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Abstract
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MORE MIXED VOLUME PRESERVING CURVATURE FLOWS

JAMES A. MCCOY

ABSTRACT. We extend the results of [28] to include several new cases where convex surfaces evolve to spheres under mixed volume preserving curvature flows, using recent results for unconstrained curvature flows and new regularity arguments in the constrained flow setting. We include results for speeds that are degree 1 homogeneous in the principal curvatures and indicate how, with sufficient curvature pinching conditions on the initial hypersurfaces, some results may be extended to speeds homogeneous of degree $\alpha > 1$. In particular, these extensions require lower speed bounds that are obtained here without using estimates for equations of porous medium type, in contrast to most previous work.

1. INTRODUCTION

Let M_0 be a compact, strictly convex hypersurface of dimension $n \geq 2$, without boundary, smoothly embedded in $\mathbb{R}^{n+1}$ and represented by some diffeomorphism $X_0 : \mathbb{S}^n \rightarrow X_0\left(\mathbb{S}^n\right) = M_0 \subset \mathbb{R}^{n+1}$. We consider the family of maps $X_t = X\left(\cdot, t\right)$ evolving according to

$$\frac{\partial}{\partial t} X\left(x,t\right) = \left\{h\left(t\right) - F\left(\mathscr{W}\left(x,t\right)\right)\right\} \nu\left(x,t\right) \; x \in U, \; 0 < t \leq T \leq \infty \qquad (1)$$
$$X\left(\cdot, 0\right) = X_0,$$

where $\mathscr{W}\left(x,t\right)$ is the matrix of the Weingarten map of $M_t = X_t\left(\mathbb{S}^n\right)$ at the point $X_t\left(x\right)$, $\nu\left(x,t\right)$ is the outer unit normal to M_t at $X_t\left(x\right)$ and $h\left(t\right)$ is a global term to be specified.

The function F should have the following properties:

Conditions 1.1.

a) *$F\left(\mathscr{W}\right) = f\left(\kappa\left(\mathscr{W}\right)\right)$ where $\kappa\left(\mathscr{W}\right)$ gives the eigenvalues of $\mathscr{W}$ and f is a smooth, symmetric function defined on the positive cone*

$$\Gamma = \left\{\kappa = \left(\kappa_1, \ldots, \kappa_n\right) \in \mathbb{R}^n : \kappa_i > 0 \text{ for all } i = 1, 2, \ldots, n\right\}.$$

b) *f is strictly increasing in each argument: $\frac{\partial f}{\partial \kappa_i} > 0$ for each $i = 1, \ldots, n$ at every point of Γ.*

c) *f is homogeneous of degree one: $f\left(k\kappa\right) = kf\left(\kappa\right)$ for any $k > 0$.*

d) *f is strictly positive on Γ and $f\left(1, \ldots, 1\right) = 1$.*

e) *Either:*
   i) *$n = 2$, or*
   ii) *f is convex, or*
   iii) *f is concave and one of the following hold*
      a) *f approaches zero on the boundary of Γ,*

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b) $\sup_{M_0}\left(\frac{H}{F}\right) < \liminf_{\kappa\to\partial\Gamma}\left(\frac{\sum_i \kappa_i}{f(\kappa)}\right)$, *where H denotes mean curvature,*

c) *f is* inverse concave*, that is, the function*

$$f_*\left(x_1,\ldots,x_n\right) = f\left(x_1^{-1},\ldots,x_n^{-1}\right)^{-1}$$

is also concave.

iv) *f is inverse concave and either*

a) *$f_* \to 0$ as $r\to\partial\Gamma$, or*

b) $\sup_{\substack{\omega\in T_z\mathbb{S}^n\\ |\omega|=1}} \frac{r(\omega,\omega)(z,0)}{f_*(r(z,0))} < \liminf_{r\to\partial\Gamma}\frac{r_{\max}}{f_*(r)}$.

v) *f satisfies no second derivative condition but either*

a) *M_0 is axially symmetric, or*

b) *M_0 satisfies a pinching condition of the form*

$$\left|A^0\right|^2 \le \sigma H^2,$$

where σ depends upon n and the second derivative bound on the preserved pinching cone.

Note we use the definition of inverse concavity as in [10] rather than that in [6], but the two are equivalent. We associate naturally to f_* a function F_* of the inverse Weingarten map as in [10] and elsewhere. Where we write Γ in part e) iv) above, we mean the cone of principal radii of curvature r_i. We further denote by r above the matrix of the inverse Weingarten map and by $r_{\max}$ its maximum eigenvalue.

We define for $k=0,1,\ldots,n$ the elementary symmetric functions of the principal curvatures of a convex hypersurface M by

$$E_k = \frac{1}{\binom{n}{k}}\sum_{1\le i_1<\ldots<i_k\le n}\kappa_{i_1}\kappa_{i_2}\ldots\kappa_{i_k}.$$

The normalised k-dimensional mean cross-sectional volumes $V_k(M)$ or *mixed volumes* of M are then given as in [9] via

$$V_k(M) = \frac{1}{|\mathbb{S}^n|}\int_M E_{n-k}\,d\mu(g)$$

for $0\le k\le n$ and by

$$V_k(M) = \frac{1}{|\mathbb{S}^n|}\int_M sE_{n+1-k}\,d\mu(g)$$

for $1\le k\le n+1$ where $s=\langle X,\nu\rangle$ is the support function of M. Particular mixed volumes are V_{n+1} the enclosed volume, V_n the surface area and V_1 the mean width of M.

As shown in [28, Corollary 4.4], if, for any $k\in\{-1,0,1,\ldots,n-1\}$, we set

$$h(t) = h_k(t) = \frac{\int_{M_t} F(\mathscr{W})\,E_{k+1}\,d\mu_t}{\int_{M_t} E_{k+1}\,d\mu_t} \tag{2}$$

then the flow (1) preserves the value of V_{n-k}, that is, under the flow (1) with $h=h_k$,

$$V_{n-k}(M_t) = V_{n-k}(M_0)$$

as long as the solution exists. Note that, in view of Condition 1.1 d), $h>0$. The term with coefficient $h(t)$ is a lower order term in the evolution equation (1) and in the evolution of various geometric quantities associated with the evolving hypersurface M_t, these terms require care particularly in obtaining regularity of the solution.

The result for such flows (1) of convex hypersurfaces may then be stated as follows:

Theorem 1.2. *Let M_0 be a smooth, closed, strictly convex, n-dimensional hypersurface without boundary, $n \geq 2$, smoothly embedded in $\mathbb{R}^{n+1}$ by $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$. Suppose that Conditions 1.1 are satisfied. Then there exists a unique family of smooth, strictly convex hypersurfaces $\{M_t = X_t(\mathbb{S}^n)\}_{0 \leq t < \infty}$ satisfying* (1), *with initial condition $X(x,0) = X_0(x)$. As $t \to \infty$ the image hypersurfaces smoothly converge uniformly and exponentially to a sphere with the same value of the fixed mixed volume V_{n-k} as M_0.*

Of Conditions 1.1, the cases in part e) that were not covered in [28] are iv) and v). Similarly to the relationship between e) iii) parts a) and b), if iv) part a) is satisfied, then necessarily iv) part b) is satisfied, so for the proof we may focus our attention on iv) b). Nevertheless it is worth stating iv) a) and, for that matter iii) a), as separate cases, since they are structure conditions on f not a pinching requirement on M_0. Parts e) i) and e) v) have the minimal requirements on f, just a dimension, symmetry or rather strong pinching requirement on M_0 and, in particular, no convexity requirement on f.

We will use similar notation as in [6, 28, 10] and elsewhere. We also refer the reader to [28] for discussions of earlier work on constrained curvature flows fitting into the above framework and for local-in-time existence of a unique solution to (1). A particular flow of note is the volume preserving mean curvature flow, the special case where $F = H$, the sum of the principal curvatures and $h(t) = \frac{\int_{M_t} H d\mu_t}{\int_{M_t} d\mu_t}$ studied by Huisken [20]. After that work, other mixed volume preserving mean curvature flows were studied by the author [26, 27]. More recently, Athanassenas and Kandanaararchchi obtained the result of Theorem 1.2 for the volume preserving mean curvature flow of axially symmetric hypersurfaces [11]. In all these cases where $F = H$, curvature derivative estimates may be obtained by an inductive maximum principle argument. On the other hand, for fully nonlinear flows, we instead appeal to regularity results for such partial differential equations. Subsequent to the discussion in [28], in the setting of evolving hypesurfaces in Euclidean space, Cabezas-Rivas and Sinestrari considered volume preserving flows of suitably curvature-pinched hypersurfaces by powers of the elementary symmetric functions of $\mathscr{W}$, such that the F term is homogeneous of degree greater than 1 [12]. Later in [33] the initial curvature pinching requirement was removed in the case of volume preserving flows where the leading order speed term is a positive power of the mean curvature. Important in these analyses was a divergence structure facilitating regularity estimates via results for porous-medium type equations. Our functions F do not necessarily have such a structure. Some other stability-type results for quite general F with initial hypersurfaces close to spheres in $C^{2,\alpha}$ may be found in [17]. A reason for interest in geometrically constrained flows is their application to isoperimetric-type inequalities (see [28, Section 10] for example). The techniques presented in this article could also be used to extend the results in [22] where general nonnegative continuous functions $h(t)$ are considered.

