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On twisted higher-rank graph C*-algebras

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ON TWISTED HIGHER-RANK GRAPH C^*-ALGEBRAS

ALEX KUMJIAN, DAVID PASK, AND AIDAN SIMS

Abstract. We define the categorical cohomology of a k-graph Λ and show that the first three terms in this cohomology are isomorphic to the corresponding terms in the cohomology defined in our previous paper. This leads to an alternative characterisation of the twisted k-graph C^*-algebras introduced there. We prove a gauge-invariant uniqueness theorem and use it to show that every twisted k-graph C^*-algebra is isomorphic to a twisted groupoid C^*-algebra. We deduce criteria for simplicity, prove a Cuntz-Krieger uniqueness theorem and establish that all twisted k-graph C^*-algebras are nuclear and belong to the bootstrap class.

1. Introduction

Higher-rank graphs, or k-graphs, are k-dimensional analogues of directed graphs which were introduced by the first two authors [10] to provide combinatorial models for the higher-rank Cuntz-Krieger algebras investigated by Robertson and Steger in [23]. The structure theory of k-graph C^*-algebras is becoming quite well understood [4, 7, 8, 9, 22], and the class of k-graph algebras has been shown to contain many interesting examples [12, 17].

In [14] we introduced a homology theory $H_*(\Lambda)$ for each k-graph Λ and the corresponding cohomology $H^*(\Lambda, A)$ with coefficients in an abelian group A. We proved a number of fundamental results providing tools for calculating homology, and showed that the homology of a k-graph is naturally isomorphic to that of its topological realisation. Of most interest to us was to show how, given a k-graph and a $\mathbb{T}$-valued 2-cocycle ϕ, one may construct a twisted k-graph C^*-algebra $C^*_\phi(\Lambda)$. Up to isomorphism, $C^*_\phi(\Lambda)$ only depends on the cohomology class of ϕ. Examples of this construction include all noncommutative tori, and also the Heegaard-type quantum 3-spheres of [2].

The purpose of this paper is to begin to analyse the structure of twisted k-graph C^*-algebras. In particular, to provide a groupoid model for twisted k-graph C^*-algebras, and to establish versions of the standard uniqueness theorems. The path groupoid of a k-graph was the basis for the description of k-graph C^*-algebras in [10], and many key theorems about k-graph C^*-algebras flow from this description and Renault's structure theory for groupoid C^*-algebras [21]. We therefore set out to show that each twisted k-graph C^*-algebra is also isomorphic to the twisted groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda, \sigma)$ associated to the path groupoid $\mathcal{G}_\Lambda$ and an appropriate continuous $\mathbb{T}$-valued 2-cocycle on $\mathcal{G}_\Lambda$. It is not immediately clear how to manufacture a groupoid cocycle from a k-graph cocycle. Part of the difficulty lies in that continuous groupoid cocycle cohomology is based on the simplicial structure of groupoids while the k-graph cohomology of [14] is based on the cubical structure of k-graphs.

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Our solution to this difficulty is to introduce another cohomology theory $\underline{H}^*(\Lambda, A)$ for k-graphs, defined by analogy with continuous groupoid cocycle cohomology using the simplicial structure of the k-graph as a small category. We call this the categorical cohomology of Λ (it is no doubt closely related to the standard notion of the cohomology of a small category, see [1]), and refer to the theory developed in [14] simply as the cohomology of Λ or, if we wish to emphasise the distinction between the two theories, as the cubical cohomology of Λ. It is relatively straightforward to see (Remark 3.9 and Theorem 3.10) that the cohomology groups $H^0(\Lambda, A)$ and $H^1(\Lambda, A)$ of [14] are isomorphic to the corresponding categorical cohomology groups $\underline{H}^0(\Lambda, A)$ and $\underline{H}^1(\Lambda, A)$.

Of most interest to us, because of its role in the definition of twisted C^*-algebras, is second cohomology. We show in Theorem 3.16 and Theorem 4.15 that there is a map between (cubical) 2-cocycles and categorical 2-cocycles on a k-graph Λ that induces an isomorphism $H^2(\Lambda, A) \cong \underline{H}^2(\Lambda, A)$. However, this result requires substantially more argument than those discussed in the preceding paragraph. The proof occupies the greater part of Section 3 and all of Section 4. Our approach is inspired by the classification of central extensions of groups by second cohomology (see [3, §IV.3]). We first construct by hand a map $\phi \mapsto c_\phi$ from cubical cocycles to categorical cocycles which determines a homomorphism $\psi : H^2(\Lambda, A) \to \underline{H}^2(\Lambda, A)$. We then define the notion of a central extension of a k-graph by an abelian group, and show that each categorical A-valued 2-cocycle c on Λ determines a central extension $\mathcal{X}_c$ of Λ by A. We show that isomorphism classes of central extensions of Λ by A form a group $\operatorname{Ext}(\Lambda, A)$, and that the assignment $c \mapsto \mathcal{X}_c$ determines an isomorphism $\underline{H}^2(\Lambda, A) \cong \operatorname{Ext}(\Lambda, A)$ (cf. [1, Theorem 2.3] and [21, Proposition I.1.14]). We show that for $c \in \underline{Z}^2(\Lambda, A)$ there is a section $\sigma : \Lambda \to \mathcal{X}_c$ which gives rise to a cubical cocycle ϕ_c such that $[c_{\phi_c}] = [c]$ and $[\phi_{c_\phi}] = [\phi]$. This shows that $\psi : H^2(\Lambda, A) \to \underline{H}^2(\Lambda, A)$ is an isomorphism.

It is, of course, natural to ask whether $H^n(\Lambda, A) \cong \underline{H}^n(\Lambda, A)$ for all n. We suspect this is so, but have not found a proof as yet, and the methods we use to prove isomorphism of the first three cohomology groups do not seem likely to extend readily to a general proof. In any case, we expect that the central extensions of k-graphs introduced here are of interest in their own right. For example, we believe that extensions of k-graphs can be used to adapt Elliott's argument [6, proof of Theorem 2.2] — which shows that the K-groups of a noncommutative torus are isomorphic to those of the corresponding classical torus — to show that the K-groups of a twisted k-graph C^*-algebra are identical to those of the untwisted algebra whenever the twisting cocycle is obtained from exponentiation of a real-valued cocycle.

In the second half of the paper we turn to the relationship between categorical cohomology and twisted C^*-algebras of k-graphs. We define the twisted C^*-algebra $C^*(\Lambda, c)$ associated to a categorical $\mathbb{T}$-valued 2-cocycle c on a row-finite k-graph Λ with no sources, and show that $C^*_\phi(\Lambda) \cong C^*(\Lambda, c_\phi)$ for each cubical $\mathbb{T}$-valued 2-cocycle ϕ. The advantage of the description of twisted k-graph C^*-algebras in terms of categorical cocycles is that it closely mirrors the usual definition of the C^*-algebra of a k-graph. This allows us to commence a study of the structure theory of twisted k-graph C^*-algebras. We prove that there is map $c \mapsto \sigma_c$ which induces a homomorphism from the second categorical cohomology of a k-graph to the second continuous cohomology of the associated path groupoid. We then prove that for a categorical $\mathbb{T}$-valued 2-cocycle c on Λ, there is a homomorphism from the twisted k-graph C^*-algebra associated to c to Renault's twisted groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda, \sigma_c)$; this shows in particular, that all the generators of every

twisted k-graph C^*-algebra are nonzero. We then prove a version of an Huef and Raeburn's gauge-invariant uniqueness theorem for twisted k-graph C^*-algebras, and use it to prove that $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$.

We finish up in Section 8 by using the results of the previous sections to establish some fundamental structure results. We use the realisation of each twisted k-graph C^*-algebra as a twisted groupoid C^*-algebra, together with Renault's theory of groupoid C^*-algebras [21] to prove a version of the Cuntz-Krieger uniqueness theorem. We also indicate how groupoid technology applies to describe twisted C^*-algebras of pullback and cartesian-product k-graphs, and to show that every twisted k-graph C^*-algebra is nuclear and belongs to the bootstrap class $\mathcal{N}$.

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2. Preliminaries

2.1. **Higher-rank graphs.** We adopt the conventions of [13, 16] for k-graphs. Given a nonnegative integer k, a *k-graph* is a nonempty countable small category Λ equipped with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. When $d(\lambda) = n$ we say λ has *degree* n. We will typically use d to denote the degree functor in any k-graph in this paper.

For $k \geq 1$, the standard generators of $\mathbb{N}^k$ are denoted $e_1, \dots, e_k$, and for $n \in \mathbb{N}^k$ and $1 \leq i \leq k$ we write n_i for the ith coordinate of n. For $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ let $|n| := \sum_{i=1}^k n_i$; for $\lambda \in \Lambda$ we define $|\lambda| := |d(\lambda)|$. For $m, n \in \mathbb{N}^k$, we write $m \leq n$ if $m_i \leq n_i$ for all $i \leq k$, and we write $m \vee n$ for the coordinatewise maximum of m and n.

For $n \in \mathbb{N}^k$, we write Λ^n for $d^{-1}(n)$. The *vertices* of Λ are the elements of Λ^0. The factorisation property implies that $o \mapsto \mathrm{id}_o$ is a bijection from the objects of Λ to Λ^0. We will use this bijection to identify $\mathrm{Obj}(\Lambda)$ with Λ^0 without further comment. The domain and codomain maps in the category Λ then become maps $s, r : \Lambda \to \Lambda^0$. More precisely, for $\alpha \in \Lambda$, the *source* $s(\alpha)$ is the identity morphism associated with the object $\mathrm{dom}(\alpha)$ and similarly, $r(\alpha) = \mathrm{id}_{\mathrm{cod}(\alpha)}$. An *edge* is a morphism f with $d(f) = e_i$ for some $i \in \{1, \dots, k\}$.

Let λ be an element of a k-graph Λ and suppose $m, n \in \mathbb{N}^k$ satisfy $0 \leq m \leq n \leq d(\lambda)$. By the factorisation property there exist unique elements $\alpha, \beta, \gamma \in \Lambda$ such that

$$\lambda = \alpha\beta\gamma, \quad d(\alpha) = m, \quad d(\beta) = n - m, \quad \text{and} \quad d(\gamma) = d(\lambda) - n.$$

We define $\lambda(m, n) := \beta$. Observe that $\alpha = \lambda(0, m)$ and $\gamma = \lambda(n, d(\lambda))$.

For $\alpha, \beta \in \Lambda$ and $E \subset \Lambda$, we write αE for $\{\alpha\lambda : \lambda \in E, r(\lambda) = s(\alpha)\}$ and $E\beta$ for $\{\lambda\beta : \lambda \in E, s(\lambda) = r(\beta)\}$. So for $u, v \in \Lambda^0$, we have $uE = E \cap r^{-1}(u)$ and $Ev = E \cap s^{-1}(v)$.

Recall from [19] that for $\mu, \nu \in \Lambda$, the set $\mu\Lambda \cap \nu\Lambda \cap \Lambda^{d(\mu)\vee d(\nu)}$ of minimal common extensions of μ and ν is denoted $\mathrm{MCE}(\mu, \nu)$.

We allow 0-graphs with the convention that $\mathbb{N}^0 = \{0\}$. A 0-graph consists only of identity morphisms, and we regard it as a countable nonempty collection of isolated vertices.

It is standard that if $E = (E^0, E^1, r, s)$ is a directed graph as in [11], then its path category is a 1-graph, and conversely, every 1-graph Λ is the path category of the directed graph with vertices Λ^0, edges Λ^1 and range and source maps inherited from Λ. In this paper we shall treat directed graphs and 1-graphs interchangeably. That is, if E is a

directed graph (E^0, E^1, r, s), then we shall also use E to denote its path category regarded as a 1-graph.

2.2. **Cohomology of k-graphs.** We now recall the (cubical) cohomology of a k-graph described in [14]: For $k \geq 0$ define

$$\mathbf{1}_k := \begin{cases} (1, \dots, 1) & \text{if } k > 0, \\ 0 & \text{if } k = 0. \end{cases}$$

Let Λ be a k-graph. For $0 \leq r \leq k$ let

$$Q_r(\Lambda) := \{\lambda \in \Lambda : d(\lambda) \leq \mathbf{1}_k, |\lambda| = r\}.$$

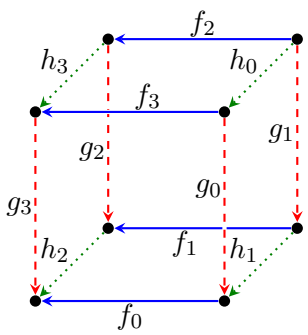
For $r > k$ let $Q_r(\Lambda) := \emptyset$.

Fix $0 < r \leq k$. The set $Q_r(\Lambda)$ consists of the morphisms of Λ which may be expressed as the composition of a sequence of r edges whose degrees are distinct generators of $\mathbb{N}^k$. The factorisation property implies that each element of $Q_r(\Lambda)$ determines a commuting diagram in Λ shaped like an r-cube. For example if $\lambda \in Q_3(\Lambda)$ with $d(\lambda) = e_i + e_j + e_l$ with $i < j < l$, then multiple applications of the factorisation property yield factorisations

$$\lambda = f_0 g_0 h_0 = f_0 h_1 g_1 = h_2 f_1 g_1 = h_2 g_2 f_2 = g_3 h_3 f_2 = g_3 f_3 h_0$$

such that $d(f_n) = e_i$, $d(g_n) = e_j$ and $d(h_n) = e_l$ for all n. So λ determines the following commuting diagram in which edges of degree e_i are blue and solid, edges of degree e_j are red and dashed and edges of degree e_l are green and dotted:

(2.1)



Each $\lambda \in Q_r(\Lambda)$ determines $2r$ elements of $Q_{r-1}(\Lambda)$ which we regard as faces of λ. Fix $\lambda \in Q_r(\Lambda)$ and express $d(\lambda) = e_{i_1} + \cdots + e_{i_r}$ where $i_1 < \cdots < i_r$. For $1 \leq j \leq r$ define $F^0_j(\lambda)$ and $F^1_j(\lambda)$ to be the unique elements of $\Lambda^{d(\lambda) - e_{i_j}}$ such that $\lambda = \alpha F^1_j(\lambda) = F^0_j(\lambda)\beta$ for some $\alpha, \beta \in \Lambda^{e_{i_j}}$. Equivalently,

$$F^0_j(\lambda) = \lambda(0, d(\lambda) - e_{i_j}) \quad \text{and} \quad F^1_j(\lambda) = \lambda(e_{i_j}, d(\lambda)).$$

In example (2.1), $F^1_1(\lambda) = g_0 h_0 = h_1 g_1$, $F^0_2(\lambda) = f_0 h_1 = h_2 f_1$ and so on.

For $r \in \mathbb{N}$ let $C_r(\Lambda) = \mathbb{Z}Q_r(\Lambda)$. For $r \geq 1$, define $\partial_r : C_r(\Lambda) \to C_{r-1}(\Lambda)$ to be the unique homomorphism such that

$$\partial_r(\lambda) = \sum_{i=1}^{r} \sum_{\ell=0}^{1} (-1)^{i+\ell} F^\ell_i(\lambda) \quad \text{for all } \lambda \in Q_r(\Lambda).$$

We write ∂_0 for the zero homomorphism $C_0(\Lambda) \to \{0\}$. By [14, Lemma 3.3] $(C_*(\Lambda), \partial_*)$ is a chain complex.

As in [14], for $r \in \mathbb{N}$ we denote by $H_r(\Lambda)$ the quotient group $H_r(\Lambda) = \ker(\partial_r)/\operatorname{Im}(\partial_{r+1})$. We call $H_r(\Lambda)$ the rth homology group of Λ.

Recall that a morphism $\phi : \Lambda \to \Gamma$ of k-graphs is a functor $\phi : \Lambda \to \Gamma$ such that $d_\Gamma(\phi(\lambda)) = d_\Lambda(\lambda)$ for all $\lambda \in \Lambda$. As in [14, Lemma 3.5] the assignment $\Lambda \mapsto H_*(\Lambda)$ is a

covariant functor from the category of k-graphs with k-graph morphisms to the category of abelian groups with homomorphisms.

Notation 2.1. Let Λ be a k-graph and let A be an abelian group. For $r \in \mathbb{N}$, we write $C^r(\Lambda, A)$ for the collection of all functions $f : Q_r(\Lambda) \to A$. We identify $C^r(\Lambda, A)$ with $\mathrm{Hom}(C_r(\Lambda), A)$ in the usual way. Define maps $\delta^r : C^r(\Lambda, A) \to C^{r+1}(\Lambda, A)$ by

$$(\delta^r f)(\lambda) := f(\partial_{r+1}(\lambda)) = \sum_{i=1}^{r+1} \sum_{\ell=0}^{1} (-1)^{i+\ell} f(F_i^\ell(\lambda)).$$

Then $(C^*(\Lambda, A), \delta^*)$ is a cochain complex.

As in [14], we define the *cohomology* $H^*(\Lambda, A)$ of the k-graph Λ with coefficients in A to be the cohomology of the complex $C^*(\Lambda, A)$; that is $H^r(\Lambda, A) := \ker(\delta^r)/\operatorname{Im}(\delta^{r-1})$. For $r \geq 0$, we write $Z^r(\Lambda, A) := \ker(\delta^r)$ for the group of r-cocycles, and for $r > 0$, we write $B^r(\Lambda, A) = \operatorname{Im}(\delta^{r-1})$ for the group of r-coboundaries. We define $B^0(\Lambda, A) := \{0\}$. For each r, $H^r(\Lambda, A)$ is a bifunctor, which is contravariant in Λ and covariant in A.

Remark 2.2. As mentioned in the introduction, in the next section we introduce a new cohomology theory, called "categorical cohomology" for k-graphs. When we wish to emphasise the distinction between the two, we will refer to the version discussed here as "cubical cohomology".

3. Categorical cohomology

Here we introduce a second notion of cohomology for k-graphs, obtained from the simplicial structure of the category Λ in a manner analogous to Renault's cohomology for groupoids (see [21, Definition I.1.11]), which he attributes to Westman (see [24]). We also follow his use of normalised cochains.

Notation 3.1. Let Λ be a k-graph, and let A be an abelian group. For each integer $r \geq 1$, let $\Lambda^{*r} := \big\{(\lambda_1, \ldots, \lambda_r) \in \prod_{i=1}^r \Lambda : s(\lambda_i) = r(\lambda_{i+1}) \text{ for each } i\big\}$ be the collection of composable r-tuples in Λ, and let $\Lambda^{*0} := \Lambda^0$. For $r \geq 0$, a function $f : \Lambda^{*r} \to A$ is said to be an *r-cochain* if $f(\lambda_1, \ldots, \lambda_r) = 0$ whenever $\lambda_i \in \Lambda^0$ for some $0 < i \leq r$. Observe that when $r = 0$ the last condition is vacuous, so every function $f : \Lambda^0 \to A$ is a 0-cochain. Let $\underline{C}^r(\Lambda, A)$ be the set of all r-cochains, regarded as a group under pointwise addition.

Definition 3.2. Fix $r \geq 1$. For $f \in \underline{C}^r(\Lambda, A)$ define $\underline{\delta}^r f : \Lambda^{*(r+1)} \to A$ by

$$\begin{aligned}(\underline{\delta}^r f)(\lambda_0, \ldots, \lambda_r) &= f(\lambda_1, \ldots, \lambda_r) \\ &\quad + \textstyle\sum_{i=1}^r (-1)^i f\big(\lambda_0, \ldots, \lambda_{i-2}, (\lambda_{i-1}\lambda_i), \lambda_{i+1}, \ldots, \lambda_r\big) \\ &\quad + (-1)^{r+1} f(\lambda_0, \ldots, \lambda_{r-1}).\end{aligned} \tag{3.1}$$

For $f \in \underline{C}^0(\Lambda, A)$, define $\underline{\delta}^0 f : \Lambda^{*1} \to A$ by

$$(\underline{\delta}^0 f)(\lambda) := f(s(\lambda)) - f(r(\lambda)). \tag{3.2}$$

Remark 3.3. It is routine to check that each $\underline{\delta}^r$ maps $\underline{C}^r(\Lambda, A)$ to $\underline{C}^{r+1}(\Lambda, A)$.

We sometimes emphasise the condition that $f(\lambda_1, \ldots, \lambda_r) = 0$ whenever $\lambda_i \in \Lambda^0$ for some i by referring to such cochains as *normalised* cochains. However, since we will not consider any other sort of cochain in this paper, we usually eschew the adjective.

Lemma 3.4. *The sequence*

$$0 \to \underline{C}^0(\Lambda, A) \xrightarrow{\underline{\delta}^0} \underline{C}^1(\Lambda, A) \xrightarrow{\underline{\delta}^1} \underline{C}^2(\Lambda, A) \xrightarrow{\underline{\delta}^2} \dots$$

is a cochain complex.

Proof. For $f \in \underline{C}^0(\Lambda, A)$ and $(\lambda_1, \lambda_2) \in \Lambda^{*2}$, we have

$$\begin{aligned}(\underline{\delta}^1 \circ \underline{\delta}^0 f)(\lambda_1, \lambda_2) &= (\underline{\delta}^0 f)(\lambda_1) - (\underline{\delta}^0 f)(\lambda_1\lambda_2) + (\underline{\delta}^0 f)(\lambda_2) \\ &= f(r(\lambda_1)) - f(s(\lambda_1)) - (f(r(\lambda_1)) - f(s(\lambda_2))) + f(r(\lambda_2)) - f(s(\lambda_2)) \\ &= 0,\end{aligned}$$

so $\underline{\delta}^1 \circ \underline{\delta}^0 = 0$.

To see that $\underline{\delta}^{i+1} \circ \underline{\delta}^i = 0$ for $i \geq 1$, we calculate:

$$(\underline{\delta}^{i+1} \circ \underline{\delta}^i f)(\lambda_0, \dots, \lambda_{i+1}) = (\underline{\delta}^i f)(\lambda_1, \dots, \lambda_{i+1}) \tag{3.3}$$

$$+ \sum_{j=1}^{i+1} (-1)^j (\underline{\delta}^i f)\big(\lambda_0, \dots, (\lambda_{j-1}\lambda_j), \dots, \lambda_{i+1}\big) \tag{3.4}$$

$$+ (-1)^{i+2} (\underline{\delta}^r f)(\lambda_0, \dots, \lambda_i). \tag{3.5}$$

We must show that the right-hand side is equal to zero. Expand each term using (3.1). For each j, the jth term in the expansion of (3.3) cancels the first term in the expansion of the jth summand of (3.4). Likewise, the jth term in the expansion of (3.5) cancels with the last term in the expansion of the jth summand of (3.4). Finally, for $2 \leq j \leq i$, the ith term in the expansion of the jth summand of (3.4) cancels with the jth term in the expansion of the $(i+1)^{\text{st}}$ summand. □

Definition 3.5. The *categorical cohomology* of Λ with coefficients in A is the cohomology $\underline{H}^*(\Lambda, A)$ of the cochain complex described above. That is,

$$\underline{H}^r(\Lambda, A) := \ker(\underline{\delta}^r)/\operatorname{Im}(\underline{\delta}^{r-1}) \quad \text{for each } r.$$

We write $\underline{B}^r(\Lambda, A)$ for the group $\operatorname{Im}(\underline{\delta}^{r-1})$ of r-coboundaries, and $\underline{Z}^r(\Lambda, A)$ for the group $\ker(\underline{\delta}^r)$ of r-cocycles.

Remark 3.6. For each r, $\underline{H}^r(\Lambda, A)$ is a bifunctor which is covariant in A and contravariant in Λ.

Remark 3.7. Definitions 3.2 and 3.5 make sense for an arbitrary small category Λ. If the category also carries a topology compatible with the structure maps, and A is a locally compact abelian group, it is natural to require A-valued n-cochains on Λ to be continuous. In this paper, we distinguish this *continuous cocycle cohomology* from its discrete cousin by denoting the cochain groups $\tilde{C}^*(\Lambda, A)$, the coboundary groups $\tilde{B}^*(\Lambda, A)$, the cocycle groups $\tilde{Z}^*(\Lambda, A)$ and the cohomology groups $\tilde{H}^*(\Lambda, A)$. If Λ is a topological groupoid $\mathcal{G}$ in the sense of Renault, then we have simply replicated Renault's continuous cocycle cohomology of $\mathcal{G}$ introduced in [21].

