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Boundary quotients of the Toeplitz algebra of the affine semigroup over the natural numbers

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Abstract. We study the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ and three quotients of this algebra: the C^*-algebra $\mathcal{Q}_\mathbb{N}$ recently introduced by Cuntz, and two new ones, which we call the additive and multiplicative boundary quotients. These quotients are universal for Nica-covariant representations of $\mathbb{N} \rtimes \mathbb{N}^\times$ satisfying extra relations, and can be realised as partial crossed products. We use the structure theory for partial crossed products to prove a uniqueness theorem for the additive boundary quotient, and use the recent analysis of KMS states on $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ to describe the KMS states on the two quotients. We then show that $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$, $\mathcal{Q}_\mathbb{N}$ and our new quotients are all interesting new examples for Larsen's theory of Exel crossed products by semigroups.

1. *Introduction*

Laca and Raeburn [**17**] have recently studied the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ of the semidirect product of the additive semigroup $\mathbb{N} = \{n \in \mathbb{Z} : n \geq 0\}$ by the natural action of the multiplicative semigroup $\mathbb{N}^\times = \{n \in \mathbb{Z} : n > 0\}$. They proved that $\mathbb{N} \rtimes \mathbb{N}^\times$ is the positive cone in a quasi-lattice ordering of the enveloping group $\mathbb{Q} \rtimes \mathbb{Q}^*_+$ [**17**, Proposition 2.2], which means that one can run the pair $(\mathbb{Q} \rtimes \mathbb{Q}^*_+, \mathbb{N} \rtimes \mathbb{N}^\times)$ through the general theory of the Toeplitz algebras of quasi-lattice ordered groups [**4, 11, 16, 20**]. Thus, we know from [**4**] that the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ has a distinguished boundary quotient, which we will call the Crisp–Laca quotient. This quotient is simple and purely infinite [**4**], so Laca and Raeburn conjectured that the Crisp–Laca quotient is the purely infinite simple algebra $\mathcal{Q}_{\mathbb{N}}$ which Cuntz had associated to $\mathbb{N} \rtimes \mathbb{N}^{\times}$ [**5**]. They verified this conjecture in [**17**, Theorem 6.3].

The Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ carries a very interesting dynamics σ arising from the dual action of $(\mathbb{Q}_{+}^{*})^{\wedge}$ and the embedding of $\mathbb{R}$ in $(\mathbb{Q}_{+}^{*})^{\wedge}$, which takes $t \in \mathbb{R}$ to the character $r \mapsto r^{it}$. The dynamical system $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times}), \mathbb{R}, \sigma)$ has a rich supply of KMS$_{\beta}$ states for $\beta \in [1, \infty]$, and exhibits a phase transition at $\beta = 2$ [**17**, Theorem 7.1]; if we distinguish between KMS$_{\infty}$ states and ground states, as in [**3**], then there is a second phase transition at $\beta = \infty$ [**17**, Theorem 7.1(4)].

The main technical tool in the analysis of [**17**] is a description of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ as a partial crossed product $C(\Omega) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_{+}^{*})$ arising from work of Exel *et al* [**11**]. The compact space Ω is the spectrum of the commutative C^{*}-subalgebra of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ generated by the range projections of the generating isometries, and the Crisp–Laca quotient is, almost by definition, the quotient $C(\partial\Omega) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_{+}^{*})$ associated to a 'minimal boundary' $\partial\Omega$ of Ω. Since the semigroup $\mathbb{N} \rtimes \mathbb{N}^{\times}$ is a product, we can go to infinity along the additive subsemigroup $\mathbb{N}$ or along the multiplicative subsemigroup $\mathbb{N}^{\times}$. This gives, respectively, an additive boundary Ω_{add} and a multiplicative boundary Ω_{mult}; the Crisp–Laca boundary $\partial\Omega$ is the intersection $\Omega_{\mathrm{add}} \cap \Omega_{\mathrm{mult}}$. In [**17**], the set Ω_{add} played a crucial role in the construction of KMS$_{\beta}$ states for $\beta \in [1, 2]$ (see [**17**, Proposition 9.1]). Both Ω_{add} and Ω_{mult} determine quotients $C(\Omega_{\mathrm{add}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_{+}^{*})$ and $C(\Omega_{\mathrm{mult}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_{+}^{*})$ of $C(\Omega) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_{+}^{*}) \cong \mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$. We call the corresponding quotients of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ the additive boundary quotient $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ and the multiplicative boundary quotient $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$.

The present project started when we noticed that these new boundary quotients are very interesting in their own right, and set out to see what we could say about them. We find that the phase transition at $\beta = 2$ arises from the quotient map of $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ onto $\mathcal{Q}_{\mathbb{N}}$, and that the phase transition at $\beta = \infty$ arises from the quotient map of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ onto $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$.

We then show that all four algebras provide interesting new examples for Larsen's theory of Exel crossed products by semigroups [**18**]. The motivating example for Exel's theory was a subshift of finite type, for which his crossed product is the associated Cuntz–Krieger algebra; he and Vershik subsequently used his construction to study other irreversible dynamical systems [**12**]. In [**8**], Exel only considered crossed products by $\mathbb{N}$, and Larsen's extension covers actions of more general abelian semigroups, including the action of $\mathbb{N}^{k}$ by subshifts on the path space of a k-graph [**1**]. (Exel's own theory of crossed products by semigroups in [**9**] seems to be quite different from Larsen's; the two are compared in [**9**, §13].)

We prove, answering a question raised by Larsen, that Cuntz's $\mathcal{Q}_{\mathbb{N}}$ is an Exel crossed product by an endomorphic action of the semigroup $\mathbb{N}^{\times}$ on $C(\mathbb{T})$. There is a parallel realisation of the multiplicative boundary quotient as an Exel crossed product for an action of $\mathbb{N}^{\times}$ on the Toeplitz algebra $\mathcal{T}(\mathbb{N})$. The additive boundary quotient and $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ itself fit into the picture as Toeplitz analogues of Exel crossed products for the same endomorphic actions of $\mathbb{N}^{\times}$ on $C(\mathbb{T})$ and $\mathcal{T}(\mathbb{N})$.

We begin in §3 by finding presentations of $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ and $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$, and identifying the Nica-covariant isometric representations V of $\mathbb{N} \rtimes \mathbb{N}^{\times}$ that give faithful representations of $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ (Theorem 3.5). Our main tools are the presentation of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ from [**17**] and the general machinery of [**11**], which we review in §2. In §4, we use the results of [**17**] to analyse the KMS states on our two boundary quotients.

In §§5–7, we relate our four algebras to Larsen's theory of Exel crossed products [**18**]. She considered dynamical systems (A, P, α, L) in which α is an action of a semigroup P by endomorphisms of a C^*-algebra A, and L is an action of P by transfer operators. Following earlier work on the case $P = \mathbb{N}$ in [**2, 8**], Larsen constructed a product system M_L of Hilbert bimodules over P, and then her crossed product is the Cuntz–Pimsner algebra $\mathcal{O}(M_L)$, as defined by Fowler in [**13**]. The motivating examples in [**18**] involve a compact abelian group Γ and the action $\alpha : \mathbb{N}^\times \to \operatorname{End} C(\Gamma)$ defined by $\alpha_a(f)(g) = f(g^a)$. However, not much is known about these crossed products, and Larsen asked in particular whether her crossed product $C(\mathbb{T}) \rtimes_{\alpha,L} \mathbb{N}^\times$ can be described in familiar terms.

We show that $C(\mathbb{T}) \rtimes_{\alpha,L} \mathbb{N}^\times$ is isomorphic to Cuntz's $\mathcal{Q}_\mathbb{N}$, and that the additive boundary quotient is another important C^*-algebra associated to the product system M_L, namely the Nica–Toeplitz algebra $\mathcal{NT}(M_L)$ (Theorem 5.2). The algebra $\mathcal{NT}(M_L)$ is universal for representations of the product system which are Nica covariant in a sense made precise by Fowler [**13**], and in our situations the Cuntz–Pimsner algebra is a proper quotient of $\mathcal{NT}(M_L)$. (Fowler wrote $\mathcal{T}_{\mathrm{cov}}$ rather than $\mathcal{NT}$—we explain in Remark 5.3 why we think the notation needs to be changed.)

To fit our other two algebras into the setup of [**18**], we construct an Exel system $(\mathcal{T}, \mathbb{N}^\times, \beta, K)$ based on the usual Toeplitz algebra $\mathcal{T} = \mathcal{T}(\mathbb{N})$. We show that the associated Nica–Toeplitz algebra $\mathcal{NT}(M_K)$ is the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$, and that the crossed product $\mathcal{T} \rtimes_{\beta,K} \mathbb{N}^\times := \mathcal{O}(M_K)$ is our multiplicative boundary quotient (Theorem 6.6). We finish by showing that all the isomorphisms we have found are compatible, and fit together nicely in a large commutative diagram (Theorem 7.4). Our results and those of [**15**] suggest that studying the KMS states on the C^*-algebras of other product systems might be very interesting indeed.

Notation. As in [**17**], $\mathbb{N}$ denotes the additive semigroup of non-negative integers, and $\mathbb{N}^\times$ the multiplicative semigroup of positive integers. We write $\mathcal{P}$ for the set of prime numbers, and $e_p(a)$ for the exponent of p in the prime factorisation of $a \in \mathbb{N}^\times$, so that $a = \prod_{p\in\mathcal{P}} p^{e_p(a)}$, and $\mathcal{N}$ for the set $\prod_{p\in\mathcal{P}} p^{\mathbb{N}\cup\{\infty\}}$ of supernatural numbers. We also write $\mathbb{Q}$ for the additive group of rational numbers, and $\mathbb{Q}^*_+$ for the multiplicative group $\mathbb{Q} \cap (0, \infty)$.

For $M, N \in \mathcal{N}$, we say that M divides N (written $M|N$) if $e_p(M) \le e_p(N)$ for all p, and then each pair $M, N \in \mathcal{N}$ has a least upper bound $\operatorname{lcm}(M, N)$ and a greatest lower bound $\gcd(M, N)$ in $\mathcal{N}$. As in [**17**], we define $\mathbb{Z}/N := \varprojlim((\mathbb{Z}/a\mathbb{Z}) : a \in \mathbb{N}^\times, a|N)$, which is consistent with the notation $\mathbb{Z}/N$ for $\mathbb{Z}/N\mathbb{Z}$. Then $\mathbb{Z}/p^\infty$ is the ring $\mathbb{Z}_p$ of p-adic integers and, if we write $\nabla := \prod_{p\in\mathcal{P}} p^\infty$, then $\mathbb{Z}/\nabla$ is the ring $\widehat{\mathbb{Z}}$ of integral adèles. If $M, N \in \mathcal{N}$ and $M|N$, we write $r(M)$ for the image of $r \in \mathbb{Z}/N$ in $\mathbb{Z}/M$.

2. *Preliminaries*

2.1. *Quasi-lattice ordered groups and their Toeplitz algebras.* Let G be a discrete group and P a subsemigroup of G such that $P \cap P^{-1} = \{e\}$, and consider the partial order on G

defined by $x \le y \iff x^{-1}y \in P$. Following Nica [**20**], we say that (G, P) is a *quasi-lattice ordered group* if any $x, y \in G$ which have a common upper bound in P have a least upper bound $x \vee y \in P$. An isometric representation $V : P \to B(\mathcal{H})$ is then *Nica covariant* if

$$V_x V_x^* V_y V_y^* = \begin{cases} V_{x\vee y} V_{x\vee y}^* & \text{if } x \vee y < \infty, \\ 0 & \text{if } x \vee y = \infty, \end{cases} \tag{2.1}$$

and the C^*-algebra $C^*(G, P)$ of (G, P) is generated by a universal Nica-covariant representation $i_P : P \to C^*(G, P)$; we write π_V for the representation of $C^*(G, P)$ such that $V = \pi_V \circ i_P$.

Every cancellative semigroup P has an isometric representation on $l^2(P)$ characterised in terms of the usual basis $\{e_x : x \in P\}$ by $T_y e_x = e_{yx}$; we call this the *Toeplitz representation* of P. The *Toeplitz algebra* $\mathcal{T}(P)$ is the C^*-subalgebra of $B(l^2(P))$ generated by the isometries $\{T_y\}$. Nica observed that, when (G, P) is quasi-lattice ordered, the Toeplitz representation T satisfies equation (2.1), and identified an amenability condition under which the corresponding representation π_T of $C^*(G, P)$ is an isomorphism onto the Toeplitz algebra $\mathcal{T}(P)$ [**20**, §4.2]. Nica's amenability hypothesis is automatic if G is an amenable group [**20**, §1.1].

Nica's algebra $C^*(G, P)$ was studied in [**16**] by viewing it as a semigroup crossed product. For $x \in P$, let 1_x denote the characteristic function of the set xP. Then the quasi-lattice property implies that $\operatorname{span}\{1_x : x \in P\}$ is closed under multiplication, and hence $B_P := \overline{\operatorname{span}}\{1_x : x \in P\}$ is a C^*-subalgebra of $l^\infty(P)$. The action τ of P by translation on $l^\infty(P)$ leaves B_P invariant, and there is an isomorphism of the semigroup crossed product $B_P \times_\tau P$ onto $C^*(G, P)$ that identifies the copies of P and carries 1_x to $i_P(x)i_P(x)^*$ [**16**, Corollary 2.4]. (We mention the algebra $B_P \times_\tau P$ here because we want to use it as motivation in the next subsection.)

2.2. *Partial crossed products and the Nica spectrum.* A *partial action* θ of a group G on a compact space X consists of open sets $\{U_t : t \in G\}$ and homeomorphisms $\theta_t : U_{t^{-1}} \to U_t$ such that θ_{st} extends $\theta_s\theta_t$ for $s, t \in G$. Each θ_t induces an isomorphism $\alpha_{t^{-1}} : f \mapsto f \circ \theta_t$ of the ideal $C_0(U_t)$ in $C(X)$ onto $C_0(U_{t^{-1}})$, and the α_t form a partial action of G on $C(X)$ as in [**11**]. The system $(C(X), G, \alpha)$ has a partial crossed product $C(X) \rtimes_\alpha G$ which is generated by a universal covariant representation (ρ, u). There is also a reduced partial crossed product $C(X) \rtimes_{\alpha,r} G$ but, when the partial action α is amenable (which is automatic if G is amenable), this reduced crossed product coincides with the full one [**7**, Proposition 4.2]. Thus, when G is amenable, as ours will be, we can apply results in [**11**] about reduced crossed products to full crossed products.