The structure of this article is as follows. In Section 2 we detail some flow independent geometric results that will be needed in the analysis, plus, for completeness we recall the parts of the proof of Theorem 1.2 that carry over directly from [28]. In Sections 3, 4 and 5 we complete the proof of Theorem 1.2 in cases e) parts iv), v) a) and v) b) respectively. In Section 6 we show how the results of Sections 4 and 5 may be extended to the case of F homogeneous of degree $\alpha > 1$.

2. Relevant previous results and geometric estimates

We recall the following flow-independent results that will be used in our subsequent analysis of the flow.

Lemma 2.1. *If $H > 0$ and $h_{ij} \geq \varepsilon H g_{ij}$ is valid for some $\varepsilon > 0$ then, using coordinates at any particular point that diagonalise the Weingarten map, we have*

(i) $HC - \left(|A|^2\right)^2 = \sum_{i<j}^{n} \kappa_i \kappa_j (\kappa_i - \kappa_j)^2 \geq n\varepsilon^2 H^2 \left|A^0\right|^2$;

(ii) $nC - H|A|^2 = \frac{1}{2}\sum_{i \neq j} (\kappa_i + \kappa_j)(\kappa_i - \kappa_j)^2 \geq 2n\varepsilon H \left|A^0\right|^2$;

(iii) $\left|H\nabla_i h_{jk} - h_{jk}\nabla_i H\right|^2 \geq \frac{n-1}{2}\varepsilon^2 H^2 |\nabla A|^2$,

where $|A|^2 = \kappa_1^2 + \ldots + \kappa_n^2$ is the squared norm of the second fundamental form, $\left|A^0\right| = |A|^2 - \frac{1}{n}H^2$ is the squared norm of the trace-free second fundamental form and $C = \kappa_1^3 + \ldots + \kappa_n^3$.

The inequalities above appears as [19, Lemma 2.3 (i)], [20, Lemma 1.4 (iii)] and Lemma [12, Lemma 4.1] respectively. The third is attributed to Huisken and is a stronger version of an inequality in [19]. It is important here because it contains the full ∇A on the right hand side rather than $|\nabla H|$.

The next fact follows from the structure conditions on F. It is shown for example in [36].

Lemma 2.2. *For any concave function F satisfying Conditions 1.1 a) to d),*

$$\sum_{k=1}^{n} \dot{f}_k = \text{trace}\left(\dot{F}^{kl}\right) \geq 1.$$

The next result will be needed for the case of Conditions 1.1 e) v) a). It is proved as [30, Lemma 5.1].

Lemma 2.3. *Let $G(\mathscr{W}) = g(\kappa(\mathscr{W}))$ be a smooth, symmetric, homogeneous of degree zero function in the principal curvatures of an axially symmetric hypersurface, where the coordinates are chosen such that x_1 is the axial direction. At any stationary point of G for which $\dot{G}$ is nondegenerate,*

$$\left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} = \frac{2f\dot{g}^1}{\kappa_2(\kappa_2 - \kappa_1)}(\nabla_1 h_{22})^2.$$

Note: By nondegenerate we mean that in coordinates that diagonalise the Weingarten map, the corresponding diagonal matrix of $\dot{G}$ has no zero entries on the diagonal.

The following result, that holds for rather tight curvature pinching, is [9, Lemma 2.2]. It will be used in the case of Conditions 1.1 e) v) b).

Lemma 2.4. *Let M be a compact hypersurface whose principal curvatures satisfy $H > 0$ and $\left|A^0\right|^2 \leq \sigma H^2$ at a point p, with $\sigma < \frac{1}{n(n-1)}$. Then M is convex at p and the principal curvatures satisfy*

$$\left(1 - \sqrt{n(n-1)\sigma}\right)\frac{H}{n} \leq \kappa_i \leq \left(1 + \sqrt{n(n-1)\sigma}\right)\frac{H}{n}. \tag{3}$$

Consequently, the curvatures of M are pinched at p in the sense that

$$\kappa_{\min} \geq \frac{\left(1 - \sqrt{n(n-1)\sigma}\right)}{\left(1 + \sqrt{n(n-1)\sigma}\right)}\kappa_{\max}.$$

Remark: For the inequality $h_{ij} \geq \varepsilon H g_{ij}$ to hold on a convex hypersurface, it follows by taking the trace that $\varepsilon \leq \frac{1}{n}$. The first inequality of (3) says that under the assumptions of Lemma 2.4 we have

$$h_{ij} \geq \varepsilon H g_{ij} \text{ with } \varepsilon = \frac{1 - \sqrt{n(n-1)\sigma}}{n}. \tag{4}$$

We recalling some important consequences of preservation of curvature pinching under a constrained curvature flow such as (1).

Corollary 2.5 (Consequences of curvature pinching)**.** *Once the curvature flow* (1) *has been shown to preserve curvature pinching, where F satisfies Conditions 1.1, we have the following consequences:*

a) *The flow* (1) *is uniformly parabolic;*
b) *The inradius of the evolving M_t is bounded below, and the circumradius is bounded above;*
c) *Continuous, symmetric functions of the principal curvatures that are homogeneous of degree zero are uniformly bounded above and below;*
d) *The gradient of the support function, $\left|\overline{\nabla} s\right|$, remains uniformly bounded;*
e) *There is a constant d such that $M_t \subset B_d(O)$.*

The constants involved above depend only on n, M_0, the particular F and the value of the preserved mixed volume V_{n-k}.

Proof: Parts a) and b) are [28, Corollaries 5.5, 3.6] respectively; part a) follows immediately since F is homogeneous of degree 1 while part b) uses also that one of the mixed volumes V_{n-k} is fixed under the flow. Part c) holds since degree zero homogeneity allows one to restrict the argument to that part of $\{|A| = 1\}$ that lies within the preserved cone of principal curvatures; a continuous function obtains a minimum and a maximum on this set. Parts d) and e) are [28, Corollary 5.6 i) and ii)] respectively, see also [26, 27]. We provide more details about the support function in Section 3. Parts d) and e) are consequences of an Aleksandrov reflection argument for parabolic equations on $\mathbb{S}^n$ [15]. □

Using an argument of Tso [35], an upper bound on F and some further important consequences now follow as in [28, Theorem 6.4]. Specifically,

Corollary 2.6. *Under the flow* (1)*, with F satisfying Conditions 1.1, F is bounded above, by a constant depending only on n and M_0. Consequently, $h(t)$, $\left|\frac{\partial X}{\partial t}\right|$ and $|A|^2$ remain bounded.*

Proof: The proof is as in [28, Section 6]. In particular, curvature pinching is used in estimating zero order terms in the evolution equation for $\frac{F}{s-\delta}$, where the constant $\delta > 0$ is chosen appropriately. Curvature pinching is also used to obtain the bound on $|A|$ from the upper bound on F, as in [28, Corollary 6.7]. □

Short-time existence of a solution to (1) and uniqueness modulo tangential diffeomorphisms holds here exactly as in [28, Section 7], where a fixed point argument is applied to the corresponding scalar equation for the support function. We refer the reader to [28] for details and note the parallels with the freezing time method used there and again here for long time regularity. Note that the equation for the support function is also used in this article in Section 3.

We conclude this section with some further consequences of curvature pinching. The first is a positive lower bound on the global term $h(t)$ which is included essentially for independent interest, because in this article we would only use it when we have a positive lower bound on F anyway, either from the Krylov-Safonov Harnack inequality in the case of F homogeneous of degree 1, or via arguments particular to our settings for F of higher homogeneity. Geometric consequences of curvature pinching related to that below, using techniques of convex geometry were also obtained in [5, 27, 28, 9].

Corollary 2.7. *If the compact hypersurface M without boundary has curvatures satisfying everywhere $h_{ij} \geq \varepsilon H g_{ij}$, then for each $k = 0, 1, \ldots, n-1$, the term h as given by* (2) *satisfies*

$$h(t) \geq \varepsilon^{\alpha} \left(\frac{\int_M E_{k+1} d\mu}{|M|} \right)^{\frac{\alpha}{k+1}},$$

where F is a symmetric, monotone increasing, normalised, degree $\alpha > 0$ homogeneous function of the principal curvatures. Consequently, if the flow (1) *preserves curvature pinching, then h is uniformly bounded below, with the bound depending only on n, k, α and ε.*

Proof: We have in view of curvature pinching that

$$\kappa_i \geq \varepsilon H = n\varepsilon E_1 \geq n\varepsilon E_{k+1}^{\frac{1}{k+1}},$$

where the last step follows by the Maclaurin inequality for the elementary symmetric functions. Using homogeneity and normalisation of F, we therefore estimate

$$\int_M F E_{k+1} d\mu \geq \int_M \kappa_{\min}^{\alpha} E_{k+1} d\mu \geq \varepsilon^{\alpha} \int_M E_{k+1}^{\frac{\alpha+k+1}{k+1}} d\mu.$$