A function from a k-graph Λ into a group G is called a *functor* if it preserves products (see [10, §5]). Such functors have sometimes been referred to informally as cocycles; the following lemma justifies this informal usage.

Lemma 3.8. *Let (Λ, d) be a k-graph, and let A be an abelian group. Then a cochain $f_0 \in \underline{C}^0(\Lambda, A)$ is a categorical 0-cocycle if and only if it is constant on connected components; a*

cochain $f_1 \in \underline{C}^1(\Lambda, A)$ is a categorical 1-cocycle if and only if it is a functor; and a cochain $f_2 \in \underline{C}^2(\Lambda, A)$ is a categorical 2-cocycle if and only if it satisfies the cocycle identity

$$f_2(\lambda_1, \lambda_2) + f_2(\lambda_1\lambda_2, \lambda_3) = f_2(\lambda_2, \lambda_3) + f_2(\lambda_1, \lambda_2\lambda_3) \tag{3.6}$$

for all $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda^{*3}$.

Proof. For the first statement, observe that f_0 is a 0-cocycle if and only if $(\underline{\delta}^0 f_0)(\lambda) = 0$ for all λ, which occurs if and only if $f_0(s(\lambda)) = f_0(r(\lambda))$ for all λ; that is, if and only if f_0 is constant on connected components.

A 1-cochain f_1 is a 1-cocycle if and only if $(\underline{\delta}^1 f_1)(\lambda_1, \lambda_2) = 0$ for all $(\lambda_1, \lambda_2) \in \Lambda^{*2}$; that is, if and only if

$$f_1(\lambda_1) - f_1(\lambda_1\lambda_2) + f_1(\lambda_2) = 0 \text{ for all } (\lambda_1, \lambda_2) \in \Lambda^{*2},$$

and this in turn is equivalent to the assertion that f_1 is a functor.

Fix $f_2 \in \underline{C}^2(\Lambda, A)$. Then $f_2 \in \underline{Z}^2(\Lambda, A)$ if and only if for all $(\lambda_1, \lambda_2, \lambda_3) \in \Lambda^{*3}$,

$$0 = (\underline{\delta}^2 f_2)(\lambda_1, \lambda_2, \lambda_3) = f_2(\lambda_2, \lambda_3) - f_2(\lambda_1\lambda_2, \lambda_3) + f_2(\lambda_1, \lambda_2, \lambda_3) - f(\lambda_2, \lambda_3).$$

Hence f_2 is a 2-cocycle if and only if it satisfies (3.6). □

We now turn to the relationship between the cubical and the categorical cohomology of a k-graph Λ. We will ultimately prove that $H^i(\Lambda, A) \cong \underline{H}^i(\Lambda, A)$ for $i \leq 2$, but sorting this out will take the remainder of this section and all of the next.

Remark 3.9. By definition of the coboundary maps on cohomology from [14], an A-valued 0-cocycle on a k-graph Λ is a function $c : \Lambda^0 \to A$ which is invariant for the equivalence relation $\sim_{\text{cub}}$ on vertices generated by $r(e) \sim_{\text{cub}} s(e)$ for each edge e. As in Lemma 3.8 an A-valued categorical 0-cocycle on Λ is a function $f_0 : \Lambda^0 \to A$ which is invariant for the equivalence relation $\sim_{\text{cat}}$ on vertices generated by $r(\lambda) \sim_{\text{cat}} s(\lambda)$ for all $\lambda \in \Lambda$. Since every path in Λ can be factorised into edges, $\sim_{\text{cub}}$ and $\sim_{\text{cat}}$ are identical. Hence

$$H^0(\Lambda, A) = \underline{H}^0(\Lambda, A) = \{f : \Lambda^0 \to A \mid f \text{ is constant on connected components}\}.$$

Our next result says that restriction of functions determines isomorphisms $\underline{Z}^1(\Lambda, A) \cong Z^1(\Lambda, A)$, $\underline{B}^1(\Lambda, A) \cong B^1(\Lambda, A)$ and hence $\underline{H}^1(\Lambda, A) \cong H^1(\Lambda, A)$.

Theorem 3.10. *Let Λ be a k-graph and let $f \in C^1(\Lambda, A)$. If $f \in Z^1(\Lambda, A)$ then there exists a unique element $\tilde{f} \in \underline{Z}^1(\Lambda, A)$ such that $\tilde{f}|_{Q_1(\Lambda)} = f$. Conversely, if $g \in \underline{Z}^1(\Lambda, A)$, then $g|_{Q_1(\Lambda)} \in Z^1(\Lambda, A)$. Finally, $f \in B^1(\Lambda, A)$ if and only if $\tilde{f} \in \underline{B}^1(\Lambda, A)$, and the map $f \mapsto \tilde{f}$ induces an isomorphism $H^1(\Lambda, A) \cong \underline{H}^1(\Lambda, A)$.*

Proof. Suppose first that $g \in \underline{Z}^1(\Lambda, A)$ and let $g_0 = g|_{Q_1(\Lambda)}$. Then for any $\lambda \in Q_2(\Lambda)$, we have

$$\begin{aligned}\delta^1(g_0)(\lambda) &= g_0(F_1^1(\lambda)) - g_0(F_1^0(\lambda)) - g_0(F_2^1(\lambda)) + g_0(F_2^0(\lambda)) \\ &= \big(g_0(F_2^0(\lambda)) + g_0(F_1^1(\lambda))\big) - \big(g_0(F_1^0(\lambda)) + g_0(F_2^1(\lambda))\big).\end{aligned}$$

Since $F_2^0(\lambda)F_1^1(\lambda) = \lambda = F_1^0(\lambda)F_2^1(\lambda)$, that g is a functor implies that $\delta^1(g_0) = 0$ so $g_0 \in Z^1(\Lambda, A)$.

Now suppose that $f \in Z^1(\Lambda, A)$. We claim that there is a well-defined functor $\tilde{f} : \Lambda \to A$ such that for any path $\lambda \in \Lambda$ and any factorisation $\lambda = \lambda_1 \cdots \lambda_{|\lambda|}$ with each $\lambda_i \in Q_1(\Lambda)$,

$$\tilde{f}(\lambda) = \sum_{i=1}^{|\lambda|} f(\lambda_i). \tag{3.7}$$

Given a path $\lambda \in \Lambda$, an edge-factorisation of λ is a decomposition $\lambda = \lambda_1 \cdots \lambda_{|\lambda|} \in \Lambda$ with each $\lambda_i \in Q_1(\Lambda)$. We say that

$$\lambda_1 \cdots \lambda_i \lambda_{i+1} \cdots \lambda_{|\lambda|} \to \lambda_1 \cdots \lambda'_i \lambda'_{i+1} \cdots \lambda_{|\lambda|}$$

is an *allowable transition of edge-factorisations of* λ if $d(\lambda_i) = d(\lambda'_{i+1}) = e_j$ and $d(\lambda_{i+1}) = d(\lambda'_i) = e_l$ for some $1 \le l < j \le k$, and $\lambda_i \lambda_{i+1} = \lambda'_i \lambda'_{i+1}$. Any edge-factorisation of a fixed path $\lambda \in \Lambda$ can be transformed into any other by a sequence of such allowable transitions and their inverses. Since f is a cocycle, the formula (3.7) is invariant under allowable transitions and so determines a well-defined function $\tilde{f}$ from Λ to A, which is a functor which extends f by definition. Moreover, any functor $\tilde{f} : \Lambda \to A$ which extends f must satisfy (3.7), and so must be equal to $\tilde{f}$.

A function $f : Q_1(\Lambda) \to A$ belongs to $B^1(\Lambda, A)$ if and only if there is a map $b : \Lambda^0 \to A$ such that $f(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in Q_1(\Lambda)$. It follows that $f \in B^1(\Lambda, A)$ if and only if there is a function $b : \Lambda^0 \to A$ such that the unique extension $\tilde{f} : \Lambda \to A$ of Theorem 3.10 satisfies $\tilde{f}(\lambda) = b(s(\lambda)) - b(r(\lambda)) = (\underline{\delta}^0 b)(\lambda)$ for all $\lambda \in \Lambda$; that is, if and only if $\tilde{f} \in \underline{B}^1(\Lambda, A)$. □

We now wish to show that each cubical 2-cocycle determines a categorical 2-cocycle, and deduce that there is a homomorphism from $Z^2(\Lambda, A)$ to $\underline{Z}^2(\Lambda, A)$ which descends to a homomorphism $\psi : H^2(\Lambda, A) \to \underline{H}^2(\Lambda, A)$. The set-up and proof of this result will occupy the remainder of this section. In the next section, we will introduce central extensions of k-graphs by abelian groups to show that ψ is an isomorphism.

So for the remainder of the section, we fix a k-graph Λ and an abelian group A. By definition of δ^2, for $\phi \in Z^2(\Lambda, A)$ and any $\lambda \in Q_3(\Lambda)$,

$$\phi(F_3^0(\lambda)) + \phi(F_2^1(\lambda)) + \phi(F_1^0(\lambda)) = \phi(F_1^1(\lambda)) + \phi(F_2^0(\lambda)) + \phi(F_3^1(\lambda)). \tag{3.8}$$

To commence our construction of the homomorphism $\psi : H^2(\Lambda, A) \to \underline{H}^2(\Lambda, A)$ we recall the notion of the skeleton, viewed as a k-coloured graph, of a k-graph Λ.

Notation 3.11. A *k-coloured graph* is a directed graph E endowed with a map $C : E^1 \to \{1, \dots, k\}$ which we regard as assigning a colour to each edge. Using our convention that the path-category of E, regarded as a 1-graph, is still denoted E, we extend C to a functor, also denoted C, from E to the free semigroup $\mathbb{F}_k^+ = \langle 1, 2, \dots, k \rangle$ on k generators.

Given a k-graph Λ we write E_Λ for the k-coloured graph such that $E_\Lambda^0 = \Lambda^0$, $E_\Lambda^1 = Q_1(\Lambda) = \bigcup_{i=1}^k \Lambda^{e_i}$, the maps $r, s : E_\Lambda^1 \to E_\Lambda^0$ are inherited from Λ, and $d(\alpha) = e_{C(\alpha)}$ for all $\alpha \in Q_1(\Lambda)$.

There is a surjective functor $\pi : E_\Lambda \to \Lambda$ such that $\pi(\alpha) = \alpha$ for all $\alpha \in Q_1(\Lambda)$. Let $q : \mathbb{F}_k^+ \to \mathbb{N}^k$ be the semigroup homomorphism such that $q(i) = e_i$ for $1 \le i \le k$. Then $q \circ C = d \circ \pi$.

We define a preferred section for π as follows. Given $\lambda \in \Lambda^n$, we denote by $\overline{\lambda} \in E_\Lambda$ the unique path $\overline{\lambda}_1 \dots \overline{\lambda}_{|n|}$ in E_Λ such that $\pi(\overline{\lambda}) = \lambda$ and $C(\overline{\lambda}_i) \le C(\overline{\lambda}_{i+1})$ for all i[1].

An *allowable transition* in E_Λ is an ordered pair $(u, w) \in E_\Lambda \times E_\Lambda$ such that $\pi(u) = \pi(w)$ and there is an i such that $u_j = w_j$ for $j \notin \{i, i+1\}$ and $C(w_{i+1}) = C(u_i) < C(u_{i+1}) = C(w_i)$. The factorisation property forces $u_i u_{i+1} = w_i w_{i+1}$ because $\pi(u) = \pi(w)$ in Λ. Informally, if (u, w) is an allowable transition, then the edges w_i and w_{i+1} are in reverse colour-order, and u is the path obtained by switching them around using the factorisation property in Λ. If (u, w) is an allowable transition we define $p(u, w) := \min\{j : u_j \neq w_j\}$

[1]The ordering on the generators of $\mathbb{F}_k^+$ is just the usual ordering of $\{1, \dots, k\}$.

Definition 3.12. Given a k-graph Λ, the *transition graph* of Λ is the 1-graph F_Λ such that $F^0_\Lambda := E_\Lambda$, $F^1_\Lambda := \{(u,w) : (u,w) \text{ is an allowable transition in } E_\Lambda\}$, and $r, s : F^1_\Lambda \to F^0_\Lambda$ are defined by $r(u,w) := u$ and $s(u,w) := w$.

Let Λ be a k-graph. Given $u \in F^0_\Lambda$, since u is a path in E_Λ, we will frequently write $\ell(u)$ for the number of edges in u regarded as a path in E_Λ. The connected components of the transition graph F_Λ are in one-to-one correspondence with elements of Λ. Specifically, given a path $\lambda \in \Lambda$, the set $\pi^{-1}(\lambda) \subset F^0_\Lambda$ is the collection of vertices in a connected component F_λ of F_Λ. We have $\ell(u) = |\lambda|$ for all $u \in F^0_\lambda$.

Each F_λ (and hence F_Λ) contains no directed cycles. Moreover, for each $\lambda \in \Lambda$, the preferred factorisation $\overline{\lambda}$ is the unique terminal vertex of F_λ.

Define $h : F^0_\Lambda \to \mathbb{N}$ by

$$h(u) = \sum_{i=1}^{\ell(u)} |\{j < i : C(u_j) > C(u_i)\}|.$$

An induction shows that $h(u)$ measures the distance from u to the terminal vertex in its connected component: that is, we have $h(u) = |\alpha|$ for any path $\alpha \in \overline{\pi(u)}F_\Lambda u$. In particular, for $\lambda \in \Lambda$, we have $h(\overline{\lambda}) = 0$, and if $u, w \in F^0_\Lambda$, and $\alpha \in uF_\Lambda w$, then $|\alpha| = h(w) - h(u)$.

Notation 3.13. Define $\tilde{\phi} : F^1_\Lambda \to A$ as follows: if $(u,w) \in F^1_\Lambda$ and $p(u,w) = i$, then $\tilde{\phi}(u,w) = \phi(\pi(u_i u_{i+1}))$. That is, $\tilde{\phi}(u,w)$ is the value of ϕ on the element of $Q_2(\Lambda)$ which is flipped when passing from w to u. We extend $\tilde{\phi}$ to a functor from F_Λ to A by $\tilde{\phi}(\alpha) := \sum_{i=1}^{|\alpha|} \tilde{\phi}(\alpha_i)$.

Lemma 3.14. *Let $\tau, \rho \in F^1_\Lambda$ with $s(\tau) = s(\rho)$. Then there exist $\mu \in F_\Lambda r(\tau)$ and $\nu \in F_\Lambda r(\rho)$ such that $r(\mu) = r(\nu)$ and $\tilde{\phi}(\mu\tau) = \tilde{\phi}(\nu\rho)$.*

Proof. Let $w := s(\tau)$, and let $n = \ell(w)$ so that $w = w_1 \cdots w_n$ with each $w_i \in E^1_\Lambda$. We assume without loss of generality that $p(\tau) \le p(\rho)$. We consider three cases.

Case 1: $p(\tau) = p(\rho)$. Then $\tau = \rho$, and $\mu = \nu = r(\tau)$ trivially have the desired properties.

Case 2: $p(\rho) \ge p(\tau) + 2$. Then

$$\begin{aligned} r(\tau) &= w_1 \cdots w_{p(\tau)-1} e f w_{p(\tau)+2} \cdots w_{p(\rho)-1} w_{p(\rho)} w_{p(\rho)+1} w_{p(\rho)+2} \cdots w_n \quad \text{and} \\ r(\rho) &= w_1 \cdots w_{p(\tau)-1} w_{p(\tau)} w_{p(\tau)+1} w_{p(\tau)+2} \cdots w_{p(\rho)-1} g h w_{p(\rho)+2} \cdots w_n \end{aligned}$$

where $w_{p(\tau)} w_{p(\tau)+1} = ef$ and $w_{p(\rho)} w_{p(\rho)+1} = gh$ are 2-cubes of Λ. Let

$$v := w_1 \cdots w_{p(\tau)-1} e f w_{p(\tau)+2} \cdots w_{p(\rho)-1} g h w_{p(\rho)+2} \cdots w_n \in F^0_\Lambda,$$

and let $\mu := (v, r(\tau))$ and $\nu := (v, r(\rho))$. Then $\mu \in F^1_\Lambda r(\tau)$ and $\nu \in F^1_\Lambda r(\rho)$ with $r(\mu) = r(\nu) = v$, and $\tilde{\phi}(\mu\tau) = \phi(gh) + \phi(ef) = \tilde{\phi}(\nu\rho)$ as required.

Case 3: $p(\rho) = p(\tau) + 1$. Then $\lambda := \pi(w_{p(\tau)} w_{p(\tau)+1} w_{p(\tau)+2})$ belongs to $Q_3(\Lambda)$. Hence we may factorise λ as in (2.1). That is,

$$\lambda = f_0 g_0 h_0 = f_0 h_1 g_1 = h_2 f_1 g_1 = h_2 g_2 f_2 = g_3 h_3 f_2 = g_3 f_3 h_0$$

where $C(f_i) = C(f_0) < C(g_j) = C(g_0) < C(g_l) = C(g_0)$ for all i, j, l. Since ρ and τ are allowable transitions, we have $w_{p(\tau)} = h_2$, $w_{p(\tau)+1} = g_2$ and $w_{p(\tau)+2} = f_2$. We have

$$r(\tau) = w_1 \cdots w_{p(\tau)-1} g_3 h_3 f_2 w_{p(\tau)+3} \cdots w_n \quad \text{and} \quad r(\rho) = w_1 \cdots w_{p(\tau)-1} h_2 f_1 g_1 w_{p(\tau)+3} \cdots w_n.$$

Define $\mu, \nu \in F_\Lambda^2$ by

$$\mu := (w_1 \cdots f_0 g_0 h_0 \cdots w_n,\ w_1 \cdots g_3 f_3 h_0 \cdots w_n)\,(w_1 \cdots g_3 f_3 h_0 \cdots w_n,\ w_1 \cdots g_3 h_3 f_2 \cdots w_n),$$

and

$$\nu := (w_1 \cdots f_0 g_0 h_0 \cdots w_n,\ w_1 \cdots f_0 h_1 g_1 \cdots w_n)\,(w_1 \cdots f_0 h_1 g_1 \cdots w_n,\ w_1 \cdots h_2 f_1 g_1 \cdots w_n).$$

Then $\mu \in F_\Lambda r(\tau)$ and $\nu \in F_\Lambda r(\beta)$ with $r(\mu) = r(\nu)$. Moreover,

$$\tilde\phi(\mu\tau) = \phi(f_0 g_0) + \phi(f_3 h_0) + \phi(g_3 h_3) = \phi(F_3^0(\lambda)) + \phi(F_2^1(\lambda)) + \phi(F_1^0(\lambda)) \quad \text{and}$$
$$\tilde\phi(\nu\rho) = \phi(g_0 h_0) + \phi(f_0 h_1) + \phi(f_1 g_1) = \phi(F_1^1(\lambda)) + \phi(F_2^0(\lambda)) + \phi(F_3^1(\lambda)),$$

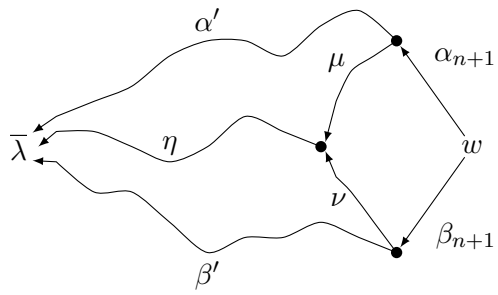
so $\tilde\phi(\mu\tau) = \tilde\phi(\nu\rho)$ by (3.8). □

Lemma 3.15. *Let Λ be a k-graph. There is a well-defined function $S_\phi : F_\Lambda^0 \to A$ defined by $S_\phi(w) = \tilde\phi(\alpha)$ for any $\alpha \in \overline{\pi(w)} F_\Lambda w$.*

Proof. Since each connected component of F_Λ has a unique sink and is finite, it suffices to fix $\lambda \in \Lambda$, a vertex $w \in F_\lambda^0$ and two paths $\alpha, \beta \in \overline{\lambda} F_\lambda w$ and show that $\tilde\phi(\alpha) = \tilde\phi(\beta)$. We proceed by induction on $h(w)$. If $h(w) = 0$ then $w = \overline{\lambda}$ and the result is trivial.

Now fix $n \in \mathbb{N}$. Suppose as an inductive hypothesis that $\tilde\phi(\alpha) = \tilde\phi(\beta)$ whenever $\alpha, \beta \in \overline{\lambda} F_\Lambda w$ with $h(w) \le n$. Fix $w \in F_\Lambda$ with $h(w) = n+1$ and $\alpha, \beta \in \overline{\lambda} F_\Lambda w$.

We have $|\alpha| = |\beta| = n+1$. Write $\alpha = \alpha'\alpha_{n+1}$ and $\beta = \beta'\beta_{n+1}$ where $\alpha_{n+1}, \beta_{n+1} \in F_\Lambda^1$. By Lemma 3.14 applied to $\tau := \alpha_{n+1}$ and $\rho := \beta_{n+1}$ in $F_\Lambda^1 w$, there exist $\mu \in F_\Lambda r(\alpha_{n+1})$ and $\nu \in F_\Lambda r(\beta_{n+1})$ such that $r(\mu) = r(\nu)$ and $\tilde\phi(\mu\alpha_{n+1}) = \tilde\phi(\nu\beta_{n+1})$. Since $\overline{\lambda}$ is the unique sink in F_λ, and since F_λ^0 is finite, there is a path η from $r(\mu)$ to $\overline{\lambda}$. The situation is summarised in the following diagram.



Since $h(r(\alpha_{n+1})) = h(r(\beta_{n+1})) = n$, the inductive hypothesis gives $\tilde\phi(\eta\mu) = \tilde\phi(\alpha')$ and $\tilde\phi(\eta\nu) = \tilde\phi(\beta')$. We then have

$$\tilde\phi(\alpha) = \tilde\phi(\alpha') + \tilde\phi(\alpha_{n+1}) = \tilde\phi(\eta\mu) + \tilde\phi(\alpha_{n+1}) = \tilde\phi(\eta) + \tilde\phi(\mu\alpha_{n+1}).$$

A symmetric calculation shows that

$$\tilde\phi(\beta) = \tilde\phi(\eta) + \tilde\phi(\nu\beta_{n+1}).$$

Since $\tilde\phi(\mu\alpha_{n+1}) = \tilde\phi(\nu\beta_{n+1})$ by choice of μ and ν, it follows that $\tilde\phi(\alpha) = \tilde\phi(\beta)$. □

Lemma 3.15 implies that $\tilde\phi$ is a 1-coboundary of F_Λ: specifically, $\tilde\phi = \underline{\delta}^0(S_\phi)$. We call S_ϕ the *shuffle function* associated with ϕ. We regard it as measuring the "cost" of shuffling the edges in a coloured path into preferred order.

Theorem 3.16. *Let (Λ, d) be a k-graph. For $\phi \in Z^2(\Lambda, A)$, define*

$$c_\phi : \Lambda^{*2} \to A \quad \text{by } c_\phi(\mu, \nu) := S_\phi(\overline{\mu}\,\overline{\nu}).$$

Then $c_\phi \in Z^2(\Lambda, A)$ and $\phi \mapsto c_\phi$ is a homomorphism from $Z^2(\Lambda, A)$ to $\underline{Z}^2(\Lambda, A)$ satisfying

(1) $c_\phi(f,g) = \phi(fg)$ *if* $d(f) = e_i$ *and* $d(g) = e_j$ *with* $i > j$, *and* $c_\phi(f,g) = 0$ *if* $d(f) = e_i$ *and* $d(g) = e_j$ *with* $i \le j$; *and*
(2) $c_\phi(\mu,\nu) = 0$ *whenever* $\overline{\mu}\,\overline{\nu} = \overline{\mu\nu}$; *in particular* $c_\phi(r(\lambda),\lambda) = c_\phi(\lambda, s(\lambda)) = 0$ *for all* $\lambda \in \Lambda$.

Moreover, if $\phi \in B^2(\Lambda, A)$, *then* $c_\phi \in \underline{B}^2(\Lambda, A)$; *hence,* $[\phi] \mapsto [c_\phi]$ *defines a homomorphism* $\psi : H^2(\Lambda, A) \to \underline{H}^2(\Lambda, A)$.

