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In this paper, we consider the pricing of credit default swaps (CDSs) with the reference asset driven by a geometric Brownian motion with a multi-scale stochastic volatility (SV), which is a two-factor volatility process with one factor controlling the fast time scale and the other representing the slow time scale. A key feature of the current methodology is to establish an equivalence relationship between the CDS and the down-and-out binary option through the discussion of "no default" probability, while balancing the two SV processes with the perturbation method. An approximate but closed-form pricing formula for the CDS contract is finally obtained, whose accuracy is in the order of $\mathcal{O}(\epsilon + \delta + \sqrt{\epsilon\delta})$.

AMS(MOS) subject classification.

Keywords. Credit default swaps, multi-scale, stochastic volatility, perturbation method, down-and-out binary option.

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1 Introduction

How to effectively manage and control credit risks is a hot topic in today's financial engineering area, because this kind of risks is one of the most adverse factors for the development of financial markets and is also the primary cause of financial crises. As a new kind of financial instruments, credit derivatives are nowadays playing a significant role in dealing with problems caused by credit risks. Among them, the most basic and successful one is the so-called credit default swap (CDS).

A CDS is a contract that allows credit risks to be traded. In specific, the buyer of the CDS pays a regular fee to the seller until the end of the contract or until a credit event occurs, whereas the seller of this contract undertakes the responsibility of compensating the buyer in case of default. Through this kind of trading mechanics, it is clear that the CDS is able to transfer credit risks from its buyer to the seller.

The accurate determination of the CDS price is fundamental for financial institutions, because it can not only help those institutions to determine capital reserves to set aside to cover for risk connected to financing and investment activities, but also mitigate credit exposures through hedging them with credit derivatives, such as the CDS. Choosing suitable models for credit risks and the reference asset is crucial in the accurate pricing of CDS. In the literature, the credit risks are usually modelled by two kinds of models, i.e., the reduced-form models and the structural models. The formers are adopted by a number of researchers including Duffie & Singleton [8], Jarrow et al. [15] and Hull & White [14]. With the flexibility in the functional form, these models are able to provide strong in-sample fitting properties. However, they fail to capture the wide range of default correlations and may result in poor out-of-sample behaviors, as suggested by several empirical studies [1, 4]. The structural models, as another alternative, use the evolution of the reference asset and the value of the debt to determine the probability of default. Typical models in this category include the Merton model [16], which characterizes the breach of

default by assuming that the default would occur if the company is insolvent. Although elegant, some of the assumptions under this model are unrealistic. For example, under this model, it is assumed that the target company would only default at the expiry date and the value of the company can drop to almost zero without default. However, nowadays, the default of a company can be triggered when its value is below a certain level away from zero at any time before or at the maturity of the bond.

As far as the modelling of the evolution of the reference asset is concerned, Merton [16] assumes that it follows a geometric Brownian motion. This assumption is also adopted by the classical Black-Scholes (B-S) model [2] for the underlying asset. Empirical studies have, however, suggested that this assumption is at odds with some of the real market conditions [9, 10], and usually leads to the mispricing of financial derivatives. There are a number of modifications to such an assumption and some of them have already been used in the CDS pricing field. For example, de Malherbe [7] replaced the geometric Brownian motion by a Poisson process, and determined the corresponding CDS price by a probabilistic approach. With stochastic intensity models adopted for the default events, Brigo and Chourdakis [3] considered the pricing of CDS when the counterparty risk is also taken into consideration. Recently, He & Chen [13] adopted the generalized mixed fractional Brownian motion for the reference asset and derived a closed-form formula for the price of the CDS. However, in most of the work mentioned above, the default of the company is assumed to be triggered only at the maturity date.

In this paper, we shall replace the constant volatility appearing in the Merton model by a stochastic volatility driven by two time scales. This so-called multi-scale stochastic volatility (SV) model has many advantages over a single time scale and is much closer to the real financial market conditions because it can not only capture the long range memory characteristic of the volatility correlations but also ensures the leverage effect to decay much faster than the volatility correlation [11, 12]. Our solution process begins by deriving an analytical expression for the CDS price under a general default model, in

which the "no default" probability still needs to be determined. A key step of the solution process is to establish an equivalence relationship between the unknown probability and the down-and-out binary option. With the perturbation method as used in [11], a sequence of simplified systems governing the price of the down-and-out binary option are obtained and solved. The price of the CDS is then obtained.