Now using the Hölder inequality

$$\int_M E_{k+1} d\mu \leq \left(\int_M E_{k+1}^{\frac{\alpha+k+1}{k+1}} d\mu \right)^{\frac{k+1}{\alpha+k+1}} |M|^{\frac{\alpha}{\alpha+k+1}},$$

from which it follows that

$$h(t) \geq (n\varepsilon)^{\alpha} \left(\frac{\int_M E_{k+1} d\mu}{|M|} \right)^{\frac{\alpha}{k+1}}.$$

As a result of curvature pinching and the Aleksandrov-Fenchel inequalities (see [27], for example), the expression on the right hand side may now be bounded below in terms of the mixed volume V_{n-k} that is fixed under the flow. □

Remark: Corollary 2.7 does not provide a lower bound on $h(t)$ directly in the case of volume preserving flows ($k = 1$). However, when $\alpha \geq 1$ we can make a small modification to the above argument as follows: we estimate

$$\int_M F\, d\mu \geq \int_M \kappa_{\min}^{\alpha} d\mu \geq (n\varepsilon)^{\alpha} \int_M E_1^{\alpha} d\mu.$$

If $\alpha = 1$ we are done, because we now have $h(t)$ bounded from below in terms of a ratio of mixed volumes, to while we can apply [27, Theorem 2.3] for a time independent lower bound. If $\alpha > 1$ we estimate using the Hölder inequality

$$\int_M E_1^{\alpha} d\mu \geq \frac{\left(\int_M E_1 d\mu\right)^{\alpha}}{|M_t|^{\alpha-1}},$$

then we may similarly apply [27, Theorem 2.3].

Finally for this section we show how curvature pinching provides direct control on the ratio of the circumradius r_+ to the inradis r_- of M. Firstly we recall [9, Lemma 3.2]:

Lemma 2.8. *For any $\eta > 0$ there exists $\delta > 0$ depending only on ε and n such that any convex body satisfying*

$$\frac{V_1^{n+1}}{V_{n+1}} \le 1 + \delta$$

has

$$\frac{r_+(M)}{r_-(M)} \le 1 + \eta.$$

It follows that to control the ratio of the radii close to 1 it suffices to use pinching to control the ratio $\frac{V_1^{n+1}}{V_{n+1}}$.

Corollary 2.9. *The curvature pinching estimate $\kappa_{\max} \le \eta\, \kappa_{\min}$ implies*

$$\frac{V_1^{n+1}}{V_{n+1}} \le \eta^{n^2+1+\frac{1}{n-1}}.$$

Remark: It follows that as $\eta \to 1$ (tighter curvature pinching), $\frac{V_1^{n+1}}{V_{n+1}} \to 1$ and from Lemma 2.8 $\frac{r_+(M)}{r_-(M)} \to 1$.

Proof of Corollary: The proof is similar to that of [9, Theorem 3.1] but here we have a different type of curvature pinching. In view of normalisation of the elementary symmetric functions, for any $\ell = 0, 1, \ldots, n$ we have

$$\kappa_{\min}^{\ell} \le E_\ell,$$

(note that $E_0 = 1$.) Moreover, in view of pinching we therefore have for any $k \le \ell$,

$$E_k \le \kappa_{\max}^k \le \eta^k \kappa_{\min}^k \le \eta^k E_\ell^{\frac{k}{\ell}}.$$

Taking $k = n-1$, $\ell = n$ and dividing by $|\mathbb{S}^n|$ we estimate using the Hölder inequality

$$V_1 = \frac{1}{|\mathbb{S}^n|}\int_M E_{n-1}\, d\mu \le \frac{\eta^{n-1}}{|\mathbb{S}^n|}\int_M E_n^{\frac{n-1}{n}}\, d\mu \le \eta^{n-1} V_n^{\frac{1}{n}}.$$

Now taking $k = 1$ and $\ell = n$ we estimate

$$\begin{aligned} V_n = \frac{1}{|\mathbb{S}^n|}\int_M E_1\, s\, d\mu \le \frac{\eta}{|\mathbb{S}^n|}\int_M E_n^{\frac{1}{n}}\, s\, d\mu \\ \le \frac{\eta}{|\mathbb{S}^n|}\left(\int_M E_n\, s\, d\mu\right)^{\frac{1}{n}}\left(\int_M s\, d\mu\right)^{1-\frac{1}{n}} = \eta V_1^{\frac{1}{n}} V_{n+1}^{\frac{n-1}{n}}. \end{aligned}$$

From these two estimates it follows that

$$V_1^n \le \eta^{n^2-n+1} V_1^{\frac{1}{n}} V_{n+1}^{\frac{n-1}{n}},$$

which is equivalent to the required inequality. □

3. THE CASE OF CONDITIONS 1.1 E) IV)

If f, a function of the principal curvatures κ_i, is inverse concave, then, by definition, the corresponding function f_* of the principal radii of curvature $r_i = \frac{1}{\kappa_i}$ is concave. It is therefore natural to add a tangential diffeomorphism to the flow (1) such that parametrisation of the evolving hypersurface by its support function is preserved under the flow and to work with the corresponding evolution equation on $\mathbb{S}^n \times [0,T)$, as in [3], for example. We refer the reader to details of this procedure in [3, 4, 10, 36], as example references. The support function is then given by

$$s(x,t) = \langle X(x,t), x\rangle \tag{5}$$

where x is the outer unit normal to M_t at $X(x,t)$ for all t in the interval of existence. The matrix of the inverse Weingarten map, denoted $\mathscr{W}^{-1}$, has entries given by

$$r_{ij} = \overline{\nabla}_i \overline{\nabla}_j s + \overline{g}_{ij} s$$

where $\overline{\nabla}$ denotes the covariant derivative on $\mathbb{S}^n$. In view of (1), with the addition of the appropriate tangential term ensuring the parametrisation (5) is preserved, we have the following evolution equations:

Lemma 3.1.

$$\frac{\partial s}{\partial t} = h(t) - F_*^{-1}(r_{ij}), \tag{6}$$

$$\frac{\partial}{\partial t} F_* = \overline{\mathscr{L}} F_* - 2F_*^{-3} \dot{F}_*^{ij} \overline{\nabla}_i F_* \overline{\nabla}_j F_* + \left(h - F_*^{-1}\right) \operatorname{trace} \dot{F}_*,$$

$$\frac{\partial}{\partial t} r_{ij} = \overline{\mathscr{L}} r_{ij} + F_*^{-2} \ddot{F}_*^{kl,pq} \overline{\nabla}_i r_{pq} \overline{\nabla}_j r_{kl} - 2F_*^{-3} \overline{\nabla}_i F_* \overline{\nabla}_j F_* - F_*^{-2}\left(\operatorname{trace} \dot{F}_*\right) r_{ij} + h\overline{g}_{ij},$$

where $F_*\left(\mathscr{W}^{-1}\right) = f_*(r_1, \ldots, r_n)$ *and* $\overline{\mathscr{L}} := F_*^{-2} \dot{F}_*^{kl} \overline{\nabla}_k \overline{\nabla}_l$.

Proof: That there is a function $F_*\left(\mathscr{W}^{-1}\right)$ such that $F_*\left(\mathscr{W}^{-1}\right) = f_*(r_1, \ldots, r_n)$ follows from the definition of f_* and Conditions 1.1, a). For a discussion of the relationship between these we refer the reader to [6]. The evolution equations are derived similarly as in [10], for example, where the only difference is the lower order $h(t)$ term in each case. □

We show preservation of curvature pinching using a similar argument as in [10, Lemma 11]. The key point here is to ensure the sign of the $h(t)$ term goes the right way for applying the maximum principle.

Lemma 3.2. *Under the flow* (6)*, the quantity* $\sup_{\substack{\omega \in T_x\mathbb{S}^n \\ |\omega|=1}} \frac{r(\omega,\omega)(x,t)}{f_*(r(x,t))}$ *is strictly decreasing in* t *unless* M_t *is a sphere.*

Proof: Set $T_{ij} = r_{ij} - CF_*\overline{g}_{ij}$, where $C > 0$ is chosen such that $T_{ij}(\cdot, 0) < 0$. Using Lemma 3.1, we find that T_{ij} evolves according to

$$\begin{aligned}\frac{\partial}{\partial t} T_{ij} = \overline{\mathscr{L}} T_{ij} + F_*^{-2} \ddot{F}_*^{kl,pq} \overline{\nabla}_i r_{pq} \overline{\nabla}_j r_{kl} - 2F_*^{-3} \overline{\nabla}_i F_* \overline{\nabla}_j F_* + 2CF_*^{-3} \dot{F}_*^{kl} \overline{\nabla}_k F_* \overline{\nabla}_l F_* \overline{g}_{ij} \\ - F_*^{-2}\left(\operatorname{trace} \dot{F}_*\right) T_{ij} + h\left(1 - C \operatorname{trace} \dot{F}_*\right) \overline{g}_{ij}.\end{aligned} \tag{7}$$

Suppose there exists a first time $t_0 > 0$ in the interval of existence where T has a null eigenvector at some point (x_0, t_0), that is, there exists a vector v such that $T_{ij} v^i = 0$ at (x_0, t_0). We show that the maximum of T does not increase using the generalisation of the maximum principle for tensors of Hamilton [16, Theorem 9.1], that is, [6, Theorem 3.2].