To prove the theorem, we need some further technical results

Notation 3.17. For $\phi \in Z^2(\Lambda, A)$, define $\tilde{c}_\phi : (E_\Lambda)^{*2} \to A$ by

$$\tilde{c}_\phi(u,w) := S_\phi(uw) - S_\phi(u) - S_\phi(w).$$

Lemma 3.18. *For all* $(u,w) \in (E_\Lambda)^{*2}$, *we have*

$$S_\phi(uw) = S_\phi(\overline{\pi(u)}\,\overline{\pi(w)}) + S_\phi(u) + S_\phi(w)$$

and hence $\tilde{c}_\phi(u,w) = S_\phi(\overline{\pi(u)}\,\overline{\pi(w)})$. *Moreover,* $\tilde{c}_\phi \in \underline{Z}^2(E_\Lambda, A)$; *that is,*

$$\tilde{c}_\phi(u,w) + \tilde{c}_\phi(uw,x) = \tilde{c}_\phi(w,x) + \tilde{c}_\phi(u,wx) \tag{3.9}$$

for all $(u,w,x) \in (E_\Lambda)^{*3}$.

Proof. Fix a path α in F_Λ from u to $\overline{\pi(u)}$ and a path β from w to $\overline{\pi(w)}$. There is a path α' from uw to $\overline{\pi(u)}w$ with $|\alpha'| = |\alpha|$ determined by $p(\alpha'_j) := p(\alpha_j)$ for all $j \le |\alpha|$. Likewise, there is a path β' from $\overline{\pi(u)}w$ to $\overline{\pi(u)}\,\overline{\pi(w)}$ with $|\beta'| = |\beta|$ such that $p(\beta'_j) = \ell(u) + p(\beta_j)$ for all j. By definition of these paths, we have

$$\tilde{\phi}(\alpha') = \tilde{\phi}(\alpha) = S_\phi(u) \quad \text{and} \quad \tilde{\phi}(\beta') = \tilde{\phi}(\beta) = S_\phi(w). \tag{3.10}$$

Since $\overline{\pi(uw)}$ is the unique sink in $F_{\pi(uw)}$, there is a path γ from $\overline{\pi(u)}\,\overline{\pi(w)}$ to $\overline{\pi(uw)}$, and we have

$$\tilde{\phi}(\gamma) = S_\phi(\overline{\pi(u)}\,\overline{\pi(w)}). \tag{3.11}$$

Then $\gamma\beta'\alpha'$ is a path from uw to $\overline{\pi(uw)}$, and hence $\tilde{\phi}(\gamma\beta'\alpha') = S_\phi(uw)$. Using that $\tilde{\phi}$ is a functor, and then equations (3.10) and (3.11), we now calculate:

$$S_\phi(uw) = \tilde{\phi}(\gamma\beta'\alpha') = \tilde{\phi}(\gamma) + \tilde{\phi}(\beta') + \tilde{\phi}(\alpha') = S_\phi(\overline{\pi(u)}\,\overline{\pi(w)}) + S_\phi(w) + S_\phi(u),$$

proving the first assertion of the lemma. The second assertion follows immediately from the definition of $\tilde{c}_\phi$.

For the final assertion, we calculate

$$\begin{aligned}\tilde{c}_\phi(u,w) + \tilde{c}_\phi(uw,x) &= \big(S_\phi(uw) - S_\phi(u) - S_\phi(w)\big) + \big(S_\phi(uwx) - S_\phi(uw) - S_\phi(x)\big)\\ &= S_\phi(uwx) - S_\phi(u) - S_\phi(w) - S_\phi(x).\end{aligned}$$

A similar calculation yields

$$\tilde{c}_\phi(w,x) + \tilde{c}_\phi(u,wx) = S_\phi(uwx) - S_\phi(u) - S_\phi(w) - S_\phi(x)$$

also. □

Proof of Theorem 3.16. Fix $\phi \in Z^2(\Lambda, A)$. We first show that c_ϕ satisfies (1) and (2). For (1), suppose that $f \in \Lambda^{e_i}$ and $g \in s(f)\Lambda^{e_j}$. If $i \le j$, then $fg = \overline{fg}$, so $S_\phi(fg) = 0$ as required. If $i > j$, we factorise $fg = g'f'$ where $d(g') = e_j$ and $d(f') = e_i$, and note that $\overline{fg} = g'f'$, and $S_\phi(fg) = \phi(fg)$ by definition.

For (2), suppose that $\overline{\mu}\,\overline{\nu} = \overline{\mu\nu}$. Then $c_\phi(\mu,\nu) = S_\phi(\overline{\mu\nu}) = 0$ by definition. Since $\overline{r(\lambda)}\,\overline{\lambda} = \overline{\lambda} = \overline{r(\lambda)\lambda}$ for all λ, and similarly for $\overline{\lambda}\,\overline{s(\lambda)}$, (2) follows.

To see that c_ϕ is a cocycle, it remains to show that it satisfies the cocycle identity

$$c_\phi(\lambda_1,\lambda_2)+c_\phi(\lambda_1\lambda_2,\lambda_3)=c_\phi(\lambda_2,\lambda_3)+c_\phi(\lambda_1,\lambda_2\lambda_3)$$

for $(\lambda_1,\lambda_2,\lambda_3)\in\Lambda^{*3}$. By Lemma 3.18, we have $c_\phi(\pi(u),\pi(w))=\tilde{c}_\phi(u,w)$ for any $(u,w)\in E_\Lambda^{*2}$, and hence

$$c_\phi(\lambda_1,\lambda_2)+c_\phi(\lambda_1\lambda_2,\lambda_3)=\tilde{c}_\phi(\overline{\lambda_1},\overline{\lambda_2})+\tilde{c}_\phi(\overline{\lambda_1}\,\overline{\lambda_2},\overline{\lambda_3}),$$

and

$$c_\phi(\lambda_2,\lambda_3)+c_\phi(\lambda_1,\lambda_2\lambda_3)=\tilde{c}_\phi(\overline{\lambda_2},\overline{\lambda_3})+\tilde{c}_\phi(\overline{\lambda_1},\overline{\lambda_2}\,\overline{\lambda_3}),$$

so the cocycle identity for c_ϕ follows from the cocycle identity (3.9) for $\tilde{c}_\phi$.

To see that $\phi\mapsto c_\phi$ is a homomorphism, observe that if $\phi_1,\phi_2\in Z^2(\Lambda,A)$, then $S_{\phi_1+\phi_2}=S_{\phi_1}+S_{\phi_2}$, and hence $\tilde{c}_{\phi_1+\phi_2}=\tilde{c}_{\phi_1}+\tilde{c}_{\phi_2}$. It then follows that $c_{\phi_1+\phi_2}=c_{\phi_1}+c_{\phi_2}$ also.

Finally, we must show that the assignment $\phi\mapsto c_\phi$ carries coboundaries to coboundaries. Fix $\phi\in B^2(\Lambda,A)$ and $f\in C^1(\Lambda,A)$ such that $\phi=\delta^1 f$. By definition of $\delta^1:C^1(\Lambda,A)\to C^2(\Lambda,A)$,

$$\phi(\lambda)=f(F_2^0(\lambda))+f(F_1^1(\lambda))-f(F_1^0(\lambda))-f(F_0^1(\lambda))$$

for all $\lambda\in Q_2(\Lambda)$. In particular, if $d(\alpha)=e_i$ and $d(\beta)=e_j$ with $i<j$ and if $\alpha\beta=\eta\zeta$ with $d(\eta)=e_j$ and $d(\zeta)=e_i$, then $\phi(\alpha\beta)=f(\alpha)+f(\beta)-f(\eta)-f(\zeta)$.

Define $b:E_\Lambda\to A$ by $b(w):=\sum_{i=1}^{\ell(w)}f(w_i)$. We show by induction on $h(w)$ that $S_\phi(w)=b(\overline{\pi(w)})-b(w)$ for all $w\in F_\Lambda^0$. It is trivial when $h(w)=0$. Now suppose that $S_\phi(w)=b(\overline{\pi(w)})-b(w)$ whenever $h(w)\le n$, and fix $w\in F_\Lambda^0$ with $h(w)=n+1$. Fix $\alpha\in\overline{\pi(w)}F_\Lambda w$, and let $w':=r(\alpha_{n+1})$. Then Lemma 3.15 implies that

$$\begin{aligned}S_\phi(w)&=\sum_{i=1}^{n+1}\tilde{\phi}(\alpha_i)=\tilde{\phi}(\alpha_{n+1})+\sum_{i=1}^{n}\tilde{\phi}(\alpha_i)\\&=S_\phi(\alpha_1,\dots,\alpha_n)+\tilde{\phi}(\alpha_{n+1})=b(\overline{\pi(w)})-b(w')+\tilde{\phi}(\alpha_{n+1}),\end{aligned}\tag{3.12}$$

where the last equality follows from the inductive hypothesis.

Let $j:=p(\alpha_{n+1})$, and let $\lambda:=\pi(w_jw_{j+1})$. We have $\tilde{\phi}(\alpha_{n+1})=\phi(\lambda)$ by definition of $\tilde{\phi}$. Since α_{n+1} is an allowed transition, we have $C(w_j)>C(w_{j+1})$, and hence

$$w_j=F_1^0(\lambda),\quad w_{j+1}=F_2^1(\lambda),\quad w'_j=F_2^0(\lambda),\quad\text{and}\quad w'_{j+1}=F_1^1(\lambda).$$

Hence $\tilde{\phi}(\alpha_{n+1})=\phi(\lambda)=f(w'_j)+f(w'_{j+1})-f(w_j)-f(w_{j+1})$. Combining this with (3.12), we have

$$\begin{aligned}S_\phi(w)&=b(\overline{\pi(w)})-b(w')+f(w'_j)+f(w'_{j+1})-f(w_j)-f(w_{j+1})\\&=b(\overline{\pi(w)})-\Big(\textstyle\sum_{i=1}^{\ell(w)}f(w'_i)\Big)+f(w'_j)+f(w'_{j+1})-f(w_j)-f(w_{j+1}).\end{aligned}$$

Since $w'_i=w_i$ for $i\notin\{j,j+1\}$, it follows that

$$S_\phi(w)=b(\overline{\pi(w)})-\textstyle\sum_{i=1}^{\ell(w)}f(w_i)=b(\overline{\pi(w)})-b(w).$$

Define $g\in\underline{C}^1(\Lambda,A)$ by $g(\lambda)=-b(\overline{\lambda})$ for $\lambda\in\Lambda$. We prove that $c_\phi=\underline{\delta}^1 g$. Fix $(\mu,\nu)\in\Lambda^{*2}$. Then

$$\begin{aligned}c_\phi(\mu,\nu)=S_\phi(\overline{\mu}\,\overline{\nu})=b(\overline{\mu\nu})-b(\overline{\mu}\,\overline{\nu})&=b(\overline{\mu\nu})-b(\overline{\mu})-b(\overline{\nu})\\&=-g(\mu\nu)+g(\mu)+g(\nu)=(\underline{\delta}^1 g)(\mu,\nu).\end{aligned}$$

Hence $c_\phi\in\underline{B}^2(\Lambda,A)$, so $[\phi]\mapsto[c_\phi]$ is a homomorphism from $H^2(\Lambda,A)$ to $\underline{H}^2(\Lambda,A)$. $\square$

4. Central extensions of k-graphs

In this section we prove that the map $[\phi] \mapsto [c_\phi]$ of Theorem 3.16 is an isomorphism (see Theorem 4.15). To prove this, we introduce the notion of a central extension of a k-graph Λ by an abelian group A. We show that the collection of isomorphism classes of central extensions forms a group $\mathrm{Ext}(\Lambda, A)$ which is isomorphic to $\underline{H}^2(\Lambda, A)$ (cf. [1, Theorem 2.3] and [21, Proposition I.1.14]). We show that each central extension is isomorphic to one obtained from a cubical 2-cocycle which is unique modulo coboundaries. We use this to prove that the map $[\phi] \mapsto [c_\phi]$ is an isomorphism $H^2(\Lambda, A) \cong \underline{H}^2(\Lambda, A)$.

Definition 4.1. Let A be an abelian group, and let Λ be a k-graph. An *extension of* Λ *by* A is a sequence

$$\mathcal{X} : \Lambda^0 \times A \xrightarrow{\iota} X \xrightarrow{q} \Lambda$$

consisting of a small category X, a functor $\iota : \Lambda^0 \times A \to X$, and a surjective functor $q : X \to \Lambda$ such that $q(\iota(v, a)) = v$ for all $v \in \Lambda^0$ and $a \in A$, and such that whenever $q(x) = q(y)$, there exists a unique $a(x, y) \in A$ such that $x = \iota(r(x), a(x, y))y$. We say that $\mathcal{X}$ is a *central extension* if it satisfies $\iota(r(x), a)x = x\iota(s(x), a)$ for all $x \in X$ and $a \in A$.

As we do for k-graphs, for $x \in X$ we write $r(x)$ for $\mathrm{id}_{\mathrm{cod}(x)}$ and $s(x)$ for $\mathrm{id}_{\mathrm{dom}(x)}$.

Remark 4.2. Let $\mathcal{X}$ be an extension of a k-graph Λ by an abelian group A. The functor ι is automatically injective and induces a bijection between Λ^0 and $\mathrm{Obj}(X)$. Since $q(\iota(v, a)) = v$ for all $(v, a) \in \Lambda^0 \times A$, we have $q(x) = q(y)$ if *and only if* there exists $a \in A$ such that $\iota(r(x), a)x = y$. We then have $q(r(x)) = q(\iota(r(x), a))q(r(x)) = q(\iota(r(x), a)r(x)) = q(r(y))$. Since q is injective on objects, it follows that $q(x) = q(y)$ implies $r(x) = r(y)$ (and similarly $s(x) = s(y)$) for all $x, y \in X$.

Notation 4.3. Given an extension $\mathcal{X}$ of Λ by A, it is unambiguous, and frequently convenient, to write $a \cdot x$ for $\iota_X(r(x), a)x$ and $x \cdot a$ for $x\iota_X(s(x), a)$. In this notation, $q(x) = q(y)$ if and only if $x = a(x, y) \cdot y$, and $\mathcal{X}$ is a central extension precisely if $a \cdot x = x \cdot a$ for all $x \in X$ and $a \in A$. We implicitly identify Λ^0 with the identity morphisms in X via the bijection $v \mapsto \iota_X(v, 0)$. This allows us to write $a \cdot v$ or $v \cdot a$ (as appropriate) for $\iota_X(v, a)$. With this convention, we also have $q_X(v) = v$.

That ι is a functor implies that $a \cdot (b \cdot x) = (a + b) \cdot x$ for all $a, b \in A$ and $x \in X$. Since composition in X is also associative, we have identities like $x(a \cdot y) = (x \cdot a)y = (a \cdot x)y = a \cdot (xy)$. In particular, the expression $a \cdot xy$ is unambiguous.

Lemma 4.4. *Let Λ be a k-graph, A an abelian group, and*

$$\mathcal{X} : \Lambda^0 \times A \to X \to \Lambda$$

a central extension of Λ by A. If $(w, x, z), (w, y, z) \in X^{*3}$ and $q(x) = q(y)$, then $a(x, y) = a(wxz, wyz)$. In particular, $a(xz, yz) = a(x, y) = a(wx, wy)$.

Moreover given elements $x_1, \dots, x_n \in X$ such that $q(x_1) = q(x_2) = \cdots = q(x_n)$, we have $a(x_1, x_n) = \sum_{i=1}^{n-1} a(x_i, x_{i+1})$.

Proof. Using that $\mathcal{X}$ is a central extension, we calculate:

$$a(x, y) \cdot wyz = w(a(x, y) \cdot y)z = wxz.$$

The first assertion of the lemma therefore follows from uniqueness of $a(wxz, wyz)$. The second assertion follows from the first applied with $w = r(x)$ and with $z = s(x)$.

The final assertion follows from a straightforward induction: it is trivial when $n = 2$. Suppose as an inductive hypothesis that it holds for $n \leq N$, and fix $x_1, \dots x_{N+1}$ with

$q(x_i) = q(x_j)$ for all i, j. Then

$$\Big(\sum_{i=1}^{N} a(x_i, x_{i+1})\Big) \cdot x_{N+1} = \Big(\sum_{i=1}^{N-1} a(x_i, x_{i+1})\Big) \cdot a(x_N, x_{N+1}) \cdot x_{N+1}$$
$$= \Big(\sum_{i=1}^{N-1} a(x_i, x_{i+1})\Big) \cdot x_N = x_1$$

by the inductive hypothesis. □

Notation 4.5. Let Λ be a k-graph, let A be an abelian group, and let

$$\mathcal{X} : \Lambda^0 \times A \xrightarrow{\iota_X} X \xrightarrow{q_X} \Lambda \quad \text{and} \quad \mathcal{Y} : \Lambda^0 \times A \xrightarrow{\iota_Y} Y \xrightarrow{q_Y} \Lambda$$

be central extensions of Λ by A. Let $X *_\Lambda Y := \{(x, y) \in X \times Y : q_X(x) = q_Y(y)\}$. Define a relation $\sim_A$ on $X *_\Lambda Y$ by $(x, y) \sim_A (-a \cdot x, a \cdot y)$ for all $(x, y) \in X *_\Lambda Y$ and $a \in A$.

Lemma 4.6. *With Λ, A, $\mathcal{X}$ and $\mathcal{Y}$ as in Notation 4.5, the relation $\sim_A$ is an equivalence relation, and satisfies*

$$(x \cdot a, y) = (a \cdot x, y) \sim_A (x, a \cdot y) = (x, y \cdot a)$$

for all $(x, y) \in X *_\Lambda Y$ and all $a \in A$.

Proof. The relation $\sim_A$ is clearly reflexive. For symmetry, observe that

$$(-a \cdot x, a \cdot y) \sim_A \big(a \cdot (-a \cdot x), -a \cdot (a \cdot y)\big) = (x, y).$$

For transitivity, observe that

$$(-b \cdot (-a \cdot x), b \cdot (a \cdot y)) = (-(a + b) \cdot x, (a + b) \cdot y) \sim_A (x, y).$$

For the final assertion, we first establish the middle equality by calculating

$$(a \cdot x, y) \sim (-a \cdot (a \cdot x), a \cdot y) = (x, a \cdot y).$$

the other equalities follow because $\mathcal{X}$ and $\mathcal{Y}$ are central extensions. □

Lemma 4.7. *With the hypotheses of Lemma 4.6, let $Z(X, Y) := X *_\Lambda Y/\sim_A$, and for $(x, y) \in X *_\Lambda Y$, let $[x, y]$ denote its equivalence class in $Z(X, Y)$. There are well-defined maps $r, s : Z(X, Y) \to Z(X, Y)$ such that*

$$r([x, y]) = \big[r(x), r(y)\big] \quad \text{and} \quad s([x, y]) = \big[s(x), s(y)\big] \quad \text{for all } (x, y) \in X *_\Lambda Y.$$

There is also a well-defined composition determined by $[x_1, y_1][x_2, y_2] = [x_1x_2, y_1y_2]$ whenever $s([x_1, y_1]) = r([x_2, y_2])$.

Proof. Suppose that $(x, y) \sim_A (x', y')$, say $(x, y) = (-a \cdot x', a \cdot y')$. Then $q_X(x) = q_X(x')$ and $q_Y(y) = q_Y(y')$, so $(r(x), r(y)) = (r(x'), r(y'))$ by Remark 4.2. So $r : Z(X, Y) \to Z(X, Y)$ is well defined. A similar argument shows that s is well defined.

To see that composition is well defined, fix $a, b \in A$. Since X is a central extension, we calculate:

$$(-a \cdot x_1)(-b \cdot x_2) = -a \cdot (-b \cdot x_1)x_2 = -(a + b) \cdot x_1x_2.$$

Similarly $(a \cdot y_1)(b \cdot y_2) = (a + b) \cdot y_1y_2$. In particular,

$$((-a \cdot x_1)(-b \cdot x_2), (a \cdot y_1)(b \cdot y_2)) \sim_A (x_1x_2, y_1y_2).$$ □

If $[x,y]=[x',y']$ in $Z(X,Y)$, then $x'=-a\cdot x$ for some $a\in A$, so $q(x')=q(x)$. So we may define $\iota=\iota_{Z(X,Y)}:\Lambda^0\times A\to Z(X,Y)$ and $q=q_{Z(X,Y)}:Z(X,Y)\to\Lambda$ by

$$\iota(v,a):=[\iota_X(v,a),\iota_Y(v,0)]\qquad\text{and}\qquad q([x,y]):=q_X(x). \tag{4.1}$$

Lemma 4.6 implies that

$$\begin{aligned}\iota(v,a)=[\iota_X(v,a),\iota_Y(v,0)]&=[\iota_X(v,a),\iota_Y(v,-a)\iota_Y(v,a)]\\&=[\iota_X(v,-a)\iota_X(v,a),\iota_Y(v,a)]=[\iota_X(v,0),\iota_Y(v,a)].\end{aligned} \tag{4.2}$$

Lemma 4.8. *Let Λ be a k-graph, let A be an abelian group, and let X and Y be central extensions of Λ by A as in Notation 4.5. Then $Z(X,Y)$ is a small category under the operations described in Lemma 4.7 and with the identity morphism corresponding to an object $v\in\Lambda^0$ given by $\mathrm{id}_v=[v,v]$.*

Let $\iota:=\iota_{Z(X,Y)}$ and $q:=q_{Z(X,Y)}$ be as in (4.1). Then $q([x,y])=q_Y(y)$ for all $[x,y]\in Z(X,Y)$, and

$$\mathcal{X}+\mathcal{Y}:\Lambda^0\times A\xrightarrow{\iota}Z(X,Y)\xrightarrow{q}\Lambda$$

is a central extension of Λ by A and satisfies

$$a([x,y],[x',y'])=a(x,x')+a(y,y')\ \textit{whenever}\ q([x,y])=q([x',y']). \tag{4.3}$$

Finally, $\iota_{Z(X,Y)}(v,a)=[\iota_X(v,b),\iota_X(v,a-b)]$ for all $v\in\Lambda^0$ and $a,b\in A$.

Remark 4.9. The rather suggestive notation $\mathcal{X}+\mathcal{Y}$ is justified by Example 4.11 and Proposition 4.13 below.

Proof of Lemma 4.8. Routine checks show that $Z(X,Y)$ is a category. It is small because X and Y are.

That $q(x,y)=q_Y(y)$ for all $(x,y)\in X*_\Lambda Y$ is just a combination of the definitions of the map q and the space $X*_\Lambda Y$. Using this it is routine to see that ι and q are functors (the operations in $X*_\Lambda Y$ being coordinate-wise). For $v\in\Lambda^0$, we have

$$q(\iota(v,0))=q([\iota_X(v,0),\iota_Y(v,0)])=q_X(\iota_X(v,0))=v$$

since $\mathcal{X}$ is an extension. Moreover, if $q([x,y])=q([x',y'])$, then $q_X(x)=q_X(x')=q_Y(y')=q_Y(y)$, so there exists a unique element $a=a(x,x')\in A$ such that $x=a\cdot x'$, and a unique $b=b(y,y')$ such that $y=b\cdot y'$. We claim that

$$[x,y]=(a+b)\cdot[x',y'].$$

Applying (4.2) in the third equality below, we calculate:

$$\begin{aligned}(a+b)\cdot[x',y']&=\iota(r(x),a+b)[x',y']\\&=\iota(r(x),a)\iota(r(x),b)[x',y']\\&=[\iota_X(r(x),a),\iota_Y(r(x),0)][\iota_X(r(x),0),\iota_Y(r(x),b)][x',y']\\&=[(a+0)\cdot x',(0+b)\cdot y']\\&=[x,y].\end{aligned}$$

To see that $a+b$ is the unique element with this property, suppose that $c\cdot[x',y']=[x,y]$. Then $(c\cdot x',y')\sim_A(x,y)$, so there exists $d\in A$ such that $c\cdot x'=-d\cdot x$ and $y'=d\cdot y$. Hence $y=-d\cdot y'$ and uniqueness of $b(y,y')$ forces $b=b(y,y')=-d$; and then

$$x=d\cdot c\cdot x'=(c-b)\cdot x',$$

and uniqueness of $a(x,x')$ forces $a=a(x,x')=c-b$. Hence $c=a(x,x')+b(y,y')$ as required. Thus $\mathcal{X}+\mathcal{Y}$ is an extension of Λ by A. It is central because each of $\mathcal{X}$ and $\mathcal{Y}$ is central.