Suppose that (G, P) is a quasi-lattice ordered group. Following [**20**] and [**11**, §6], we consider the *Nica spectrum* Ω of P, which is the set of non-empty directed hereditary subsets ω of P, viewed as a subset of the compact space $\{0, 1\}^P$. As in [**14**, §2], for $g \in G$ and $\omega \in \Omega$, we set $g\omega := \{gy : y \in \omega\}$, and define $\theta_g(\omega)$ to be the hereditary closure $\operatorname{Her}((g\omega) \cap P)$. The partially defined maps θ_g form a partial action of G on Ω; the domain of θ_g is $U_{g^{-1}} = \{\omega : (g\omega) \cap P \neq \emptyset\}$, which is non-empty if and only if $g \in PP^{-1}$. Lifting this partial action to $C(\Omega)$ gives a partial dynamical system $(C(\Omega), G, \alpha)$, and it was shown in [**11**, Theorem 6.4] that $C^*(G, P)$ is isomorphic to the partial crossed product $C(\Omega) \rtimes_\alpha G$. We need to understand how this isomorphism works.

The Nica spectrum Ω enters into the picture because it is the spectrum of the commutative C^*-algebra B_P appearing in [**16**]: the functional $\hat{\omega}$ corresponding to $\omega \in \Omega$ is defined by $\hat{\omega}(f) = \lim_{x \in \omega} f(x)$, which makes sense because ω is directed. The Gelfand transform carries the generating function $1_x \in B_P$ into the characteristic function of the set $\{\omega \in \Omega : x \in \omega\}$, which is the domain U_x of $\theta_{x^{-1}}$. The isomorphism of [**11**, Theorem 6.4] carries the generating isometries $i_P(x)$ into the generators u_x, and the functions $1_x \in B_P \subset B_P \times_\tau P = C^*(G, P)$ into $\rho(\chi_{U_x})$. From now on, we write 1_x for χ_{U_x}.

The *spectrum* $\Omega_\mathcal{R}$ of a subset $\mathcal{R}$ of $C(\Omega)$ is

$$\Omega_\mathcal{R} := \{\omega \in \Omega : \theta_{t^{-1}}(\omega)^\wedge(f) = f(\theta_{t^{-1}}(\omega)) = 0 \text{ for all } t \in \omega,\ f \in \mathcal{R}\}. \tag{2.2}$$

Proposition 4.1 of [**11**] says that $\Omega_\mathcal{R}$ is a closed invariant subset of Ω, and [**11**, Theorem 4.4] says that the $C(\Omega_\mathcal{R}) \rtimes G$ is the quotient of $C(\Omega) \rtimes_\alpha G$ obtained by imposing the relations $\{f = 0 : f \in \mathcal{R}\}$. The boundary $\partial\Omega$ is the spectrum of a maximal set of relations for which the quotient is non-trivial; $\partial\Omega$ is the closure in Ω of the set of maximal hereditary directed subsets. (See [**14**, Definition 3.3] or [**4**, Lemma 3.5] for more detail, including a description of the set $\mathcal{R}$ for which $\partial\Omega = \Omega_\mathcal{R}$.) The boundary $\partial\Omega$ gives rise to the Crisp–Laca quotient $C(\partial\Omega) \rtimes G$ [**4**, Theorem 6.3].

2.3. *The Toeplitz algebra of* $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$. We now consider the semidirect product $\mathbb{Q} \rtimes \mathbb{Q}_+^*$, where

$$(r, a)(q, b) = (r + aq, ab) \quad \text{for } r, q \in \mathbb{Q} \text{ and } a, b \in \mathbb{Q}_+^*$$

and

$$(r, a)^{-1} = (-a^{-1}r, a^{-1}) \quad \text{for } r \in \mathbb{Q} \text{ and } a \in \mathbb{Q}_+^*.$$

Laca and Raeburn proved that $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ is quasi-lattice ordered [**17**, Proposition 2.1]. Since $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ is amenable, Nica's theory implies that the Toeplitz representation T gives an isomorphism π_T of $C^*(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ onto the $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$, and hence $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), T)$ is universal for Nica-covariant representations of $\mathbb{N} \rtimes \mathbb{N}^\times$. In [**17**, Theorem 4.1], this universal property is used to present $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ as the universal C^*-algebra generated by isometries s and $\{v_p : p \in \mathcal{P}\}$ satisfying:

(T1) $v_p s = s^p v_p$;
(T2) $v_p v_q = v_q v_p$;
(T3) $v_p^* v_q = v_q v_p^*$ for $p \neq q$;
(T4) $s^* v_p = s^{p-1} v_p s^*$; and
(T5) $v_p^* s^k v_p = 0$ for $1 \leq k < p$.

Thus, if S and $\{V_p : p \in \mathcal{P}\}$ are isometries satisfying (T1)–(T5), there is a representation $\pi_{S,V}$ of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ such that $\pi_{S,V}(s) = S$ and $\pi_{S,V}(v_p) = V_p$. The relations (T1) and (T2) imply that there is an isometric representation w of $\mathbb{N} \rtimes \mathbb{N}^\times$ satisfying $w_{(m,a)} = s^m \prod_{p \in \mathcal{P}} v_p^{e_p(a)}$, and then (T3)–(T5) imply that w is Nica covariant (see [**17**, §4]).

Applying [**16**, Theorem 3.7] to $(\mathbb{Q} \rtimes \mathbb{Q}_+^*, \mathbb{N} \rtimes \mathbb{N}^\times)$ gives conditions on a Nica-covariant representation W of $\mathbb{N} \rtimes \mathbb{N}^\times$ which ensure that π_W is faithful. Since $\mathbb{N} \rtimes \mathbb{N}^\times$ has many minimal elements, the hypotheses of [**16**, Theorem 3.7] simplify as follows.

THEOREM 2.1. *Suppose that S and $\{V_p : p \in \mathcal{P}\}$ are isometries satisfying (T1)–(T5). Then the corresponding representation $\pi_{S,V}$ of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ is faithful if and only if*

$$(1 - SS^*) \prod_{p\in F} \prod_{k=0}^{p-1} (1 - S^k V_p V_p^* S^{*k}) \neq 0 \quad \textit{for every finite set } F \textit{ of primes.} \tag{2.3}$$

Proof. Let $W : (m, a) \mapsto S^m \prod_{p\in\mathcal{P}} V_p^{e_p(a)}$ be the associated Nica-covariant representation of $\mathbb{N} \rtimes \mathbb{N}^\times$. We fix a finite subset E of $\mathbb{N} \rtimes \mathbb{N}^\times \backslash \{(0, 1)\}$, and aim to show that

$$\prod_{(m,a)\in E} (1 - W_{(m,a)} W^*_{(m,a)}) \neq 0,$$

which is the hypothesis of [**16**, Theorem 3.7].

If we make E larger, then we make the product smaller, so we may as well assume that $(1, 1) \in E$ and that E has at least one element (m, a) with $a > 1$. Then for every m we have

$$1 - W_{(m,1)} W^*_{(m,1)} = 1 - S^m S^{*m} \geq 1 - SS^*. \tag{2.4}$$

For each $(m, a) \in E$ with $a > 1$, we choose a prime $p = p_{(m,a)}$ in the prime factorisation of a and $n = n_{(m,a)}$ between 0 and $p - 1$ such that $n \equiv m \pmod p$. Then $(n, p) \leq (m, a)$, so $W_{(m,a)} W^*_{(m,a)} \leq W_{(n,p)} W^*_{(n,p)}$ and

$$1 - W_{(m,a)} W^*_{(m,a)} \geq 1 - W_{(n,p)} W^*_{(n,p)} = 1 - S^n V_p (S^n V_p)^*. \tag{2.5}$$

Equations (2.4) and (2.5) imply that

$$\begin{aligned}\prod_{(m,a)\in E} (1 - W_{(m,a)} W^*_{(m,a)}) &= \prod_{(m,1)\in E} (1 - W_{(m,1)} W^*_{(m,1)}) \prod_{(m,a)\in E, a>1} (1 - W_{(m,a)} W^*_{(m,a)}) \\ &\geq (1 - SS^*) \prod_{(m,a)\in E, a>1} (1 - S^{n_{(m,a)}} V_{p_{(m,a)}} V^*_{p_{(m,a)}} S^{*n_{(m,a)}}) \\ &\geq (1 - SS^*) \prod_{(m,a)\in E, a>1} \prod_{k=0}^{p_{(m,a)}-1} (1 - S^k V_{p_{(m,a)}} V^*_{p_{(m,a)}} S^{*k}),\end{aligned}$$

which is non-zero by hypothesis with $F = \{p_{(m,a)} : (m, a) \in E\}$. The result now follows from [**16**, Theorem 3.7]. □

2.4. *The Nica spectrum of* $\mathbb{N} \rtimes \mathbb{N}^\times$. As in [**17**, §5], for a supernatural number $N \in \mathcal{N}$, $m \in \mathbb{N}$ and $r \in \mathbb{Z}/N$, we define

$$A(m, N) := \{(k, a) \in \mathbb{N} \rtimes \mathbb{N}^\times : a|N \text{ and } a^{-1}(m - k) \in \mathbb{N}\}$$

and

$$B(r, N) := \{(k, a) \in \mathbb{N} \rtimes \mathbb{N}^\times : a|N \text{ and } k \in r(a)\}.$$

These are hereditary, directed subsets of $\mathbb{N} \rtimes \mathbb{N}^\times$, and [**17**, Corollary 5.6] says that the Nica spectrum Ω of $\mathbb{N} \rtimes \mathbb{N}^\times$ is

$$\Omega = \{A(m, M) : M \in \mathcal{N}, m \in \mathbb{N}\} \cup \{B(r, N) : N \in \mathcal{N}, r \in \mathbb{Z}/N\}.$$

From [**11**, Theorem 6.4] and [**20**, §4.2], we obtain an isomorphism of $C(\Omega) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$ onto $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ which maps $u_{(1,1)}$ to s, $u_{(0,p)}$ to v_p for all $p \in \mathcal{P}$, $\rho(1_{(1,1)})$ to ss^* and $\rho(1_{(0,p)})$ to $v_p v_p^*$. By [**17**, Proposition 6.1], Cuntz's $\mathcal{Q}_\mathbb{N}$ is the universal C^*-algebra generated by isometries satisfying (T1), (T2) and the relations:
(Q5) $\sum_{k=0}^{p-1} s^k v_p (s^k v_p)^* = 1$ for every $p \in \mathcal{P}$; and
(Q6) $ss^* = 1$.
Note that (T1), (T2), (Q5) and (Q6) imply (T3) and (T4), so $\mathcal{Q}_\mathbb{N}$ can be viewed as a quotient of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$.

In this paper, we investigate the *additive* and *multiplicative boundaries*

$$\Omega_{\mathrm{add}} := \{B(r, N) : N \in \mathcal{N}, r \in \mathbb{Z}/N\}$$

and

$$\Omega_{\mathrm{mult}} := \{A(m, \nabla) : m \in \mathbb{N}\} \cup \{B(r, \nabla) : r \in \widehat{\mathbb{Z}}\}.$$

We reach the additive boundary by letting the m in a pair (m, a) go to infinity along an arithmetic progression, and the multiplicative boundary by letting a go to infinity in the semigroup $\mathbb{N}^\times$ directed by $a \leq b \Longleftrightarrow a|b$. (The set Ω_{mult} is not quite the same as the set described as the multiplicative boundary in [**17**, Remark 5.9], which contains also the $B(r, N)$ associated to $N \in \mathcal{N} \backslash \mathbb{N}$.)

3. *The additive and multiplicative boundary quotients*

We will show in Lemma 3.1 that both Ω_{add} and Ω_{mult} are the spectra of subsets of $C(\Omega)$. It then follows from [**11**, Proposition 4.1] that they are closed invariant subsets of Ω, and from [**11**, Theorem 4.4] that $C(\Omega_{\mathrm{add}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$ and $C(\Omega_{\mathrm{mult}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$ are quotients of $C(\Omega) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$. We define the additive and multiplicative boundary quotients $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ and $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ to be the corresponding quotients of the Toeplitz algebra $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ (see Proposition 3.4).

LEMMA 3.1. *Let* $\mathcal{R}_{\mathrm{add}} = \{1 - 1_{(1,1)}\}$ *and*

$$\mathcal{R}_{\mathrm{mult}} = \left\{1 - \sum_{k=0}^{p-1} 1_{(k,p)} : p \in \mathcal{P}\right\}.$$

Then $\Omega_{\mathrm{add}} = \Omega_{\mathcal{R}_{\mathrm{add}}}$ *and* $\Omega_{\mathrm{mult}} = \Omega_{\mathcal{R}_{\mathrm{mult}}}$.

Proof. According to the definition of $\Omega_{\mathcal{R}_{\mathrm{add}}}$ in (2.2), for the first assertion it suffices to show that for each $\omega \in \Omega$,

$$\theta_{(k,a)^{-1}}(\omega)^\wedge(1 - 1_{(1,1)}) = 0 \quad \text{for all } (k, a) \in \omega \Longleftrightarrow \omega \in \Omega_{\mathrm{add}}. \tag{3.1}$$

To prove (3.1), note that

$$\begin{aligned}\theta_{(k,a)^{-1}}(\omega)^\wedge(1 - 1_{(1,1)}) = 0 &\Longleftrightarrow (1, 1) \in \theta_{(k,a)^{-1}}(\omega)\\ &\Longleftrightarrow (k, a)(1, 1) = (k + a, a) \in \omega.\end{aligned}$$

Now suppose that $\omega = B(r, N)$ is in Ω_{add}. Then

$$(k, a) \in B(r, N) \Longleftrightarrow (k + a, a) \in B(r, N) \Longrightarrow \theta_{(k,a)^{-1}}(\omega)^\wedge(1 - 1_{(1,1)}) = 0,$$

so the left-hand side of (3.1) holds. Conversely, suppose that $\omega \notin \Omega_{\mathrm{add}}$. Then $\omega = A(l, N)$ for some $N \in \mathcal{N}$ and $l \in \mathbb{N}$. Since $(l, 1) \in A(l, N)$ but $(l+1, 1) = (l, 1)(1, 1) \notin A(l, N)$, the left-hand side of (3.1) fails. This proves (3.1), and $\Omega_{\mathrm{add}} = \Omega_{\mathcal{R}_{\mathrm{add}}}$.