The rest of the paper is organized as follows. In Section 2, the multi-scale SV models are reviewed. In Section 3, the CDS contract considered in this paper is specified and the general expression for the fair price of the CDS is derived. After that, the partial differential equation (PDE) system governing the key part of the CDS price is established, based on which the approximation solution is derived by the perturbation methods. Concluding remarks are given in the last section.

2 Multi-scale volatility models

In this section, the multi-scale SV models are briefly revisited for the sake of completeness of the paper. This kind of models are introduced by Fouque et al. [5, 11] based upon various empirical studies. Under these models, the underlying S_t is assumed to follow a geometric Brownian motion with SV controlled by a fast and a slow time scale. In specific, S_t satisfies

$$\frac{dS_t}{S_t} = \mu dt + f(Y_t, Z_t)dB_{1,t},$$

where μ is the drift rate, $B_{1,t}$ is a standard Brownian motion, and f is a bounded positive function representing the SV. Moreover, f is driven by two other factors, Y_t and Z_t, which are governed by the following two processes as

$$
\begin{aligned}
dY_t &= [\frac{1}{\epsilon}(m - Y_t)]dt + \frac{\sqrt{2}v}{\sqrt{\epsilon}}dB_{2,t}, \\
dZ_t &= \delta c(Z_t) + \sqrt{\delta}g(Z_t)dB_{3,t},
\end{aligned}
$$

where the functions $c(z)$ and $g(z)$ are smooth and grow at most linearly as $z \to \infty$. v^2 represents the variance of the invariant distribution of Y_t. It determines the long-run level of the volatility fluctuations. Moreover, $\dfrac{1}{\epsilon}$ is the mean-reverting rate of Y_t controlling the reversion speed to the long-term mean m. By assuming that ϵ is a positive small parameter, Y_t is referred to as the fast volatility factor because its autocorrelation now decays exponentially on the time scale ϵ. On the other hand, for the process Z_t, it is assumed that δ is also a small positive parameter, and Z_t is referred to as the slow volatility factor. We remark that the independence of ϵ and δ is consistent with market observations [6]. Recent studies also suggest that ϵ and δ can be different by an order of magnitude (roughly $\epsilon \backsim \mathcal{O}(0.005)$ and $\delta \backsim \mathcal{O}(0.05)$). Therefore, following a major assumption made in [5, 11, 12], we shall focus on the case of $\epsilon \neq \delta$ in the current paper. It should also be remarked that the three Brownian motions are not necessarily independent, and the correlation among them can be expressed as

$$
\mathbf{B_t} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \rho_3 & \sqrt{1-\rho_2^2-\rho_3^2} \end{pmatrix} \mathbf{W_t},
$$

where $\mathbf{B_t} = \begin{pmatrix} B_{1,t} \\ B_{2,t} \\ B_{3,t} \end{pmatrix}$ and $\mathbf{W_t}$ is a standard three-dimensional Brownian motion.

We further remark that when the SV is introduced, the market is no longer complete, implying that the equivalent martingale measure is not unique any more. In the current work, we adopt the same combined "risk premiums" as used in [5, 11], i.e.,

$$
\begin{aligned}
\Lambda_1(y,z) &= \rho_1 \frac{(u-r)}{f(y,z)} + \sqrt{1-\rho_1^2}\, l_1(y,z), \\
\Lambda_2(y,z) &= \rho_2 \frac{(u-r)}{f(y,z)} + \rho_3 l_1(y,z) + \sqrt{1-\rho_1^2-\rho_3^2}\, l_2(y,z),
\end{aligned}
$$