The final assertion follows from (4.2). □

We next construct from each $c \in \underline{Z}^2(\Lambda, A)$ a central extension of Λ by A by twisting the composition in $\Lambda \times A$.

Notation 4.10. Let Λ be a k-graph, let A be an abelian group, and fix $c \in \underline{Z}^2(\Lambda, A)$. Let $X_c(\Lambda, A)$ be the small category with underlying set and structure maps identical to the cartesian-product category $A \times \Lambda$ and with composition defined by

$$(\mu, a)(\nu, b) := (\mu\nu, c(\mu, \nu) + a + b).$$

We will usually suppress the Λ and A in our notation, and write X_c for $X_c(\Lambda, A)$.

Example 4.11. Let Λ be a k-graph, let A be an abelian group, and fix $c \in \underline{Z}^2(\Lambda, A)$. Define $\iota : \Lambda^0 \times A \to X_c$ by inclusion of sets, and define $q : X_c \to \Lambda$ by $q(\lambda, a) := \lambda$. Then

$$\mathcal{X}_c : \Lambda^0 \times A \xrightarrow{\iota} X_c \xrightarrow{q} \Lambda$$

is a central extension of Λ by A.

In particular, the trivial cocycle $0 : \Lambda^{*2} \to A$ given by $0(\mu, \nu) = 0$ for all μ, ν gives rise to the *trivial extension* $\mathcal{X}_0 : \Lambda^0 \times A \to X_0 \to \Lambda$, where $X_0 = \Lambda \times A$ is the cartesian-product category (with un-twisted composition).

Definition 4.12. Let Λ be a k-graph, and let A be an abelian group. We say that two extensions

$$\mathcal{X} : \Lambda^0 \times A \xrightarrow{\iota_X} X \xrightarrow{q_X} \Lambda \quad \text{and} \quad \mathcal{Y} : \Lambda^0 \times A \xrightarrow{\iota_Y} Y \xrightarrow{q_Y} \Lambda$$

of Λ by A are *isomorphic* if there is a bijective functor $f : X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd}
 & X \arrow[dd, "f"] \arrow[dr, "q_X"] & \\
\Lambda^0 \times A \arrow[ur, "\iota_X"] \arrow[dr, "\iota_Y"'] & & \Lambda \\
 & Y \arrow[ur, "q_Y"'] &
\end{tikzcd}$$

We call f an isomorphism from of $\mathcal{X}$ with $\mathcal{Y}$. We write $\mathrm{Ext}(\Lambda, A)$ for the set of isomorphism classes of extensions of Λ by A.

Fix a central extension

$$\mathcal{X} : \Lambda^0 \times A \xrightarrow{\iota} X \xrightarrow{q} \Lambda.$$

Let $\overline{X} := \{\overline{x} : x \in X\}$ be a copy of the category X. Define $\overline{\iota} : \Lambda^0 \times A \to \overline{X}$ by $\overline{\iota}(v, a) := \overline{\iota(v, -a)}$. Define $\overline{q} : \overline{X} \to \Lambda$ by $\overline{q}(\overline{x}) = q(c)$. Then

$$-\mathcal{X} : \Lambda^0 \times A \xrightarrow{\overline{\iota}} \overline{X} \xrightarrow{\overline{q}} \Lambda$$

is also a central extension of Λ by A. Observe that $a \cdot \overline{x} = \overline{-a \cdot x}$ for $a \in A$ and $x \in X$.

Proposition 4.13. *Let Λ be a k-graph, and let A be an abelian group. Then the formula $[\mathcal{X}] + [\mathcal{Y}] := [\mathcal{X} + \mathcal{Y}]$ determines a well-defined operation under which* $\mathrm{Ext}(\Lambda, A)$ *is an abelian group with identity element $[\mathcal{X}_0]$, the class of the trivial extension. Moreover, $-[\mathcal{X}] = [-\mathcal{X}]$ for each extension $\mathcal{X}$ of Λ by A.*

Proof. We must first check that $[\mathcal{X}] + [\mathcal{Y}]$ is well-defined.

Suppose that $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{Y}$ and $\mathcal{Y}'$ are central extensions of Λ by A, and that f_X is an isomorphism of $\mathcal{X}$ with $\mathcal{X}'$ and f_Y is an isomorphism of $\mathcal{Y}$ with $\mathcal{Y}'$.

Then $(f_X, f_Y) : X *_\Lambda Y \to X' *_\Lambda Y'$ determined by $(f_X, f_Y)(x, y) = (f_X(x), f_Y(y))$ is bijective. For $a \in A$ and $x \in X$,

$$f_X(a \cdot x) = f_X(\iota_X(r(x), a)x) = f_X(\iota_X(r(x), a))f_X(x) = \iota_{X'}(r(x), a) \cdot f_X(x) = a \cdot f_X(x)$$

and similarly $f_Y(a \cdot y) = a \cdot f_Y(y)$ for each $y \in Y$. Hence

$$(f_X, f_Y)(-a \cdot x, a \cdot y) = (f_X(-a \cdot x), f_Y(a \cdot y)) = (-a \cdot f_X(x), a \cdot f_Y(y)) \sim_A (f_X(x), f_Y(y)).$$

It follows that there is a well-defined map $(f_X, f_Y)^\sim : Z(X, Y) \to Z(X', Y')$ determined by $(f_X, f_Y)^\sim([x, y]) = [f_X(x), f_Y(y)]$. This map is bijective because $[x', y'] \mapsto [f_X^{-1}(x), f_Y^{-1}(y)]$ is a well-defined inverse. It is routine to see that $(f_X, f_Y)^\sim$ is an isomorphism between $\mathcal{X} + \mathcal{Y}$ and $\mathcal{X}' + \mathcal{Y}'$, so $[\mathcal{X}] + [\mathcal{Y}]$ is well-defined.

It is routine to check that $[[x, y], z] \mapsto [x, [y, z]]$ determines an isomorphism $([\mathcal{X}]+[\mathcal{Y}])+[\mathcal{Z}] \cong [\mathcal{X}] + ([\mathcal{Y}] + [\mathcal{Z}])$ so addition in $\operatorname{Ext}(\Lambda, A)$ is associative. Likewise $[x, y] \mapsto [y, x]$ determines an isomorphism $[\mathcal{X}] + [\mathcal{Y}] \cong [\mathcal{Y}] + [\mathcal{X}]$, so the operation is commutative.

To see that $[\mathcal{X}] + [\mathcal{X}_0] = [\mathcal{X}]$ for all $[\mathcal{X}] \in \operatorname{Ext}(\Lambda, A)$, we show that $[x, (\lambda, a)] \mapsto a \cdot x$ determines an isomorphism of $Z(X, X_0)$ onto X with inverse given by $x \mapsto [x, (q(x), 0)]$. We must show first that the formula $[x, (\lambda, a)] \mapsto a \cdot x$ is well-defined. If $[x, (\lambda, a)] = [y, (\mu, b)]$ then there exists $c \in A$ such that $y = -c \cdot x$ and $(\mu, b) = c \cdot (\lambda, a) = (\lambda, a + c)$. In particular, $q(x) = \lambda = \mu = q(y)$, and $c = b - a$ is then the unique element $a(x, y)$ of A such that $x = a(x, y) \cdot y$. Hence

$$a \cdot x = a \cdot ((b - a) \cdot y) = b \cdot y,$$

so the formula $[x, (\lambda, a)] \mapsto a \cdot x$ is well-defined. The map $x \mapsto [x, (q(x), 0)]$ is an inverse, and these maps determine an isomorphism of $\mathcal{X} + \mathcal{X}_0$ with $\mathcal{X}$.

Finally, we must show that $[\mathcal{X}] + [-\mathcal{X}] = [\mathcal{X}_0]$. For this we show that the map $[x, \overline{y}] \mapsto (q(x), a(x, y))$ is an isomorphism with inverse $(\lambda, a) \mapsto [x, -a \cdot \overline{x}]$ for any x such that $q(x) = \lambda$. We first check that $[x, \overline{y}] \mapsto (q(x), a(x, y))$ is well-defined. Since $[x, \overline{y}] = [x', \overline{y'}]$ in $Z(X, \overline{X})$ implies that $q(x) = q(x') = q(y) = q(y')$, it suffices to show that $a(x, y) = a(x', y')$. To see this, observe that since $[x, \overline{y}] = [x', \overline{y'}]$, there exists a unique $b \in A$ such that $x = -b \cdot x'$ and $\overline{y} = b \cdot \overline{y'}$, so $y = -b \cdot y'$. Hence

$$a(x', y') \cdot y = \big(a(x', y') - b\big) \cdot y' = -b \cdot x' = x.$$

So uniqueness of $a(x, y)$ forces $a(x', y') = a(x, y)$. Thus $[x, \overline{y}] \mapsto (q(x), a(x, y))$ is well-defined. We claim that the formula $(\lambda, a) \mapsto [x, -a \cdot \overline{x}]$ does not depend on the choice of x such that $q(x) = \lambda$. To see this, suppose $q(y) = \lambda$ also. Then $x = a(x, y) \cdot y$. Hence, using once again that $-a(x, y) \cdot \overline{y} = \overline{a(x, y) \cdot y} = \overline{x}$, we see that

$$[x, -a \cdot \overline{x}] = [a(x, y) \cdot y, -a \cdot (-a(x, y) \cdot \overline{y})] = [a(x, y) \cdot y, -a(x, y) \cdot (-a \cdot \overline{y})] = [y, -a \cdot \overline{y}].$$

It is now routine to see that $[x, \overline{y}] \mapsto [q(x), a(x, y)]$ determines an isomorphism from $\mathcal{X} + (-\mathcal{X})$ to $\mathcal{X}_0$. □

Our next result shows that every central extension of Λ by A is isomorphic to one of the form $\mathcal{X}_c$ described in Example 4.11, and that the assignment $c \mapsto \mathcal{X}_c$ determines an isomorphism from $\underline{H}^2(\Lambda, A)$ to $\operatorname{Ext}(\Lambda, A)$.

Let Λ be a k-graph, and let A be an abelian group. Let

$$\mathcal{X} : \Lambda^0 \times A \xrightarrow{\iota} X \xrightarrow{q} \Lambda$$

be a central extension of Λ by A. A *normalised section for* q is a function $\sigma : \Lambda \to X$ such that $q \circ \sigma$ is the identity map on Λ and such that $\sigma(v) = \iota(v, 0)$ for all $v \in \Lambda^0$. A normalised section for q is typically not multiplicative.

Theorem 4.14. *Let Λ be a k-graph, let A be an abelian group, and let $\mathcal{X}$ be an extension of Λ by A. For each normalised section σ for $q : X \to \Lambda$, define $c_\sigma : \Lambda^{*2} \to A$ by $c_\sigma(\mu,\nu)\cdot\sigma(\mu\nu) = \sigma(\mu)\sigma(\nu)$; that is, $c_\sigma(\mu,\nu) = a(\sigma(\mu)\sigma(\nu),\sigma(\mu\nu))$. Then c_σ is a 2-cocycle. If σ' is any other normalised section for q, then c_σ and $c_{\sigma'}$ are cohomologous. Finally, the assignment $[\mathcal{X}] \mapsto [c_\sigma]$ for any normalised section σ for q is an isomorphism $\theta : \operatorname{Ext}(\Lambda, A) \cong \underline{H}^2(\Lambda, A)$ with inverse given by $\theta^{-1}([c]) = [\mathcal{X}_c]$.*

Proof. We first check that c_σ is a 2-cocycle. Fix $(\mu,\nu) \in \Lambda^{*2}$. Then $\sigma(\mu)\sigma(\nu) = c_\sigma(\mu,\nu)\cdot\sigma(\mu\nu)$. If either μ or ν is in Λ^0, then $\sigma(\mu)\sigma(\nu) = \sigma(\mu\nu)$ so $c_\sigma(\mu,\nu) = 0$. Hence, $c_\sigma \in \underline{C}^2(\Lambda, A)$. Fix $(\lambda,\mu,\nu) \in \Lambda^{*3}$. By uniqueness of $a(\sigma(\lambda)\sigma(\mu)\sigma(\nu),\sigma(\lambda\mu\nu))$ (see Lemma 4.4) it suffices to show that

$$\big(c_\sigma(\lambda,\mu) + c_\sigma(\lambda\mu,\nu)\big)\cdot\sigma(\lambda\mu\nu) = \sigma(\lambda)\sigma(\mu)\sigma(\nu) = \big(c_\sigma(\mu,\nu) + c_\sigma(\lambda,\mu\nu)\big)\cdot\sigma(\lambda\mu\nu).$$

We just verify the first equality; the second follows from similar considerations. We calculate

$$\sigma(\lambda)\sigma(\mu)\sigma(\nu) = c_\sigma(\lambda,\mu)\cdot\sigma(\lambda\mu)\sigma(\nu) = \big(c_\sigma(\lambda,\mu) + c_\sigma(\lambda\mu,\nu)\big)\cdot\sigma(\lambda\mu\nu).$$

Hence, $c_\sigma \in \underline{Z}^2(\Lambda, A)$.

Now suppose that σ' is another normalised section for q. For each $\lambda \in \Lambda$, we have $q(\sigma(\lambda)) = q(\sigma'(\lambda))$, so there is a unique $b(\lambda) := a(\sigma(\lambda),\sigma'(\lambda)) \in A$ such that $\sigma(\lambda) = b(\lambda)\cdot\sigma'(\lambda)$. Since σ and σ' are normalised, $b(v) = 0$ for all $v \in \Lambda^0$, so $b \in C^1(\Lambda, A)$.

If $s(\mu) = r(\nu)$ in Λ, then

$$\begin{aligned}(b(\mu\nu) - b(\mu) - b(\nu))\cdot\sigma(\mu)\sigma(\nu) &= b(\mu\nu)\cdot\sigma'(\mu)\sigma'(\nu)\\ &= (b(\mu\nu) + c_{\sigma'}(\mu,\nu))\cdot\sigma'(\mu\nu)\\ &= c_{\sigma'}(\mu,\nu)\cdot\sigma(\mu\nu)\\ &= (c_{\sigma'}(\mu,\nu) - c_\sigma(\mu,\nu))\cdot\sigma(\mu)\sigma(\nu).\end{aligned}$$

Hence $c_{\sigma'}(\mu,\nu) - c_\sigma(\mu,\nu) = (\underline{\delta}^1 b)(\mu,\nu)$, so c_σ and $c_{\sigma'}$ are cohomologous.

For the final assertion, we must first check that $[\mathcal{X}] \mapsto [c_\sigma]$ is well-defined. If f is an isomorphism of extensions $\mathcal{X}$ and $\mathcal{Y}$, and if σ is a section for q_X, then $\sigma' := f\circ\sigma$ is a section for q_Y. Since

$$a(x,x')\cdot f(x') = f(a(x,x')\cdot x') = f(x)$$

for all $x, x' \in X$ with $q(x) = q(x')$, we have $a(f(x), f(x')) = a(x,x')$ for all $x, x' \in X$. In particular, since f is a functor,

$$\begin{aligned}c_{\sigma'}(\mu,\nu) &= a(\sigma'(\mu)\sigma'(\nu),\sigma'(\mu\nu))\\ &= a(f(\sigma(\mu)\sigma(\nu)), f(\sigma(\mu\nu))) = a(\sigma(\mu)\sigma(\nu),\sigma(\mu\nu)) = c_\sigma(\mu,\nu).\end{aligned}$$

Thus $c_\sigma = c_{\sigma'}$. Since we already proved that distinct normalised sections for the same central extension yield cohomologous categorical 2-cocycles, it follows that for any pair of sections σ for q_X and ρ for q_Y we have $[c_\sigma] = [c_\rho]$ in $\underline{H}^2(\Lambda, A)$. Hence $[\mathcal{X}] \mapsto [c_\sigma]$ (for any section σ for q) is well defined. This map is additive by (4.3), and hence a homomorphism.

To see that it is an isomorphism, it suffices to show that the map $c \mapsto [\mathcal{X}_c]$ from $\underline{Z}^2(\Lambda, A)$ to $\operatorname{Ext}(\Lambda, A)$ determines a well-defined map $[c] \mapsto [\mathcal{X}_c]$ from $\underline{H}^2(\Lambda, A)$ to $\operatorname{Ext}(\Lambda, A)$, and that this map is an inverse for θ. Fix $c \in \underline{Z}^2(\Lambda, A)$ and $b \in \underline{C}^1(\Lambda, A)$ and let $c' = c - (\underline{\delta}^1 b)$ so that $[c] = [c'] \in \underline{H}^2(\Lambda, A)$. Define $f : X_c \to X_{c'}$ by $f(\lambda, a) := (\lambda, a + b(\lambda))$. To see that f is a functor, we calculate:

$$f((\lambda,a)(\mu,a')) = f(\lambda\mu, c(\lambda,\mu) + a + a') = (\lambda\mu, c(\lambda,\mu) + b(\lambda\mu) + a + a'),$$

and

$$f(\lambda,a)f(\mu,a') = (\lambda, a+b(\lambda))(\mu, a'+b(\mu)) = (\lambda\mu, c'(\lambda,\mu)+b(\lambda)+b(\mu)+a+a').$$

Since $c - c' = \underline{\delta}^1 b$, we have

$$c(\lambda,\mu) = c'(\lambda,\mu) + b(\lambda) + b(\mu) - b(\lambda\mu),$$

and hence $c'(\lambda,\mu)+b(\lambda)+b(\mu) = c(\lambda,\mu)+b(\lambda\mu)$, giving $f((\lambda,a)(\mu,a')) = f(\lambda,a)f(\mu,a')$. The functor f is bijective because $(\lambda,a)\mapsto(\lambda,a-b(\lambda))$ is an inverse. Hence $[c]\mapsto[\mathcal{X}_c]$ is a well-defined map from $\underline{H}^2(\Lambda,A)$ to $\operatorname{Ext}(\Lambda,A)$.

To see that $[c]\mapsto[\mathcal{X}_c]$ is an inverse for θ, fix an extension

$$\mathcal{X} : \Lambda^0\times A \xrightarrow{\iota} X \xrightarrow{q} \Lambda$$

and a section σ for q. We must show that $\mathcal{X}_{c_\sigma}$ is isomorphic to $\mathcal{X}$. We define $f : X\to X_{c_\sigma}$ by $f(x) := \big(q(x), a(x,\sigma(q(x)))\big)$ and $g : X_{c_\sigma}\to X$ by $g(\lambda,a) := a\cdot\sigma(\lambda)$. Then

$$f\big(g(\lambda,a)\big) = f(a\cdot\sigma(\lambda)) = \big(q(a\cdot\sigma(\lambda)), a(a\cdot\sigma(\lambda),\sigma(q(a\cdot\sigma(\lambda))))\big) = \big(\lambda, a(a\cdot\sigma(\lambda),\sigma(\lambda))\big).$$

For all $x\in X$ and $b\in A$, we have $a(b\cdot x,x) = b$ by definition. Thus $f\circ g = \mathrm{id}_{X_{c_\sigma}}$. Likewise, for $x\in X$, we have

$$g\circ f(x) = g\big(q(x), a(x,\sigma(q(x)))\big) = a(x,\sigma(q(x)))\cdot\sigma(q(x)) = x.$$

So f and g are mutually inverse, and we just need to show that f preserves composition. We fix $(x,y)\in X^{*2}$ and calculate:

$$f(xy) = \big(q(xy), a(xy,\sigma(q(xy)))\big) = \big(q(x)q(y), a(xy,\sigma(q(x)q(y)))\big),$$

and

$$\begin{aligned} f(x)f(y) &= \big(q(x), a(x,\sigma(q(x)))\big)\big(q(y), a(y,\sigma(q(y)))\big) \\ &= \big(q(x)q(y), c_\sigma(q(x),q(y)) + a(x,\sigma(q(x))) + a(y,\sigma(q(y)))\big) \\ &= \big(q(x)q(y), a\big(\sigma(q(x))\sigma(q(y)),\sigma(q(x)q(y))\big) + a(x,\sigma(q(x))) + a(y,\sigma(q(y)))\big). \end{aligned}$$

Since $\mathcal{X}$ is a central extension, we have

$$\begin{aligned} &\big(a\big(\sigma(q(x))\sigma(q(y)),\sigma(q(x)q(y))\big) + a(x,\sigma(q(x))) + a(y,\sigma(q(y)))\big)\cdot\sigma(q(x)q(y)) \\ &\qquad = \big(a(x,\sigma(q(x))) + a(y,\sigma(q(y)))\big)\sigma(q(x))\sigma(q(y)) \\ &\qquad = \big(a(x,\sigma(q(x)))\cdot\sigma(q(x))\big)\big(a(y,\sigma(q(y)))\cdot\sigma(q(y))\big) \\ &\qquad = xy. \end{aligned}$$

Hence $a\big(\sigma(q(x))\sigma(q(y)),\sigma(q(x)q(y))\big) + a(x,\sigma(q(x))) + a(y,\sigma(q(y))) = a(xy,\sigma(q(x)q(y)))$, giving $f(xy) = f(x)f(y)$ as claimed. This shows that f is a functor, and hence an isomorphism of extensions. So $[c]\mapsto[\mathcal{X}_c]$ is a left inverse for θ. To see that it is a right inverse also, fix a cocycle $c\in\underline{Z}^2(\Lambda,A)$. Then the normalised section $\sigma_c : \lambda\mapsto(\lambda,0)$ for $q : X_c\to\Lambda$ satisfies $c_{\sigma_c} = c$. □

Theorem 4.15. *The map $\psi : H^2(\Lambda,A)\to\underline{H}^2(\Lambda,A)$ given by $[\phi]\mapsto[c_\phi]$ is an isomorphism (see Theorem 3.16).*

Proof. Fix a central extension

$$\mathcal{X} : \Lambda^0\times A \xrightarrow{\iota} X \xrightarrow{q} \Lambda$$

of Λ by A. For each $v\in\Lambda^0$, let $\sigma(v)\in X$ be the unique identity morphism such that $q(\sigma(v)) = v$; that is, $\sigma(v) = v$. For each edge $e\in E^1_\Lambda$ in the skeleton of Λ, fix an element $\sigma(e)\in X$ such that $q(\sigma(e)) = e$. Extend σ to a section for q by setting

$\sigma(\lambda) := \sigma(\overline{\lambda}_1)\sigma(\overline{\lambda}_2)\cdots\sigma(\overline{\lambda}_{|\lambda|})$ where $\lambda \mapsto \overline{\lambda}$ is the preferred section for the quotient map $\pi : E_\Lambda \to \Lambda$ as in Notation 3.11.

Define $\phi : Q_2(\Lambda) \to A$ by

$$\phi(\lambda) := a(\sigma(\lambda), \sigma(F_1^0(\lambda))\sigma(F_2^1(\lambda)));$$

that is, if $d(\lambda) = e_i + e_j$ with $i < j$, and if $\lambda = fg = g'f'$ where $f, f' \in \Lambda^{e_i}$ and $g, g' \in \Lambda^{e_j}$, then

$$\phi(\lambda) \cdot \big(\sigma(g')\sigma(f')\big) = \sigma(f)\sigma(g).$$

We must first check that ϕ is a cubical 2-cocycle. Fix $\lambda \in Q_3(\Lambda)$, say $d(\lambda) = e_i + e_j + e_l$ where $i < j < l$, and factorise

$$\lambda = f_0g_0h_0 = f_0h_1g_1 = h_2f_1g_1 = h_2g_2f_2 = g_3h_3f_2 = g_3f_3h_0$$

as in (2.1), so $d(f_n) = e_i$, $d(g_n) = e_j$ and $d(h_n) = e_l$ for all n.