The argument is the same as in the proof of [10, Lemma 11] except we just need to check that the h term in (7) has the right sign. We have

$$h\left(1-C\,\mathrm{trace}\,\dot{F}_*\right)\overline{g}_{ij}v^iv^j \leq 0$$

since at (x_0,t_0),

$$r_{\max} = Cf_* = C\sum_i \dot{f}_*^i\, r_i \leq C\sum_i \dot{f}_*^i\, r_{\max}$$

so

$$1-C\,\mathrm{trace}\,\dot{f}_* \leq 0.$$

It follows by the maximum principle for tensors that $T_{ij} \leq 0$ everywhere as long as the solution exists and therefore $r_{ij} \leq CF_*\overline{g}_{ij}$. In view of Conditions 1.1, e) iv) b), the quotients $\frac{r_i}{f_*}$ remain contained within a compact subset of the part of the unit sphere within Γ showing the weak monotonicity of the quantity $\sup_{\substack{\omega\in T_x\mathbb{S}^n\\ |\omega|=1}} \frac{r(\omega,\omega)(x,t)}{f_*(r(x,t))}$ under the flow. That this quantity is strictly decreasing unless M_t is a sphere follows by the strong maximum principle as in [10, Lemma 11]. □

Remark: In view of Lemma 3.2 it follows from Condition 1.1, e) iii) that there exists $C > 0$ such that $r_{\max} \leq Cr_{\min}$ holds under the flow (6). This in turn implies that $h_{ij} \geq \varepsilon H g_{ij}$ is maintained under the flow, for a small ε depending on n, M_0 and the particular F.

Completion of the proof of Theorem 1.2 in this case: Long time existence and regularity now follows by appropriate adjustments to the corresponding argument in [28]. Writing the evolving hypersurface locally as a graph with bounded gradient, that is, letting $X : U \subset \mathbb{R}^n \times [0,T) \to \mathbb{R}^{n+1}$ be given by

$$X(x,t) = (x, z(x,t)), \tag{8}$$

the graph height function z satisfies a uniformly parabolic evolution equation with bounded measurable coefficients:

$$\frac{\partial z}{\partial t} = \sqrt{1+|Dz|^2}\,\{h(t) - F(\mathscr{W})\} = g^{ik}\dot{F}_{ij}(\mathscr{W})\,D_kD_jz + \sqrt{1+|Dz|^2}h(t)\,; \tag{9}$$

the evolution equation for F is likewise uniformly parabolic with bounded measurable coefficients. Indeed, uniform parabolicity and boundedness of the coefficients follow from Corollaries 2.5 and 2.6. It follows by a result of Krylov and Safonov [21] that the quantities z and F are $C^{0,\alpha}$ in spacetime.

For higher regularity, we again use the freezing time idea of [28, Section 8] but in view of the property that f is inverse concave, we adopt the parametrisation of [10, Section 6]. Specifically, set $e_0 = (0,1) \in \mathbb{S}^n \subset \mathbb{R}^{n+1} \simeq \mathbb{R}^n \times \mathbb{R}$. For points $z = (\tilde{z}, z_0)$ in the upper hemisphere, write $s(z) = z_0\,\sigma\left(\frac{\tilde{z}}{z_0}\right)$. As discussed in [10], positivity of the matrix (r_{ij}) is equivalent to convexity of $\sigma : \mathbb{R}^n \to \mathbb{R}$. Similarly as in [10, Equation (21)], we may derive the evolution equation

$$\frac{\partial}{\partial t}\sigma(x) = \sqrt{1+|x|^2}\left[h(t) - F_*\left(\mathscr{W}^{-1}\right)^{-1}\right] = \sqrt{1+|x|^2}h(t) + \frac{\Theta\left(Q\circ D^2\sigma\circ Q\right)}{1+|x|^2}, \tag{10}$$

where $Q^{ij} = \delta^{ij} + \frac{x^ix^j}{1+\sqrt{1+|x|^2}}$ and $\Theta(A) = \theta(a)$ is a symmetric function of the eigenvalues a of A.

At a fixed time t_0, equation (10) is uniformly elliptic and the operator Θ is concave, so [13, Theorem 3]; gives that σ is locally $C^{2,\alpha}$ at time t_0. We note that these spatial

$C^{2,\alpha}$ estimates depend on the pinching cone; since this is preserved via Lemma 3.2, the spatial $C^{2,\alpha}$ estimates are independent of t_0. The remainder of the argument to show $X \in C^{k,\alpha}(\mathbb{S}^n \times [0,\infty))$ is as in [28, Section 8], using in particular time regularity via the maximum principle argument in [7] and Schauder estimates (see, eg [23]) for higher regularity. In view of Lemma 3.2, a stability argument as in [27, Section 5] and [28, Section 9] may be used to show exponential convergence of the solution image hypersurfaces to the sphere, with the same value of the fixed V_{n-k} as M_0. This completes the proof of Theorem 1.2 in this case. □

4. THE CASE OF CONDITIONS 1.1 E) V) A)

In the next two sections, instead of considering the evolution equations on $\mathbb{S}^n$ to establish pinching, we instead compute directly on the evolving hypersurface M_t, as in [19], for example.

Lemma 4.1. *Under the flow* (1) *any smooth, symmetric function of the principal curvatures* $G(\mathscr{W}) = g(\kappa(\mathscr{W}))$ *evolves according to*

$$\frac{\partial}{\partial t} G = \mathscr{L} G + \left(\dot{G}^{ij} \ddot{F}^{kl,rs} - \dot{F}^{ij} \ddot{G}^{kl,rs}\right) \nabla_i h_{kl} \nabla_j h_{rs} + \dot{F}^{kl} h_k^{\ m} h_{ml} \dot{G}^{ij} h_{ij} - h \dot{G}^{ij} h_i^{\ m} h_{mj},$$

where $\mathscr{L} = \dot{F}^{kl} \nabla_k \nabla_l$ *and* ∇ *denotes the covariant derivative on* M_t.

Proposition 4.2. *Under the flow* (1), *with* M_0 *axially symmetric, the pinching ratio of the different principal curvatures,* $r = \frac{\kappa_1}{\kappa_2}$ *does not deteriorate, that is*

$$\min_{M_0} \frac{\kappa_1}{\kappa_2} \le \left. \frac{\kappa_1}{\kappa_2} \right|_{(x,t)} \le \max_{M_0} \frac{\kappa_1}{\kappa_2}.$$

In particular, the evolving hypersurface M_t *remains strictly convex. Moreover, the pinching ratio strictly improves (ie. gets closer to* 1*) unless* M_t *is a sphere.*

Remark: Rewriting the above as $\underline{C} \le \left. \frac{\kappa_1}{\kappa_2} \right|_{(x,t)} \le \overline{C}$, we have $\kappa_i \ge \min\left\{\underline{C}, \frac{1}{\overline{C}}\right\} \kappa_j$ so

$$n\, \kappa_i \ge \min\left\{\underline{C}, \frac{1}{\overline{C}}\right\} \left[\kappa_1 + (n-1)\, \kappa_2\right] = \min\left\{\underline{C}, \frac{1}{\overline{C}}\right\} H,$$

that is, $h_{ij} \ge \varepsilon H g_{ij}$ with $\varepsilon = \frac{1}{n} \min\left\{\underline{C}, \frac{1}{\overline{C}}\right\}$.

Proof of Proposition: The proof is similar to that of [30, Theorem 7.1]; again we need to check that the h term in the evolution equation for the appropriate pinching function has the right sign. We use Lemma 4.1 with the degree zero homogeneous quantity

$$G = \frac{n \left|A^0\right|^2}{H^2}$$

and note that G is positive on M_0 since M_0 is convex, unless M_0 is a sphere. Therefore, unless M_0 is a sphere, G remains positive for at least a short time by continuity. Let us restrict initially to a short time in which M_t remains convex. At a maximum point (x_0, t_0) of G we must have $G > 0$ and therefore $\kappa_1 \ne \kappa_2$ since otherwise $G|_{(x,t_0)} \equiv 0$ implying $\kappa_1 \equiv \kappa_2$ and M_{t_0} is a sphere.

With a slight abuse of notation, we may write

$$g(\kappa_1, \kappa_2) = \frac{n\left[\kappa_1^2 + (n-1)\, \kappa_2^2\right] - \left[\kappa_1^2 + (n-1)\, \kappa_2^2\right]}{\kappa_1^2 + (n-1)\, \kappa_2^2}$$

and we calculate

$$\dot{g}_1 = \frac{2n(n-1)\kappa_2(\kappa_1-\kappa_2)}{H^3} \text{ and } \dot{g}_2 = \frac{2n\kappa_1(\kappa_2-\kappa_1)}{H^3}.$$

From these we observe using diagonal coordinates that $\dot{G}$ is nondegenerate at this maximum point (x_0,t_0). Further

$$\sum_i \dot{g}^i\kappa_i^2 = \frac{2n}{H^3}\sum_i\left(H\kappa_i^3 - |A|^2\kappa_i^2\right) = \frac{2n}{H^3}\left[HC - \left(|A|^2\right)^2\right] \geq 0 \tag{11}$$

using Lemma 2.1 (i) and our assumption that M_{t_0} is convex. Moreover, as in [30], the gradient term in Lemma 4.1 is, in view of Lemma 2.3 equal to

$$\frac{4n(n-1)f}{H^3\kappa_2(\kappa_2-\kappa_1)}\kappa_2(\kappa_1-\kappa_2)(\nabla_1 h_{22})^2 = -\frac{4n(n-1)f}{H^3}(\nabla_1 h_{22})^2 \leq 0,$$

where again we have used that M_{t_0} is convex so $H>0$. It follows that the maximum of G does not increase and therefore, as in [30], the pinching ratio does not deteriorate under the flow (1). Since M_0 was strictly convex, that the pinching ratio does not deteriorate implies that M_t remains strictly convex, as long as the solution to (1) exists.