5

where $l_1(y, z)$ and $l_2(y, z)$ are smooth and bounded functions. Therefore, under the risk-neutral measure, the three original processes described above can be expressed as

$$
\begin{cases}
\dfrac{dS}{S} &= rdt + \sigma_t dB_{1,t}^Q, \\
dY_t &= [\dfrac{1}{\epsilon}(m - Y_t) - \dfrac{\sqrt{2}v}{\sqrt{\epsilon}}\Lambda_1(Y_t, Z_t)]dt + \dfrac{\sqrt{2}v}{\sqrt{\epsilon}}dB_{2,t}^Q, \\
dZ_t &= [\delta c(Z_t) - \sqrt{\delta}g(Z_t)\Lambda_2(Y_t, Z_t)]dt + \sqrt{\delta}g(Z_t)dB_{3,t}^Q,
\end{cases}
\tag{2.1}
$$

where $\begin{pmatrix} B_{1,t}^Q \\ B_{2,t}^Q \\ B_{3,t}^Q \end{pmatrix} = \mathbf{B_t} + \int_0^t \begin{pmatrix} (\mu - r)f(Y_s, Z_s) \\ l_1(Y_s, Z_s) \\ l_2(Y_s, Z_s) \end{pmatrix} ds.$

3   Approximation formula for the CDS

In this section, we shall focus on deriving the price of the CDS when its reference asset follows a multi-scale SV process. According to three issues to be addressed, this section is further divided into three subsections. In the first subsection, the general expression of the CDS price containing the unknown "default probability" will be derived, whereas in the second subsection, the unknown "default probability" will be further determined. In the last subsection, the accuracy of the approximated solution will be discussed. We remark that the price of the CDS is different from the value of the contract. In fact, its price is defined as the spread, which is the regular fee specified in the CDS contract that the buyer should pay to the seller to ensure a compensation in case of default.

3.1   The CDS contract

Before deriving the approximation, a brief description of the CDS contract considered in the current work is needed, because it is closely related to our solution procedure that will be described later. Let M, R, and T be the face value of reference asset, the recovery rate, and the expiration date, respectively. In addition, let $p(S, t)$ be the probability of no-

default before the current time t. It is clear that $p(S, 0) = 1$ for any reasonable company. We remark that the CDS considered in the current work is more realistic than others in the literature [13, 16]. While the latter only allow the default to happen at the maturity date, the former assumes that the company could default at any time before or at the maturity of the bond. It should also be remarked that the function p might also have variables other than S and t, depending on the particular model one chooses for modelling the price of the reference asset. However, this will not affect the determination of a general expression for the fair price of the CDS, as will be shown below. Therefore, for simplicity, in this subsection where a general default model is considered, we assume p is a function of S and t only.

As pointed out earlier, the target of the current work is to determine the spread c. To achieve this, we need to investigate the cash flow of the CDS contract. If the default is not triggered, the CDS seller does not need to pay anything to the buyer but receives an amount of $cMdt$ per unit time dt. Therefore, the present value of the cash flow of the buyer can be calculated as

$$V_1 = \sum_t [e^{-rt}cMp(S,t)dt] = cM \int_0^T e^{-rt}p(S,t)dt.$$

However, if the reference company defaults, the seller has to pay an amount of $(1 - R)M$ as the compensation fee to the buyer immediately. Thus, the present value of the cash flow of the seller can be found as

$$\begin{aligned} V_2 &= \sum_t [-e^{-rt}(1-R)Mdp(S,t)], \\ &= -(1-R)M \int_0^T e^{-rt}dp(S,t). \end{aligned}$$

It should be noticed that when a CDS is initiated, it should be fair to both parties, which implies that the net value of the CDS should be zero when it is entered into. As a result,

we have $V_1 - V_2 = 0$, and thus

$$cM \int_0^T e^{-rt} p(S, t) dt + (1 - R)M \int_0^T e^{-rt} dp(S, t) = 0,$$

from which, the general expression of the CDS spread can be found as

$$
\begin{aligned}
c &= \frac{(1 - R) \int_0^T -e^{-rt} dp(S, t)}{\int_0^T e^{-rt} p(S, t) dt}, \\
&= \frac{(1 - R)[-e^{-rt} p(S, t)|_0^T - r \int_0^T e^{-rt} p(S, t) dt]}{\int_0^T e^{-rt} p(S, t) dt}, \\
&= \frac{(1 - R)[1 - e^{-rT} p(S, T)]}{\int_0^T e^{-rt} p(S, t) dt} - r(1 - R),
\end{aligned}
\tag{3.1}
$$

where the above derivation is based on the fact that $p(S, 0) = 1$ and $\displaystyle\int_0^T e^{-rt} p(S, t) dt \neq 0$.