By definition of ϕ, we have

$$\phi(F_3^0(\lambda)) + \phi(F_2^1(\lambda)) + \phi(F_1^0(\lambda)) = \phi(f_0g_0) + \phi(f_3h_0) + \phi(g_3h_3).$$

Moreover,

$$\phi(f_0g_0) \cdot \sigma(g_3)\sigma(f_3) = \sigma(f_0)\sigma(g_0), \quad \phi(f_3h_0) \cdot \sigma(h_3)\sigma(f_2) = \sigma(f_3)\sigma(h_0), \quad \text{and}$$
$$\phi(g_3h_3) \cdot \sigma(h_2)\sigma(g_2) = \sigma(g_3)\sigma(h_3).$$

Using this and three applications of Lemma 4.4, we deduce that

$$\big(\phi(f_0g_0) + \phi(f_3h_0) + \phi(g_3h_3)\big) \cdot \sigma(h_2)\sigma(g_2)\sigma(f_2) = \sigma(f_0)\sigma(g_0)\sigma(h_0) = \sigma(\lambda),$$

and hence

$$\phi(F_3^0(\lambda)) + \phi(F_2^1(\lambda)) + \phi(F_1^0(\lambda)) = a(\sigma(\lambda), \sigma(h_2)\sigma(g_2)\sigma(f_2)).$$

Symmetric reasoning also shows that

$$\phi(F_1^1(\lambda)) + \phi(F_2^0(\lambda)) + \phi(F_3^1(\lambda)) = a(\sigma(\lambda), \sigma(h_2)\sigma(g_2)\sigma(f_2)).$$

In particular, ϕ satisfies (3.8), so it is a cubical 2-cocycle.

Next, we claim that the cohomology class $[\phi]$ is independent of the choices made (on the skeleton). Let $\tilde{\sigma}$ be another section of q constructed as above. We show that the resulting cubical 2-cocycle is cohomologous to ϕ. Let $b(e) = a(\tilde{\sigma}(e), \sigma(e))$ for each $e \in Q_1(\Lambda) = E_\Lambda^1$. Then the cubical 2-cocycle $\tilde{\phi}$ built from $\tilde{\sigma}$ is defined so that if $d(\lambda) = e_i + e_j$ with $i < j$, and if $\lambda = fg = g'f'$ where $f, f' \in \Lambda^{e_i}$ and $g, g' \in \Lambda^{e_j}$, then

$$\tilde{\phi}(\lambda) = a(\tilde{\sigma}(f)\tilde{\sigma}(g), \tilde{\sigma}(g')\tilde{\sigma}(f')).$$

A routine computation then shows

$$\phi(\lambda) - \tilde{\phi}(\lambda) = b(g') + b(f') - b(f) - b(g) = (\delta^1 b)(\lambda).$$

Hence $[\phi] = [\tilde{\phi}]$.

We now claim that the cocycle c_σ obtained from the section σ via Theorem 4.14 is equal to $-c_\phi$. To see this, fix $(\mu, \nu) \in \Lambda^{*2}$. By definition, $c_\phi(\mu, \nu) = S_\phi(\overline{\mu}\,\overline{\nu})$. Fix a sequence of allowable transitions from $\overline{\mu}\,\overline{\nu}$ to $\overline{\mu\nu}$; that is, a path $\alpha_1 \cdots \alpha_n$ in F_Λ with $r(\alpha_1) = \overline{\mu\nu}$ and $s(\alpha_n) = \overline{\mu}\,\overline{\nu}$. Fix $i \le n$ and let $u = r(\alpha_i)$ and $w = s(\alpha_i)$. Recall from Notation 3.13 the definition of $\tilde{\phi} : F_\Lambda^1 \to A$. For each i, we have $\tilde{\phi}(\alpha_i) = \phi(w_{p(\alpha_i)}w_{p(\alpha_i)+1}) = a(u_{p(\alpha_i)}u_{p(\alpha_i)+1}, w_{p(\alpha_i)}w_{p(\alpha_i)+1})$. Let $\ell := |\mu\nu|$. Using the first assertion of Lemma 4.4, we see that

$$\tilde{\phi}(\alpha_i) = a\big(\sigma(u_1)\sigma(u_2)\cdots\sigma(u_\ell), \sigma(w_1)\sigma(w_2)\cdots\sigma(w_\ell)\big).$$

Hence, by definition of S_ϕ, we have

$$S_\phi(\overline{\mu}\,\overline{\nu}) = \sum_{i=1}^{n} a\big(\sigma(r(\alpha_i)_1)\cdots\sigma(r(\alpha_i)_\ell), \sigma(s(\alpha_i)_1)\cdots\sigma(s(\alpha_i)_\ell)\big).$$

Now using the second assertion of Lemma 4.4, we deduce that

$$S_\phi(\overline{\mu}\,\overline{\nu}) = a\big(\sigma((\overline{\mu\nu})_1\cdots\sigma(\overline{\mu\nu})_\ell, \sigma(\overline{\mu}_1)\cdots\sigma(\overline{\mu}_{|\mu|})\sigma(\overline{\nu}_1)\cdots\sigma(\overline{\nu}_{|\nu|})\big) = a(\sigma(\mu\nu), \sigma(\mu)\sigma(\nu)).$$

Hence $c_\phi(\mu,\nu) = a(\sigma(\mu\nu), \sigma(\mu)\sigma(\nu)) = -c_\sigma(\mu,\nu)$ as claimed.

Now fix $c \in \underline{Z}^2(\Lambda, A)$. By Theorem 4.14, we have $[c] = [c_\sigma]$ for any section σ for $q : X_c \to \Lambda$. By the preceding paragraphs, there exists a section σ for $q : X_c \to \Lambda$, and a cubical 2-cocycle ϕ on Λ such that $c_\sigma = -c_\phi$. In particular, we have $[c] = [c_\sigma] = [-c_\phi]$, and it follows that the map $[\phi] \mapsto [c_\phi]$ is surjective from $H^2(\Lambda, A)$ to $\underline{H}^2(\Lambda, A)$. Since the class $[\phi]$ does not depend on the choice of section σ, the map is also injective. □

5. Twisted k-graph C^*-algebras

In this section, unless otherwise noted, we restrict attention to row-finite k-graphs with no sources and consider twisted k-graph C^*-algebras. We recall the definition of a twisted k-graph C^*-algebra from [14], and then introduce the notion of a twisted Cuntz-Krieger (Λ, c)-family associated to a categorical cocycle $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. We show that given a cubical cocycle $\phi \in Z^2(\Lambda, \mathbb{T})$, if $c_\phi \in \underline{Z}^2(\Lambda, \mathbb{T})$ is the 2-cocycle of Theorem 3.16 then the twisted C^*-algebra $C^*_\phi(\Lambda)$ introduced in [14, Section 7] is universal for twisted Cuntz-Krieger c_ϕ-families for Λ.

For the abelian group $\mathbb{T}$ we break with our conventions earlier in the paper and write the group operation multiplicatively, write $\overline{z}$ for the inverse of $z \in \mathbb{T}$ and write 1 for the identity element.

For the following, recall from [19] that a k-graph Λ is said to be *locally convex* if, whenever e_i, e_j are distinct generators of $\mathbb{N}^k$ and $\mu \in \Lambda^{e_i}$ and $\nu \in \Lambda^{e_j}$ satisfy $r(\mu) = r(\nu)$, both $s(\mu)\Lambda^{e_j}$ and $s(\nu)\Lambda^{e_i}$ are nonempty.

Definition 5.1 (see [14, Definitions 7.4, 7.5])**.** Let Λ be a row-finite locally convex k-graph and fix $\phi \in Z^2(\Lambda, \mathbb{T})$. A Cuntz-Krieger ϕ-representation of Λ in a C^*-algebra A is a set $\{p_v : v \in \Lambda^0\} \subseteq A$ of mutually orthogonal projections and a set $\{s_\lambda : \lambda \in \bigcup_{i=1}^k \Lambda^{e_i}\} \subseteq A$ satisfying

(1) for all $i \le k$ and $\lambda \in \Lambda^{e_i}$, $s_\lambda^* s_\lambda = p_{s(\lambda)}$;
(2) for all $1 \le i < j \le k$ and $\mu, \mu' \in \Lambda^{e_i}$, $\nu, \nu' \in \Lambda^{e_j}$ such that $\mu\nu = \nu'\mu'$,
$$s_{\nu'} s_{\mu'} = \phi(\mu\nu) s_\mu s_\nu; \text{ and}$$
(3) for all $v \in \Lambda^0$ and all $i \in \{1, \dots, k\}$ such that $v\Lambda^{e_i} \neq \emptyset$,
$$p_v = \sum_{\lambda \in v\Lambda^{e_i}} s_\lambda s_\lambda^*.$$

We define $C^*_\phi(\Lambda)$ to be the universal C^*-algebra generated by a Cuntz-Krieger ϕ-representation of Λ.

The following is much closer to the usual definition of a Cuntz-Krieger Λ-family. Notice, however, that we now restrict attention to k-graphs with no sources: that is, $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. Every k-graph with no sources is locally convex. Versions of the following definition for row-finite locally convex k-graphs or for finitely aligned k-graphs incorporating the ideas of [19] or [20] seem likely to produce reasonable notions of twisted k-graph C^*-algebras but we do not pursue this level of generality here.

Definition 5.2. Let (Λ, d) be a row-finite k-graph with no sources, and fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. A Cuntz-Krieger (Λ, c)-family in a C^*-algebra B is a function $t : \lambda \mapsto t_\lambda$ from Λ to B such that

(CK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $t_\mu t_\nu = c(\mu, \nu) t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(CK4) $t_v = \sum_{\lambda \in v\Lambda^n} t_\lambda t_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

We first show that given a $\mathbb{T}$-valued 2-cocycle ϕ, the universal C^*-algebra $C^*_\phi(\Lambda)$ of Definition 5.1 is universal for Cuntz-Krieger (Λ, c_ϕ)-families.

Recall from Notation 3.11 that for $\lambda \in \Lambda$, we write $\overline{\lambda}$ for the path in E_Λ corresponding to the factorisation $\lambda = \overline{\lambda}_1 \cdots \overline{\lambda}_n$ of λ in which edges of degree e_1 appear leftmost, then those of degree e_2 and so on.

Proposition 5.3. *Let Λ be a row-finite k-graph with no sources, and let $\phi \in Z^2(\Lambda, \mathbb{T})$. Let $c_\phi \in \underline{Z}^2(\Lambda, \mathbb{T})$ be the categorical 2-cocycle obtained from Theorem 3.16. Let $\{p_v : v \in \Lambda^0\}$ and $\{s_\lambda : \lambda \in \bigsqcup_{i=1}^k \Lambda^{e_i}\}$ be the universal generating Cuntz-Krieger ϕ-representation of Λ in $C^*_\phi(\Lambda)$. For $v \in \Lambda^0$, let $t_v := p_v$ and for $\lambda \in \Lambda \backslash \Lambda^0$, set $t_\lambda = s_{\overline{\lambda}_1} \cdots s_{\overline{\lambda}_{|\lambda|}}$. Then $t : \lambda \mapsto t_\lambda$ constitutes a Cuntz-Krieger (Λ, c_ϕ)-family in $C^*_\phi(\Lambda)$. Moreover, this family is universal in the sense that given any Cuntz-Krieger (Λ, c_ϕ)-family $\{t'_\lambda : \lambda \in \Lambda\}$ in a C^*-algebra B, there is a homomorphism $\pi : C^*_\phi(\Lambda) \to B$ such that $\pi(t_\lambda) = t'_\lambda$.*

Proof. Recall from Definition 3.12 that F_Λ denotes the transition graph associated to Λ, and that for each $\lambda \in \Lambda$, the preferred factorisation $\overline{\lambda}$ of λ is the terminal vertex in the component F_λ of F_Λ corresponding to λ. Since each $u \in F^0_\Lambda$ is a path $u = u_1 \cdots u_n \in E_\Lambda$, we may define partial isometries $\{\tau_u : u \in F^0_\Lambda, \ell(u) \geq 1\}$ by $\tau_u := s_{u_1} \cdots s_{u_n}$. Thus for $\lambda \in \Lambda$ with $d(\lambda) \neq 0$, the definition of t_λ given in the statement of the proposition can be restated as $t_\lambda = \tau_{\overline{\lambda}}$. For $v \in \Lambda^0$, we define $t_v := p_v$.

Suppose that (u, v) is an allowable transition in λ, say $u = u_1 \cdots u_{i-1} u_i \cdots u_n$ and $v = u_1 \cdots u_{i-2} v_{i-1} v_i u_{i+1} \cdots u_n$ with $d(u_i) = d(v_{i-1}) = e_j$ and $d(u_{i-1}) = d(v_i) = e_l$ with $j > l$. Then relation (2) of Definition 5.1 gives

$$s_{u_1} \cdots s_{v_{i-1}} s_{v_i} \cdots s_{u_n} = \phi(u_{i-1} u_i) s_{u_1} \cdots s_{u_{i-1}} s_{u_i} \cdots s_{u_n}$$

Hence, using the map $\tilde{\phi} : F^1_\Lambda \to \mathbb{T}$ defined as in Notation 3.13, we have

$$\tau_v = \tilde{\phi}(u, v) \tau_u.$$

So if α is a path in F_Λ, then $\tau_{s(\alpha)} = \tilde{\phi}(\alpha) \tau_{r(\alpha)}$.

Fix $(\mu, \nu) \in \Lambda^{*2}$ and $\alpha \in F_\Lambda$ with $r(\alpha) = \overline{\mu\nu}$ and $s(\alpha) = \overline{\mu}\,\overline{\nu}$. By the above

$$\tilde{\phi}(\alpha) t_{\mu\nu} = \tilde{\phi}(\alpha) \tau_{\overline{\mu\nu}} = \tau_{\overline{\mu}\,\overline{\nu}} = \tau_{\overline{\mu}} \tau_{\overline{\nu}} = t_\mu t_\nu.$$

Lemma 3.15 implies that $\tilde{\phi}(\alpha) = S_\phi(\overline{\mu}\,\overline{\nu})$, so, by the definition of c_ϕ in Theorem 3.16, $c_\phi(\mu, \nu) t_{\mu\nu} = t_\mu t_\nu$. Thus $t : \Lambda \to C^*_\phi(\Lambda)$ satisfies relation (CK2) for the cocycle c_ϕ. It trivially satisfies (CK1), and standard induction arguments establish (CK3) and (CK4). Hence t is a Cuntz-Krieger (Λ, c_ϕ)-family.

Let $\{t'_\lambda : \lambda \in \Lambda\}$ be a Cuntz-Krieger (Λ, c_ϕ)-family in a C^*-algebra B. We claim that $\{t'_v : v \in \Lambda^0\}$ and $\{t'_\lambda : \lambda \in \bigsqcup_{i=1}^k \Lambda^{e_i}\}$ constitute a Cuntz-Krieger ϕ-representation of Λ in B. Relations (1) and (3) are special cases of (CK3) and (CK4) respectively. Let μ, ν, μ', ν' be as in Definition 5.1(2). By definition of c_ϕ, we have

$$c_\phi(\nu', \mu') = \phi(\mu\nu) \quad \text{and} \quad c_\phi(\mu, \nu) = 1.$$

Hence

$$t'_{\nu'}t'_{\mu'} = c_\phi(\nu',\mu')t'_{\nu'\mu'} = c_\phi(\nu',\mu')\overline{c_\phi(\mu,\nu)}t'_\mu t'_\nu = \phi(\mu\nu)t'_\mu t'_\nu.$$

So the elements $\{t'_v : v \in \Lambda^0\}$ and $\{t'_\lambda : \lambda \in \bigsqcup_{i=1}^k \Lambda^{e_i}\}$) satisfy the relations (1)–(3). By the universal property of $C^*_\phi(\Lambda)$ there is a homomorphism $\pi : C^*_\phi(\Lambda) \to B$ such that $\pi(t_\lambda) = t'_\lambda$ for $\lambda \in \Lambda^0 \cup \bigsqcup_{i=1}^k \Lambda^{e_i}$. An induction shows that $\pi(t_\lambda) = t'_\lambda$ for all $\lambda \in \Lambda$. □

Proposition 5.3 shows that the twisted C^*-algebras associated to $\mathbb{T}$-valued 2-cocycles in [14] can be regarded as twisted C^*-algebras associated to the corresponding categorical cocycles. But, while every categorical 2-cocycle c is cohomologous to c_ϕ for some $\phi \in Z^2(\Lambda,\mathbb{T})$, it is not clear that every categorical 2-cocycle is equal to c_ϕ for some ϕ.

Notation 5.4. Let (Λ, d) be a row-finite k-graph with no sources, and fix $c \in \underline{Z}^2(\Lambda,\mathbb{T})$. Relations (CK1) and (CK3) imply that the images of elements of a Cuntz-Krieger (Λ, c)-family under any $*$-homomorphism are partial isometries and hence have norm 1 (or 0). A standard argument (see, for example, [18, Propositions 1.20 and 1.21]) then shows that there is a C^*-algebra $C^*(\Lambda, c)$ generated by a Cuntz-Krieger (Λ, c)-family $s : \Lambda \to C^*(\Lambda, c)$ which is universal in the sense that given any other Cuntz-Krieger (Λ, c)-family t, there is a homomorphism $\pi_t : C^*(\Lambda, c) \to C^*(t)$ such that $\pi_t \circ s = t$. This universal property determines $C^*(\Lambda, c)$ up to canonical isomorphism.

The following Remark reconciles the use of s to denote the universal family in $C^*(\Lambda, c_\phi)$ with the use of the same symbol to denote the universal family in $C^*_\phi(\Lambda)$.

Remark 5.5. Let Λ be a row-finite k-graph with no sources, and fix $\phi \in Z^2(\Lambda,\mathbb{T})$. Let $c_\phi \in \underline{Z}^2(\Lambda,\mathbb{T})$ be the corresponding categorical 2-cocycle. Proposition 5.3 and that $C^*(\Lambda, c)$ is determined by its universal property imply that there is an isomorphism $C^*(\Lambda, c_\phi) \to C^*_\phi(\Lambda)$ which carries each generator of $C^*(\Lambda, c_\phi)$ associated to a vertex or an edge to the corresponding generator of $C^*_\phi(\Lambda)$. We will henceforth identify $C^*(\Lambda, c_\phi)$ and $C^*_\phi(\Lambda)$ via this isomorphism without comment.

Proposition 5.6. *Let Λ be a row-finite k-graph with no sources, fix $c_1, c_2 \in \underline{Z}^2(\Lambda,\mathbb{T})$ and suppose that c_1 and c_2 are cohomologous, that is, $c_1 = (\underline{\delta}^1 b)c_2$ for some $b \in \underline{C}^1(\Lambda,\mathbb{T})$. Denote by s^{c_i} the universal Cuntz-Krieger (Λ, c_i)-family in $C^*(\Lambda, c_i)$ for $i = 1, 2$. Then there is an isomorphism $\pi : C^*(\Lambda, c_1) \to C^*(\Lambda, c_2)$ satisfying $\pi(s^{c_1}_\lambda) = b(\lambda)s^{c_2}_\lambda$ for all $\lambda \in \Lambda$. In particular, if $c \in \underline{B}^2(\Lambda,\mathbb{T})$, then $C^*(\Lambda, c) \cong C^*(\Lambda)$.*

Proof. For $(\mu,\nu) \in \Lambda^{*2}$ we have $c_1(\mu,\nu) = b(\mu)b(\nu)\overline{b(\mu\nu)}c_2(\mu,\nu)$, and hence

$$b(\mu)b(\nu)c_2(\mu,\nu) = b(\mu\nu)c_1(\mu,\nu).$$

Hence relation (CK2) gives

$$b(\mu)s^{c_2}_\mu b(\nu)s^{c_2}_\nu = b(\mu)b(\nu)\big(c_2(\mu,\nu)s^{c_2}_{\mu\nu}\big) = c_1(\mu,\nu)\big(b(\mu\nu)s^{c_2}_{\mu\nu}\big).$$

Since relations (CK1), (CK3) and (CK4) do not depend on the cocycle c_1, it follows that the function $t : \Lambda \to C^*(\Lambda, c_2)$ defined by $\lambda \mapsto b(\lambda)s^{c_2}_\lambda$ is a Cuntz-Krieger (Λ, c_1)-family, so the universal property of $C^*(\Lambda, c_1)$ yields a homomorphism $\pi : C^*(\Lambda, c_1) \to C^*(\Lambda, c_2)$ satisfying $\pi(s^{c_1}_\lambda) = b(\lambda)s^{c_2}_\lambda$ for all $\lambda \in \Lambda$. The symmetric argument yields a homomorphism $\phi : C^*(\Lambda, c_2) \to C^*(\Lambda, c_1)$ which is an inverse for π on generators. Hence π is an isomorphism as claimed.

For the final assertion, note that if c is a coboundary, then it is cohomologous to the trivial cocycle $1 \in \underline{Z}^2(\Lambda,\mathbb{T})$ given by $1(\mu,\nu) = 1$ for all $(\mu,\nu) \in \Lambda^{*2}$. Since $C^*(\Lambda, 1)$ is universal for the same relations as the k-graph C^*-algebra $C^*(\Lambda)$ of [10], we have $C^*(\Lambda, 1) \cong C^*(\Lambda)$. Hence $C^*(\Lambda, c) \cong C^*(\Lambda)$ by the preceding paragraph. □

Recall that by Theorem 4.15 the map $[\phi] \mapsto [c_\phi]$ yields an isomorphism $H^2(\Lambda, A) \cong \underline{H}^2(\Lambda, A)$. Combining Proposition 5.3 and Proposition 5.6 we obtain the following.

Corollary 5.7. *Let Λ be a row-finite k-graph with no sources, and fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $\phi \in Z^2(\Lambda, \mathbb{T})$ be a 2-cocycle such that c_ϕ is cohomologous to c. Then $C^*(\Lambda, c) \cong C^*_\phi(\Lambda)$.*

The preceding corollary gives another proof that if $\phi, \psi \in Z^2(\Lambda, \mathbb{T})$ are cohomologous, then $C^*_\psi(\Lambda) \cong C^*_\phi(\Lambda)$ (see [14, Proposition 7.6]).

6. Twisted groupoid C^*-algebras

Let Λ be a row-finite k-graph with no sources. Let $\mathcal{G}_\Lambda$ be the k-graph groupoid of [10] (see Definition 6.1 below) and let $\tilde{H}^2(\mathcal{G}_\Lambda, \cdot)$ denote the continuous cocycle cohomology used in [21] (see Remark 3.7). Given a categorical 2-cocycle c on Λ we construct a 2-cocycle σ_c on the groupoid $\mathcal{G}_\Lambda$. Given a locally compact abelian group A we show that $c \mapsto \sigma_c$ determines a homomorphism $\underline{H}^2(\Lambda, A) \to \tilde{H}^2(\mathcal{G}_\Lambda, A)$. If c is $\mathbb{T}$-valued, we show that there is a canonical homomorphism from the twisted k-graph C^*-algebra $C^*(\Lambda, c)$ to Renault's twisted groupoid algebra $C^*(\mathcal{G}_\Lambda, \sigma_c)$ (see [21]). We show in §7 that this homomorphism is an isomorphism.

We denote by $\Lambda\, {}_s{*}_s\, \Lambda$ the set $\{(\mu, \nu) \in \Lambda \times \Lambda : s(\mu) = s(\nu)\}$. Recall the definition of Λ^∞ given in [10, Definition 2.1]: we write Ω_k for the k-graph $\{(m, n) \in \mathbb{N}^k : m \leq n\}$ with $r(m, n) = m$, $s(m, n) = n$, $(m, n)(n, p) = (m, p)$ and $d(m, n) = n - m$, and we define Λ^∞ to be the collection of all k-graph morphisms $x : \Omega_k \to \Lambda$. For $p \in \mathbb{N}^k$, we define $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $(\sigma^p x)(m, n) := x(m + p, n + p)$ for all $(m, n) \in \Omega_k$. For $x \in \Lambda^\infty$ we denote $x(0)$ by $r(x)$.

Definition 6.1 ([10, Definition 2.7])**.** Let Λ be a row-finite k-graph with no sources. Let

$$\mathcal{G}_\Lambda := \{(x, \ell - m, y) : \ell, m \in \mathbb{N}^k, \sigma^\ell x = \sigma^m y\} \subset \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty.$$

For $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ define $Z(\mu, \nu) \subset \mathcal{G}_\Lambda$ by

$$Z(\mu, \nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) : x \in \Lambda^\infty, r(x) = s(\mu)\}.$$

For $\lambda \in \Lambda$, we define $Z(\lambda) := Z(\lambda, \lambda)$.