To show the strict improvement of the pinching ratio unless M_t is a sphere we use the strong parabolic maximum principle. Suppose G attained a new extremum at some (x_0,t_0), $t_0>0$. The strong maximum principle then implies that G is identically constant. If this constant is 0 then M_{t_0} is a sphere and we are done. So suppose on the other hand G is identically equal to a positive constant. From the evolution equation of Lemma 4.1 and the fact that $h>0$, we must have

$$\sum_i \dot{g}^i\kappa_i^2 = \frac{2n}{H^3}\left[HC - \left(|A|^2\right)^2\right] \equiv 0.$$

In view of Lemma 2.1 (i), the strictly convex axially symmetric hypersurface M_{t_0} must have everywhere $\kappa_1=\kappa_2$ and is therefore a sphere. □

Remark: In the case of an unconstrained flow, a similar strong maximum principle argument as above using instead the ∇A term and Lemma 2.3 again shows the pinching ratio strictly improves unless M_t is a sphere. Such an argument could be used together with a linearisation of the rescaled flow and a stability argument to show exponential convergence to an asymptotically round point for the convex, axially symmetric contracting hypersurfaces in [30, Theorem 7.1].

Completion of the proof of Theorem 1.2 in this case: Our argument for long time existence in this case is simplified by axial symmetry, which reduces our problem to the setting of a scalar parabolic PDE with one spatial direction. While there are various results particular to such parabolic PDEs (see, for example, [14, 24] and the references therein), here we use an argument more closely related to that in the previous section; when we fix time the resulting evolution equation is an ODE.

Specifically, let us parametrise the evolving hypersurface as a radial graph by $X:\mathbb{S}^n\times[0,T)\to\mathbb{R}^{n+1}$ where $X(\theta,\omega,t)=\ell(\theta,t)\,\omega$, where $\omega\in\mathbb{S}^{n-1}$, $\theta\in[0,2\pi]$ and in view of symmetry $\ell(\pi+\delta,t)=\ell(\pi-\delta,t)$ for $\delta\in[0,\pi]$. The construction is such that the multiply-covered points $X(0,\omega,t)$ and $X(\pi,\omega,t)$ lie on the axis of symmetry of the hypersurface, while for other fixed θ, the image $X(\theta,\omega,t)$ gives a 'slice' of M_t perpendicular to the axis of symmetry. The initial hypersurface M_0 corresponds to a given initial positive function $\ell_0 : [0,2\pi] \to \mathbb{R}$ with the property that $\ell_0(\pi+\delta,t) = \ell_0(\pi-\delta,t)$ for $\delta \in [0,\pi]$. As in the previous section, we again need to add a tangential term to (1) such that this parametrisation is preserved. It is straightforward to check that the resulting evolution equation for ℓ is

$$\frac{\partial \ell}{\partial t} = \frac{\sqrt{\ell^2 + (\ell')^2}}{\ell}\left[h(t) - F(\mathscr{W})\right], \tag{12}$$

where the Weingarten map matrix is diagonal with

$$h^1{}_1 = -\frac{1}{\left(\ell^2+(\ell')^2\right)^{\frac{3}{2}}}\left(\ell\ell'' - 2\left(\ell'\right)^2 - \ell^2\right), h^j{}_j = \frac{1}{\sqrt{\ell^2+(\ell')^2}}, j = 2,\ldots,n.$$

Above subscripts 1 denotes the θ direction and $'$ to denote differentiation with respect to θ. We may consider spatially periodic solutions of (12) with period 2π.

Observed from (12) that at any fixed time t_0, spatial Hölder continuity of the quantity

$$v_{t_0} := \frac{\ell}{\sqrt{\ell^2+(\ell')^2}}\left.\frac{\partial \ell}{\partial t}\right|_{t_0}$$

follows from Hölder continuity of $F(\mathscr{W})$. In view of Corollary 2.5, Hölder continuity of F follows from the result of Krylov and Safonov [21].

Moreover, since the Weingarten map is everywhere diagonal, using the homogeneity of F we have from (12)

$$f\left(-\frac{1}{\ell^2+(\ell')^2}\left(\ell\ell''-2\left(\ell'\right)^2-\ell^2\right),1,\ldots,1\right) = \sqrt{\ell^2+(\ell')^2}\left[h(t_0) - v_{t_0}(\theta)\right]$$

and we have continued to suppress the argument (θ, t_0) of ℓ and its derivatives. Recalling Conditions 1.1, the above left hand side is a smooth, positive, strictly increasing function, say $\hat{f}$, of one positive variable. Employing the inverse function for $\hat{f}$, which we will denote $\hat{f}^{-1}$, the above ODE may be written more explicitly as

$$\ell'' = \ell + \frac{2\ell'}{\ell} - \frac{\left(\ell^2+(\ell')^2\right)}{\ell}\hat{f}^{-1}\left(\sqrt{\ell^2+(\ell')^2}\left[h(t_0)-v_{t_0}(\theta)\right]\right). \tag{13}$$

In view of Corollary 2.5 b), ℓ has a positive lower bound, moreover, recall $v_{t_0} \in C^{2,\alpha}$. Therefore the solution of (13) is $C^{2,\alpha}$ by standard theory for ordinary differential equations (see, for example, [18]). Again we note that the $C^{2,\alpha}$ estimate depends only on the pinching cone, which is preserved under the flow by Proposition 4.2; therefore these estimates are independent of t_0.

Time regularity of first and second spatial derivatives of ℓ may now be deduced using time difference quotients and the parabolic maximum principle similarly as in [7], for example. Together with Hölder continuity of ℓ and $F \circ \mathscr{W}$ we have all the ingredients to deduce Hölder continuity of $h(t)$. Therefore, from (12) we see that $\frac{\partial \ell}{\partial t} \in C^{0,\alpha}\left(\mathbb{S}^1 \times [\delta, T)\right)$. In view of short-time existence, the regularity estimates can be extended to $\mathbb{S}^1 \times [0,T)$.

The remainder of the proof for long time existence and exponential convergence to the sphere as $t \to \infty$ is the same as in [28]. □

5. The case of Conditions 1.1 e) v) b)

First note, as in [9], since Γ is open, there exists a $\delta_0 \in \left(0, \frac{1}{n(n-1)}\right)$ such that

$$\Gamma_0 = \left\{ \kappa \in \Gamma : \left|A^0\right|^2 \le \delta_0 H^2 \right\} \subset \Gamma.$$

Secondly, since f is smooth, we have as in [9] that there is a constant $\mu \ge 0$ such that, for arbitrary A with $\kappa(A) \in \Gamma_0$ and arbitrary symmetric 2-tensors B and C,

$$\left|\ddot{F}^{kl,rs}(A) B_{kl} C_{rs}\right| \le \frac{\mu}{H} |B| |C|, \tag{14}$$

from which corresponding control on $\dot{F}$ and on F itself follows by integration. Specifically, we have as in [9]

$$\left(1 - \frac{\mu \left|A^0\right|}{H}\right) I \le \dot{F} \le \left(1 + \frac{\mu \left|A^0\right|}{H}\right) I \tag{15}$$

and

$$|F - H| \le \frac{\mu}{2} \frac{\left|A^0\right|^2}{H}.$$

We show in this case sufficiently strong curvature pinching is preserved under (1) analogous to the unconstrained case in [9, Theorem 5.1]. The result is

Theorem 5.1. *Let F satisfy Conditions 1.1 including part v) b) of e). There exists a $\delta_1 \in \left(0, \frac{1}{n(n-1)}\right)$, depending on n and μ such that, if $0 < \sigma \le \min\{\delta_0, \delta_1\}$ and $\left|A^0\right|^2 < \sigma H^2$ at every point of M_0, then this inequality remains true for $t > 0$ under the flow* (1).