From (3.1), one can determine directly the spread c if all the parameters and functions in (3.1) are known in advance. Unfortunately, $p(S, t)$, which denotes the probability of no default before the current time t, is unknown. If the default can only occur at the maturity date, one can solve for $p(S, t)$ from the stochastic differential equation (SDE) that S satisfies, just as He & Chen did in [13]. The assumption that the default can happen at any time during the life span of the bond has made the determination of $p(S, t)$ much more complicated. However, in the following, we have managed to establish a relationship between $p(S, t)$ and some kind of barrier option so that once the latter is determined the former can be found directly.

Recall that $p(S, t)$ is defined as the probability of no default before the current time t. Therefore, $p(S, t)$ can be expressed as

$$p(S, t) = Prob(\min_{0 \leq t^* \leq t} S_{t^*} > L) = E[I_{\{\min_{0 \leq t^* \leq t} S_{t^*} > L\}}],$$

where S_t is the price of the reference asset at time t, $Prob$ is the probability notation, and

L is the default barrier. In addition, $I_{\{.\}}$ is the indicator function and $\min\limits_{0 \leq t^* \leq t} S_{t^*} > L$ means that the smallest value of S_t over $[0, t]$ is greater than L. Consequently, we have

$$e^{-rt}p(S,t) = e^{-rt}E[I_{\{\min\limits_{0 \leq t^* \leq t} S_{t^*} > L\}}].$$

Now, denote

$$P(S, T - t) = e^{-rt}p(S, t). \tag{3.2}$$

We have

$$P(S,t) = e^{-r(T-t)}p(S, T - t) = e^{-r(T-t)}E[I_{\{\min\limits_{T-t \leq \tau^* \leq T} S_{\tau^*} > L\}}],$$

from which, one can conclude that $P(S, t)$ is in fact the price of a down-and-out binary option written on the asset S with '$T - t$' being the time to maturity and $h(S)$ being the payoff function defined as

$$h(S) = \begin{cases} 1, & \min\limits_{T-t \leq \tau^* \leq T} S_{\tau^*} > L, \\ 0, & \text{otherwise.} \end{cases}$$

At this stage, it is clear that the pricing of the CDS is equivalent to that of a down-and-out binary option with barrier L. Once the price of this particular barrier option is calculated, the spread of the CDS can be found through (3.1) and (3.2). In particular, with P available, we have

$$c = \frac{(1 - R)[1 - P(S, 0)]}{\int_0^T P(S, y)dy} - r(1 - R). \tag{3.3}$$

In the following work, through the determination of $P(S, t)$, we shall concentrate on deriving an approximation for the price of the CDS under the multi-scaled SV framework.

3.2 Solution process

In this subsection, we shall concentrate on solving for the price of the down-and-out binary option specified in the last subsection, based on which, the price of the CDS can then be determined straightforwardly. Since the underlying price under the multi-scale SV model is closely related to both the fast and slow time factors, the no default probability is now a function of S, y, z and t, and so does P.

To determine the price of the down-and-out binary option, the PDE system governing the option price should be established first. Let $P(S, y, z, t)$ be the price of a down-and-out binary option. According to the Feynman-Kac theorem [18], the pricing system for such an option under the multi-scale SV model can be derived as

$$
\begin{cases}
\mathscr{L}^{\epsilon,\delta} P = 0, \ S > L, \ y \in (-\infty, \infty), \ z \in (-\infty, \infty), \ t \in [0, T), \\
P(S, y, z, T) = 1, \ S > L, \ y \in (-\infty, \infty), \ z \in (-\infty, \infty), \\
P(L, y, z, t) = 0, \ y \in (-\infty, \infty), \ z \in (-\infty, \infty), \ t \in [0, T), \\
P(+\infty, y, z, t) = e^{-r(T-t)}, \ y \in (-\infty, \infty), \ z \in (-\infty, \infty), \ t \in [0, T),
\end{cases}
\tag{3.4}
$$

where $\mathscr{L}^{\epsilon,\delta} = \mathscr{M}_0 + \sqrt{\delta}(\mathscr{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathscr{M}_3) + \delta\mathscr{M}_2$, with