To see that $\Omega_{\mathrm{mult}} = \Omega_{\mathcal{R}_{\mathrm{mult}}}$, fix $p \in \mathcal{P}$. It suffices to show that for $\omega \in \Omega$, we have $\omega \in \Omega_{\mathrm{mult}}$ if and only if

$$\theta_{(j,a)^{-1}}(\omega)^{\wedge}\left(1 - \sum_{k=0}^{p-1} 1_{(k,p)}\right) = 0 \quad \text{for all } (j, a) \in \omega. \tag{3.2}$$

Since the $1_{(k,p)}$ are mutually orthogonal projections, $\theta_{(j,a)^{-1}}(\omega)^{\wedge}(1_{(k,p)}) = 1$ for at most one k. Suppose that $\omega = A(m, \nabla)$. Then

$$\begin{aligned}(j, a) \in \omega \Longrightarrow\ & a^{-1}(m - j) \in \mathbb{N} \\ \Longrightarrow\ & \text{there exists } k \in \{0, \ldots, p-1\} \text{ such that } p^{-1}(a^{-1}(m-j) - k) \in \mathbb{N} \\ \Longrightarrow\ & \theta_{(j,a)}(k, p) = (j + ak, ap) \in A(m, \nabla) \\ \Longrightarrow\ & \theta_{(j,a)^{-1}}(A(m, \nabla))^{\wedge}(1_{(k,p)}) = 1,\end{aligned}$$

so $\omega = A(m, \nabla)$ satisfies equation (3.2). Now suppose that $\omega = B(r, \nabla)$. Then

$$\begin{aligned}(j, a) \in \omega \Longrightarrow\ & j \in r(a) \\ \Longrightarrow\ & \text{there exists } k \in \{0, \ldots, p-1\} \text{ such that } j + ak \in r(ap) \\ \Longrightarrow\ & \theta_{(j,a)}(k, p) = (j + ak, ap) \in B(r, \nabla),\end{aligned}$$

so $\omega = B(r, \nabla)$ satisfies equation (3.2). This proves the 'only if' part. If $w \notin \Omega_{\mathrm{mult}}$, then $\omega = A(m, N)$ or $B(r, N)$, where $N \neq \nabla$. For $\omega = A(m, N)$, we can choose a and p such that $a|N$ and $ap \nmid N$. Then $(m, a) \in \omega$ and $\theta_{(m,a)}(k, p) \notin \omega$ for all k; so $\theta_{(m,a)^{-1}}(\omega)^{\wedge}(1_{(k,p)}) = 0$ for all k, and hence the left-hand side of equation (3.2) equals 1. For $\omega = B(r, N)$, choose a and p as above and $j \in r(a)$; then the left-hand side of equation (3.2) is again 1. This proves the 'if' part. Thus, $\Omega_{\mathrm{mult}} = \Omega_{\mathcal{R}_{\mathrm{mult}}}$. □

The following definition is justified by Proposition 3.4 below.

Definition 3.2. Let I be the ideal of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ generated by the element $1 - ss^*$, and let J be the ideal of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ generated by

$$\left\{1 - \sum_{k=0}^{p-1} s^k v_p v_p^* s^{*k} : p \in \mathcal{P}\right\}.$$

The *additive boundary quotient* is $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^{\times}) := \mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})/I$ and the *multiplicative boundary quotient* is $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^{\times}) := \mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})/J$.

We immediately have the following proposition.

PROPOSITION 3.3. *The additive boundary quotient $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ is the universal C^*-algebra with presentation (T1)–(T3), (T5) and (Q6), and $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ is the universal C^*-algebra with presentation (T1)–(T4) and (Q5).*

Proof. By [**17**, Theorem 4.1], $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ is the universal C^*-algebra with presentation (T1)–(T5). Since (T1) and (Q6) together imply (T4), $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^{\times})$ is by definition the universal C^*-algebra with presentation (T1)–(T3), (T5) and (Q6). The presentation of $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ follows immediately from [**17**, Theorem 4.1], since (Q5) implies (T5) by [**17**, Proposition 6.1]. □

We next check that these quotients match up with the quotients of $C(\Omega) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$ which interest us. We denote by $(\rho^{\mathrm{add}}, u^{\mathrm{add}})$ and $(\rho^{\mathrm{mult}}, u^{\mathrm{mult}})$ the universal covariant representations that generate the partial crossed products $C(\Omega_{\mathrm{add}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$ and $C(\Omega_{\mathrm{mult}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$, respectively.

PROPOSITION 3.4. *There are isomorphisms*

$$C(\Omega_{\mathrm{add}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*) \to \mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$$

and

$$C(\Omega_{\mathrm{mult}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*) \to \mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$$

such that $u^{\mathrm{add}}_{(1,1)} \mapsto s$, $u^{\mathrm{add}}_{(0,p)} \mapsto v_p$ *and* $u^{\mathrm{mult}}_{(1,1)} \mapsto s$, $u^{\mathrm{mult}}_{(0,p)} \mapsto v_p$.

Proof. The isomorphism of $C(\Omega) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$ onto $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ sends $u_{(1,1)}$ to s, $u_{(0,p)}$ to v_p for all $p \in \mathcal{P}$, $\rho(1_{(1,1)})$ to ss^* and $\rho(1_{(0,p)})$ to $v_p v_p^*$. In particular, ss^* corresponds to the function $1_{(1,1)}$, so the isomorphism of $C(\Omega_{\mathrm{add}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}_+^*)$ onto $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ follows from Lemma 3.1.

For the multiplicative boundary quotient, note that for $p \in \mathcal{P}$ and $0 \leq k < p$, $s^k v_p v_p^* s^{*k} \in \mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ corresponds to the function $1_{(k,p)}$ in $C(\Omega)$. Now proceed as for the additive quotient. □

The next result describes the faithful representations of $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$.

THEOREM 3.5. *Suppose that S and $\{V_p : p \in \mathcal{P}\}$ are isometries satisfying (T1)–(T3), (T5) and (Q6). Then the corresponding representation $\pi_{S,V}$ of $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ is faithful if and only if*

$$\prod_{p \in F} \prod_{l=0}^{p-1} (1 - S^l V_p V_p^* S^{*l}) \neq 0 \quad \text{for every finite set } F \subset \mathcal{P}. \tag{3.3}$$

To prove this theorem, we want to apply [**11**, Theorem 2.6], and hence we need to know that the partial action of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ on Ω_{add} is topologically free. The action of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ on Ω_{mult}, on the other hand, is not topologically free—indeed, the stability subgroups are large. Thus we do not expect an analogue of Theorem 3.5 for the multiplicative boundary quotient.

Recall from [**14**, Proposition 2.1] that the sets

$$V((m, c), K) := \{\omega \in \Omega : (m, c) \in \omega \text{ and } (m, c)h \notin \omega \text{ for all } h \in K\}, \tag{3.4}$$

where K is a finite subset of $\mathbb{N} \rtimes \mathbb{N}^\times \backslash \{(0, 1)\}$ and $(m, c) \in \mathbb{N} \rtimes \mathbb{N}^\times$, form a basis of open and closed sets for the topology on Ω.

PROPOSITION 3.6. *The partial action of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$ is topologically free on Ω_{add} but not on Ω_{mult}.*

Proof. Recall that a partial action θ of a group G on a space X is topologically free when $\{x \in X : \theta_t(x) = x\}$ has empty interior for every $t \in G \backslash \{e\}$. (When we write $\theta_t(x)$, we implicitly assert that x is in the domain $U_{t^{-1}}$ of $x \mapsto \theta_t(x)$.) Equivalently, θ is topologically free if and only if each $X_t := (X \backslash U_{t^{-1}}) \cup \{x : \theta_t(x) \neq x\}$ is dense.

Now consider $X = \Omega_{\mathrm{add}}$ and $G = \mathbb{Q} \rtimes \mathbb{Q}_+^*$. Fix $(w, y) \in \mathbb{Q} \rtimes \mathbb{Q}_+^*$. Let $N \in \mathbb{N}^\times$ and $r \in \mathbb{Z}/N$; a calculation similar to one in the proof of [**17**, Proposition 5.7] shows that $\theta_{(w,y)}(B(r, N)) = B(w + ry, yN)$. So, if $y \neq 1$, then $\theta_{(w,y)}(B(r, N)) \neq B(r, N)$ and

$$X_{(w,y)} \supset \{B(r, N) : N \in \mathbb{N}^\times, r \in \mathbb{Z}/N\}. \tag{3.5}$$

Now consider $y = 1$. If $w \notin \mathbb{Z}$, then $\theta_{(w,1)}$ has domain $\emptyset$; if $w \in \mathbb{Z}$, then $\theta_{(w,1)}(B(r, N)) = B(r, N)$ if and only if $w \in N\mathbb{Z}$. So

$$X_{(w,1)} \supset \{B(r, N) : N \in \mathbb{N}^\times, r \in \mathbb{Z}/N, N \nmid w\}. \tag{3.6}$$

In view of (3.5) and (3.6), it suffices to fix $B(s, M)$ in Ω_{add} and prove that we can approximate $B(s, M)$ by elements of the form $B(r, N)$ with $N \in \mathbb{N}^\times$ and $N \nmid w$.

First suppose that $M \in \mathbb{N}^\times$, and suppose that $M|w$ (for otherwise (3.6) implies that there is nothing to prove). Choose an increasing sequence $\{p_n\}$ of primes p_n such that $p_n \nmid w$, and $s_n \in \mathbb{Z}/p_nM$ such that $s_n(M) = s$. We claim that $B(s_n, p_nM) \to B(s, M)$ in Ω_{add}. To see this, let $(k, a) \in \mathbb{N} \rtimes \mathbb{N}^\times$, and recall that

$$B(s_n, p_nM)^\wedge(1_{(k,a)}) = \begin{cases} 1 & \text{if } a|p_nM \text{ and } k \in s_n(a), \\ 0 & \text{otherwise.} \end{cases}$$

If $a \nmid M$, then for large n we have $a \nmid p_nM$, and hence

$$B(s_n, p_nM)^\wedge(1_{(k,a)}) \to 0 = B(s, M)^\wedge(1_{(k,a)});$$

if $a|M$, then $s_n(a) = s_n(M)(a) = s(a)$ and

$$B(s_n, p_nM)^\wedge(1_{(k,a)}) = 1 \Longleftrightarrow B(s, M)^\wedge(1_{(k,a)}) = 1.$$

Either way,

$$B(s_n, p_nM)^\wedge(1_{(k,a)}) \to B(s, M)^\wedge(1_{(k,a)}),$$

which says that $B(s_n, p_nM) \to B(s, M)$ in Ω_{add}.

Second, suppose that M has infinitely many prime factors. We choose $\{M_n\}$ in $\mathbb{N}^\times$ such that $M_n \nmid w$, $M_n|M_{n+1}$ and, for every $a \in \mathbb{N}^\times$, $a|M \Longleftrightarrow a|M_n$ for large n. Then an argument like that in the preceding paragraph shows that for every $(k, a) \in \mathbb{N} \rtimes \mathbb{N}^\times$, we have

$$B(s(M_n), M_n)^\wedge(1_{(k,a)}) \to B(s, M)^\wedge(1_{(k,a)}),$$

and $B(s(M_n), M_n) \to B(s, M)$ in Ω_{add}. Thus, $B(s, M)$ belongs to the closure of $X_{(w,y)}$, as required. So the action on Ω_{add} is topologically free.

Now consider the action on Ω_{mult}. Let $(s, t) \in \mathbb{Q} \rtimes \mathbb{Q}_+^*$ and $A(m, \nabla) \in \operatorname{Dom} \theta_{(s,t)}$. We claim that $\theta_{(s,t)}(A(m, \nabla)) = A(s + tm, \nabla)$. Let $(n, c) \in \theta_{(s,t)}(A(m, \nabla))$. Since $\theta_{(s,t)}(A(m, \nabla))$ is a hereditary closure, there exist $(j, b) \in A(m, \nabla)$ such that $(n, c) \leq (s, t)(j, b) = (s + tj, tb) \in \mathbb{N} \rtimes \mathbb{N}^\times$. Now $(tb)^{-1}(s + tm - (s + tj)) = b^{-1}(m - j) \in \mathbb{N}$. So $(s + tj, tb)$, and hence (n, c), are in $A(s + tm, \nabla)$. So $\theta_{(s,t)}(A(m, \nabla)) \subset A(s + tm, \nabla)$. Conversely, let $(k, a) \in A(s + tm, \nabla)$. Choose $b \in \mathbb{N}$ such that $a^{-1}tb \in \mathbb{N}$. Then $(a^{-1}(s + tm - k), a^{-1}tb) \in \mathbb{N} \rtimes \mathbb{N}^\times$, which says that $(k, a) \leq (s + tm, tb) = (s, t)(m, b) \in \theta_{(s,t)}(A(m, \nabla))$. Thus, $(k, a) \in \theta_{(s,t)}(A(m, \nabla))$, and $\theta_{(s,t)}(A(m, \nabla)) = A(s + tm, \nabla)$, as claimed.

We now choose $(s, t) \in (\mathbb{Q} \rtimes \mathbb{Q}_+^*)\backslash\{(0, 1)\}$ such that $s/(1-t)$ is in $\mathbb{N}$. Then

$$\begin{aligned}\{\omega \in \Omega_{\mathrm{mult}} : \theta_{(s,t)}(\omega) = \omega\} &\supset \{A(m, \nabla) \in \operatorname{Dom}\theta_{(s,t)} : A(m, \nabla) = A(s + tm, \nabla)\}\\ &= \{A(s/(1-t), \nabla)\}.\end{aligned}$$

We claim that $\{A(m, \nabla)\} = V((m, 1), \{(1, 1)\}) \cap \Omega_{\mathrm{mult}}$, which implies that $\{A(m, \nabla)\}$ is open in Ω_{mult}. (To see the claim, note that if $A(n, \nabla) \in V((m, 1), \{(1, 1)\})$, then $n - m \in \mathbb{N}$ and $n - (m + 1) \notin \mathbb{N}$ imply that $n = m$, and if $B(r, \nabla) \in V((m, 1), \{(1, 1)\})$, then $m \in r(1)$ and $m + 1 \notin r(1)$, which is impossible.) Thus, for our choice of (s, t), the set $\{\omega \in \Omega_{\mathrm{mult}} : \theta_{(s,t)}(\omega) = \omega\}$ has non-empty interior, and the action on Ω_{mult} is not topologically free. □

LEMMA 3.7. *Suppose that U is a non-empty open set in Ω_{add}. Then there exist $(k, a) \in \mathbb{N} \rtimes \mathbb{N}^\times$ and a finite set F of primes such that*

$$W((k, a), F) := \{\omega \in \Omega : (k, a) \in \omega \text{ and } (k, a)(l, p) \notin \omega \text{ for all } p \in F, 0 \le l < p\}$$

is non-empty and satisfies $W((k, a), F) \cap \Omega_{\mathrm{add}} \subset U$.