The sets $Z(\mu, \nu)$ form a basis of compact open sets for a locally compact Hausdorff topology on $\mathcal{G}_\Lambda$ under which it is an étale groupoid with structure maps $r(x, \ell - m, y) = (x, 0, x)$, $s(x, \ell - m, y) = (y, 0, y)$, and $(x, \ell - m, y)(y, p - q, z) = (x, \ell - m + p - q, z)$. (see [10, Proposition 2.8]). The $Z(\lambda)$ are then a basis for the relative topology on $\mathcal{G}_\Lambda^{(0)}$. We shall identify $\mathcal{G}_\Lambda^{(0)} = \{(x, 0, x) : x \in \Lambda^\infty\}$ with Λ^∞.

Notation 6.2. We write $\tilde{d}$ for the continuous $\mathbb{Z}^k$-valued 1-cocycle on $\mathcal{G}_\Lambda$ induced by the degree map on Λ. That is, $\tilde{d}(x, m, y) = m$.

Our next two results show how to use an appropriate partition of $\mathcal{G}_\Lambda$ to construct a continuous 2-cocycle on $\mathcal{G}_\Lambda$ (see Remark 3.7) from a categorical 2-cocycle on Λ.

Lemma 6.3. *Let Λ be a row-finite k-graph with no sources, let A be an abelian group and let $c \in \underline{Z}^2(\Lambda, A)$. Suppose that $\mathcal{P}$ is a countable subset of $\Lambda\, {}_s{*}_s\, \Lambda$ such that $\{Z(\mu, \nu) : (\mu, \nu) \in \mathcal{P}\}$ is a partition of $\mathcal{G}_\Lambda$. For each $g \in \mathcal{G}_\Lambda$, let (μ_g, ν_g) be the unique element of $\mathcal{P}$ such that $g \in Z(\mu_g, \nu_g)$.*

(i) *For each* $(g,h) \in \mathcal{G}_\Lambda^{(2)}$, *there exist* $\alpha \in s(\mu_g)\Lambda$, $\beta \in s(\mu_h)\Lambda$, $\gamma \in s(\mu_{gh})\Lambda$ *and* $y \in \Lambda^\infty$ *such that* $r(y) = s(\alpha) = s(\beta) = s(\gamma)$ *and*

$$g = (\mu_g\alpha y, d(\mu_g) - d(\nu_g), \nu_g\alpha y), \quad h = (\mu_h\beta y, d(\mu_h) - d(\nu_h), \nu_h\beta y), \quad \text{and} \quad gh = (\mu_{gh}\gamma y, d(\mu_{gh}) - d(\nu_{gh}), \nu_{gh}\gamma y). \tag{6.1}$$

In particular, $\mu_g\alpha = \mu_{gh}\gamma$, $\nu_h\beta = \nu_{gh}\gamma$ *and* $\nu_g\alpha = \mu_h\beta$.

(ii) *Fix* $(g,h) \in \mathcal{G}_\Lambda^{(2)}$ *and* α, β, γ *satisfying* (6.1). *The formula*

$$\big(c(\mu_g,\alpha) - c(\nu_g,\alpha)\big) + \big(c(\mu_h,\beta) - c(\nu_h,\beta)\big) - \big(c(\mu_{gh},\gamma) - c(\nu_{gh},\gamma)\big) \tag{6.2}$$

does not depend on the choice of α, β, γ.

(iii) *For* $(g,h) \in \mathcal{G}_\Lambda^{(2)}$, *define* $\sigma_c(g,h)$ *to be the value of 6.2 for any choice of* α, β, γ *satisfying* (6.1). *Then* σ_c *is a continuous groupoid* 2*-cocycle.*

Proof. Recall from Notation 6.2 the definition of $\tilde{d} : \mathcal{G}_\Lambda \to \mathbb{Z}^k$. Let $N := d(\mu_g) \vee d(\mu_{gh}) \vee (d(\mu_h) - \tilde{d}(g))$. Then routine calculations show that $\alpha := r(g)(d(\mu_g), N)$, $\gamma := r(gh)(d(\mu_{gh}), N)$, $\beta := r(h)(d(\mu_h), N + \tilde{d}(g))$ and $y = \sigma^N(r(g))$ have the desired properties.

Fix α, β, γ and α', β', γ' satisfying (6.1). Let

$$\delta := r(g)(d(\mu_g\alpha), d(\mu_g\alpha) \vee d(\mu_g\alpha')) \quad \text{and} \quad \varepsilon := r(g)(d(\mu_g\alpha'), d(\mu_g\alpha) \vee d(\mu_g\alpha'))$$

Then $\alpha\delta = \alpha'\varepsilon$, $\beta\delta = \beta'\varepsilon$ and $\gamma\delta = \gamma'\varepsilon$ satisfy (6.1). So by symmetry, it suffices to show that replacing α with $\alpha\delta$, β with $\beta\delta$ and γ with $\gamma\delta$ in (6.2) yields the same value. Since c is a categorical 2-cocycle,

$$\begin{aligned} c(\mu_g,\alpha\delta) - c(\nu_g,\alpha\delta) &= c(\mu_g\alpha,\delta) - c(\nu_g\alpha,\delta) + c(\mu_g,\alpha) - c(\nu_g,\alpha), \\ c(\mu_h,\beta\delta) - c(\nu_h,\beta\delta) &= c(\mu_h\beta,\delta) - c(\nu_h\beta,\delta) + c(\mu_h,\beta) - c(\nu_h,\beta), \quad \text{and} \\ c(\mu_{gh},\gamma\delta) - c(\nu_{gh},\gamma\delta) &= c(\mu_{gh}\gamma,\delta) - c(\nu_{gh}\gamma,\delta) + c(\mu_{gh},\gamma) - c(\nu_{gh},\gamma). \end{aligned}$$

Hence

$$\begin{aligned} &\big(c(\mu_g,\alpha\delta) - c(\nu_g,\alpha\delta)\big) + \big(c(\mu_h,\beta\delta) - c(\nu_h,\beta\delta)\big) - \big(c(\mu_{gh},\gamma\delta) - c(\nu_{gh},\gamma\delta)\big) \\ &\quad = c(\mu_g\alpha,\delta) - c(\nu_g\alpha,\delta) + c(\mu_g,\alpha) - c(\nu_g,\alpha) \\ &\qquad + c(\mu_h\beta,\delta) - c(\nu_h\beta,\delta) + c(\mu_h,\beta) - c(\nu_h,\beta) \\ &\qquad\quad - c(\mu_{gh}\gamma,\delta) + c(\nu_{gh}\gamma,\delta) - c(\mu_{gh},\gamma) + c(\nu_{gh},\gamma). \end{aligned}$$

Since $\mu_g\alpha = \mu_{gh}\gamma$, $\nu_h\beta = \nu_{gh}\gamma$ and $\nu_g\alpha = \mu_h\beta$, this simplifies to

$$\begin{aligned} &\big(c(\mu_g,\alpha\delta) - c(\nu_g,\alpha\delta)\big) + \big(c(\mu_h,\beta\delta) - c(\nu_h,\beta\delta)\big) - \big(c(\mu_{gh},\gamma\delta) - c(\nu_{gh},\gamma\delta)\big) \\ &\quad = \big(c(\mu_g,\alpha) - c(\nu_g,\alpha)\big) + \big(c(\mu_h,\beta) - c(\nu_h,\beta)\big) - \big(c(\mu_{gh},\gamma) - c(\nu_{gh},\gamma)\big). \end{aligned}$$

So (6.2) does not depend on the choice of α, β, γ.

To prove that σ_c is a 2-cocycle, fix $(g_1, g_2, g_3) \in \mathcal{G}_\Lambda^{(3)}$. Let $(\mu_i, \nu_i) = (\mu_{g_i}, \nu_{g_i})$ for $i = 1, 2, 3$. Let $(\mu_{ij}, \nu_{ij}) = (\mu_{g_ig_j}, \nu_{g_ig_j})$ for $ij = 12, 23$ and let $(\mu_{123}, \nu_{123}) = (\mu_{g_1g_2g_3}, \nu_{g_1g_2g_3})$. Fix $z \in \Lambda^\infty$, and for each symbol $\star \in \{1, 2, 3, 12, 23, 123\}$, fix $\alpha_\star \in \Lambda$ such that

$$g_\star = (\mu_\star\alpha_\star z, d(\mu_\star) - d(\nu_\star), \nu_\star\alpha_\star z).$$

Then (6.2) yields

$$\begin{aligned} \sigma_c(g_1,g_2) + \sigma_c(g_1g_2,g_3) &= \big(c(\mu_1,\alpha_1) - c(\nu_1,\alpha_1)\big) + \big(c(\mu_2,\alpha_2) - c(\nu_2,\alpha_2)\big) \\ &\quad - \big(c(\mu_{12},\alpha_{12}) - c(\nu_{12},\alpha_{12})\big) + \big(c(\mu_{12},\alpha_{12}) - c(\nu_{12},\alpha_{12})\big) \\ &\quad + \big(c(\mu_3,\alpha_3) - c(\nu_3,\alpha_3)\big) - \big(c(\mu_{123},\alpha_{123}) - c(\nu_{123},\alpha_{123})\big) \end{aligned}$$

and

$$\begin{aligned}\sigma_c(g_2,g_3)+\sigma_c(g_1,g_2g_3) &= \big(c(\mu_2,\alpha_2)-c(\nu_2,\alpha_2)\big)+\big(c(\mu_3,\alpha_3)-c(\nu_3,\alpha_3)\big)\\ &\quad-\big(c(\mu_{23},\alpha_{23})-c(\nu_{23},\alpha_{23})\big)+\big(c(\mu_1,\alpha_1)-c(\nu_1,\alpha_1)\big)\\ &\quad+\big(c(\mu_{23},\alpha_{23})-c(\nu_{23},\alpha_{23})\big)-\big(c(\mu_{123},\alpha_{123})-c(\nu_{123},\alpha_{123})\big).\end{aligned}$$

Cancelling and comparing gives

$$\sigma_c(g_1,g_2)+\sigma_c(g_1g_2,g_3)=\sigma_c(g_2,g_3)+\sigma_c(g_1,g_2g_3).$$

Hence, σ_c satisfies the groupoid 2-cocycle identity. A straightforward calculation shows that $c(g,h)=0$ if either g or h is a unit. Since σ_c is locally constant, it is continuous, so $\sigma_c\in\tilde{Z}^2(\mathcal{G}_\Lambda,A)$. □

Remark 6.4. The cocycle σ_c constructed in Lemma 6.3(iii) depends both on c and on the collection $\mathcal{P}$.

Theorem 6.5. *Let Λ be a row-finite k-graph with no sources and let A be an abelian group. Suppose that $\mathcal{P}$ is a countable subset of $\Lambda\ {}_s{*}_s\ \Lambda$ such that $\{Z(\mu,\nu):(\mu,\nu)\in\mathcal{P}\}$ is a partition of $\mathcal{G}_\Lambda$. Fix $c\in\underline{Z}^2(\Lambda,A)$, and let $\sigma_c\in\tilde{Z}^2(\mathcal{G}_\Lambda,A)$ be the continuous cocycle of Lemma 6.3. The cohomology class $[\sigma_c]$ is independent of the choice of $\mathcal{P}$ and depends only on the cohomology class of c. Moreover, $[c]\mapsto[\sigma_c]$ is a homomorphism $\underline{H}^2(\Lambda,A)\to\tilde{H}^2(\mathcal{G}_\Lambda,A)$.*

Proof. Suppose that c is a categorical 2-coboundary on Λ. We show that σ_c is a groupoid 2-coboundary on $\mathcal{G}_\Lambda$. Since c is a categorical 2-coboundary, there is a cochain $b\in\underline{C}^1(\Lambda,A)$, such that $c=\underline{\delta}^1b$. Hence, for $(\lambda_1,\lambda_2)\in\Lambda^{*2}$ we have

$$c(\lambda_1,\lambda_2)=(\underline{\delta}^1b)(\lambda_1,\lambda_2)=b(\lambda_1)-b(\lambda_1\lambda_2)+b(\lambda_2).$$

Define $a:\mathcal{G}_\Lambda\to A$ by $a(g)=b(\mu_g)-b(\nu_g)$. Then a is continuous because it is locally constant. For $g\in\mathcal{G}_\Lambda^{(0)}$ we have $\mu_g=\nu_g$, so $a(g)=0$. Hence $a\in\tilde{C}^1(\mathcal{G}_\Lambda,A)$.

Fix $(g,h)\in\mathcal{G}_\Lambda^{(2)}$. With notation as in Lemma 6.3(i), equation (6.2) gives

$$\begin{aligned}\sigma_c(g,h) &= \big(c(\mu_g,\alpha)-c(\nu_g,\beta)\big)+\big(c(\mu_h,\beta)-c(\nu_h,\beta)\big)-\big(c(\mu_{gh},\gamma)-c(\nu_{gh},\gamma)\big)\\ &=(b(\mu_g)-b(\mu_g\alpha)+b(\alpha))-(b(\nu_g)-b(\nu_g\alpha)+b(\alpha))\\ &\quad+\big(b(\mu_h)-b(\mu_h\beta)+b(\beta)\big)-\big(b(\nu_h)-b(\nu_h\beta)+b(\beta)\big)\\ &\quad-\big(b(\mu_{gh})-b(\mu_{gh}\gamma)+b(\gamma)\big)+\big(b(\nu_{gh})-b(\nu_{gh}\gamma)+b(\gamma)\big).\end{aligned}$$

Since $\mu_g\alpha=\mu_{gh}\gamma$, $\nu_h\beta=\nu_{gh}\gamma$ and $\nu_g\alpha=\mu_h\beta$, we obtain

$$\begin{aligned}\sigma_c(g,h) &= b(\mu_g)-b(\nu_g)+b(\mu_h)-b(\nu_h)-b(\mu_{gh})+b(\nu_{gh})\\ &=a(g)+a(h)-a(gh)=\tilde{\delta}^1(a)(g,h).\end{aligned}$$

Hence, $\sigma_c=\tilde{\delta}^1(a)$ is a coboundary. Since the map $c\mapsto\sigma_c$ is a homomorphism (see formula (6.2)) which maps coboundaries to coboundaries, the map $[c]\mapsto[\sigma_c]$ is a well-defined homomorphism $\underline{H}^2(\Lambda,A)\to\tilde{H}^2(\mathcal{G}_\Lambda,A)$.

It remains to verify that $[\sigma_c]$ does not depend on the choice of $\mathcal{P}$. Fix countable subsets $\mathcal{P}$ and $\mathcal{Q}$ of $\Lambda\ {}_s{*}_s\ \Lambda$ yielding partitions of $\mathcal{G}_\Lambda$. For $(\mu,\nu),(\sigma,\tau)\in\Lambda\ {}_s{*}_s\ \Lambda$, if $d(\mu)-d(\nu)\neq d(\sigma)-d(\tau)$, then $Z(\mu,\nu)\cap Z(\sigma,\tau)=\emptyset$. Otherwise, setting

$$(\mu,\nu)\wedge(\sigma,\tau):=\{(\mu\alpha,\nu\alpha):\mu\alpha\in\mathrm{MCE}(\mu,\sigma)\text{ and }\nu\alpha\in\mathrm{MCE}(\nu,\tau)\},$$

we have

$$Z(\mu,\nu)\cap Z(\sigma,\tau)=\bigsqcup_{(\eta,\zeta)\in(\mu,\nu)\wedge(\sigma,\tau)} Z(\eta,\zeta). \tag{6.3}$$

Let $\mathcal{P}\vee\mathcal{Q}:=\bigcup_{(\mu,\nu)\in\mathcal{P}}\bigcup_{(\sigma,\tau)\in\mathcal{Q}}(\mu,\nu)\wedge(\sigma,\tau)$. Then $\{Z(\eta,\zeta):(\eta,\zeta)\in\mathcal{P}\vee\mathcal{Q}\}$ is a common refinement of $\{Z(\mu,\nu):(\mu,\nu)\in\mathcal{P}\}$ and $\{Z(\sigma,\tau):(\sigma,\tau)\in\mathcal{Q}\}$ such that if $g\in\mathcal{G}_\Lambda$ satisfies $g\in Z(\mu,\nu)$ for $(\mu,\nu)\in\mathcal{P}$ and $g\in Z(\eta,\zeta)$ for $(\eta,\zeta)\in(\mathcal{P}\vee\mathcal{Q})$, then $\eta=\mu\lambda$ and $\zeta=\nu\lambda$ for some λ, and similarly for $\mathcal{Q}$. So by replacing $\mathcal{Q}$ with $\mathcal{P}\vee\mathcal{Q}$ we may assume that $\{Z(\eta,\zeta):(\eta,\zeta)\in\mathcal{Q}\}$ is a refinement of $\{Z(\mu,\nu):(\mu,\nu)\in\mathcal{P}\}$ and that for each element (η,ζ) of $\mathcal{Q}$, there is a unique element $(\mu,\nu)\in\mathcal{P}$ and a unique $\lambda\in\Lambda$ such that $\eta=\mu\lambda$ and $\zeta=\nu\lambda$.

For $g\in\mathcal{G}_\Lambda$, let $(\mu_g,\nu_g)\in\mathcal{P}$ and $(\eta_g,\zeta_g)\in\mathcal{Q}$ be the unique elements such that $g\in Z(\eta_g,\zeta_g)\subseteq Z(\mu_g,\nu_g)$ and let λ_g be the unique path such that $(\eta_g,\zeta_g)=(\mu_g\lambda_g,\nu_g\lambda_g)$.

Fix $(g,h)\in\mathcal{G}_\Lambda^{(2)}$. By Lemma 6.3(i), we may fix α',β',γ' and y satisfying

$$g=(\eta_g\alpha'y,d(\eta_g)-d(\zeta_g),\zeta_g\alpha'y),\quad h=(\eta_h\beta'y,d(\eta_h)-d(\zeta_h),\zeta_h\beta'y),\quad\text{and}$$
$$gh=(\eta_{gh}\gamma'y,d(\eta_{gh})-d(\zeta_{gh}),\zeta_{gh}\gamma'y).$$

The triple $\alpha=\lambda_g\alpha'$, $\beta=\lambda_h\beta'$, $\gamma=\lambda_{gh}\gamma'$ then satisfies (6.1). So

$$\begin{aligned} g&\in Z(\eta_g\alpha',\zeta_g\alpha')=Z(\mu_g\alpha,\nu_g\alpha)\\ h&\in Z(\eta_h\beta',\zeta_h\beta')=Z(\mu_h\beta,\nu_h\beta),\ \text{and}\\ gh&\in Z(\eta_{gh}\gamma',\zeta_{gh}\gamma')=Z(\mu_{gh}\gamma,\nu_{gh}\gamma).\end{aligned}$$

Fix $c\in\underline{Z}^2(\Lambda,A)$. Let $\sigma_c^{\mathcal{P}}$ denote the groupoid 2-cocycle obtained from Lemma 6.3 applied to c and $\mathcal{P}$ and let $\sigma_c^{\mathcal{Q}}$ denote the groupoid 2-cocycle obtained in the same way from c and $\mathcal{Q}$. By Lemma 6.3(ii), we have

$$\begin{aligned}\sigma_c^{\mathcal{P}}(g,h)&=\big(c(\mu_g,\alpha)-c(\nu_g,\alpha)\big)+\big(c(\mu_h,\beta)-c(\nu_h,\beta)\big)-\big(c(\mu_{gh},\gamma)-c(\nu_{gh},\gamma)\big),\\ &=\big(c(\mu_g,\lambda_g\alpha')-c(\nu_g,\lambda_g\alpha')\big)+\big(c(\mu_h,\lambda_h\beta')-c(\nu_h,\lambda_h\beta')\big)\\ &\qquad-\big(c(\mu_{gh},\lambda_{gh}\gamma')-c(\nu_{gh},\lambda_{gh}\gamma')\big),\quad\text{and}\\ \sigma_c^{\mathcal{Q}}(g,h)&=\big(c(\eta_g,\alpha')-c(\zeta_g,\alpha')\big)+\big(c(\eta_h,\beta')-c(\zeta_h,\beta')\big)-\big(c(\eta_{gh},\gamma')-c(\zeta_{gh},\gamma')\big)\\ &=\big(c(\mu_g\lambda_g,\alpha')-c(\nu_g\lambda_g,\alpha')\big)+\big(c(\mu_h\lambda_h,\beta')-c(\nu_h\lambda_h,\beta')\big)\\ &\qquad-\big(c(\mu_{gh}\lambda_{gh},\gamma')-c(\nu_{gh}\lambda_{gh},\gamma')\big).\end{aligned}$$

Define $b:\mathcal{G}_\Lambda\to A$ by $b(g)=c(\mu_g,\lambda_g)-c(\nu_g,\lambda_g)$. Then b is continuous because it is locally constant. If $g\in\mathcal{G}_\Lambda^{(0)}$, then $\mu_g=\nu_g$, and hence $b(g)=0$. So $b\in\tilde{C}^1(\mathcal{G}_\Lambda,A)$. The categorical 2-cocycle identity for c implies that

$$\begin{aligned}&\big(c(\mu_g,\lambda_g\alpha')-c(\nu_g,\lambda_g\alpha')\big)-\big(c(\mu_g\lambda_g,\alpha')-c(\nu_g\lambda_g,\alpha')\big)\\ &\qquad=\big(c(\mu_g,\lambda_g)-c(\lambda_g,\alpha')\big)-\big(c(\nu_g,\lambda_g)-c(\lambda_g,\alpha')\big)=b(g).\end{aligned}$$

This and a symmetric calculation yield

$$\sigma_c^{\mathcal{P}}(g,h)-\sigma_c^{\mathcal{Q}}(g,h)=b(g)+b(h)-b(gh)=(\tilde{\delta}^1b)(g,h).$$

Hence σ_c and $\tilde{\sigma}_c$ are cohomologous. □

We now show that there always exists a set $\mathcal{P}$ producing a partition of $\mathcal{G}_\Lambda$ as hypothesised in the preceding results.

Lemma 6.6. *Let Λ be a row-finite k-graph with no sources. Then there exists a countable subset $\mathcal{P} \subset \Lambda\ {}_s{*}_s\ \Lambda$ such that $(\lambda, s(\lambda)) \in \mathcal{P}$ for all $\lambda \in \Lambda$ and $\{Z(\mu,\nu) : (\mu,\nu) \in \mathcal{P}\}$ is a partition of $\mathcal{G}_\Lambda$.*

Proof. First observe that $Z(\lambda, s(\lambda)) \cap Z(\mu,\nu)$ is nonempty if and only if $\mu = \lambda\nu$, in which case, $Z(\mu,\nu) \subset Z(\lambda, s(\lambda))$. Hence, basic open sets of the form $Z(\lambda, s(\lambda))$ are pairwise disjoint. The union

$$X := \bigsqcup_{\lambda\in\Lambda} Z(\lambda, s(\lambda))$$

of these sets is clopen in $\mathcal{G}_\Lambda$. So there is a countable collection $\mathcal{U}$ of basic open sets of the form $Z(\mu,\nu)$ whose union is $\mathcal{G}_\Lambda \setminus X$.