$$
\begin{aligned}
\mathscr{M}_0 &= \frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2, \quad \mathscr{M}_1 = \rho_2 g(z)f(y,z)S\frac{\partial^2}{\partial S \partial z} - g(z)\Lambda_2(y,z)\frac{\partial}{\partial z}, \\
\mathscr{M}_2 &= \frac{1}{2}g^2(z)\frac{\partial^2}{\partial z^2} + c(z)\frac{\partial}{\partial z}, \quad \mathscr{M}_3 = \sqrt{2}\rho 12 v g(z)\frac{\partial^2}{\partial y \partial z}, \\
\mathscr{L}_0 &= (m-y)\frac{\partial}{\partial y} + v^2\frac{\partial^2}{\partial y^2}, \quad \mathscr{L}_1 = \sqrt{2}v[\rho_1 S f(y,z)\frac{\partial^2}{\partial S \partial y} - \Lambda_1(y,z)\frac{\partial}{\partial y}], \\
\mathscr{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}f^2(y,z)S^2\frac{\partial^2}{\partial S^2} + r[S\frac{\partial}{\partial S} - I].
\end{aligned}
$$

The existence and uniqueness of the solution of (3.4) can be shown by variational inequalities, which may require additional constraints along the y or z direction. Following a

major discussion as shown in [5], we shall not take the boundary conditions along the y or z directions into consideration, even if they are mathematically needed. In other words, we consider a solution that satisfies (3.4), but may or may not satisfy the boundary conditions at the end points of the two directions. But, the solution we try to find is financially meaningful, because it is at least valid for volatility levels not being extremely high or low.

As pointed out in Section 2, there are two independent small parameters, i.e., ϵ and δ, in the current case. According to the standard asymptotic analysis theory, one should explore a solution in powers of both ϵ and δ. To make the analysis convenient, an equivalent way is to find a series solution in powers of one small parameter first, while treating the other one as a normal parameter. Then, the series solution is further expanded with respect to (w.r.t) the previously fixed small parameter [11]. In the following work, we shall first seek the solution of (3.4) in the powers of $\sqrt{\delta}$, i.e.,

$$P(S, y, z, t) = \sum_{n=0}^{+\infty} \delta^{\frac{n}{2}} P_n^\epsilon. \tag{3.5}$$

By substituting (3.5) into the governing PDE contained in (3.4), we have

$$\mathscr{M}_0 P_0^\epsilon + \sqrt{\delta}[\mathscr{M}_0 P_1^\epsilon + (\mathscr{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathscr{M}_3)P_0^\epsilon] = \mathcal{O}(\delta). \tag{3.6}$$

3.2.1   The zeroth-order solution with respect to the slow scale

By setting the coefficient in front of those $\mathcal{O}(1)$ terms contained in (3.6) to zero and taking the boundary conditions into consideration, it is found that the zeroth-order term w.r.t

the slow scale δ, i.e., P_0^ϵ, satisfies

$$
\begin{cases}
\mathcal{M}_0 P_0^\epsilon = 0, \\
P_0^\epsilon(S, y, z, T) = 1, \\
P_0^\epsilon(L, y, z, t) = 0, \\
\lim_{S \to \infty} P_0^\epsilon(S, y, z, t) = e^{-r(T-t)}.
\end{cases}
\tag{3.7}
$$

One can easily observe that P_0^ϵ is nothing but the price of a down-and-out binary option with a fast-mean reversion in y. Moreover, z is no longer a variable but a parameter of P_0^ϵ, because the operator $\mathcal{M}_0$ does not contain any partial differentiations w.r.t z at all. This makes sense, as in the zeroth-order solution w.r.t the slow scale, the slowly varying factor Z_t should be frozen at its initial value z.

To derive P_0^ϵ, we shall adopt the technique developed by Fouque et al. in [11]. We expand it in the order of $\epsilon^{\frac{1}{2}}$ as

$$
P_0^\epsilon(S, y, t) = \sum_{n=0}^{+\infty} \epsilon^{\frac{n}{2}} P_{0,n}^\epsilon.
\tag{3.8}
$$