Proof. Since the sets $V((m, c), K)$ defined by (3.4) form a basis for the topology on Ω, there exist $(k, a) \in \mathbb{N} \rtimes \mathbb{N}^\times$ and a finite subset H of $\mathbb{N} \rtimes \mathbb{N}^\times\backslash\{(0, 1)\}$ such that $V((k, a), H) \cap \Omega_{\mathrm{add}}$ is non-empty and contained in U. Since

$$(k, a)(l, 1) = (k + al, a) \in B(r, N) \Longleftrightarrow (k, a) \in B(r, N),$$

every $(l, b) \in H$ has $b > 1$, and there is a prime p_h such that $p_h|b$. Set $F := \{p_h : h \in H\}$. Note that $W((k, a), F)$ is non-empty: if q is a prime which is not in F and $r \in \mathbb{Z}/aq$ satisfies $k \in r(a)$, then $B(r, aq)$ belongs to $W((k, a), F)$.

We claim that $W((k, a), F) \cap \Omega_{\mathrm{add}} \subset V((k, a), H)$. Suppose that $B(r, N) \in W((k, a), F)$. Then $(k, a) \in B(r, N)$ and, for $p \in F$ and each $0 \le l < p$, we have $(k + al, ap) \notin B(r, N)$. Since $(k, a) \in B(r, N)$, we have $a|N$ and $k \in r(a)$. We claim that ap does not divide N for every $p \in F$. To see this, suppose that ap divides N for some $p \in F$. Then $k \in r(a)$ implies that $k + al \in r(ap)$ for some l, and then $(k + al, ap) \in B(r, N)$ contradicts $B(r, N) \in W((k, a), F)$. So ap does not divide N for every $p \in F$, and p does not divide N for every $p \in F$. Now fix $h = (l, b) \in H$. There exists $p_h \in F$ such that $p_h|b$ and, since p_h does not divide N, it follows that b does not divide N. Thus ab does not divide N, and hence $(k, a)(l, b) = (k + al, ab) \notin B(r, N)$. Thus $B(r, N) \in V((k, a), H)$, as required. □

Next, we need to convert the hypothesis that the representation $\pi_{S,V}$ is non-zero on $C(\Omega_{\mathrm{add}})$ into the hypothesis (3.3) appearing in Theorem 3.5.

PROPOSITION 3.8. *Suppose that I is a non-zero ideal in $C(\Omega_{\mathrm{add}})$ and that I is invariant for the partial action θ of $\mathbb{Q} \rtimes \mathbb{Q}_+^*$. Then there is a finite set F of primes such that*

$$\Big(\prod_{p\in F}\prod_{l=0}^{p-1}(1 - 1_{(l,p)})\Big)\Big|_{\Omega_{\mathrm{add}}}$$

belongs to I.

Proof. Since I is non-zero, it contains a non-zero function f, and then $|f|^2 = ff^*$ is a non-negative function in I. Since f is continuous, there exist $\epsilon > 0$ and an open set $U \subset \Omega_{\mathrm{add}}$ such that $|f|^2 > \epsilon$ on U. By Lemma 3.7, there exist $(k, a) \in \mathbb{N} \rtimes \mathbb{N}^\times$ and a finite set F of primes such that $W((k, a), F) \cap \Omega_{\mathrm{add}} \subset U$. Then $0 \le \epsilon\chi_{W((k,a),F)\cap\Omega_{\mathrm{add}}} \le |f|^2$, and, since I is hereditary, we deduce that $\chi_{W((k,a),F)\cap\Omega_{\mathrm{add}}}$ belongs to I. Since I is invariant under the partial action of $\mathbb{Q} \rtimes \mathbb{Q}^*_+$ and

$$\chi_{W((k,a),F)} = \prod_{p\in F}\prod_{l=0}^{p-1}(1_{(k,a)} - 1_{(k,a)(l,p)}) = \theta_{(k,a)}\Bigg(\prod_{p\in F}\prod_{l=0}^{p-1}(1 - 1_{(l,p)})\Bigg),$$

applying $\theta_{(k,a)^{-1}}$ gives the result. □

Proof of Theorem 3.5. Example 3.9 shows that there are families S and $\{V_p : p \in \mathcal{P}\}$ satisfying (T1)–(T3), (T5), (Q6) and equation (3.3), and thus equation (3.3) must be satisfied in the universal algebra $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ and in any faithful representation of it.

For the converse, we use Proposition 3.4 to view $\pi_{S,V}$ as a representation of the partial crossed product $C(\Omega_{\mathrm{add}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}^*_+)$. By Proposition 3.6, the partial action on Ω_{add} is topologically free. Since $\mathbb{Q} \rtimes \mathbb{Q}^*_+$ is amenable, the reduced and full partial crossed products coincide. Thus [**11**, Theorem 2.6] implies that $\pi_{S,V}$ is faithful on

$$C(\Omega_{\mathrm{add}}) \rtimes_r (\mathbb{Q} \rtimes \mathbb{Q}^*_+) = C(\Omega_{\mathrm{add}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}^*_+)$$

if and only if it is faithful on $C(\Omega_{\mathrm{add}})$.

Suppose that $\pi_{S,V}$ is not faithful on $C(\Omega_{\mathrm{add}})$. We have $\pi_{S,V}(1_{(l,p)}) = S^l V_p V_p^* S^{*l}$ for each $p \in \mathcal{P}$ and $0 \le l \le p-1$. Since $\ker(\pi_{S,V}|_{C(\Omega_{\mathrm{add}})})$ is an invariant ideal in $C(\Omega_{\mathrm{add}})$, Proposition 3.8 gives a finite set F of primes such that

$$0 = \pi_{S,V}\Bigg(\prod_{p\in F}\prod_{l=0}^{p-1}(1 - 1_{(l,p)})\Bigg) = \prod_{p\in F}\prod_{l=0}^{p-1}(1 - S^l V_p V_p^* S^{*l}).$$

But this contradicts the hypothesis (3.3). So $\pi_{S,V}$ is faithful on $C(\Omega_{\mathrm{add}})$, and hence also on $C(\Omega_{\mathrm{add}}) \rtimes (\mathbb{Q} \rtimes \mathbb{Q}^*_+)$. □

Example 3.9. Define S on $\ell^2(\mathbb{Z} \rtimes \mathbb{N}^\times)$ by $Se_{(m,a)} = e_{(m+1,a)}$ and, for each $p \in \mathcal{P}$, define V_p on $\ell^2(\mathbb{Z} \rtimes \mathbb{N}^\times)$ by $V_p e_{(m,a)} = e_{(pm,pa)}$. Routine calculations on basis vectors show that the isometries S and $\{V_p\}$ generate the Toeplitz algebra of $\mathbb{Z} \rtimes \mathbb{N}^\times$, and that they satisfy (T1)–(T3), (T5) and (Q6). Equation (3.3) holds because $S^l V_p V_p^* S^{*l} e_{(0,1)} = 0$ for all l and p, so Theorem 3.5 implies that $\pi_{S,V}$ is faithful on $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$. Thus $\pi_{S,V}$ is an isomorphism of $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ onto $\mathcal{T}(\mathbb{Z} \rtimes \mathbb{N}^\times)$.

4. *KMS states on the boundary quotients of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$*

We now study the dynamics $\sigma : \mathbb{R} \to \mathrm{Aut}\,\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ characterised, using the presentation of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ as $C^*(s, v_p)$, by $\sigma_t(s) = s$ and $\sigma_t(v_p) = p^{it} v_p$. We consider the following diagram of quotient maps:

$$\begin{tikzcd} & \mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times) \arrow[dl, "q_{\mathrm{add}}"'] \arrow[dr, "q_{\mathrm{mult}}"] & \\ \mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times) \arrow[dr] & & \mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times) \arrow[dl] \\ & \mathcal{Q}_{\mathbb{N}} & \end{tikzcd}$$

where q_{mult} is the quotient map by the relations $1 = \sum_{k=0}^{p-1} (s^k v_p)(s^k v_p)^*$ and q_{add} is the one by the relation $1 = ss^*$. Since these relations are invariant under σ, the quotients carry induced dynamics (all of which we will denote by σ).

Cuntz proved in [**5**] that $(\mathcal{Q}_{\mathbb{N}}, \sigma)$ has a unique KMS state, and that this state has inverse temperature 1. Laca and Raeburn proved in [**17**, Lemma 10.4] that every KMS state of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ vanishes on the ideal generated by $1 - ss^*$, and hence factors through the quotient map q_{add} to give a KMS state of the additive boundary quotient $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$. Thus, parts (1), (2) and (3) of [**17**, Theorem 7.1] describe the KMS states of $(\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$, and imply in particular that this system has a phase transition at inverse temperature $\beta = 2$.

As in [**3, 17**], we distinguish between the KMS_∞ states, which are by definition weak* limits of KMS_β states as $\beta \to \infty$, and the ground states, which are by definition the states ϕ such that $z \mapsto \phi(c\sigma_z(d))$ is bounded on the upper half-plane for every pair of analytic elements c, d.

PROPOSITION 4.1. *Every* KMS_β *state of* $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ *factors through* q_{add}. *A ground state of* $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ *factors through* q_{add} *if and only if it is a* KMS_∞ *state.*

Proof. For $\beta < \infty$, [**17**, Lemma 10.4] implies that all the KMS_β states vanish on the ideal generated by $1 - ss^*$, which by Proposition 3.4 is the kernel of q_{add}. Thus, all these states factor through q_{add}, and so does any weak* limit of such states. This proves the first assertion and the 'if' direction of the second assertion.

Suppose that ϕ is a ground state of $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ which factors through q_{add}. Then ϕ vanishes on the ideal in $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ generated by $1 - ss^*$, and hence $\phi|_{C^*(s)}$ vanishes on the ideal J in $C^*(s)$ generated by $1 - ss^*$. Thus $\phi|_{C^*(s)}$ factors through a state of $C^*(s)/J$, which is isomorphic to $C(\mathbb{T})$. Thus there is a probability measure μ on $\mathbb{T}$ such that $\phi(s^m s^{*n}) = \int_{\mathbb{T}} z^{m-n}\, d\mu(z)$. But then $\phi|_{C^*(s)}$ coincides with the restriction of the KMS_∞ state $\psi_{\infty,\mu}$ (see [**17**, §9, Proof of Theorem 7.1(4)]). The equality $\phi = \psi_{\infty,\mu}$ now follows from [**17**, Lemma 8.4, Formula (8.6)]. □

COROLLARY 4.2. *Every ground state of* $(\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ *is a* KMS_∞ *state.*

Proof. Suppose that ϕ is a ground state of $(\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$. Then $\phi \circ q_{\mathrm{add}}$ is a ground state of $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ which factors through q_{add}, and hence by the second assertion in Proposition 4.1 is a KMS_∞ state of $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$. Thus $\phi \circ q_{\mathrm{add}}$ is the weak* limit of a sequence $\{\psi_n\}$ of KMS_{β_n} states. Now the first assertion of Proposition 4.1 says that each $\psi_n = \phi_n \circ q_{\text{add}}$ for a unique state ϕ_n of $\mathcal{T}_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$, and the states ϕ_n are KMS$_{\beta_n}$ states which converge weak* to ϕ. In other words, ϕ is a KMS$_\infty$ state. □

Theorem 7.1(4) of [**17**] says that the map $\phi \mapsto \phi|_{C^*(s)}$ is an affine homeomorphism of the set of ground states of $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ onto the state space of $C^*(s) = \mathcal{T}(\mathbb{N})$, and hence there are many ground states of $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ which do not vanish on the ideal generated by $1 - ss^*$. Thus there are many more ground states than KMS$_\infty$ states. We interpret this as saying that $(\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ exhibits a second phase transition at infinity. Corollary 4.2, on the other hand, says that $(\mathcal{T}_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ does not have a phase transition at infinity.

Since a KMS$_\beta$ state ϕ satisfies $\phi(s^k v_p v_p^* s^{*k}) = p^{-\beta}$ (see [**17**, Lemma 8.3]), it satisfies

$$\phi\left(\sum_{k=0}^{p-1} s^k v_p v_p^* s^{*k}\right) = pp^{-\beta} = p^{1-\beta}.$$

Since $q_{\text{mult}}(\sum_{k=0}^{p-1} s^k v_p v_p^* s^{*k}) = 1$, this means that no KMS$_\beta$ state with $\beta > 1$ can factor through the quotient map q_{mult}, and the system $(\mathcal{T}_{\text{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ has only the one KMS$_1$ state lifted from $(\mathcal{Q}_\mathbb{N}, \sigma)$. Lemma 8.4 of [**17**] implies that every ground state ϕ satisfies $\phi(s^k v_p v_p^* s^{*k}) = p^{-\beta} = 0$, and hence does not factor through q_{mult}. Thus $(\mathcal{T}_{\text{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times), \sigma)$ does not have any ground states.

5. *Cuntz's $\mathcal{Q}_\mathbb{N}$ as an Exel crossed product*

For each $a \in \mathbb{N}^\times$ and $f \in C(\mathbb{T})$, define $\alpha_a(f)(z) = f(z^a)$. Then α_a is an endomorphism of $C(\mathbb{T})$, and the function $L_a : C(\mathbb{T}) \to C(\mathbb{T})$ defined by $L_a(f)(z) = a^{-1} \sum_{w^a = z} f(w)$ is a *transfer operator* for α_a, in the sense that L_a is a positive linear map from $C(\mathbb{T})$ to $C(\mathbb{T})$ satisfying the *transfer-operator identity*

$$L(\alpha_a(f)g) = f L_a(g). \tag{5.1}$$

We have $\alpha_a \alpha_b = \alpha_{ab}$ and $L_b L_a = L_{ab}$, and hence the $(C(\mathbb{T}), \alpha_a, L_a)$ combine to give an *Exel system* $(C(\mathbb{T}), \mathbb{N}^\times, \alpha, L)$ of the sort studied by Larsen in [**18**] (see [**18**, Proposition 5.1]). In this section, we prove that our boundary quotient $\mathcal{T}_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ and Cuntz's $\mathcal{Q}_\mathbb{N}$ are C^*-algebras naturally associated to the Exel system $(C(\mathbb{T}), \mathbb{N}^\times, \alpha, L)$. Before making this precise, we need to review Larsen's construction.

