0}	horizontal time factor calculated from initial parameters
$\tilde{T}$	modified time factor
$\tilde{T}_{Darcy}$	Darcian time factor
$\tilde{T}_m$	modified time factor at application of m th instantaneous load
$\tilde{T}_p$	modified time factor at preconsolidation pressure
t	time
U	degree of consolidation
U_h	average degree of consolidation in horizontal direction
U_{hs}	degree of consolidation calculated with settlement data
u	pore water pressure
u_m^-	pore pressure immediately before application of m th instantaneous load
u_m^+	pore pressure immediately after application of m th instantaneous load
u_s	smear zone pore pressure
$\bar{u}$	average excess pore pressure
$\bar{u}_0$	initial excess pore pressure
W	normalised pore pressure
W_p	normalised pore pressure at preconsolidation pressure

v	velocity of flow
y	transformed integration variable
α	function parameter
β	non-Darcian radial consolidation parameter
γ_w	unit weight of water
ε	strain
$\partial\varepsilon/\partial t$	volumetric strain rate
θ	function parameter
κ	ratio of undisturbed permeability to permeability at drain/soil interface
μ	smear zone parameter for Darcian flow
ρ	settlement
σ	total stress
$\Delta\sigma$	change in total stress
$\bar{\sigma}$	average total stress
σ'	effective stress
σ'_0	initial effective stress
σ'_p	preconsolidation stress

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