Proof: The proof is similar to that in [9]; we consider the evolution equation for $Z_\sigma = \left|A^0\right|^2 - \sigma H^2$. In view of (1), the Weingarten map now evolves according to

$$\frac{\partial}{\partial t} h^i{}_j = \mathscr{L} h^i{}_j + \ddot{F}^{kl,pq} \nabla^i h_{kl} \nabla_j h_{pq} + \dot{F}^{kl} h_k^{\ p} h_{pl} h^i{}_j - h\, h^{im} h_{mj}.$$

It follows that $Z_\sigma = \left|A^0\right|^2 - \sigma H^2$ evolves according to

$$\begin{aligned} \frac{\partial}{\partial t} Z_\sigma = {} & \mathscr{L} Z_\sigma + 2\left[h^{ij} - \left(\frac{1}{n} + \sigma\right) H g^{ij}\right] \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} \\ & - 2\dot{F}^{ij}\left[\nabla_i h_{kl} \nabla_j h^{kl} - \frac{1}{n} \nabla_i H \nabla_j H\right] 2\sigma \dot{F}^i \nabla_i H \nabla_j H \\ & + 2\dot{F}^{kl} h_k^{\ p} h_{pl} Z_\sigma - \frac{2}{n} h \left[nC - (1 + n\sigma) H |A|^2\right]. \end{aligned}$$

Of course, this equation has the same form as in the unconstrained case, equation (5.1) in [9] in the case F is homogeneous of degree $\alpha = 1$, but with the addition of the h term. In view of Lemma 2.1 (ii), the h term above has the same sign as the zero order $(1 - \alpha) F$ term that appears in equation (5.1) of [9], so the same maximum principle argument as in [9] applies. □

Remark: It is possible to show, by taking a slightly smaller σ above, still depending only on n and μ that the quantity Z_σ is in fact strictly decreasing. We use this later to show that convergence to the sphere is exponential.

Completion of the proof of Theorem 1.2 in this case: Similarly as in Section 3 we adopt a local graph representation for the evolving hypersurface with graph height z such that $|Dz|$ and $\left|D^2 z\right|$ are locally bounded. Recalling (8) and using the so-called 'square root matrix' $g^{-1\backslash 2}$ of the inverse metric, we can write the evolution equation for z as

$$\frac{\partial z}{\partial t} = g_{kp}^{-1/2} \dot{F}^{pq} g_{ql}^{-1/2} D_{kl} z + \sqrt{1+|Dz|^2}\, h(t)\,. \tag{16}$$

Taking an even smaller σ in Theorem 5.1 if necessary, we can ensure in view of (15) that (16) is not only uniformly parabolic but the coefficient matrix of the second derivatives of z is as close as we need to the identity. Since $h(t)$ is also bounded, by Corollary 2.6, the equation (16) satisfies the conditions required to infer $z \in C^{1,\alpha}$ in space-time, by the parabolic Cordes-Nirenberg type estimate (see [23, Lemma 12.13], for example, or the statement [9, Theorem 7.3])).

Next, we consider the equation for first spatial derivatives of z: for each i,

$$\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x_i}\right) = g_{kp}^{-1/2} \dot{F}^{pq} g_{ql}^{-1/2} D_{kl}\left(\frac{\partial z}{\partial x_i}\right) + \dot{F}^{kl} \frac{\partial}{\partial x_i}\left(g_{kp}^{-1}\right) D_{pl} z + h(t) \frac{D_m z}{\sqrt{1+|Dz|^2}} D_m\left(\frac{\partial z}{\partial x_i}\right).$$

Again, the equation is uniformly parabolic and the coefficient matrix of the second derivatives of $\frac{\partial z}{\partial x_i}$ can be made sufficiently close to the identity. The spatial derivative of g_{kp}^{-1} involves only first and second derivatives of z, which are locally bounded, so the lower order terms are bounded and the parabolic Cordes-Nirenberg estimate again applies to give $\frac{\partial z}{\partial x_i}$ is $C^{1,\alpha}$ in space-time. It follows that $\mathscr{W}$, E_{k+1} and μ are each Hölder continuous in time, so $h(t)$ is also Hölder continuous and we observe therefore from (9) that $\frac{\partial z}{\partial t} \in C^{0,\alpha}$. Hence $z \in C^{2,\alpha}$ and this estimate can be made global by a standard argument. Higher, global $C^{k,\alpha}$ regularity now follows by a standard inductive argument using Schauder estimates (for the parabolic Schauder estimate see [23, Theorem 4.9], for example). The independence of the estimates on the maximal time T implies $T = \infty$.

To show that convergence to the sphere is exponential, we may use the evolution equation for $f_0 = \frac{|A^0|^2}{H^2}$ in a similar way as in [20] equation (10), for example. The evolution equation for f_0 has extra terms in this case since F is fully nonlinear, nevertheless, it is straightforward to compute as

$$\begin{aligned} \frac{\partial}{\partial t} f_0 = \mathscr{L} f_0 + \frac{1}{H^2} \dot{F}^{kl} \nabla_k f_0 \nabla_l H^2 + \frac{2}{H^3}\left(H h^{ij} - |A|^2 g^{ij}\right) \ddot{F}^{kl,rs} \nabla_i h_{kl} \nabla_j h_{rs} \\ - \frac{2}{H^4} \dot{F}^{kl} \left(H \nabla_k h_{ij} - h_{ij} \nabla_k H\right)\left(H \nabla_l h^{ij} - h^{ij} \nabla_l H\right) - \frac{2h}{H^3}\left[HC - \left(|A|^2\right)^2\right]. \end{aligned} \tag{17}$$

We estimate the gradient terms that do not contain the ∇f_0 factor as follows: firstly we calculate using diagonal coordinates

$$\begin{aligned} \frac{1}{H^2}\left|H h^{ij} - |A|^2 g_{ij}\right|^2 &= \frac{1}{H^2} \sum_i \left(H\kappa_i - |A|^2\right)^2 \\ &= \frac{1}{H^2}\left[H^2 |A|^2 - 2H^2 |A|^2 + n\left(|A|^2\right)^2\right] = n\left|A^0\right|^2 |A|^2 \leq \sigma(n\sigma+1) H^2, \end{aligned}$$

where we have invoked the estimate of Theorem 2.5. Therefore, using (14),

$$\frac{2}{H^2}\left(h^{ij}-\frac{1}{n}Hg^{ij}-f_0Hg^{ij}\right)\ddot{F}^{kl,rs}\nabla_i h_{kl}\nabla_j h_{rs}\leq\frac{2\mu}{H^4}\left|Hh^{ij}-|A|^2g_{ij}\right||\nabla A|^2$$
$$\leq 2\mu\sqrt{(1+n\sigma)\,\sigma}\frac{|\nabla A|^2}{H^2}.$$

Secondly we use (15) and then Lemma 2.1 (iii), in a similar way as in the proof of [29, Theorem 1.7] to estimate

$$\dot{F}^{kl}\left(H\nabla_k h_{ij}-h_{ij}\nabla_k H\right)\left(H\nabla_l h^{ij}-h^{ij}\nabla_l H\right)\geq\left(1-\frac{\mu\left|A^0\right|}{H}\right)\left|H\nabla_k h_{ij}-h_{ij}\nabla_k H\right|^2$$
$$\geq\left(1-\mu\sqrt{\sigma}\right)\left(\frac{n-1}{2}\right)\varepsilon^2H^2|\nabla A|^2,$$

where we have again used the curvature pinching of Theorem 5.1 and ε is given by (4). Therefore, at an extremum of f_0, the gradient terms in (17) may be estimated by

$$\left[2\mu\sqrt{(1+n\sigma)\,\sigma}+\left(\mu\sqrt{\sigma}-1\right)(n-1)\left(\frac{1-\sqrt{n\,(n-1)\,\sigma}}{n}\right)^2\right]\frac{|\nabla A|^2}{H^2}.$$

This expression is clearly nonpositive on $0\leq\sigma\leq\sigma_0$ for some σ_0 depending on n and μ.

Now, in view of Corollary 2.6 and the Krylov-Safonov Harnack inequality [21], the minimum of F is bounded below away from zero; using this or Lemma 2.7 we have that $h(t)$ is also bounded below away from zero and thus it follows using Lemma 2.1 (i) that for almost every t,

$$\frac{d}{dt}\max_{M_t}f_0\leq-\delta\max_{M_t}f_0.$$

This implies

$$f_0(x,t)\leq\max_{M_0}f_0\cdot e^{-\delta t}.$$

The limiting value as $t\to\infty$, $f_0(\cdot,\infty)=0$ is attained only on a sphere, whose radius is determined via the value of the preserved mixed volume under (1). Hence, in view of the definition of f_0, the principal curvatures of M_t decay exponentially to their value on this sphere. Exponential convergence of all curvature derivatives to zero follows by interpolation, since the derivatives of A^0 control those of A (see, for example, [31]). The stability argument in [28, Section 9], using [25, Theorem 9.1.2], gives that the solution hypersurfaces M_t converge exponentially to the sphere modulo translations. A standard argument (see, for example [5]) gives exponential convergence of the embeddings $X(\mathbb{S}^n,t)$ to the limit embedding of a sphere without any correction for translations. □

6. Speeds of higher homogeneity

Typically, the analysis of curvature contraction type flows (ie the leading order term corresponds to contraction) becomes more delicate when the degree of homogeneity of the speed is not equal to 1. In particular, the first derivative $\dot{F}$ is no longer homogeneous of degree zero, so uniform parabolicity of evolution equations does not follow directly from a curvature pinching estimate. A lower speed bound becomes more important, and no longer follows directly from the Harnack inequality. Moreover, evolution equations for curvature pinching and other quantities become more complicated, typically introducing additional terms whose sign depends upon the degree of homogeneity, sometimes in opposite ways.