Suppose that (A, P, α, L) is an Exel system as in [**18**]. We make the simplifying assumptions that A is unital and that $\alpha_x(1) = 1 = L_x(1)$ for $x \in P$ (which hold for our system above). For each $x \in P$, we make $A_{L_x} := A$ into a bimodule over A by $a \cdot m \cdot b = am\alpha_x(b)$ for $a, b \in A$ and $m \in A_{L_x}$, we define a pre-inner product on A_{L_x} by $\langle m, n \rangle_{L_x} = L_x(m^* n)$ and we complete A_{L_x} to get a Hilbert bimodule M_{L_x} (see [**8**] or [**2**, §3]). To help keep the copies of A straight, we write $q_x(a)$ for the image of $a \in A_{L_x}$ in M_{L_x}, and $\phi_x : A \to \mathcal{L}(M_{L_x})$ for the homomorphism implementing the left action of A on M_{L_x}. These bimodules combine to give a product system in the sense of Fowler [**13**, §2]: the maps $q_x(a) \otimes q_y(b) \mapsto q_{xy}(a\alpha_x(b))$ extend to bimodule isomorphisms of $M_{L_x} \otimes_A M_{L_y}$ onto $M_{L_{xy}}$, and these isomorphisms give the disjoint union $M_L := \bigsqcup M_{L_x}$ the structure of a semigroup. The bimodule M_{L_e} over the identity e of P is the bimodule ${}_A A_A$ in which all the operations are given by multiplication in A, and the products of $a \in M_{L_e}$ and $m \in M_{L_x}$ are given by the module actions.

A *representation*† ψ of a product system M in a C^*-algebra B consists of linear maps $\psi_x : M_x \to B$ such that $\psi_A := \psi_e$ is a homomorphism of C^*-algebras, $\psi_x(m)\psi_y(n) = \psi_{xy}(mn)$ and $\psi_A(\langle q_x(a), q_x(b)\rangle_x) = \psi_x(q_x(a))^*\psi_x(q_x(b))$. We are interested in two special classes of representations which reflect extra properties of the setup.

Suppose that ψ is a representation of $M = \bigsqcup M_x$ in B. For each $x \in P$, there is a representation $\psi^{(x)}$ of $\mathcal{K}(M_x)$ in B such that $\psi^{(x)}(\Theta_{m,n}) = \psi_x(m)\psi_x(n)^*$ for $m, n \in M_x$. Following Fowler [**13**], we say that ψ is *Cuntz–Pimsner covariant* if

$$\psi_A(a) = \psi^{(x)}(\phi_x(a)) \quad \text{for every } a \in A,\, x \in P \text{ such that } \phi_x(a) \in \mathcal{K}(M_x).$$

The *Cuntz–Pimsner algebra* $\mathcal{O}(M)$ of the product system M is generated by a universal Cuntz–Pimsner covariant representation j_M of M in $\mathcal{O}(M)$.

Suppose that ψ is a representation of M in B, and P is the positive cone in a quasi-lattice ordered group (G, P). If $x, y \in P$ satisfy $x \le y$, then $y = xp$ for some $p \in P$, the product structure gives an isomorphism of $M_x \otimes_A M_p$ onto M_y and we use this isomorphism to define a homomorphism $\iota_x^y : \mathcal{L}(M_x) \to \mathcal{L}(M_y)$ such that $\iota_x^y(T)(mn) = (Tm)n$ for $m \in M_x$, $n \in M_p$. Suppose that the product system is compactly aligned in the sense that

$$R \in \mathcal{K}(M_x) \text{ and } T \in \mathcal{K}(M_y) \Longrightarrow \iota_x^{x\vee y}(R)\iota_y^{x\vee y}(T) \in \mathcal{K}(M_{x\vee y});$$

we then say that ψ is *Nica covariant* if

$$\psi^{(x)}(R)\psi^{(y)}(T) = \begin{cases} \psi^{(x\vee y)}(\iota_x^{x\vee y}(R)\iota_y^{x\vee y}(T)) & \text{if } x \vee y < \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

The *Nica–Toeplitz algebra* $\mathcal{NT}(M)$ is generated by a universal Nica-covariant representation i_M. (Fowler calls this $\mathcal{T}_{\mathrm{cov}}(M)$—see Remark 5.3 below.) As a point of notation, we will write $i_{M,x}$ instead of i_{M_x} or $i_{M_{L_x}}$.

The relationship between $\mathcal{NT}(M)$ and $\mathcal{O}(M)$ is a bit murky, and there has been some debate about whether Fowler found the optimal definition of $\mathcal{O}(M)$. For example, Sims and Yeend argue convincingly that Nica covariance should have been built into the definition of $\mathcal{O}(M)$, so that $\mathcal{O}(M)$ is a quotient of $\mathcal{NT}(M)$ [**22**]. However, the systems of interest to us have extra features which make the debate irrelevant.

Example 5.1. Consider the Exel system $(C(\mathbb{T}), \mathbb{N}^\times, \alpha, L)$ described at the beginning of the section, and Larsen's product system M_L over $\mathbb{N}^\times$. We know from [**19**, Lemma 3.3] that each $C(\mathbb{T})_{L_a}$ is already complete in the inner product defined by L_a, so $M_{L_a} = \{q_a(f) : f \in C(\mathbb{T})\}$. It follows from work of Packer and Rieffel [**21**, Proposition 1] that if ι is the usual generator $\iota : z \mapsto z$, then $\{q_a(\iota^k) : 0 \le k < a\}$ is an orthonormal basis for M_{L_a} (see [**10**, Lemma 2.6]). The reconstruction formula for this basis says that the identity operator 1 on M_{L_a} is the finite-rank operator $\sum_{k=0}^{a-1} \Theta_{q_a(\iota^k),q_a(\iota^k)}$, and hence every adjointable operator $T = \sum_{k=0}^{a-1} \Theta_{q_a(T(\iota^k)),q_a(\iota^k)}$ also has finite rank. In particular, every $\phi(f)$ is compact, and the product system is compactly aligned (by [**13**, Proposition 5.8]). (Essentially the same product system is studied in [**23**] as an example of a topological k-graph with $k = \infty$.)

† These were called 'Toeplitz representations' in [**13**], and we have deliberately changed the name for the reasons we discuss in Remark 5.3.

The semigroup $\mathbb{N}^\times$, which is the positive cone in $(\mathbb{Q}_+^*, \mathbb{N}^\times)$, also has some particularly nice properties. It is not only quasi-lattice ordered, it is lattice ordered in the sense that *every* pair $a, b \in \mathbb{N}^\times$ has a least upper bound $a \vee b = \mathrm{lcm}(a, b)$.

When the left action of A on each M_x is by compact operators and the semigroup is lattice ordered, [**13**, Theorem 6.3] implies that the Cuntz–Pimsner algebra $\mathcal{O}(M)$ is a quotient of $\mathcal{NT}(M)$. We write Q for the quotient map, and identify $Q \circ i_M$ with the universal Cuntz–Pimsner covariant representation j_M. Larsen works explicitly with abelian semigroups, for which the notions of quasi-lattice ordered and lattice ordered coincide. She does not explicitly assume that $\phi_x(A) \subset \mathcal{K}(M_x)$, but this is true in all her examples.

Larsen defines her crossed product $A \rtimes_{\alpha,L} P$ following Exel's path in [**8**], and then proves that, under our hypotheses, it is isomorphic to the Cuntz–Pimsner algebra $\mathcal{O}(M_L)$ [**18**, Proposition 4.3]. We define $i_P : P \to \mathcal{NT}(M_L)$ by $i_P(a) = i_{M,a}(q_a(1))$, and then define the *Nica–Toeplitz algebra* $\mathcal{NT}(A, P, \alpha, L)$ of the Exel system to be the triple $(\mathcal{NT}(M_L), i_M, i_P)$. Similarly, we define $j_P : P \to \mathcal{O}(M_L)$ by $j_P(a) = j_{M,a}(q_a(1))$, and then $(\mathcal{O}(M_L), j_M, j_P)$ is the crossed product $A \rtimes_{\alpha,L} P$ of the Exel system.

THEOREM 5.2. *Let $(C(\mathbb{T}), \mathbb{N}^\times, \alpha, L)$ be the Exel system discussed at the beginning of §5 and in Example 5.1. Then there are isomorphisms:*

(a) *ϕ_1 of $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ onto $\mathcal{NT}(M_L)$ such that $\phi_1(s) = i_{C(\mathbb{T})}(\iota)$ and $\phi_1(v_p) = i_{\mathbb{N}^\times}(p)$ for p prime; and*

(b) *ϕ_2 of $\mathcal{Q}_\mathbb{N}$ onto $C(\mathbb{T}) \rtimes_{\alpha,L} \mathbb{N}^\times = \mathcal{O}(M_L)$ such that $\phi_2(s) = j_{C(\mathbb{T})}(\iota)$ and $\phi_2(v_p) = j_{\mathbb{N}^\times}(p)$ for p prime.*

Larsen has told us that she, Hong and Szymanski have obtained Theorem 5.2 by other methods.

Proof. We define $S := i_{C(\mathbb{T})}(\iota)$ and $V_p := i_{\mathbb{N}^\times}(p) = i_{M,p}(q_p(1))$ for p prime, and prove that (S, V_p) satisfy the relations (T1)–(T3), (T5) and (Q6) in the presentation of $\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ of Proposition 3.3. Since $1 \cdot q_a(f) = q_a(f)$ and $q_a(f) \cdot 1 = q_a(f\alpha_a(1)) = q_a(f)$ for all $f \in C(\mathbb{T})$ and $a \in \mathbb{N}^\times$, $i_{C(\mathbb{T})}(1)$ is an identity for $C^*(i_{M_L}(m), i_{C(\mathbb{T})}(f)) = \mathcal{NT}(M_L)$, and $i_{C(\mathbb{T})}$ is unital. This implies, first, that $V_p^* V_p = i_{C(\mathbb{T})}(\langle q_p(1), q_p(1)\rangle) = i_{C(\mathbb{T})}(1) = 1$, so that each V_p is an isometry, and, second, that $S = i_{C(\mathbb{T})}(\iota)$ is unitary, which is (Q6).

A quick calculation shows that $\alpha_p(\iota) = \iota^p$, and hence

$$\begin{aligned} V_p S &= i_{M,p}(q_p(1)) i_{C(\mathbb{T})}(\iota) = i_{M,p}(q_p(1) \cdot \iota) = i_{M,p}(q_p(\alpha_p(\iota))) \\ &= i_{M,p}(q_p(\iota^p)) = i_{C(\mathbb{T})}(\iota^p) i_{M,p}(q_p(1)) = S^p V_p, \end{aligned}$$

which is (T1). For distinct primes p and r, we have $q_p(1)q_r(1) = q_{pr}(1\alpha_p(1)) = q_{pr}(1)$, and hence

$$\begin{aligned} V_p V_r &= i_{M,p}(q_p(1)) i_{M,r}(q_r(1)) = i_{M,pr}(q_p(1)q_r(1)) \\ &= i_{M,pr}(q_{pr}(1)) = i_{M,rp}(q_{rp}(1)) = V_r V_p, \end{aligned}$$

which is (T2). For (T5), we recall that $\{q_p(\iota^k) : 0 \le k < p\}$ is an orthonormal basis for M_{L_p}. Thus, for k satisfying $1 \le k < p$, we have

$$\begin{aligned} V_p^* S^k V_p &= i_{M,p}(q_p(1))^* i_{C(\mathbb{T})}(q_p(\iota^k)) i_{M,p}(q_p(1)) \\ &= i_{M,p}(q_p(1))^* i_{M,p}(q_p(\iota^k)) \\ &= i_{C(\mathbb{T})}(\langle q_p(1), q_p(\iota^k)\rangle_{L_p}) = 0, \end{aligned}$$

which is (T5).

To check (T3), we need to invoke Nica covariance of the representation i_{M_L}. Suppose that p and r are distinct primes. Then $p \vee r = pr$, and Nica covariance says that

$$i_{M_L}^{(p)}(R) i_{M_L}^{(r)}(T) = i_{M_L}^{(pr)}(\iota_p^{pr}(R)\iota_r^{pr}(T)) \quad \text{for } R \in \mathcal{K}(M_{L_p}) \text{ and } T \in \mathcal{K}(M_{L_r}). \tag{5.3}$$

We aim to apply this with $R = \Theta_{q_p(1),q_p(1)}$ and $T = \Theta_{q_r(1),q_r(1)}$, and we need to compute the product appearing on the right-hand side of equation (5.3). Since the endomorphisms α_p and α_r are unital, we can realise each $q_{pr}(f) \in M_{L_{pr}}$ as a product $q_p(f)q_r(1)$ or as $q_r(f)q_p(1)$. Thus, recalling that $\iota_p^{pr}(R)(xy) = (Rx)y$ for $x \in M_{L_p}$ and $y \in M_{L_r}$, we have

$$\begin{aligned} &\iota_p^{pr}(\Theta_{q_p(1),q_p(1)})\iota_r^{pr}(\Theta_{q_r(1),q_r(1)})(q_{pr}(f)) \\ &\quad = \iota_p^{pr}(\Theta_{q_p(1),q_p(1)})(\Theta_{q_r(1),q_r(1)}(q_r(f))q_p(1)) \\ &\quad = \iota_p^{pr}(\Theta_{q_p(1),q_p(1)})((q_r(1)\cdot\langle q_r(1), q_r(f)\rangle)q_p(1)) \\ &\quad = \iota_p^{pr}(\Theta_{q_p(1),q_p(1)})(q_r(\alpha_r(L_r(f)))q_p(1)) \\ &\quad = \iota_p^{pr}(\Theta_{q_p(1),q_p(1)})(q_p(\alpha_r(L_r(f)))q_r(1)) \\ &\quad = q_p(\alpha_p L_p \alpha_r L_r(f))q_r(1) \\ &\quad = q_{pr}(\alpha_p L_p \alpha_r L_r(f)). \end{aligned} \tag{5.4}$$

Since p and r are distinct primes, and in particular coprime, the sets $\{w \in \mathbb{T} : w^p = z^r\}$ and $\{v^r \in \mathbb{T} : v^p = z\}$ are the same, and a calculation using this shows that $L_p\alpha_r = \alpha_r L_p$. Thus

$$\begin{aligned} \iota_p^{pr}(\Theta_{q_p(1),q_p(1)})\iota_r^{pr}(\Theta_{q_r(1),q_r(1)})(q_{pr}(f)) &= q_{pr}(\alpha_{pr}L_{pr}(f)) \\ &= \Theta_{q_{pr}(1),q_{pr}(1)}(q_{pr}(f)). \end{aligned} \tag{5.5}$$

Now Nica covariance in the form of equation (5.3) implies that

$$\begin{aligned} V_p V_p^* V_r V_r^* &= i_{M_L}^{(p)}(\Theta_{q_p(1),q_p(1)}) i_{M_L}^{(r)}(\Theta_{q_r(1),q_r(1)}) \\ &= i_{M_L}^{(pr)}(\iota_p^{pr}(\Theta_{q_p(1),q_p(1)})\iota_r^{pr}(\Theta_{q_r(1),q_r(1)})) \\ &= i_{M_L}^{(pr)}(\Theta_{q_{pr}(1),q_{pr}(1)}) \\ &= V_{pr}V_{pr}^* = V_{pr}V_{rp}^* = V_p V_r V_p^* V_r^*, \end{aligned}$$

which implies (T3) because V_p and V_r are isometries.