It remains to show that there is a collection of pairwise disjoint basic open sets $\mathcal{V}$ such that $\bigcup\mathcal{V} = \bigcup\mathcal{U}$. We first show that, given two basic open sets $Z(\mu_1,\nu_1)$, $Z(\mu_2,\nu_2)$, both $Z(\mu_1,\nu_1) \cap Z(\mu_2,\nu_2)$ and $Z(\mu_1,\nu_1) \setminus Z(\mu_2,\nu_2)$ may be expressed as finite disjoint unions of such basic open sets. We may assume that $d(\mu_1) - d(\nu_1) = d(\mu_2) - d(\nu_2)$, since $Z(\mu_1,\nu_1) \cap Z(\mu_2,\nu_2)$ is empty otherwise. Recall from (6.3) that

$$Z(\mu_1,\nu_1) \cap Z(\mu_2,\nu_2) = \bigsqcup_{(\eta,\zeta)\in(\mu_1,\nu_1)\wedge(\mu_2,\nu_2)} Z(\eta,\zeta).$$

Since $Z(\mu_1,\nu_1) = \bigsqcup_{d(\mu\alpha)=d(\mu_1)\vee d(\nu_1)} Z(\mu_1\alpha,\nu_1\alpha)$, it follows that

$$\begin{aligned} Z(\mu_1,\nu_1) \setminus Z(\mu_2,\nu_2) = \bigsqcup \big\{ & Z(\mu_1\alpha,\nu_1\alpha) : d(\mu_1\alpha) = d(\mu_1) \vee d(\nu_1), \\ & (\mu_1\alpha,\nu_1\alpha) \notin (\mu_1,\nu_1) \wedge (\mu_2,\nu_2) \big\}. \end{aligned} \tag{6.4}$$

Now by the standard inclusion-exclusion decomposition, $Z(\mu_1,\nu_1) \cup Z(\mu_2,\nu_2)$ is also a finite disjoint union of basic open sets. The collection $\mathcal{U} = \{U_1, U_2, \dots\}$ may now be replaced by a pairwise disjoint collection $\mathcal{V}$ recursively. Set $\mathcal{V}_1 = \{U_1\}$. Now suppose that $\mathcal{V}_i$ is a collection of mutually disjoint basic open sets such that $\bigcup_{j=1}^i U_j = \bigsqcup\mathcal{V}_i$. Use (6.4) to write $U_{i+1} \setminus \bigcup\mathcal{V}_i = \bigsqcup\mathcal{W}_{i+1}$ where $\mathcal{W}_{i+1}$ is a finite collection of mutually disjoint sets of the form $Z(\eta,\zeta)$, and let $\mathcal{V}_{i+1} := \mathcal{V}_i \cup \mathcal{W}_{i+1}$. Then $\bigcup_{j=1}^{i+1} U_j = \bigsqcup\mathcal{V}_{i+1}$. By induction we obtain the desired family $\mathcal{V}$ of mutually disjoint basic open sets such that $\bigsqcup\mathcal{V} = \bigcup\mathcal{U}$. □

We can now prove that every twisted k-graph algebra admits a homomorphism into a twisted C^*-algebra, in the sense of Renault (see [21]), of the path groupoid of the corresponding k-graph. It follows that all of the generators of $C^*(\Lambda, c)$ are nonzero. We will use our gauge-invariant uniqueness theorem in the next section to see that this homomorphism is an isomorphism.

Recall from [21] that involution and convolution in the dense *-subalgebra $C_c(\mathcal{G}_\Lambda, \sigma_c)$ of $C^*(\mathcal{G}_\Lambda, \sigma_c)$ are given by

$$\big(f * g\big)(\gamma) = \sum_{\alpha\beta=\gamma} \sigma_c(\alpha,\beta) f(\alpha) g(\beta) \quad \text{and} \quad f^*(\gamma) = \overline{\sigma_c(\gamma^{-1},\gamma) f(\gamma^{-1})}.$$

Theorem 6.7. *Let Λ be a row-finite k-graph with no sources. Let $\mathcal{P}$ be a countable subset of $\Lambda\ {}_s{*}_s\ \Lambda$ containing $\{(\lambda, s(\lambda)) : \lambda \in \Lambda\}$ such that $\{Z(\mu,\nu) : (\mu,\nu) \in \mathcal{P}\}$ is a partition of $\mathcal{G}_\Lambda$. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, and let $\sigma_c \in \tilde{Z}^2(\mathcal{G}_\Lambda, \mathbb{T})$ be the cocycle constructed from $\mathcal{P}$ as in Lemma 6.3. Then there is a surjective homomorphism $\pi : C^*(\Lambda, c) \to C^*(\mathcal{G}_\Lambda, \sigma_c)$ such that $\pi(s_\lambda) = 1_{Z(\lambda, s(\lambda))}$ for all $\lambda \in \Lambda$. Moreover, for each $(\mu,\nu) \in \Lambda\ {}_s{*}_s\ \Lambda$, there is a finite subset $F \subseteq s(\mu)\Lambda$ such that $Z(\mu,\nu) = \bigsqcup_{\tau\in F} Z(\mu\tau,\nu\tau)$ and a function $a : F \to \mathbb{T}$ such that $\pi(s_\mu s_\nu^*) = \sum_{\tau\in F} a_\tau 1_{Z(\mu\tau,\nu\tau)}$.*

Proof. By the universal property of $C^*(\Lambda, c)$, to prove the first statement it suffices to show that $t : \lambda \mapsto 1_{Z(\lambda, s(\lambda))}$ is a Cuntz-Krieger (Λ, c)-family in $C^*(\mathcal{G}_\Lambda, \sigma_c)$. Calculations like those of [11, Lemma 4.3] verify (CK1), (CK3) and (CK4). It remains only to verify that $t_\mu t_\nu = c(\mu, \nu) t_{\mu\nu}$ whenever $r(\nu) = s(\mu)$. Fix $h \in \mathcal{G}_\Lambda$. We have

$$(t_\mu t_\nu)(h) = 1_{Z(\mu, s(\mu))} * 1_{Z(\nu, s(\nu))}(h) = \sum_{g \in \mathcal{G}_\Lambda} \sigma_c(g, g^{-1}h) 1_{Z(\mu, s(\mu))}(g) 1_{Z(\nu, s(\nu))}(g^{-1}h). \tag{6.5}$$

For fixed g, putting $h' = g^{-1}h$, the product $1_{Z(\mu, s(\mu))}(g) 1_{Z(\nu, s(\nu))}(h')$ is equal to 1 if $h' = (\nu z, d(\nu), z)$ and $g = (\mu\nu z, d(\mu), \nu z)$ for some $z \in \Lambda^\infty$, and to 0 otherwise. So: there is at most one nonzero term in the sum on the right-hand side of (6.5); there is such a term precisely when $h = (\mu\nu z, d(\mu\nu), z)$; and then the nonzero term occurs with $g = (\mu\nu z, d(\mu), \nu z)$ and $h' = (\nu z, d(\nu), z)$. Setting $z := s(h)$, $g := (\mu\nu z, d(\mu), \nu z)$ and $h' := (\nu z, d(\nu), z)$, we have

$$(t_\mu t_\nu)(h) = \sigma_c(g, h') 1_{Z(\mu\nu, s(\nu))}(h).$$

We have $g \in Z(\mu, s(\mu))$, $h \in Z(\nu, s(\nu))$, and $gh' \in Z(\mu\nu, s(\mu\nu))$. Since $(\mu, s(\mu))$, $(\nu, s(\nu))$, $(\mu\nu, s(\mu\nu))$ belong to $\mathcal{P}$, we have $\mu_g = \mu$, $\nu_g = s(\mu)$, $\mu_{h'} = \nu$, $\nu_{h'} = s(\nu)$, $\mu_{gh'} = \mu\nu$, and $\nu_{gh'} = s(\mu\nu)$. Furthermore $\alpha := \nu$ and $\beta = \gamma := s(\nu)$ satisfy (6.1), and hence (6.2) yields

$$\sigma_c(g, h') = c(\mu, \nu) - c(\nu, s(\nu)) + c(\nu, s(\nu)) - c(s(\nu), s(\nu)) - \big(c(\mu\nu, s(\nu)) - c(s(\nu), s(\nu)\big),$$

which, since c is normalised, collapses to $\sigma_c(g, h') = c(\mu, \nu)$. Hence

$$(t_\mu t_\nu)(x, m, y) = c(\mu, \nu) 1_{Z(\mu\nu, s(\nu))}(x, m, y) = c(\mu, \nu) t_{\mu\nu}(x, m, y),$$

establishing (CK2). Hence there is a homomorphism $\pi : C^*(\Lambda, c) \to C^*(\mathcal{G}_\Lambda, \sigma_c)$ such that $\pi(s_\lambda) = 1_{Z(\lambda, s(\lambda))}$. We postpone the proof that this map is surjective until we have established the final statement.

Fix μ, ν with $s(\mu) = s(\nu)$. For $g \in \mathcal{G}_\Lambda$, we have

$$\begin{aligned}\pi(s_\mu s_\nu^*)(g) = \pi(s_\mu)\pi(s_\nu)^*(g) &= 1_{\mu, s(\mu)} * 1^*_{\nu, s(\nu)}(g) \\ &= \sum_{hk = g} \sigma_c(h, k) \overline{\sigma_c(k^{-1}, k)} 1_{Z(\mu, s(\mu))}(h) 1_{Z(\nu, s(\nu))}(k^{-1}).\end{aligned}$$

There is at most one nonzero term in the sum, and this occurs when $g = (\mu x, d(\mu) - d(\nu), \nu x)$, in which case $h = (\mu x, d(\mu), x)$ and $k = (x, -d(\nu), \nu x)$ for some $x \in \Lambda^\infty$. So $\pi(s_\mu s_\nu^*)$ is supported on $Z(\mu, \nu)$, and for $g = (\mu x, d(\mu) - d(\nu), \nu x)$ we have

$$\pi(s_\mu s_\nu^*)(g) = \sigma_c\big((\mu x, d(\mu), x), (x, -d(\nu), \nu x)\big) \sigma_c\big((\nu x, d(\nu), x), (x, -d(\nu), \nu x)\big).$$

Since σ_c is locally constant and $\mathbb{T}$-valued by construction, it follows that $\pi(s_\mu s_\nu^*)$ can be written as a linear combination of the desired form.

For surjectivity, observe as in the proof of [10, Corollary 3.5(i)] that $C^*(\mathcal{G}_\Lambda, \sigma_c) = \overline{\operatorname{span}}\{1_{Z(\mu, \nu)} : s(\mu) = s(\nu)\}$, so it suffices to show that each $1_{Z(\mu, \nu)}$ is in the image of π. Fix $(\mu, \nu) \in \Lambda_s *_s \Lambda$ and express $\pi(s_\mu s_\nu^*) = \sum_{\tau \in F} a_\tau 1_{Z(\mu\tau, \nu\tau)}$ as above.

A routine calculation using the definitions of convolution and involution in $C_c(\mathcal{G}_\Lambda, \sigma_c)$ shows that

$$\pi(s_{\nu\tau} s_{\nu\tau}^*) = 1_{Z(\nu\tau, s(\tau))} * 1^*_{Z(\nu\tau, s(\tau))} = 1_{Z(\nu\tau)} \quad \text{for each } \tau \in F.$$

That each $1_{Z(\nu\tau)} \in C_0(\mathcal{G}_\Lambda^{(0)})$, and that σ_c is a normalised cocycle imply that $f * 1_{Z(\nu\tau)}(g) = f(g) 1_{Z(\nu\tau)}(s(g))$ for all $f \in C_c(\mathcal{G}_\Lambda, \sigma_c)$. Hence that the range map on $\mathcal{G}_\Lambda$ is bijective on $Z_{(\mu, \nu)}$ and that the $Z_{(\mu\tau, \nu\tau)}$ are mutually disjoint force

$$1_{Z(\mu\tau, \nu\tau)} 1_{Z(\nu\tau')} = \delta_{\tau, \tau'} 1_{Z(\mu\tau, \nu\tau)} \quad \text{for all } \tau, \tau' \in F.$$

Thus

$$1_{Z(\mu,\nu)} = \Big(\sum_{\tau\in F} a_\tau 1_{Z(\mu\tau,\nu\tau)}\Big)\Big(\sum_{\tau\in F} \overline{a}_\tau 1_{Z(\nu\tau)}\Big) = \pi(s_\mu s_\nu^*)\sum_{\tau\in F}\overline{a}_\tau \pi(s_{\nu\tau}s_{\nu\tau}^*). \qquad \square$$

Corollary 6.8. *Let Λ be a row-finite k-graph with no sources, and fix $c \in \underline{Z}^2(\Lambda,\mathbb{T})$. Let $\{s_\lambda : \lambda \in \Lambda\}$ be the universal generating Cuntz-Krieger (Λ, c)-family in $C^*(\Lambda, c)$. Then each $s_\lambda \neq 0$.*

Proof. By Theorem 6.7, there exist $\sigma \in \tilde{Z}^2(\mathcal{G}_\Lambda,\mathbb{T})$ and a homomorphism $\pi : C^*(\Lambda, c) \to C^*(\mathcal{G}_\Lambda, \sigma)$ such that $\pi(s_\lambda) = 1_{Z(\lambda,s(\lambda))}$ for all $\lambda \in \Lambda$. Since Λ has no sources, each $Z(\lambda, s(\lambda))$ is nonempty, and hence each $1_{Z(\lambda,s(\lambda))}$ is nonzero. So each s_λ is nonzero. $\square$

We conjecture that for each $r \geq 0$, there is an injective homomorphism from $\underline{H}^r(\Lambda, A)$ to $\tilde{H}^r(\mathcal{G}_\Lambda, A)$. For $r = 0$ it is clear that there is such a homomorphism determined by the map from $\underline{C}^0(\Lambda, A)$ to $\tilde{C}^0(\mathcal{G}_\Lambda, A)$ which takes $f : \Lambda^0 \to A$ to the function $x \mapsto f(r(x))$ from $\mathcal{G}_\Lambda^{(0)}$ to A. The following remark indicates how to define such a homomorphism for $r = 1$. It is not clear to us how to proceed for $r \geq 3$.

Remark 6.9. Let Λ be a row-finite k-graph with no sources and let A be an abelian group. Let $c \in \underline{Z}^1(\Lambda, A)$. As observed in [10, §5] the function $\tilde{c} : \mathcal{G}_\Lambda \to A$ given by

$$\tilde{c}(x, \ell - m, y) = c(x(0, \ell)) - c(y(0, m))$$

defines an element of $\tilde{Z}^1(\mathcal{G}_\Lambda, A)$. It is straightforward to check that $c \mapsto \tilde{c}$ is a homomorphism of abelian groups and that it preserves coboundaries (indeed, if $c = \underline{\delta}^1 b$, then $\tilde{c} = \tilde{\delta}^1\tilde{b}$ where $\tilde{b}(x) = b(x(0))$). Hence, the map $[c] \mapsto [\tilde{c}]$ defines a homomorphism $\underline{H}^1(\Lambda, A) \to \tilde{H}^1(\mathcal{G}_\Lambda, A)$. We now show by example that this map need not be surjective

Recall that B_2 is the path category, regarded as a 1-graph, of the directed graph with a single vertex $\star$ and two edges, $B_2^1 = \{f_1, f_2\}$. We have $H_0(B_2) \cong \mathbb{Z}$, and [14, Examples 4.11(1)] shows that $H_1(B_2) \cong \mathbb{Z}^2$. Hence by the universal coefficient theorem [14, Theorem 7.3] and Theorem 3.10 we have $\underline{H}^1(B_2, \mathbb{Z}) \cong \mathbb{Z}^2$. The path space B_2^∞ may be identified with the sequence space $X = \{(x_i)_{i=1}^\infty : x_i \in \{f_1, f_2\} \text{ for all } i\}$. By [5, Theorem 2.2], $\tilde{Z}^1(\mathcal{G}_{B_2}, \mathbb{Z}) \cong H^0(X, \mathbb{Z})$ (which may be identified with the group of continuous maps $h : X \to \mathbb{Z}$) and

$$\tilde{H}^1(\mathcal{G}_{B_2}, \mathbb{Z}) \cong \operatorname{coker}\big(1 - \sigma^* : H^0(X, \mathbb{Z}) \to H^0(X, \mathbb{Z})\big)$$

where $\sigma : X \to X$ is the shift and $(\sigma^* h)(x) = h(\sigma x)$. Given a path $\lambda \in B_2$, let $h_\lambda = 1_{Z(\lambda)}$ be the characteristic function of the cylinder set $Z(\lambda)$. Then $H^0(X, \mathbb{Z})$ is spanned by $\{h_\lambda : \lambda \in \Lambda\}$. Observe that $\sigma^* h_\lambda = h_{f_1\lambda} + h_{f_2\lambda}$. It follows that the map $\varphi : H^0(X, \mathbb{Z}) \to \mathbb{Z}[1/2]$ determined by $\varphi(h_\lambda) = 2^{-d(\lambda)}$ satisfies the condition $\varphi(1 - \sigma^*) = 0$. Since φ is surjective and vanishes on coboundaries it determines a surjective map $\tilde{\varphi} : \tilde{H}^1(\mathcal{G}_\Lambda, A) \to \mathbb{Z}[1/2]$. Moreover, the range of the map

$$c \in \underline{Z}^1(B_2, \mathbb{Z}) \mapsto \tilde{c} \in \tilde{Z}^1(\mathcal{G}_{B_2}, \mathbb{Z}) \cong H^0(X, \mathbb{Z})$$

is $\mathbb{Z}h_{f_1} + \mathbb{Z}h_{f_2}$. Hence, $\tilde{\varphi}([\tilde{c}]) \in \mathbb{Z}$ and the induced map $\underline{H}^1(\Lambda, A) \to \tilde{H}^1(\mathcal{G}_\Lambda, A)$ is not surjective.

7. The gauge-invariant uniqueness theorem

As for untwisted k-graph C^*-algebras, each twisted k-graph C^*-algebra carries a gauge action of the k-torus. We establish a gauge-invariant uniqueness theorem and deduce that the homomorphism of Theorem 6.7 is an isomorphism. Where arguments in this section

closely parallel proofs of the corresponding results for untwisted k-graph C^*-algebras, we have sometimes outlined proofs without giving full details.

Remark 7.1. Let t be a Cuntz-Krieger (Λ, c)-family. Since a sum of projections is itself a projection if and only if the original projections are mutually orthogonal, relations (CK1), (CK3) and (CK4) ensure that for fixed $n \in \mathbb{N}^k$ and distinct $\mu, \nu \in \Lambda^n$, we have $t_\mu t_\mu^* t_\nu t_\nu^* = 0$.

Lemma 7.2. *Let Λ be a row-finite k-graph with no sources, fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, and let t be a Cuntz-Krieger (Λ, c)-family. For $\mu, \nu, \eta, \zeta \in \Lambda$ and $n \geq d(\nu) \vee d(\eta)$,*

$$t_\nu^* t_\eta = \sum_{\nu\alpha=\eta\beta\in\Lambda^n} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\alpha t_\beta^* \quad \textit{and} \tag{7.1}$$

$$t_\mu t_\nu^* t_\eta t_\zeta^* = \sum_{\nu\alpha=\eta\beta\in\Lambda^n} c(\mu,\alpha)\overline{c(\nu,\alpha)} c(\eta,\beta)\overline{c(\zeta,\beta)} t_{\mu\alpha} t_{\zeta\beta}^*. \tag{7.2}$$

Consequently, $\operatorname{span}\{s_\mu s_\nu^* : (\mu,\nu) \in \Lambda\, {}_s{*}_s\, \Lambda\}$ *is a dense $*$-subalgebra of $C^*(\Lambda, c)$.*

Proof. Using (CK4), we have

$$t_\nu^* t_\eta = \sum_{\lambda\in\Lambda^n r(\eta)} t_\nu^* t_\lambda t_\lambda^* t_\eta.$$

Since $n \geq d(\nu) \vee d(\eta)$, given $\lambda \in \Lambda^n$, the factorisation property allows us to factor $\lambda = \lambda^1\lambda^2 = \lambda^3\lambda^4$ where $d(\lambda^1) = d(\nu)$ and $d(\lambda^3) = d(\eta)$, and then by (CK2), (CK3) and Remark 7.1, we have

$$t_\nu^* t_\eta = \sum_{\lambda\in\Lambda^n r(\eta)} \overline{c(\lambda^1,\lambda^2)} c(\lambda^3,\lambda^4) t_\nu^* t_{\lambda^1} t_{\lambda^2} t_{\lambda^4}^* t_{\lambda^3}^* t_\eta = \sum_{\nu\alpha=\eta\beta\in\Lambda^n} \overline{c(\nu,\alpha)} c(\eta,\beta) t_{r(\alpha)} t_\alpha t_\beta^* t_{r(\beta)}.$$

Equation 7.1 follows from (CK2) because c is normalised. Left-multiplying by t_μ, right-multiplying by t_ζ^* and applying (CK2) establishes (7.2). □

Remark 7.3. If we apply Lemma 7.2 with $n = d(\nu) \vee d(\eta)$, then we obtain

$$t_\nu^* t_\eta = \sum_{\nu\alpha=\eta\beta\in\mathrm{MCE}(\nu,\eta)} \overline{c(\nu,\alpha)} c(\eta,\beta) t_\alpha t_\beta^*. \quad \text{and}$$

$$t_\mu t_\nu^* t_\eta t_\zeta^* = \sum_{\nu\alpha=\eta\beta\in\mathrm{MCE}(\mu,\nu)} c(\mu,\alpha)\overline{c(\nu,\alpha)} c(\eta,\beta)\overline{c(\zeta,\beta)} t_{\mu\alpha} t_{\zeta\beta}^*.$$

Lemma 7.4. *Let Λ be a row-finite k-graph with no sources and $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. There is a strongly continuous action $\gamma : \mathbb{T}^k \to \operatorname{Aut} C^*(\Lambda, c)$ such that*

$$\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda \quad \textit{for all } \lambda \in \Lambda.$$

Moreover $C^*(\Lambda, c)^\gamma = \overline{\operatorname{span}}\{s_\mu s_\nu^* : (\mu,\nu) \in \Lambda\, {}_s{*}_s\, \Lambda, d(\mu) = d(\nu)\}$.

Proof. Fix $z \in \mathbb{T}^k$ and set $t_\lambda^z = z^{d(\lambda)} s_\lambda$ for all $\lambda \in \Lambda$. It is routine to verify that $\lambda \mapsto t_\lambda^z$ satisfies (CK1)–(CK4) of Definition 5.2, so it is a Cuntz-Krieger (Λ, c)-family. Therefore the universal property of $C^*(\Lambda, c)$ yields an endomorphism γ_z of $C^*(\Lambda, c)$ satisfying $\gamma_z(s_\lambda) = t_\lambda^z = z^{d(\lambda)} s_\lambda$ for all $\lambda \in \Lambda$ with inverse $\gamma_{\overline{z}}$; hence, γ_z is an automorphism. Since $\gamma_{wz}(s_\lambda) = \gamma_w(\gamma_z(s_\lambda))$ for all $\lambda \in \Lambda$, the map $\gamma : \mathbb{T} \to \operatorname{Aut}(C^*(\Lambda, c))$ is a homomorphism. That $z \mapsto \gamma_z(a)$ is continuous for each a follows from an $\varepsilon/3$-argument using Lemma 7.2. Thus, γ defines a strongly continuous action of $\mathbb{T}^k$ on $C^*(\Lambda, c)$ with the desired property.

That $C^*(\Lambda, c)^\gamma = \overline{\operatorname{span}}\{t_\mu t_\nu^* : (\mu,\nu) \in \Lambda {}_s{*}_s \Lambda, d(\mu) = d(\nu)\}$ is standard: the containment "$\supseteq$" is clear and the reverse containment follows from the observation that the faithful

conditional expectation $\Phi^c(a) := \int_{\mathbb{T}^k} \gamma_z(a)\,dz$ onto $C^*(\Lambda, c)^\gamma$ annihilates $s_\mu s_\nu^*$ whenever $d(\mu) \neq d(\nu)$. □

Notation 7.5. Let X be a countable set. We write $\mathcal{K}_X$ for the universal C^*-algebra generated by matrix units $\{\theta_{x,y} : x, y \in X\}$ indexed by X (see [18, Remark A.10]). That is the $\theta_{x,y}$ satisfy $\theta_{x,y}^* = \theta_{y,x}$ and $\theta_{w,x}\theta_{y,z} = \delta_{x,y}\theta_{w,z}$. The algebra $\mathcal{K}_X$ is simple, and is canonically isomorphic to $\mathcal{K}(\ell^2(X))$. If X is finite, then any enumeration of X induces an isomorphism $\mathcal{K}_X \cong M_{|X|}(\mathbb{C})$.