In view of the result of [33] and since the relevant results of [9, 30] continue to hold where the degree of homogeneity of the speed is $\alpha > 1$ it is natural to consider whether the corresponding constrained flows with F homogeneous of degree $\alpha > 1$ also evolve suitable initial hypersurfaces to spheres. In this section we show that this is indeed the case. The main results are that suitable pinching estimates continue to hold and it is possible to establish a useful lower bound on F, at least on finite time intervals after a short time. In contrast to previous work on evolving hypersurfaces by higher homogeneity speeds [32, 1, 12, 33], we do not require here any estimates for equations of porous medium type.

Under the flow (1), where F is now homogeneous of degree α, a symmetric function G of the principal curvatures evolves according to

$$\frac{\partial}{\partial t}G = \mathscr{L}G + \left(\dot{G}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{G}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} + \dot{F}^{kl}h_k^{\ m}h_{ml}\dot{G}^{ij}h_{ij} + (1-\alpha)F\dot{G}^{ij}h_i^{\ m}h_{mj} - h\dot{G}^{ij}h_i^{\ m}h_{mj},$$

where $\mathscr{L}G = \dot{F}^{ij}\nabla_i\nabla_j G$. Compared with the proof of Proposition 4.2, the extra term in the evolution equation for $G = \frac{n\left|A^0\right|^2}{H^2}$ is

$$(1-\alpha)F\dot{G}^{ij}h_i^{\ m}h_{mj} = \frac{2n(1-\alpha)F}{H^3}\left[HC - \left(|A|^2\right)^2\right], \tag{18}$$

where we have used the same calculation as for (11). For $\alpha \geq 1$ this term is clearly nonpositive. Now, as in [30], for the gradient terms to have the right sign requires the pinching ratio to be not greater than $1 + \frac{2}{\alpha - 1}$. The same argument as in the proof of Proposition 4.2 then shows that the pinching ratio does not deteriorate in the case of evolving axially symmetric hypersurfaces.

In the other case we consider, where $G = Z_\sigma = \left|A^0\right|^2 - \sigma H^2$ for suitably small σ, the additional term in the evolution equation is now exactly as in [9, Equation (5.1)]:

$$\frac{2}{n}(1-\alpha)F\left[nC - (1+n\sigma)H|A|^2\right].$$

By the same argument as in the proof of [9, Theorem 5.1] using in particular Lemma 2.3 of [9], this extra term is nonpositive for $\alpha \geq 1$ at points where $Z_\sigma = 0$. So we again conclude $Z_\sigma < 0$ and thus pinching is preserved.

An upper bound on F continues to hold if F is homogeneous of degree $\alpha \geq 1$, again similarly to the argument of Tso [35]. Here we again use an absolute lower bound for the inradius in terms of the preserved mixed volume [28, Corollary 3.6]; this is a flow-independent consequence of curvature pinching. For lower speed bounds, we consider the cases of strong curvature pinching and axially symmetric hypersurfaces separately.

In the case of strong curvature pinching, we needed sufficiently strong curvature control that the Cordes-Nirenberg estimates could be applied. In this case, for a lower bound on F on finite time intervals, after a sufficient time, we adapt an idea of Smoczyk [34] (see also [9]); here this involves placing another requirement on the sufficiently strong pinching.

Lemma 6.1. *Set $\varphi_{t_0}(t) = \int_{t_0}^t h(s)\,ds$ and let $\beta \in \mathbb{R}$ be a constant. Under the flow* (1)*, for any parameter $\beta \in \mathbb{R}$ and any fixed point $p \in \mathbb{R}^{n+1}$ we have the evolution equation*

$$\begin{aligned}\frac{\partial}{\partial t}&\left[\langle X-p,\nu\rangle + \beta(t-t_0)F - \varphi_{t_0}\right] \\ &= \mathscr{L}\left[\langle X-p,\nu\rangle + \beta(t-t_0)F - \varphi_{t_0}\right] + \dot{F}^{kl}h_k^{\ m}h_{ml}\left[\langle X-p,\nu\rangle + \beta(t-t_0)F - \varphi_{t_0}\right] \\ &\quad + [\beta - (\alpha+1)]F + \left[\varphi_{t_0} - \beta(t-t_0)h(t)\right]\dot{F}^{kl}h_k^{\ m}h_{ml}.\end{aligned} \tag{19}$$

We wish to show that the spatial minimum of $\left[\langle X-p,\nu\rangle+\beta\,(t-t_0)\,F-\varphi_{t_0}\right]$ does not decrease, at least for a short time after $t=t_0$. Certainly, we will need to choose $\beta\geq\alpha+1$, moreover, we need to show

$$[\beta-(\alpha+1)]\,F+\left[\varphi_{t_0}-\beta\,(t-t_0)\,h\,(t)\right]\dot{F}^{kl}h_k^{\ m}h_{ml}\geq 0$$

at least for a short time after $t=t_0$. Let us neglect the nonnegative φ_{t_0} term, so the remaining coefficient of $\dot{F}^{kl}h_k^{\ m}h_{ml}$ is clearly nonpositive for any $\beta>0$. In view of preserved curvature pinching, there is a constant $\overline{C}>0$ such that

$$\frac{\dot{F}^{kl}h_k^{\ m}h_{ml}}{F^{\alpha+1}}\leq\overline{C},$$

that is

$$\dot{F}^{kl}h_k^{\ m}h_{ml}\leq\overline{C}F^{\alpha+1},$$

so we estimate

$$\begin{aligned}
&[\beta-(\alpha+1)]\,F+\left[\varphi_{t_0}-\beta\,(t-t_0)\,h\,(t)\right]\dot{F}^{kl}h_k^{\ m}h_{ml}\\
&\geq[\beta-(\alpha+1)]\,F-\beta\,(t-t_0)\,\overline{F}\,\overline{C}\,F^{1+\frac{1}{\alpha}}\\
&\geq\left\{[\beta-(\alpha+1)]-\beta\,(t-t_0)\,\overline{C}\,\overline{F}^{1+\frac{1}{\alpha}}\right\}F=\left\{\beta\left[1-(t-t_0)\,\overline{C}\,\overline{F}^{1+\frac{1}{\alpha}}\right]-(\alpha+1)\right\}F,
\end{aligned}$$

where we have also used the upper bound on F from Corollary 2.6, denoted $\overline{F}$, to estimate both F and h. The expression in braces above is clearly nonnegative provided

$$\beta\geq\frac{\alpha+1}{1-\overline{C}\,\overline{F}^{1+\frac{1}{\alpha}}\,(t-t_0)}.$$

Suppose we take $t-t_0\leq\frac{1}{2\overline{C}\,\overline{F}^{1+\frac{1}{\alpha}}}$. Then we may take $\beta=2\,(\alpha+1)$ and for $t\in\left[t_0,t_0+\frac{1}{2\overline{C}\,\overline{F}^{1+\frac{1}{\alpha}}}\right]$, the above shows that the minimum of $\left[\langle X-p,\nu\rangle+\beta\,(t-t_0)\,F-\varphi_{t_0}\right]$ does not decrease.

Corollary 6.2. *Under the flow* (1)*, for any t_0 we have for a short time later*

$$F\,(\mathscr{W}\,(x,t))\geq\frac{r_-+\varphi_{t_0}-R_{t_0}\,(t)}{\beta\,(t-t_0)},$$

where r_- is the uniform inradius bound on M_t and $R_{t_0}\,(t)$ is the radius of a ball that encloses M_{t_0} and evolves under (1)*, with the same $h\,(t)$ as that of M_t. In particular, for curvature pinching ratio η close enough to* 1*, we have a positive lower speed bound after a given waiting time.*

Proof: The proof is similar to that of [9, Proposition 12.1]. Under the flow (1), while $\left[\langle X-p,\nu\rangle+\beta\,(t-t_0)\,F-\varphi_{t_0}\right]$ remains positive, that is, while $t-t_0<\frac{1}{\overline{C}\,\overline{F}^{1+\frac{1}{\alpha}}}$, we have equivalently that

$$F\,(\mathscr{W}\,(x,t))\geq\frac{\langle p-X\,(x,t)\,,\nu\,(x,t)\rangle+\varphi_{t_0}\,(t)}{\beta\,(t-t_0)}. \tag{20}$$

For a given x, choose $p\in M_{t_0}$ such that $\langle p,\nu\,(x,t)\rangle$ is maximised. By this choice, and definition of the support function, $\langle p,\nu\,(x,t)\rangle$ is the support function of M_{t_0} at some point, but not necessarily at $X\,(x,t_0)$. Nevertheless, $\langle p,\nu\,(x,t)\rangle\geq r_-$, since M_{t_0} encloses $B_{r_-}\,(q)$ for some q enclosed by M_{t_0}.

Let $R_{t_0}(t_0) = r_+$, the outer radius of M_{t_0}, so there is a sphere of radius $R_{t_0}(t_0)$ that encloses M_{t_0}. Let this sphere evolve by (1), with the $h(t)$ determined by the evolving M_t. By the comparison principle, M_t remains enclosed by the evolving sphere, whose radius satisfies

$$\frac{d}{dt}R_{t_0}(t) = h(t) - R_{t_0}^{-\alpha}(t). \tag{21}$$

Note that R_{t_0} is increasing, that is, the corresponding sphere is expanding, because its radius is greater than the radius of the sphere with the same value of V_{n-k} as M_t, so $h(t) > R_{t_0}^{-\alpha}(t)$. Therefore, in particular, $R_{t_0}(t) \geq r_+$.