Thus (S, V_p) satisfy the relations (T1)–(T3), (T5) and (Q6), and Proposition 3.4 gives us a homomorphism $\pi_{S,V} : \mathcal{T}_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times) \to \mathcal{NT}(M_L)$ taking (s, v_p) to (S, V_p). To prove that $\pi_{S,V}$ is faithful, we verify that it satisfies the hypothesis (3.3) of Theorem 3.5. For p prime and $0 \le k < p$, we have

$$\begin{aligned} 1 - S^k V_p V_p^* S^{*k} &= 1 - i_{C(\mathbb{T})}(\iota)^k i_{M,p}(q_p(1)) i_{M,p}(q_p(1))^* i_{C(\mathbb{T})}(\iota)^{*k} \\ &= 1 - i_{M,p}(q_p(\iota^k)) i_{M,p}(q_p(\iota^k))^*. \end{aligned}$$

Let $l : M_L \to \mathcal{L}(\mathcal{F}(M_L))$ be the Fock representation of [**13**, §2], so that for $m \in M_{L_p}$, $l_p(m)$ is multiplication by m in the sense of the product system. Proposition 2.8 of [**13**] gives a homomorphism $l_* : \mathcal{NT}(M_L) \to \mathcal{L}(\mathcal{F}(M_L))$ such that $l_* \circ i_{M_L} = l$. In particular, we have

$$l_*(1 - S^k V_p V_p^* S^{*k}) = l_{C(\mathbb{T})}(1) - l_p(q_p(\iota^k)) l_p(q_p(\iota^k))^*.$$

For each $a \in \mathbb{N}^\times$, $l_p(q_p(\iota^k)) : M_{L_a} \to M_{L_{ap}}$, and the adjoint $l_p(q_p(\iota^k))^*$ maps $M_{L_{ap}}$ to M_{L_a} and vanishes on M_{L_b} when $b \notin p\mathbb{N}^\times$. In particular, $l_p(q_p(\iota^k))^*$ is zero on $M_{L_1} = C(\mathbb{T})$. So $l_*(1 - S^k V_p V_p^* S^{*k})$ is the identity on M_{L_1}, and so is each finite product

$$l_*\Bigg(\prod_{p\in F}\prod_{k=0}^{p-1}(1 - S^k V_p V_p^* S^{*k})\Bigg) = \prod_{p\in F}\prod_{k=0}^{p-1} l_*(1 - S^k V_p V_p^* S^{*k}).$$

This implies in particular that $\prod_{p\in F}\prod_{k=0}^{p-1}(1 - S^k V_p V_p^* S^{*k}) \neq 0$ for any finite subset F of primes. Thus Theorem 3.5 implies that $\pi_{S,V}$ is faithful.

To see that $\pi_{S,V}$ is surjective, we note first that $i_{C(\mathbb{T})}(\iota^k) = S^k$ belongs to the range of $\pi_{S,V}$, and hence so does $i_{C(\mathbb{T})}(f)$ for every $f \in C(\mathbb{T})$. Next consider a typical element $q_a(f)$ of M_{L_a}. From the definition of multiplication in the product system (and remembering that the α_b are unital), we have

$$q_a(f) = f \cdot q_a(1) = f \cdot \Bigg(\prod_{p|a} q_{p^{e_p(a)}}(1)\Bigg) = f \cdot \Bigg(\prod_{p|a} q_p(1)^{e_p(a)}\Bigg),$$

and hence $i_{M,a}(q_a(f)) = i_{C(\mathbb{T})}(f)\prod_{p|a} V_p^{e_p(a)}$ belongs to the range of $\pi_{S,V}$. Thus the range of $\pi_{S,V}$ contains all the generators of $\mathcal{NT}(M_L)$, and $\pi_{S,V}$ is surjective. Now $\phi_1 := \pi_{S,V}$ has the properties described in part (a).

For (b), we consider the composition $Q \circ \phi_1$ with the quotient map $Q : \mathcal{NT}(M_L) \to C(\mathbb{T}) \rtimes_{\alpha,L} \mathbb{N}^\times$. Since $(Q \circ i_{M_L}, Q \circ i_{C(\mathbb{T})})$ is the universal representation $(j_{M_L}, j_{C(\mathbb{T})})$ generating $C(\mathbb{T}) \rtimes_{\alpha,L} \mathbb{N}^\times$, $Q \circ \phi_1$ maps s into $j_{C(\mathbb{T})}(\iota)$ and v_p into $j_{\mathbb{N}^\times}(p) = j_{M,p}(q_p(1))$. For each prime p, the pair $(j_{M,p}, j_{C(\mathbb{T})})$ is Cuntz–Pimsner covariant and, since $\{q_p(\iota^k) : 0 \le k < p\}$ is an orthonormal basis for M_{L_p},

$$\begin{aligned}
1 = j_{C(\mathbb{T})}(1) = j_{M_L}^{(p)}(\phi_p(1)) &= j_{M_L}^{(p)}\Bigg(\sum_{k=0}^{p-1}\Theta_{q_p(\iota^k),q_p(\iota^k)}\Bigg) \\
&= \sum_{k=0}^{p-1} j_{M,p}(q_p(\iota^k)) j_{M,p}(q_p(\iota^k))^* \\
&= \sum_{k=0}^{p-1} j_{C(\mathbb{T})}(\iota)^k j_{M,p}(q_p(1))(j_{C(\mathbb{T})}(\iota)^k j_{M,p}(q_p(1)))^* \\
&= \sum_{k=0}^{p-1} Q \circ \phi_1(s^k v_p (s^k v_p)^*).
\end{aligned} \tag{5.6}$$

Since the relations $1 = \sum_{k=0}^{p-1} s^k v_p (s^k v_p)^*$ are the extra relations (Q5) satisfied in the quotient $\mathcal{Q}_\mathbb{N}$ of $\mathcal{T}_{\text{add}}(\mathbb{N} \rtimes \mathbb{N}^\times)$, this calculation implies that $Q \circ \phi_1$ factors through a surjection ϕ_2 of $\mathcal{Q}_\mathbb{N}$ onto $C(\mathbb{T}) \rtimes_{\alpha,L} \mathbb{N} = \mathcal{O}(M_L)$; ϕ_2 is an isomorphism because $\mathcal{Q}_\mathbb{N}$ is simple. □

Remark 5.3. Fowler also associated a 'Toeplitz algebra' $\mathcal{T}(M)$ to each product system M, which is universal for representations that are not necessarily Nica covariant. Larsen then analogously defines the Toeplitz algebra $\mathcal{T}(A, P, \alpha, L)$ to be Fowler's $\mathcal{T}(M_L)$. This algebra is in general substantially larger than $\mathcal{NT}(M_L)$. For example, consider the trivial system $(\mathbb{C}, \mathbb{N}^2, \mathrm{id}, \mathrm{id})$. The associated product system M_{id} of bimodules over $\mathbb{C}$ is also trivial. Any pair of commuting isometries V, W gives a representation of the product system such that $\psi_{(m,n)}(z) = zV^m W^n$, and hence a representation of $\mathcal{T}(\mathbb{C}, \mathbb{N}^2, \mathrm{id}, \mathrm{id})$ taking $i_{\mathbb{N}^2}(m, n)$ to $V^m W^n$. But $\mathcal{NT}(\mathbb{C}, \mathbb{N}^2, \mathrm{id}, \mathrm{id})$ is universal for $*$-commuting isometries, so it is $\mathcal{NT}(\mathbb{C}, \mathbb{N}^2, \mathrm{id}, \mathrm{id})$ rather than $\mathcal{T}(\mathbb{C}, \mathbb{N}^2, \mathrm{id}, \mathrm{id})$ which is isomorphic to the Toeplitz algebra $\mathcal{T}(\mathbb{N}^2) \subset B(l^2(\mathbb{N}^2))$.

We think it is unfortunate that Fowler chose to call his $\mathcal{T}(M)$ the Toeplitz algebra of the system. Nica's covariance relation for isometric representations of P is a property of the Toeplitz representation on $l^2(P)$, which, under Nica's amenability hypothesis, characterises the Toeplitz algebra among C^*-algebras generated by isometric representations. The analogue of the Toeplitz representation for a product system is the Fock representation, and it is automatically Nica covariant in Fowler's sense [**13**, Lemma 5.3]. So $\mathcal{NT}(M)$ might have been a better choice for *the* Toeplitz algebra of M.

6. *The multiplicative boundary quotient as an Exel crossed product*

We know from [**15**] that for the Toeplitz algebra of a single bimodule (or, equivalently, a product system over $\mathbb{N}$), the phase transition of ground states is indexed by the states of the coefficient algebra. Our results on the additive boundary quotient and $\mathcal{Q}_{\mathbb{N}}$, viewed as algebras associated to the product system of bimodules over $C(\mathbb{T})$, suggest that the phenomenon in [**15**] may hold for product systems over other semigroups. Since the ground states of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ are indexed by the states of the usual Toeplitz algebra $\mathcal{T} = \mathcal{T}(\mathbb{N})$, we were led to conjecture that $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ and $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ might be realisable as the algebras associated to a product system of bimodules with coefficients in $\mathcal{T}$. In this section, we confirm this conjecture. We find it intriguing that we can apparently get useful hints about the structure of an algebra from an analysis of its KMS states.

In this section, S denotes the unilateral shift on $l^2(\mathbb{N})$ and V is the isometric representation of $\mathbb{N}^\times$ on $l^2(\mathbb{N})$ characterised in terms of the usual basis by $V_a e_n = e_{an}$. We recall that $\mathcal{T} = \overline{\mathrm{span}}\{S^m S^{*n} : m, n \in \mathbb{N}\}$.

LEMMA 6.1. *There is an Exel system $(\mathcal{T}, \mathbb{N}^\times, \beta, K)$ such that $\beta_a(S) = S^a$ and $K_a(T) = V_a^* T V_a$, and K_a is characterised by*

$$K_a(S^n S^j S^{*j}) = \begin{cases} 0 & \text{if } a \text{ does not divide } n, \\ S^{a^{-1}n} S^i S^{*i} & \text{if } a|n \text{ and } i \in \mathbb{N} \text{ satisfies } j \in (a(i-1), ai]. \end{cases} \tag{6.1}$$

Proof. Since V_a is an isometry, $\mathrm{Ad}\, V_a^*$ is a bounded linear operator on $B(l^2(\mathbb{N}))$, and it is trivially positive and unital. We claim that $\mathrm{Ad}\, V_a^*$ maps $\mathcal{T}$ into $\mathcal{T}$. Since $\mathrm{Ad}\, V_a^*$ is continuous and adjoint preserving, it suffices to show that every $\mathrm{Ad}\, V_a^*(S^n S^j S^{*j})$ is in $\mathcal{T}$. We now take $k \in \mathbb{N}$ and compute:

$$\mathrm{Ad}\, V_a^*(S^n S^j S^{*j}) e_k = V_a^* S^n S^j S^{*j} V_a e_k = \begin{cases} e_{k+a^{-1}n} & \text{if } j \le ak \text{ and } a|n, \\ 0 & \text{else.} \end{cases}$$

Now let $i \in \mathbb{N}$ be the unique integer such that $j \in (a(i-1), ai]$. Then

$$S^i S^{*i} e_k = \begin{cases} e_k & \text{if } k \geq i, \\ 0 & \text{else,} \end{cases} = \begin{cases} e_k & \text{if } j \leq ak, \\ 0 & \text{else.} \end{cases}$$

So, if $a|n$, then $\operatorname{Ad} V_a^*(S^n S^j S^{*j})e_k = S^{a^{-1}n} S^i S^{*i} e_k$, and $\operatorname{Ad} V_a^*(S^n S^j S^{*j}) = S^{a^{-1}n} S^i S^{*i}$ belongs to $\mathcal{T}$. Now $K_a := \operatorname{Ad} V_a^*|_{\mathcal{T}}$ has the required properties.

To establish the transfer-operator identity, we check that $\beta_a(S^n S^{*m})V_a = V_a S^n S^{*m}$, and then

$$\begin{aligned} K_a(\beta_a(S^m S^{*n})T) &= V_a^* \beta_a(S^m S^{*n}) T V_a = (\beta_a(S^n S^{*m})V_a)^* T V_a \\ &= (V_a S^n S^{*m})^* T V_a = S^m S^{*n} K_a(T). \end{aligned}$$

It is easy to check that both K and β are multiplicative. $\square$

We now investigate the product system associated to the Exel system $(\mathcal{T}, \mathbb{N}^\times, \beta, K)$ of Lemma 6.1. For this product system, the canonical maps $q_a : \mathcal{T} \to M_{K_a}$ have non-trivial kernel, unlike those for the system M_L in the last section. Part (c) of Lemma 6.2 will be used in place of the identity $L_p \alpha_r = \alpha_r L_p$ used in the proof of Theorem 5.2 (we can see that $K_p \beta_r$ is not the same as $\beta_r K_p$ by applying them both to SS^*).

LEMMA 6.2.