Proposition 7.6. *Let Λ be a row-finite k-graph with no sources. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$.*

(1) *For $n \in \mathbb{N}^k$, $\overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\}$ is a C^*-subalgebra of $C^*(\Lambda, c)^\gamma$, and the assignment $s_\mu s_\nu^* \mapsto \theta_{\mu,\nu}$ determines an isomorphism*

$$\overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\} \cong \bigoplus_{v \in \Lambda^0} \mathcal{K}_{\Lambda^n v}.$$

(2) *If $m \leq n \in \mathbb{N}^k$, then $\overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^m\} \subseteq \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\}$. Moreover,*

$$C^*(\Lambda)^\gamma = \overline{\bigcup_{n \in \mathbb{N}^k} \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\}} \qquad \text{is AF.}$$

(3) *Given a Cuntz-Krieger (Λ, c)-family t, the homomorphism π_t induced by the universal property restricts to an injection on $C^*(\Lambda, c)^\gamma$ if and only if each t_v is nonzero.*

(4) *For any two cocycles $c_1, c_2 \in \underline{Z}^2(\Lambda, \mathbb{T})$, the fixed-point algebras $C^*(\Lambda, c_1)^\gamma$ and $C^*(\Lambda, c_2)^\gamma$ are isomorphic.*

Proof. (1) Lemma 7.4 implies that

$$C^*(\Lambda, c)^\gamma = \overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\} = \overline{\bigcup_{n \in \mathbb{N}^k} \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\}}. \tag{7.3}$$

Furthermore, equation (7.2) and that $c(\mu, s(\mu)) = 1$ implies that if $\mu, \nu, \eta, \zeta \in \Lambda^n$ with $s(\mu) = s(\nu)$ and $s(\eta) = s(\zeta)$, then $s_\mu s_\nu^* s_\eta s_\zeta^* = \delta_{\mu,\nu} s_\mu s_\zeta^*$. So for each $n \in \mathbb{N}^k$, the subspace $\overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\}$ is closed under multiplication. Since it is clearly closed under involution, it is a C^*-subalgebra of $C^*(\Lambda, c)^\gamma$. That $s_\mu s_\nu^* \mapsto \theta_{\mu,\nu}$ determines the desired isomorphism with $\bigoplus_{v \in \Lambda^0} \mathcal{K}_{\Lambda^n v}$ follows from the uniqueness of the latter.

(2) Relation (CK4) implies that if $m \leq n \in \mathbb{N}^k$, then $\overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^m\} \subseteq \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\}$. That $C^*(\Lambda, c)^\gamma = \overline{\bigcup_{n \in \mathbb{N}^k} \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\}}$ follows from Lemma 7.4. Since $C^*(\Lambda, c)^\gamma$ is an inductive limit of AF algebras it is also AF.

(3 The "only if" follows from Corollary 6.8. For the "if" implication, observe that each minimal projection $s_\mu s_\mu^*$ in $\overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda^n\}$ is equivalent to $s_\mu^* s_\mu = s_{s(\mu)}$. So if each s_v is nonzero, then each $s_\mu s_\mu^*$ is nonzero. The result then follows from the direct-limit decomposition (7.3), the simplicity of each $\mathcal{K}_{\Lambda^n v}$ and the ideal structure of direct sums of C^*-algebras.

(4) Fix $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, and fix cocycles $c_1, c_2 \in \underline{Z}^2(\Lambda, \mathbb{T})$. The assignments $s_\mu^{c_1}(s_\nu^{c_1})^* \mapsto \theta_{\mu,\nu} \mapsto s_\mu^{c_2}(s_\nu^{c_2})^*$ determine isomorphisms

$$\overline{\operatorname{span}}\{s_\mu^{c_1}(s_\nu^{c_1})^* : \mu, \nu \in \Lambda^n\} \cong \mathcal{K}_{\Lambda^n v} \cong \overline{\operatorname{span}}\{s_\mu^{c_2}(s_\nu^{c_2})^* : \mu, \nu \in \Lambda^n\}.$$

Moreover, for each pair $v, w \in \Lambda^0$ and $m \leq n \in \mathbb{N}^k$, the multiplicity of the partial inclusion

$$\mathcal{K}_{\Lambda^m v} \cong \overline{\operatorname{span}}\{s_\mu^{c_1}(s_\nu^{c_1})^* : \mu, \nu \in \Lambda^m v\} \hookrightarrow \overline{\operatorname{span}}\{s_\mu^{c_1}(s_\nu^{c_1})^* : \mu, \nu \in \Lambda^n w\} \cong \mathcal{K}_{\Lambda^n w}$$

is $|v\Lambda^{n-m}w|$ which does not depend on the cocycle c_1. Since AF algebras are completely determined by the dimensions of the summands of the approximating subalgebras and by the multiplicities of the partial inclusions, this proves the result. □

With the preceding analysis in hand, we can prove a version of an Huef and Raeburn's gauge-invariant uniqueness theorem for twisted k-graph C^*-algebras.

Corollary 7.7 (The gauge-invariant uniqueness theorem)**.** *Let Λ be a row-finite k-graph with no sources, and fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let $t : \Lambda \to B$ be a Cuntz-Krieger (Λ, c)-family in a C^*-algebra B. Suppose that there is a strongly continuous action β of $\mathbb{T}^k$ on B satisfying $\beta_z(t_\lambda) = z^{d(\lambda)}t_\lambda$ for all $\lambda \in \Lambda$ and $z \in \mathbb{T}^k$. Then the induced homomorphism $\pi_t : C^*(\Lambda, c) \to B$ is injective if and only if $t_v \neq 0$ for all $v \in \Lambda^0$.*

Proof. The "only if" direction follows from Corollary 6.8.

The "if" direction follows from the following standard argument. Let $\Phi^\gamma : C^*(\Lambda, c) \to C^*(\Lambda, c)^\gamma$ and $\Phi^\beta : C^*(t) \to C^*(t)^\gamma$ be the conditional expectations obtained by averaging over γ and β. Then Φ^γ is a faithful conditional expectation, and $\pi \circ \Phi^\gamma = \Phi^\beta \circ \pi$. So for $a \in C^*(\Lambda, c)$, we have

$$\pi(a) = 0 \implies \pi(a^*a) = 0 \implies \Phi^\beta(\pi_t(a^*a)) = 0 \implies \pi(\Phi^\gamma(a^*a)) = 0.$$

This forces $\Phi^\gamma(a^*a) = 0$ because π restricts to an injection on $C^*(\Lambda, c)^\gamma$ by Proposition 7.6(3). Hence $a^*a = 0$ because Φ^γ is faithful on positive elements, and then $a = 0$ by the C^*-identity. □

Corollary 7.8. *Let Λ be a row-finite k-graph with no sources. Let $\mathcal{P}$ be a countable subset of $\Lambda\ {}_s{*}_s\ \Lambda$ such that $\{(\lambda, s(\lambda)) : \lambda \in \Lambda\} \subseteq \mathcal{P}$ and $\{Z(\mu, \nu) : (\mu, \nu) \in \mathcal{P}\}$ is a partition of $\mathcal{G}_\Lambda$. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Then the homomorphism $\pi : C^*(\Lambda, c) \to C^*(\mathcal{G}_\Lambda, \sigma_c)$ of Theorem 6.7 satisfying $\pi(s_\lambda) = 1_{Z(\lambda, s(\lambda))}$ for all $\lambda \in \Lambda$ is an isomorphism.*

Proof. We showed in Theorem 6.7 that π is a surjective homomorphism, so it remains to show that it is injective. There is a strongly continuous action β of $\mathbb{T}^k$ on $C^*(\mathcal{G}_\Lambda, \sigma_c)$ satisfying

$$\beta_z(f)(x, \ell - m, y) = z^{\ell - m} f(x, \ell - m, y)$$

for all $f \in C_c(\mathcal{G}_\Lambda, \sigma_c)$, all $z \in \mathbb{T}^k$ and all $(x, \ell - m, y) \in \mathcal{G}_\Lambda$. Moreover, $\beta_z \circ \pi = \pi \circ \gamma_z$ for all z: for $z \in \mathbb{T}^k$ and $\lambda \in \Lambda$,

$$\pi(\gamma_z(s_\lambda)) = z^{d(\lambda)} 1_{Z(\lambda, s(\lambda))} = \beta_z(1_{Z(\lambda, s(\lambda))}) = \beta_z(\pi(s_\lambda)).$$

Since each $Z(v) \neq \emptyset$, each $\pi(s_v) = 1_{Z(v)}$ is nonzero, so Corollary 7.7 implies that π is injective. □

Corollary 7.9. *Let Λ be a row-finite k-graph with no sources. Let $\mathcal{P}$ be a countable subset of $\Lambda\ {}_s{*}_s\ \Lambda$ such that $\{Z(\mu, \nu) : (\mu, \nu) \in \mathcal{P}\}$ is a partition of $\mathcal{G}_\Lambda$. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, and let $\sigma_c \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ be the cocycle constructed from $\mathcal{P}$ as in Lemma 6.3. Then $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c) \cong C^*_r(\mathcal{G}_\Lambda, \sigma_c)$.*

Proof. By Lemma 6.6 there exists a countable set $\mathcal{Q} \subseteq \Lambda\ {}_s{*}_s\ \Lambda$ such that $\{Z(\mu, \nu) : (\mu, \nu) \in \mathcal{Q}\}$ is a partition of $\mathcal{G}_\Lambda$ and such that $(\lambda, s(\lambda)) \in \mathcal{Q}$ for all $\lambda \in \Lambda$. Let $\sigma_c^{\mathcal{Q}} \in Z^2(\mathcal{G}_\Lambda, \mathbb{T})$ be the cocycle constructed from $\mathcal{Q}$ as in Lemma 6.3. Corollary 7.8 implies that $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c^{\mathcal{Q}})$. Moreover, Theorem 6.5 implies that $\sigma_c^{\mathcal{Q}}$ and σ_c are cohomologous in $\tilde{Z}^2(\mathcal{G}_\Lambda, \mathbb{T})$, and then [21, Proposition II.1.2] implies that $C^*(\mathcal{G}_\Lambda, \sigma_c^{\mathcal{Q}}) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$.

For the assertion that $C^*(\mathcal{G}_\Lambda, \sigma_c) = C^*_r(\Lambda, \sigma_c)$, let $\psi : C^*(\Lambda, c) \to C^*_r(\mathcal{G}_\Lambda, \sigma_c^{\mathcal{Q}})$ be the homomorphism obtained by composing the quotient map $q : C^*(\mathcal{G}_\Lambda, \sigma_c^{\mathcal{Q}}) \to C^*_r(\mathcal{G}_\Lambda, \sigma_c^{\mathcal{Q}})$ with the isomorphism $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c^{\mathcal{Q}})$ of Corollary 7.8. The $\mathbb{Z}^k$-grading of $\mathcal{G}_\Lambda$

induces a strongly continuous $\mathbb{T}^k$-action on $C_r^*(\mathcal{G}_\Lambda, \sigma_c^{\mathcal{Q}})$ which is compatible under ψ with the gauge action on $C^*(\Lambda, c)$. So the argument of the preceding paragraph also applies to the reduced C^*-algebra, giving $C^*(\mathcal{G}_\Lambda, \sigma_c) \cong C^*(\Lambda, c) \cong C_r^*(\mathcal{G}_\Lambda, \sigma_c)$. □

8. Structure theory

In this section we establish some structure theorems for twisted k-graph C^*-algebras. We begin with a version of the Cuntz-Krieger uniqueness theorem and a simplicity result that follow from Renault's structure theory for groupoid C^*-algebras [21] and Corollary 7.9.

Recall from [15, Definition 3.1] that a row-finite k-graph with no sources is said to be *aperiodic* if for every pair α, β of distinct elements of Λ such that $s(\alpha) = s(\beta)$, there exists $\tau \in s(\alpha)\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$.

Remark 8.1. The original aperiodicity condition (A) of [10] insists that for each $v \in \Lambda^0$ there exists $x \in \Lambda^\infty$ with $r(x) = v$ such that for all $p \neq q \in \mathbb{N}^k$, we have $\sigma^p x \neq \sigma^q x$. Proposition 3.6 of [15] implies that condition (A) is equivalent to aperiodicity of Λ in the sense described above, and [22, Lemma 3.2] implies that this is also equivalent to the condition of "no local periodicity" described there.

Recall from [10, Definition 4.7] that a row-finite k-graph Λ with no sources is *cofinal* if for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda x(n) \neq \emptyset$.

Corollary 8.2. *Let Λ be a row-finite k-graph with no sources. Suppose that Λ is aperiodic. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and a Cuntz-Krieger (Λ, c)-family t in a C^*-algebra B. If each $t_v \neq 0$ then the homomorphism $\pi_t : C^*(\Lambda, c) \to B$ is injective (so $C^*(\Lambda, c) \cong C^*(t)$). Moreover, $C^*(\Lambda, c)$ is simple if and only if Λ is cofinal.*

Proof. By Remark 8.1, Λ satisfies Condition (A). Hence [10, Proposition 4.5] implies that $\mathcal{G}_\Lambda$ is topologically free in the sense that the units with trivial isotropy are dense in $\mathcal{G}_\Lambda^{(0)}$. Since $C^*(\Lambda, c) \cong C^*(\mathcal{G}_\Lambda, \sigma_c)$, the result now follows from [21, Proposition II.4.6] and the arguments of [10, Theorem 4.6 and Proposition 4.8]. □

Remark 8.3. Let Λ be a row-finite k-graph with no sources. Combining Remark 8.1 with [22, Theorem 3.1] shows that the untwisted C^*-algebra $C^*(\Lambda)$ is simple if *and only if* Λ is both aperiodic and cofinal. This is not the case in general for twisted k-graph algebras: [14, Example 7.7] shows how to recover the irrational rotation algebras, which are simple, as twisted C^*-algebras of a 2-graph which fails the aperiodicity condition quite spectacularly. So in general, simplicity of the untwisted C^*-algebra $C^*(\Lambda)$ implies simplicity of each $C^*(\Lambda, c)$ but the converse does not hold.

We show next that each $C^*(\Lambda, c)$ is nuclear. Our argument follows closely that of [10, Theorem 5.5]. We first establish a technical result.

Lemma 8.4 (cf. [10, Lemma 5.4])**.** *Let Λ be a row-finite k-graph with no sources, and suppose that the degree map on Λ is a coboundary. For each $c \in \underline{Z}^2(\Lambda, \mathbb{T})$, the twisted C^*-algebra $C^*(\Lambda, c)$ is AF, and is isomorphic to $C^*(\Lambda)$.*

Proof. Since d is a coboundary, there exists $b \in \underline{C}^0(\Lambda, \mathbb{Z}^k)$ such that $d(\lambda) = (\underline{\delta}^0 b)(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Lambda$.

Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. For $n \in \mathbb{N}^k$, let

$$A_n := \overline{\mathrm{span}}\{s_\mu s_\nu^* : (\mu, \nu) \in \Lambda \,{}_s{*}_s\, \Lambda, b(s(\mu)) = n\} \subseteq C^*(\Lambda, c),$$

and for $v \in \Lambda^0$ with $b(v) = n$, let $A_{n,v} := \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu) = v\} \subseteq A_n$. Arguing as in the proof of [10, Lemma 5.4] we see: that $A_m \subseteq A_n$ for $m \leq n \in \mathbb{N}^k$; that $C^*(\Lambda, c) = \varinjlim_{n\in\mathbb{N}^k} A_n$; that $A_n = \bigoplus_{b(v)=n} A_{n,v}$ for each n; and that $s_\mu s_\nu^* \mapsto \theta_{\mu,\nu}$ determines isomorphisms $A_{n,v} \cong \mathcal{K}_{\Lambda v}$ for each n, v. So $C^*(\Lambda, c)$ is AF.

To calculate the multiplicities of the partial inclusions $A_{m,v} \to A_{n,w}$, fix $m \leq n$ and $v \in b^{-1}(m)$, and observe that if $s(\mu) = v$ then

$$s_\mu s_\mu^* = \sum_{\alpha\in v\Lambda^{n-m}} s_\mu s_\alpha s_\alpha^* s_\mu^* = \sum_{b(w)=n} \sum_{\alpha\in v\Lambda w} s_{\mu\alpha} s_{\mu\alpha}^*.$$

So for $w \in b^{-1}(n)$, the multiplicity of the partial inclusion of $A_{m,v}$ in $A_{n,w}$ is $|v\Lambda w|$ and in particular does not depend on the cocycle c. Since AF algebras are completely determined by the dimensions of the summands of the approximating subalgebras and by the multiplicities of the partial inclusions, the isomorphism class of $C^*(\Lambda, c)$ is independent of c. □

Suppose that Λ is a k-graph, A is a discrete abelian group, and $f : \Lambda \to A$ is a functor. The *skew-product* k-graph $\Lambda \times_f A$ is the cartesian product $\Lambda \times A$ with operations $r(\mu, a) = (r(\mu), a)$, $s(\mu, a) = (s(\mu), a + f(\mu))$, $(\mu, a)(\nu, a + f(\mu)) = (\mu\nu, a)$ and $d(\mu, a) = d(\mu)$ (see [10, Definition 5.1]).

Lemma 8.5 (cf. [10, Corollary 5.3])**.** *Let Λ be a row-finite k-graph and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Let A be a locally compact abelian group and $f : \Lambda \to A$ a functor. There is a strongly continuous action α^f of $\widehat{A}$ on $C^*(\Lambda, c)$ such that $\alpha^f_\chi(s_\lambda) = \chi(f(\lambda))s_\lambda$ for all $\chi \in \widehat{A}$ and $\lambda \in \Lambda$. There is a cocycle $\tilde{c} \in \underline{Z}^2(\Lambda \times_f A, \mathbb{T})$ given by $\tilde{c}((\mu, a), (\nu, a + f(\mu)) = c(\mu, \nu)$. If A is discrete, then $C^*(\Lambda \times_f A, \tilde{c})$ is isomorphic to the crossed-product $C^*(\Lambda, c) \rtimes_{\alpha^f} \widehat{A}$.*

Proof. Our proof follows that of [10, Corollary 5.3] except that we must take into account the cocycles c and $\tilde{c}$.

The existence of the action α^f follows from the universal property of $C^*(\Lambda, c)$: for each $\chi \in \widehat{A}$, the map $t : \lambda \mapsto \chi(f(\lambda))s_\lambda$ determines a Cuntz-Krieger (Λ, c)-family in $C^*(\Lambda, c)$. Continuity follows from an $\varepsilon/3$-argument.

The map $\tilde{c}$ is a 2-cocycle because c is a 2-cocycle and $(\mu, a) \mapsto \mu$ is a functor.

Suppose that A is discrete. Let $\mathcal{P}$ be a countable subset of $\Lambda_s *_s \Lambda$ containing $\{(\lambda, s(\lambda)) : \lambda \in \Lambda\}$ and such that $\{Z(\mu, \nu) : (\mu, \nu) \in \mathcal{P}\}$ is a partition of $\mathcal{G}_\Lambda$ as in Lemma 6.6. Then $\mathcal{Q} := \{((\mu, a + f(\mu)), (\nu, a + f(\nu))) : (\mu, \nu) \in \mathcal{P}, a \in A\}$ gives a partition of $\mathcal{G}_{\Lambda\times_f A}$ with the same properties. Let $\overline{f}$ be the 1-cocycle on $\mathcal{G}_\Lambda$ given by $\overline{f}(\alpha x, d(\alpha) - d(\beta), \beta x) = f(\alpha) - f(\beta)$. Let $\mathcal{G}_\Lambda(\overline{f})$ be the skew-product groupoid of [21], and let $\sigma_{\tilde{c}}$ be the cocycle on $\mathcal{G}_{\Lambda\times_f A}$ obtained from Lemma 6.3 applied to $\tilde{c} \in \underline{Z}^2(\Lambda\times_f A, \mathbb{T})$ and $\mathcal{Q}$. Let σ_c be the cocycle on $\mathcal{G}_\Lambda$ obtained in the same way from $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and $\mathcal{P}$. If q denotes the quotient map $\mathcal{G}_\Lambda(\overline{f}) \to \mathcal{G}_\Lambda$ then $\sigma_c \circ q$ is a continuous 2-cocycle on $\mathcal{G}_\Lambda(\overline{f})$. By [10, Theorem 5.2], the groupoid $\mathcal{G}_{\Lambda\times_f A}$ is isomorphic to $\mathcal{G}_\Lambda(\overline{f})$. Moreover, this isomorphism carries $\sigma_{\tilde{c}}$ to $\sigma_c \circ q$. We can now apply [21, Theorem II.5.7] as in the proof of [10, Corollary 5.3]. □

Remark 8.6. Note that $\tilde{c}$ is the pull-back of c under the functor $\Lambda \times_f A \to \Lambda$ (given by $(\lambda, a) \mapsto \lambda$).

Corollary 8.7 (cf. [10, Theorem 5.5])**.** *Let Λ be a row-finite k-graph and let $c \in \underline{Z}^2(\Lambda, \mathbb{T})$. Then $C^*(\Lambda, c)$ belongs to the bootstrap class $\mathcal{N}$, and in particular is nuclear.*

Proof. We follow the proof of [10, Theorem 5.5]. By Takai duality, we have

$$C^*(\Lambda, c) \sim_{Me} C^*(\Lambda, c) \times_\gamma \mathbb{T}^k \times_{\hat{\gamma}} \mathbb{Z}^k.$$

Lemma 8.5 implies that $C^*(\Lambda, c) \times_\gamma \mathbb{T}^k \cong C^*(\Lambda \times_d \mathbb{Z}^k, \tilde{c})$. Define $b : (\Lambda \times_d \mathbb{Z}^k)^0 \to \mathbb{Z}^k$ by $b(v, m) = m$. Then the degree map on $\Lambda \times_d \mathbb{Z}^k$ is equal to $\underline{\delta}^0 b$, so Lemma 8.4 implies that $C^*(\Lambda, c) \times_\gamma \mathbb{T}^k$ is AF. Hence $C^*(\Lambda, c)$ is Morita equivalent to a crossed product of an AF algebra by $\mathbb{Z}^k$, which proves the result. □

Finally, we consider pullbacks and cartesian products of k-graphs. Recall from [10] that if Λ is a k-graph and $f : \mathbb{N}^l \to \mathbb{N}^k$ is a homomorphism, then the pullback l-graph $f^*\Lambda$ is defined by $f^*\Lambda = \{(\lambda, m) \in \Lambda \times \mathbb{N}^l : d(\lambda) = f(m)\}$ with coordinatewise operations and degree map $d(\lambda, m) = m$. Recall also that if Λ_1 is a k_1-graph and Λ_2 is a k_2-graph, then $\Lambda_1 \times \Lambda_2$ is a $(k_1 + k_2)$-graph with coordinatewise operations and degree map $d(\lambda_1, \lambda_2) = (d(\lambda_1), d(\lambda_2))$.

Corollary 8.8. (1) *Let Λ be a row-finite k-graph with no sources. Fix $c \in \underline{Z}^2(\Lambda, \mathbb{T})$ and a homomorphism $f : \mathbb{N}^l \to \mathbb{N}^k$. There is a cocycle f^*c on $f^*\Lambda$ given by $f^*c(\lambda, m) = c(\lambda)$, and there is a homomorphism $\pi_f : C^*(f^*\Lambda, f^*c) \to C^*(\Lambda, c)$ given by $\pi_f(s_{\lambda,m}) = s_\lambda$. If f is injective, so is π_f. If f is surjective, then π_f is also, and $C^*(f^*\Lambda, f^*c) \cong C^*(\Lambda, c) \otimes C(\mathbb{T}^{l-k})$.*

(2) *For each $i \in \{1, 2\}$, let Λ_i be a row-finite k_i-graph and fix $c_i \in \underline{Z}^2(\Lambda_i, \mathbb{T})$. Then $(c_1 \times c_2)(\lambda_1, \lambda_2) := c_1(\lambda_1)c_2(\lambda_2)$ determines an element $c_1 \times c_2 \in \underline{Z}^2(\Lambda_1 \times \Lambda_2)$. Moreover, the formula $(s_{\lambda_1}, s_{\lambda_2}) \mapsto s_{\lambda_1} \otimes s_{\lambda_2}$ determines an isomorphism*

$$C^*(\Lambda_1 \times \Lambda_2, c_1 \times c_2) \cong C^*(\Lambda_1, c_1) \otimes C^*(\Lambda_2, c_2).$$

Proof. The arguments are more or less identical to those of [10, Proposition 1.11] and [10, Corollary 3.5(iii) and (iv)]. □

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