Now $\langle X(x,t), \nu(x,t)\rangle$ is the support function of M_t, so $\langle X(x,t), \nu(x,t)\rangle \leq R_{t_0}(t)$ and from (20) we estimate

$$F(\mathscr{W}(x,t)) \geq \frac{r_- + \varphi_{t_0}(t) - R_{t_0}(t)}{\beta(t-t_0)}.$$

Clearly, for $t - t_0$ very small, the right hand side of the above is negative. But we wish to obtain from the above a positive lower bound on F after a certain waiting time, that itself must fall within $\left[t_0, t_0 + \frac{1}{\overline{C}\overline{F}^{1+\frac{1}{\alpha}}}\right]$.

Consider $R_{t_0}^+(t) := r_- + \varphi_{t_0}(t)$, the solution of the ODE

$$\frac{d}{dt}R_{t_0}^+ = h(t)$$

corresponding to a sphere expanding from $B_{r_-}(q)$ at $t = t_0$. In view of (21),

$$\frac{d}{dt}\left(R_{t_0}^+ - R_{t_0}\right) = R_{t_0}^{-\alpha} \geq r_+^{-\alpha}.$$

It follows that for $t > t_0$,

$$R_{t_0}^+(t) - R_{t_0}(t) \geq R_{t_0}^+(t_0) - R_{t_0}(t_0) + r_+^{-\alpha}(t-t_0),$$

that is,

$$r_- + \varphi_{t_0}(t) - R_{t_0}(t) \geq r_- - r_+ + r_+^{-\alpha}(t-t_0).$$

For a positive lower speed bound, it suffices to ensure the above right hand side is positive. This will be true provided $t - t_0 > r_+^{\alpha}(r_+ - r_-)$. Using Corollary 2.9 we can ensure that this falls within the time on which we have $\left[\langle X-p, \nu\rangle + \beta(t-t_0)F - \varphi_{t_0}\right]$ positive by requiring the pinching ratio sufficiently small. Specifically, it suffices to have

$$\frac{1}{2\overline{C}\overline{F}^{1+\frac{1}{\alpha}}} > r_+^{\alpha}(r_+ - r_-);$$

this can be ensured by requiring the pinching ratio to be sufficiently close to 1, depending via $\overline{C}$ and $\overline{F}$ only on n, α and M_0. □

In the case of axially symmetric hypersurfaces, we instead use a geometric argument together with curvature pinching and a lower barrier to obtain a lower speed bound. For any t, let us parametrise the generating curve of the axially symmetric hypersurface away from the poles by $Y(\theta)$, for $\theta \in (0,\pi)$, and denote by ω the rotational directions. In terms of the support function s of the generating curve (see, for example [2]), we may choose coordinates such that

$$Y(\theta) = s(\theta)(\sin\theta, \cos\theta) + s_\theta(\theta)(\cos\theta, -\sin\theta).$$

The inverse of the second fundamental form is diagonal; its components $r_{\omega\omega}$ corresponding to the rotational directions are then given by

$$r_{\omega\omega} = \overline{\nabla}_\omega \overline{\nabla}_\omega s + \overline{\sigma}_{\omega\omega} s = -\Gamma^\theta_{\omega\omega} s_\theta + \sin^2\theta\, s = \sin\theta\cos\theta\, s_\theta + \sin^2\theta\, s,$$

where Γ denotes the Christoffel symbols on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. Observe that this may be rewritten in terms of $Y(\theta)$ as

$$r_{\omega\omega} = \sin\theta\,[Y(\theta)\cdot(1,0)].$$

Therefore the corresponding entries of the inverse Weingarten map are given by

$$r^\omega_\omega = \frac{1}{\sin\theta}[Y(\theta)\cdot(1,0)].$$

Since M_t is strictly convex, we know that the above inner product is positive. In view of Corollary 2.5 e), we may therefore estimate the rotational curvatures of M_t:

$$\kappa_\omega = h^\omega_\omega = \frac{\sin\theta}{[Y(\theta)\cdot(1,0)]} \geq \frac{\sin\theta}{d} \tag{22}$$

where $d > 0$ is constant. This implies that away from the poles, the rotational curvatures have a positive lower bound. In view of curvature pinching, $\kappa_i \geq \eta\,\kappa_j$ (where $\eta = \frac{\alpha-1}{\alpha+1}$), the axial curvatures also have a positive lower bound and hence F is bounded below away from zero, away from the poles.

$$f(\kappa_\theta, \kappa_\omega, \ldots, \kappa_\omega) \geq f(\eta\,\kappa_\omega, \kappa_\omega, \ldots, \kappa_\omega) = \kappa^\alpha_\omega f(\eta, 1, \ldots, 1) \geq \left(\frac{\sin\theta}{d}\right)^\alpha f_\eta,$$

where $f_\eta = f(\eta, 1, \ldots, 1) > 0$ in view of Conditions 1.1, b).

To show that F is bounded away from zero near the poles, we may use stationary barriers. We give the details of the case for θ close to zero; the other case is similar. As in Section 3 it is convenient here to use coordinates on the sphere. Similarly as in Lemma 3.1, and as in the unconstrained case [10, Lemma 10], F now evolves according to

$$\frac{\partial F}{\partial t} = \alpha F^{1+\frac{1}{\alpha}} \dot{F}^{kl}_* \overline{\nabla}_k \overline{\nabla}_l F + [h(t) - F]\,\mathrm{tr}\,\dot{F}_*,$$

where $F_*\left(\mathscr{W}^{-1}\right) = F^{\frac{1}{\alpha}}(\mathscr{W})$ are homogeneous of degree 1 in their respective arguments. In view of curvature pinching,

$$\underline{C}\,\mathrm{Id} \leq \dot{F}_* \leq \overline{C}\,\mathrm{Id}$$

where Id is the $n\times n$ identity matrix. It follows that

$$\frac{\partial F}{\partial t} \geq \alpha F^{1+\frac{1}{\alpha}} \dot{F}^{kl}_* \overline{\nabla}_k \overline{\nabla}_l F - \mathrm{tr}\,\dot{F}_*\, F.$$

In view of axial symmetry, let us set $\hat{F}(\theta) := F(\mathscr{W})$ where θ will be the first coordinate direction. For the purpose of constructing a lower barrier we may extend $\hat{F}$ to be an even function and construct the stationary barrier on $[-\theta_*, \theta_*]$ for some $\theta_* > 0$. Computing the above derivatives on the sphere explicitly, we find

$$\frac{\partial \hat{F}}{\partial t} \geq \alpha \hat{F}^{1+\frac{1}{\alpha}} \left[\dot{F}^{11}_* \hat{F}_{\theta\theta} + \frac{(n-1)^2}{2} \dot{F}^{22}_* \sin 2\theta\, \hat{F}_\theta\right] - \mathrm{tr}\,\dot{F}_*\, \hat{F}.$$

It is therefore sufficient for a lower stationary barrier $V(\theta, t) = v(\theta)$ to satisfy

$$0 \leq \alpha v^{1+\frac{1}{\alpha}} \left[\dot{F}^{11}_* v'' + \frac{(n-1)^2}{2} \dot{F}^{22}_* \sin 2\theta\, v'\right] - \mathrm{tr}\,\dot{F}_*\, v,$$

where we have kept the same coefficients as for the equation for $\hat{F}$. Assuming $\theta_* \leq \frac{\pi}{4}$ say, and v is a positive, increasing, convex function, it is sufficient for v to satisfy

$$0 \leq \alpha \underline{C} v^{1+\frac{1}{\alpha}} v'' - n\overline{C} v, \tag{23}$$

while for the parabolic boundary condition it is sufficient to require

$$v(\theta_*) \leq \delta := \min\left(\min_{M_0} F, \left(\frac{\sin\theta_*}{d}\right)^{\alpha}\right) f_\eta.$$

It is straightforward to check that the function $v : [-\theta_*, \theta_*]$ given by

$$v^{\frac{\alpha+1}{2\alpha}}(\theta) = \left(\frac{\alpha+1}{2\alpha}\right)\sqrt{\frac{2n\overline{C}}{(\alpha-1)\underline{C}}}\,\theta + \frac{1}{2}\delta^{\frac{\alpha+1}{2\alpha}},$$

where $\theta_* = \min\left(\frac{\alpha}{\alpha+1}\sqrt{\frac{(\alpha-1)\underline{C}}{2n\overline{C}}}\delta^{\frac{\alpha+1}{2\alpha}}, \frac{\pi}{4}\right)$, satisfies the required conditions. Therefore, $F \geq v$ continues to hold under the flow on $[-\theta_*, \theta_*]$, while away from the pole we use the lower bound from (22).

Upper and lower speed bounds together with curvature pinching now imply curvature bounds and the flow with F homogeneous of degree $\alpha > 1$ is uniformly parabolic. Regularity and exponential convergence to the sphere now follow by similar arguments as in the previous sections (see also [7, 8, 9]).

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