(a) *For $j, n \in \mathbb{N}$ and $a \in \mathbb{N}^\times$, we have*

$$q_a(S^n S^j S^{*j}) = q_a(S^n \beta_a K_a(S^j S^{*j})).$$

(b) *For $a \in \mathbb{N}^\times$, we have $q_a(S^{*m}) = q_a(S^j S^{*j} S^{*m})$ whenever m and $m + j$ belong to the same $(a(i-1), ai]$.*

(c) *If p and r are distinct primes, then*

$$q_{pr}(\beta_p K_p \beta_r K_r(T)) = q_{pr}(\beta_{pr} K_{pr}(T)) \quad \textit{for all } T \in \mathcal{T}. \tag{6.2}$$

Proof. The hardest part is (c), and we will do it first. By linearity and continuity, it suffices to prove equation (6.2) for T of the form $S^n S^j S^{*j}$ or $S^j S^{*j} S^{*n}$. Both sides of equation (6.2) vanish unless $pr|n$, so we suppose that $n = prk$. First we consider $T = S^n S^j S^{*j}$. Note that $S^n = \beta_p \beta_r(S^k)$, so the transfer-operator identity implies that $\beta_p K_p \beta_r K_r(T) = S^n \beta_p K_p \beta_r K_r(S^j S^{*j})$ and $\beta_{pr} K_{pr}(T) = S^n \beta_{pr} K_{pr}(S^j S^{*j})$. For the next calculation, set $R := \beta_p K_p \beta_r K_r(S^j S^{*j}) - \beta_{pr} K_{pr}(S^j S^{*j})$. Then

$$\begin{aligned} (6.2) \iff &\langle q_{pr}(\beta_p K_p \beta_r K_r(T) - \beta_{pr} K_{pr}(T)), \\ &\qquad q_{pr}(\beta_p K_p \beta_r K_r(T) - \beta_{pr} K_{pr}(T))\rangle_{K_{pr}} = 0 \\ \iff &K_{pr}((\beta_p K_p \beta_r K_r(T) - \beta_{pr} K_{pr}(T))^* \\ &\qquad \times (\beta_p K_p \beta_r K_r(T) - \beta_{pr} K_{pr}(T))) = 0 \\ \iff &K_{pr}(R^* S^{*n} S^n R) = 0 \\ \iff &K_{pr}(R^* R) = 0. \end{aligned} \tag{6.3}$$

The formula (6.1) implies that both $\beta_p K_p \beta_r K_r(S^j S^{*j})$ and $\beta_{pr} K_{pr}(S^j S^{*j})$ have the form $S^i S^{*i}$, so R is the difference of two projections, one of which dominates the other. Thus, there is a projection P such that $R = \pm P$, and

$$\begin{aligned}(6.2) \iff K_{pr}(R^*R) = 0 \iff K_{pr}(P) = 0 \iff K_{pr}(R) = 0 \\ \iff K_{pr}(\beta_p K_p \beta_r K_r(S^j S^{*j})) = K_{pr}(\beta_{pr} K_{pr}(S^j S^{*j})).\end{aligned} \tag{6.4}$$

Since each K_a is unital, the transfer-operator identity (5.1) implies that $K_a \beta_a$ is the identity map. So

$$\begin{aligned}K_{pr}\beta_p K_p \beta_r K_r &= (K_r K_p)\beta_p K_p \beta_r K_r = K_r(K_p\beta_p)K_p\beta_r K_r = K_r K_p \beta_r K_r \\ &= K_p K_r \beta_r K_r = K_p K_r = K_{pr} = K_{pr}\beta_{pr}K_{pr},\end{aligned}$$

the right-hand side of (6.4) is true, and we have proved equation (6.2) for $T = S^n S^j S^{*j}$.

Next we consider $T = S^j S^{*j} S^{*n} = S^j S^{*j} S^{*prk}$. We proceed as above, except that at step (6.3) we find that

$$(6.2) \iff K_{pr}(S^{prk} R^* R S^{*prk}) = 0 \iff S^k K_{pr}(R^*R) S^{*k} = 0 \iff K_{pr}(R^*R) = 0,$$

and the rest of the argument is the same. This gives (c).

For part (a), we run the argument of the first paragraph with $R = S^j S^{*j} - \beta_a K_a(S^j S^{*j})$. For part (b), we write out the inner product of $q_a(S^{*m} - S^j S^{*j} S^{*m})$ with itself, getting $K_a(S^m S^{*m} - S^{m+j} S^{*(m+j)})$, and calculate this using the formula (6.1) for K_a. □

The bimodules M_{K_a} are free as right modules, just as are the M_{L_a}.

PROPOSITION 6.3. *Let $a \in \mathbb{N}^\times$. Then $\{q_a(S^k) : 0 \le k < a\}$ is an orthonormal basis for M_{K_a} as a right Hilbert $\mathcal{T}$-module.*

Proof. For $n \in \mathbb{N}$ and $0 \le j, k < a$, we have

$$\langle q_a(S^j), q_a(S^k)\rangle_{K_a} e_n = V_a^* S^{*j} S^k V_a e_n = \begin{cases} e_n & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

because a cannot divide $k - j$ unless $j = k$. Thus $\{q_a(S^k)\}$ is orthonormal.

To check that the $q_a(S^k)$ generate M_{K_a}, it suffices to see that each $q_a(S^n S^j S^{*j})$ and each $q_a(S^j S^{*j} S^{*n})$ can be written as $q_a(S^k \beta_a(R)) = q_a(S^k) \cdot R$. The first is easy: we write $n = ai + k$ with $0 \le k < a$, and then Lemma 6.2(a) gives

$$\begin{aligned}q_a(S^n S^j S^{*j}) &= q_a(S^n \beta_a K_a(S^j S^{*j})) \\ &= q_a(S^k S^{ai} \beta_a K_a(S^j S^{*j})) = q_a(S^k \beta_a K_a(S^{ai} S^j S^{*j})).\end{aligned}$$

For $q_a(S^j S^{*j} S^{*n})$, we choose k to be the smallest element of $\mathbb{N}$ such that a divides $n + k$. If $j \ge k$, then another application of Lemma 6.2(a) gives

$$\begin{aligned}q_a(S^j S^{*j} S^{*n}) &= q_a(S^k S^{j-k} S^{*(j-k)} S^{*(k+n)}) = q_a(S^k S^{j-k} S^{*(j-k)}) \cdot S^{*a^{-1}(k+n)} \\ &= q_a(S^k \beta_a K_a(S^{j-k} S^{*(j-k)})) \cdot S^{*a^{-1}(k+n)} \\ &= q_a(S^k \beta_a(K_a(S^{j-k} S^{*(j-k)}) S^{*a^{-1}(k+n)})).\end{aligned}$$

If $j < k$, then we observe first that, because k is the *smallest* element of $\mathbb{N}$ such that $a|(n+k)$, we have $S^j S^{*j} S^{*n} V_a e_i = S^k S^{*k} S^{*n} V_a e_i$ for all $i \in \mathbb{N}$; this implies that

$$\langle q_a(T), q_a(S^j S^{*j} S^{*n})\rangle_{K_a} = \langle q_a(T), q_a(S^k S^{*k} S^{*n})\rangle_{K_a} \quad \text{for all } T \in \mathcal{T},$$

and hence that

$$q_a(S^j S^{*j} S^{*n}) = q_a(S^k S^{*k} S^{*n}) = q_a(S^k \beta_a(S^{*a^{-1}(n+k)})). \qquad \square$$

PROPOSITION 6.4. *The homomorphisms $\phi_a : \mathcal{T} \to \mathcal{L}(M_{K_a})$ are injective, and have range in the compact operators. Indeed, we have*

$$\phi_a(T) = \sum_{k=0}^{a-1} \Theta_{q_a(TS^k), q_a(S^k)} \quad \textit{for all } T \in \mathcal{T}.$$

Proof. We claim that K_a is almost faithful in the sense of [**2**]. To see this, suppose that $X \in \mathcal{T}$ satisfies $K_a((XY)^*(XY)) = 0$ for all $Y \in \mathcal{T}$. Let $n \in \mathbb{N}$ and take $Y = S^{*(a-1)n}$. Then

$$\begin{aligned} 0 &= (K_a((XY)^*(XY))e_n|e_n) = (XYV_a e_n|XYV_a e_n) \\ &= (XS^{*(a-1)n} e_{an}|XS^{*(a-1)n} e_{an}) = (Xe_n|Xe_n); \end{aligned}$$

since this is true for all n, we deduce that $X = 0$, as required. Thus K_a is almost faithful, and the argument of [**2**, Theorem 4.2] implies that ϕ_a is injective.

The formula for $\phi_a(T)$ follows from the reconstruction formula for the orthonormal basis of Proposition 6.3. $\square$

LEMMA 6.5. *Let J be the ideal of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ generated by*

$$\left\{1 - \sum_{k=0}^{p-1} s^k v_p (s^k v_p)^* : p \in \mathcal{P}\right\}.$$

Let $a \in \mathbb{N}^\times$ and set $v_a := \prod_{p\in\mathcal{P}} v_p^{e_p(a)}$. Then

$$1 - \sum_{k=0}^{a-1} s^k v_a (s^k v_a)^* \in J. \tag{6.5}$$

Proof. The crucial observation is that (T1) implies that $v_b s = s^b v_b$ for all $b \in \mathbb{N}^\times$ (see the proof of [**17**, Proposition 6.1]). We will prove (6.5) by induction on the number n of prime factors of a, counted with multiplicity. If a is prime, so that $n = 1$, then (6.5) holds by definition of J. Let $n \geq 2$, and assume that (6.5) holds for all $b \in \mathbb{N}^\times$ with less than n prime factors. Consider $a = bp$, where $p \in \mathcal{P}$ and b has fewer than n prime factors. Note that

$$\sum_{k=0}^{bp-1} s^k v_{bp} (s^k v_{bp})^* = \sum_{j=0}^{b-1} \sum_{l=0}^{p-1} s^{j+lb} v_{bp} (s^{j+lb} v_{bp})^* = \sum_{j=0}^{b-1} \sum_{l=0}^{p-1} s^j v_b s^l v_p (s^j v_b s^l v_p)^*.$$

Thus

$$1-\sum_{k=0}^{bp-1} s^k v_{bp}(s^k v_{bp})^*$$
$$=\sum_{j=0}^{b-1} s^j v_b\Big(1-\sum_{l=0}^{p-1} s^l v_p(s^l v_p)^*\Big)(s^j v_b)^* + \Big(1-\sum_{j=0}^{b-1} s^j v_b(s^j v_b)^*\Big)$$

belongs to J by the induction hypothesis. □

THEOREM 6.6. *Let $(\mathcal{T}, \mathbb{N}^\times, \beta, K)$ be the Exel system described in Lemma 6.1. Then there are isomorphisms:*

(a) *θ_1 of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ onto $\mathcal{NT}(M_K)$ such that $\theta_1(s) = i_{\mathcal{T}}(S)$ and $\theta_1(v_p) = i_{\mathbb{N}^\times}(p)$ for p prime; and*

(b) *θ_2 of $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ onto $\mathcal{T} \rtimes_{\beta,K} \mathbb{N}^\times = \mathcal{O}(M_K)$ such that $\theta_2(s) = j_{\mathcal{T}}(S)$ and $\theta_2(v_p) = j_{\mathbb{N}^\times}(p)$ for p prime.*

Proof. Since $i_{\mathcal{T}}$ is unital, $T := i_{\mathcal{T}}(S)$ and $W_p := i_{M,p}(q_p(1))$ are isometries. Calculations like those in the second paragraph of the proof of Theorem 5.2 show that T and the W_p satisfy (T1), (T2) and (T5). Lemma 6.2(b) implies that $q_p(S^*) = q_p(S^{p-1}S^{*p})$, and hence

$$\begin{aligned}
T^*W_p &= i_{\mathcal{T}}(S)^* i_{M,p}(q_p(1)) = i_{M,p}(q_p(S^*)) = i_{M,p}(q_p(S^{p-1}S^{*p}))\\
&= i_{M,p}(q_p(S^{p-1}\beta_p(S^*))) = i_{M,p}(q_p(S^{p-1})\cdot S^*) = i_{M,p}(q_p(S^{p-1}))i_{\mathcal{T}}(S^*)\\
&= i_{M,p}(S^{p-1}\cdot q_p(1))i_{\mathcal{T}}(S^*) = i_{\mathcal{T}}(S^{p-1})i_{M,p}(q_p(1))i_{\mathcal{T}}(S^*) = T^{p-1}W_pT^*,
\end{aligned}$$

which is (T4). The pair (β, K) does not satisfy the analogue of the relation $L_p\alpha_r = \alpha_r L_p$ used at equation (5.5), but part (c) of Lemma 6.2 is exactly what we need to jump from equation (5.4) to equation (5.5). So the proof of (T3) also carries over.

Theorem 4.1 of [**17**] now gives a homomorphism $\theta_1 := \pi_{T,W} : \mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times) \to \mathcal{NT}(M_K)$ which does the correct things on generators. As in the proof of Theorem 5.2, we can see using the Fock representation of M_K that the elements $\{T, W_p\}$ satisfy (2.3), and hence Theorem 2.1 implies that θ_1 is injective. The argument in the second-last paragraph of the proof of Theorem 5.2 shows that θ_1 is surjective.

For (b), let $Q : \mathcal{NT}(M_K) \to \mathcal{T} \rtimes_{\beta,K} \mathbb{N}^\times$ be the quotient map, and recall that $\ker Q$ is generated by $\{i_{\mathcal{T}}(T) - i_{M_K}^{(a)}(\phi_a(T)) : a \in \mathbb{N}^\times, T \in \mathcal{T}\}$. Let J be the ideal of $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ in Lemma 6.5, so that $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)/J = \mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$. Replacing the basis $\{q_p(\iota^k) : 0 \le k < p\}$ of M_{L_p} with the basis $\{q_p(S^k) : 0 \le k < p\}$ of M_{K_p} in the calculation (5.6) shows that $1 = \sum_{k=0}^{p-1} Q \circ \theta_1(s^k v_p(s^k v_p)^*)$. Thus $J \subset \ker(Q \circ \theta_1)$, and $Q \circ \theta_1$ factors through a surjection θ_2 of $\mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times)$ onto $\mathcal{T} \rtimes_{\beta,K} \mathbb{N}^\times$ satisfying $\theta_2(s) = j_{\mathcal{T}}(S)$ and $\theta_2(v_p) = j_{\mathbb{N}^\times}(p)$, as required. To see that θ_2 is injective, we need to see that $J = \ker(Q \circ \theta_1)$. Using Proposition 6.4, we have

$$\begin{aligned}
i_{M_K}^{(a)}(\phi_a(T)) &= i_{M_K}^{(a)}\Big(\sum_{k=0}^{a-1} \Theta_{q_a(TS^k), q_a(S^k)}\Big) = \sum_{k=0}^{a-1} i_{M,a}(q_a(TS^k))i_{M,a}(q_a(S^k))^*\\
&= i_{\mathcal{T}}(T)\sum_{k=0}^{a-1} i_{\mathcal{T}}(S^k)i_{M,a}(q_a(1))(i_{\mathcal{T}}(S^k)i_{M,a}(q_a(1)))^*.
\end{aligned}$$

Now

$$i_{\mathcal{T}}(T) - i_{M_K}^{(a)}(\phi_a(T)) = i_{\mathcal{T}}(T)\left(1 - \sum_{k=0}^{a-1} i_{\mathcal{T}}(S^k) i_{M,a}(q_a(1))(i_{\mathcal{T}}(S^k) i_{M,a}(q_a(1)))^*\right)$$
$$= i_{\mathcal{T}}(T)\theta_1\left(1 - \sum_{k=0}^{a-1} s^k v_a (s^k v_a)^*\right)$$

belongs to $\theta_1(J)$ by Lemma 6.5. Thus $\ker(Q \circ \theta_1) \subset J$, and θ_2 is injective. □

7. *Compatibility of our isomorphisms*

Our final Theorem 7.4 says that all our algebras and maps fit into a commutative cube. At this stage we are missing two of the maps. However, since $C(\mathbb{T})$ is a quotient of $\mathcal{T} = \mathcal{T}(\mathbb{N})$, it is reasonable to guess that there are natural 'quotient maps' from $\mathcal{NT}(M_K)$ to $\mathcal{NT}(M_L)$ and $\mathcal{O}(M_K)$ to $\mathcal{O}(M_L)$. The next result describes the data we need to build such homomorphisms: for individual Hilbert bimodules, we need a triple of homomorphisms satisfying the axioms described in [**6**, Definition 1.16]; for a product system, we need one of these triples for each fibre. Proposition 7.1 and the following Proposition 7.2 can be viewed as partial functoriality results for the various constructions discussed in the last two sections.

PROPOSITION 7.1. *Suppose that (G, P) is a quasi-lattice ordered group, that A and B are C^*-algebras, that M is a compactly aligned product system of A–A bimodules over P and that N is a compactly aligned product system of B–B bimodules over P. Suppose that $\rho : A \to B$ is a homomorphism and that for each $x \in P$ we have a linear map $\pi_x : M_x \to N_x$ such that (ρ, π_x, ρ) is a morphism of right-Hilbert bimodules, such that $\pi_x(m)\pi_y(n) = \pi_{xy}(mn)$, and such that $\pi_x(M_x)$ generates N_x as a right Hilbert B-module. Then there are homomorphisms:*

(a) *$\pi_{\mathcal{NT}} : \mathcal{NT}(M) \to \mathcal{NT}(N)$ such that $\pi_{\mathcal{NT}} \circ i_A = i_B \circ \rho$ and $\pi_{\mathcal{NT}} \circ i_{M,x} = i_{N,x} \circ \pi_x$ for $x \in P$; and*

(b) *$\pi_{\mathcal{O}} : \mathcal{O}(M) \to \mathcal{O}(N)$ such that $\pi_{\mathcal{O}} \circ j_A = j_B \circ \rho$ and $\pi_{\mathcal{O}} \circ j_{M,x} = j_{N,x} \circ \pi_x$ for $x \in P$.*

Proof. Using that (ρ, π_x, ρ) is a homomorphism of Hilbert bimodules and $\pi_x(m)\pi_y(n) = \pi_{xy}(mn)$, one can check that the $i_{N,x} \circ \pi_x$ form a representation $i_N \circ \pi$ of the product system M. We claim that this representation is Nica covariant. For this, we recall from [**6**, Remark 1.19] that the π_x induce homomorphisms $\mu_x : \mathcal{K}(M_x) \to \mathcal{K}(N_x)$ satisfying $\mu_x(\Theta_{m,n}) = \Theta_{\pi_x(m),\pi_x(n)}$ and $\mu_x(R)(\pi_x(m)) = \pi_x(Rm)$. (Indeed, since the range of π_x generates N_x, this last equation completely determines $\mu_x(R)$.) Now we can check on rank-one operators that $(i_N \circ \pi)^{(x)} = i_N^{(x)} \circ \mu_x$. This will allow us to compute the left-hand side of the Nica-covariance relation (5.2).

For the right-hand side of equation (5.2), we also need to handle things like $\iota_x^{x\vee y}(\mu_x(R))$. Since the ranges of the π_x generate N_x, $\iota_x^{x\vee y}(\mu_x(R))$ is determined by its values on elements of the form $\pi_x(m)\pi_{x^{-1}(x\vee y)}(n) = \pi_{x\vee y}(mn)$. Then

$$\begin{aligned}
\iota_x^{x\vee y}(\mu_x(R))(\pi_x(m)\pi_{x^{-1}(x\vee y)}(n)) &= (\mu_x(R)\pi_x(m))\pi_{x^{-1}(x\vee y)}(n) \\
&= \pi_x(Rm)\pi_{x^{-1}(x\vee y)}(n) \\
&= \pi_{x\vee y}((Rm)n) \\
&= \pi_{x\vee y}(\iota_x^{x\vee y}(R)(mn)) \\
&= (\mu_{x\vee y} \circ \iota_x^{x\vee y}(R))(\pi_{x\vee y}(mn)) \\
&= (\mu_{x\vee y} \circ \iota_x^{x\vee y}(R))(\pi_x(m)\pi_{x^{-1}(x\vee y)}(n)),
\end{aligned}$$

so $\iota_x^{x\vee y} \circ \mu_x = \mu_{x\vee y} \circ \iota_x^{x\vee y}$. Thus, for $R \in \mathcal{K}(M_x)$ and $T \in \mathcal{K}(M_y)$, we have

$$\begin{aligned}
(i_N \circ \pi)^{(x)}(R)(i_N \circ \pi)^{(y)}(T) &= i_N^{(x)}(\mu_x(R))i_N^{(y)}(\mu_y(T)) \\
&= i_N^{(x\vee y)}(\iota_x^{x\vee y}(\mu_x(R))\iota_y^{x\vee y}(\mu_y(T))) \\
&= i_N^{(x\vee y)} \circ \mu_{x\vee y}(\iota_x^{x\vee y}(R)\iota_y^{x\vee y}(T)) \\
&= (i_N \circ \pi)^{(x\vee y)}(\iota_x^{x\vee y}(R)\iota_y^{x\vee y}(T)),
\end{aligned}$$

which is Nica covariance of $i_N \circ \pi$. Now the universal property of $(\mathcal{NT}(M), i_M)$ gives a homomorphism $\pi_{\mathcal{NT}}$ with the required properties.

For (b), we consider the universal representation (j_N, j_B) in $\mathcal{O}(N)$. As in the first paragraph of the proof of part (a), $(j_N \circ \pi_x, j_N \circ \rho)$ is a representation of M_x in $\mathcal{O}(N)$; we claim that this representation is Cuntz–Pimsner covariant. Suppose that $\phi_x^M(a) \in \mathcal{K}(M_x)$. With μ_x as above,

$$\mu_x(\phi_x^M(a))(\pi_x(m)) = \pi_x(\phi_x^M(a)(m)) = \pi_x(a \cdot m) = \rho(a) \cdot \pi_x(m) = \phi_x^N(\rho(a))(\pi_x(m)).$$

This implies, first, that $\phi_x^N(\rho(a))$ is compact and, second, that

$$j_B(\rho(a)) = j_N^{(x)}(\phi_x^N(\rho(a))) = j_N^{(x)}(\mu_x(\phi_x^M(a))) = (j_N \circ \pi)^{(x)}(\phi_x^M(a)).$$

Thus, $(j_N \circ \pi_x, j_N \circ \rho)$ is Cuntz–Pimsner covariant as claimed, and induces a homomorphism $\pi_{\mathcal{O}} : \mathcal{O}(M) \to \mathcal{O}(N)$ with the required properties. □

We now apply Proposition 7.1 to the product systems arising from Exel systems. The odd-looking hypothesis on $\rho(A)\alpha_x(B)$ in Proposition 7.2 is there to ensure that the range of the maps π_x generate; we do not know whether it is necessary, but it is trivially satisfied in our application.

PROPOSITION 7.2. *Suppose that (A, P, β, K) and (B, P, α, L) are Exel systems such that the associated product systems M_K and M_L are compactly aligned. Suppose that $\rho : A \to B$ is a unital homomorphism such that $\rho \circ \beta_x = \alpha_x \circ \rho$ and $\rho \circ K_x = L_x \circ \rho$, and such that products of the form $\rho(a)\alpha_x(b)$ span a dense subspace of B for $x \in P$. Then there are homomorphisms:*

(a) $\rho_{\mathcal{NT}} : \mathcal{NT}(A, P, \beta, K) \to \mathcal{NT}(B, P, \alpha, L)$ *such that* $\rho_{\mathcal{NT}} \circ i_A = i_B \circ \rho$ *and* $\rho_{\mathcal{NT}}(i_P(x)) = i_P(x)$ *for* $x \in P$*; and*

(b) $\rho \rtimes \mathrm{id} : A \rtimes_{\beta,K} P \to B \rtimes_{\alpha,L} P$ *such that* $(\rho \rtimes \mathrm{id}) \circ j_A = j_B \circ \rho$ *and* $(\rho \rtimes \mathrm{id})(j_P(x)) = j_P(x)$ *for* $x \in P$.

Proof. Since $\rho \circ K = L \circ \rho$, we have $\|q_x(\rho(a))\| \le \|q_x(a)\|$ for $a \in A$, and there are well-defined maps $\pi_x : M_{K_x} \to M_{L_x}$ such that $\pi_x(q_x(a)) = q_x(\rho(a))$. Straightforward calculations show that the (ρ, π_x, ρ) are morphisms of Hilbert bimodules, and the hypothesis on $\rho(A)$ implies that $\pi_x(M_{K_x})$ generates M_{L_x}. For $a, b \in A$, we have

$$\begin{aligned}\pi_x(q_x(a))\pi_y(q_y(b)) &= q_x(\rho(a))q_y(\rho(b))\\ &= q_{xy}(\rho(a)\alpha_x(\rho(b)))\\ &= q_{xy}(\rho(a\beta_x(b)))\\ &= \pi_{xy}(q_{xy}(a\beta_x(b)))\\ &= \pi_{xy}(q_x(a)q_y(b)).\end{aligned}$$

Thus the π_x satisfy the hypotheses of Proposition 7.1, and we get maps $\rho_{\mathcal{NT}}$ and $\rho \rtimes \mathrm{id}$ on $\mathcal{NT}(A, P, \beta, K) := \mathcal{NT}(M_K)$ and $A \rtimes_{\beta,K} P := \mathcal{O}(M_K)$. More calculations show that, because ρ is unital, these maps do the right thing on generators. □

Example 7.3. We aim to apply Proposition 7.2 to the systems $(\mathcal{T}, \mathbb{N}^\times, \beta, K)$ and $(C(\mathbb{T}), \mathbb{N}^\times, \alpha, L)$. We saw in Proposition 6.4 and Example 5.1 that the left actions on M_{K_a} and M_{L_a} are by compact operators, and hence both are compactly aligned by [**13**, Proposition 5.8]. Since the Toeplitz algebra $\mathcal{T}$ is the universal C^*-algebra generated by an isometry, there is a homomorphism $\rho : \mathcal{T} \to C(\mathbb{T})$ such that $\rho(S)$ is the identity function $\iota : z \mapsto z$. We claim that, with (β, K) as in §6 and (α, L) as in §5, ρ satisfies the hypotheses of Proposition 7.2. Since $\beta_a(S) = S^a$ and $\alpha_a(\iota) = \iota^a$, we have $\rho \circ \beta_a = \alpha_a \circ \rho$; since the range of ρ is a C^*-subalgebra of $C(\mathbb{T})$ containing the generator ι, ρ is surjective, and since α_a is unital every element of $C(\mathbb{T})$ has the form $\rho(T)\alpha_a(1)$.

It remains to check that $\rho \circ K_a = L_a \circ \rho$. Since both sides of this equation are linear and $*$-preserving, it suffices to check it on elements of the form $S^n S^j S^{*j}$. Equation (6.1) implies that $\rho(K_a(S^n S^j S^{*j})) = \iota^{a^{-1}n}$ if $a|n$ and 0 otherwise. So we need to compute $L_a(\rho(S^n S^j S^{*j}))(z) = L_a(\iota^n)(z)$ for $z \in \mathbb{T}$. Choose an ath root w_0 of z. Then

$$L_a(\iota^n)(z) = \frac{1}{a}\sum_{w^a=z}\iota^n(w) = \frac{1}{a}\sum_{w^a=z} w^n = \frac{w_0^n}{a}\sum_{l=0}^{a-1} e^{2\pi i l n/a}.$$

If a does not divide n, then $e^{2\pi i n/a} \neq 1$, and the sum is zero; if $a|n$, then $e^{2\pi i l n/a} = 1$ for all l, the sum equals a, and we get $L_a(\iota^n)(z) = w_0^n = z^{a^{-1}n} = \iota^{a^{-1}n}(z)$.

Thus, Proposition 7.2 gives us homomorphisms $\rho_{\mathcal{NT}}$ of $\mathcal{NT}(M_K)$ onto $\mathcal{NT}(M_L)$ such that $\rho_{\mathcal{NT}}(i_{\mathcal{T}}(S)) = i_{C(\mathbb{T})}(\iota)$ and $\rho_{\mathcal{NT}}(i_{\mathbb{N}^\times}(p)) = i_{\mathbb{N}^\times}(p)$, and $\rho \rtimes \mathrm{id} : \mathcal{T} \rtimes_{\beta,K} \mathbb{N}^\times \to C(\mathbb{T}) \rtimes_{\alpha,L} \mathbb{N}^\times$ such that $\rho \rtimes \mathrm{id}(j_{\mathcal{T}}(S)) = j_{C(\mathbb{T})}(\iota)$ and $\rho \rtimes \mathrm{id}(j_{\mathbb{N}^\times}(p)) = j_{\mathbb{N}^\times}(p)$.

THEOREM 7.4. *Let* $\rho : \mathcal{T} \to C(\mathbb{T})$ *be the homomorphism such that* $\rho(S) = \iota$*, and let* $\rho_{\mathcal{NT}}$ *and* $\rho \rtimes \mathrm{id}$ *be the homomorphisms found in Example 7.3. Then the isomorphisms* ϕ_1, ϕ_2

of Theorem 5.2 and θ_1, θ_2 of Theorem 6.6 fit into the following commutative diagram.

$$\begin{tikzcd}
\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times) \arrow[rrr, "\theta_1"] \arrow[dr, "q_{\mathrm{mult}}"] \arrow[dd, "q_{\mathrm{add}}"'] & & & \mathcal{NT}(M_K) \arrow[dr, "Q_K"] \arrow[dd, "\rho_{\mathcal{NT}}"] & \\
& \mathcal{T}_{\mathrm{mult}}(\mathbb{N} \rtimes \mathbb{N}^\times) \arrow[rrr, "\theta_2"] \arrow[dd] & & & \mathcal{T} \rtimes_{\beta,K} \mathbb{N}^\times \arrow[dd, "\rho \rtimes \mathrm{id}"] \\
\mathcal{T}_{\mathrm{add}}(\mathbb{N} \rtimes \mathbb{N}^\times) \arrow[rrr, "\phi_1"] \arrow[dr] & & & \mathcal{NT}(M_L) \arrow[dr, "Q_L"'] & \\
& \mathcal{Q}_{\mathbb{N}} \arrow[rrr, "\phi_2"] & & & C(\mathbb{T}) \rtimes_{\alpha,L} \mathbb{N}^\times
\end{tikzcd}$$

Indeed, because we have been careful to describe what all our maps do to generators, it is easy to check that each of the six faces commutes.

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