By substituting (3.8) into the governing equation contained in (3.7), we obtain

$$
\begin{aligned}
\mathcal{L}_0 P_{0,0} \frac{1}{\epsilon} \ &+ \ [\mathcal{L}_1 P_{0,0} + \mathcal{L}_0 P_{0,1}] \frac{1}{\sqrt{\epsilon}} + [\mathcal{L}_2 P_{0,0} + \mathcal{L}_1 P_{0,1} + \mathcal{L}_0 P_{0,2}] \\
&+ \ [\mathcal{L}_2 P_{0,1} + \mathcal{L}_1 P_{0,2} + \mathcal{L}_0 P_{0,3}] \sqrt{\epsilon} = \mathcal{O}(\epsilon).
\end{aligned}
$$

By setting the coefficient in front of those $\mathcal{O}(\frac{1}{\epsilon})$ term to zero, we have $\mathcal{L}_0 P_{0,0} = 0$, which implies that $P_{0,0}$ is independent w.r.t y, i.e., $P_{0,0}(S, y, t) = P_{0,0}(S, t)$. This is because $\mathcal{L}_0$ is the generator of an ergodic Markov process acting on y only. For the order $\mathcal{O}(\frac{1}{\sqrt{\epsilon}})$, we have

$$
\mathcal{L}_1 P_{0,0} + \mathcal{L}_0 P_{0,1} = 0.
$$

Since $P_{0,0}$ is a constant with respect to y, we have $\mathcal{L}_1 P_{0,0} = 0$ and thus $\mathcal{L}_0 P_{0,1} = 0$.

Similarly, one can deduce that $P_{0,1}$ does not depend on y as well, and thus $P_{0,1}(S, y, t) = P_{0,1}(S, t)$. At this stage, the explicit expression of $P_{0,0}$ is still unknown and the $\mathscr{O}(1)$ term should be taken into consideration. We have

$$\mathscr{L}_2 P_{0,0} + \mathscr{L}_1 P_{0,1} + \mathscr{L}_0 P_{0,2} = 0,$$

which can be simplified as

$$\mathscr{L}_2 P_{0,0} + \mathscr{L}_0 P_{0,2} = 0. \tag{3.9}$$

It should be noticed that (3.9) is in fact a Poisson equation of $P_{0,2}$ w.r.t the operator $\mathscr{L}_0$ in the variable y if $P_{0,0}$ is known. According to the Fredholm alternative theorem [17], one can deduce that there will be no solution to (3.9) unless $\mathscr{L}_2 P_{0,0}$ is centered w.r.t the invariant distribution of Y_t, i.e.,

$$< \mathscr{L}_2 P_{0,0} >= 0,$$

where $< \cdot >$ represents $\int_{-\infty}^{+\infty} \cdot \Phi(y) dy$ and $\Phi(y) = \dfrac{1}{v\sqrt{2\pi}} e^{-\frac{(y-m)^2}{2v^2}}$ is the invariant distribution of Y_t. Since $P_{0,0}$ is independent w.r.t y, the PDE system governing $P_{0,0}$ can be further simplified as

$$\begin{cases} < \mathscr{L}_2 > P_{0,0} = 0, \\ P_{0,0}(S, T) = 1, \\ P_{0,0}(L, t) = 0. \end{cases} \tag{3.10}$$

According to the definition of $\mathscr{L}_2$, it is not difficult to show that $< \mathscr{L}_2 >$ is nothing but the B-S operator with the constant volatility replaced by $\bar{f}(z)$, which is named as the effective volatility and is defined as $\bar{f}(z) =< f(y, z) >$. Therefore, $P_{0,0}$ is exactly the price of a down-and-out binary option under the B-S framework and can be solved with the method of images as

$$P_{0,0}(S, t) = e^{-r(T-t)}[N(d_1) - N(d_2) e^{-(\frac{2r}{\bar{f}^2(z)} - 1) \ln(S/L)}], \tag{3.11}$$

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where $d_1 = \dfrac{\ln(S/L) + [r - \frac{1}{2}\bar{f}^2(z)](T - t)}{\bar{f}(z)\sqrt{T - t}}$, $d_2 = \dfrac{-\ln(S/L) + [r - \frac{1}{2}\bar{f}^2(z)](T - t)}{\bar{f}(z)\sqrt{T - t}}$ and $N(\cdot)$ denotes the distribution function of the standard normal distribution.

Now, we consider the first-order correction w.r.t the fast scale ϵ. By setting the coefficient in front of $\mathcal{O}(\sqrt{\epsilon})$ to zero, we obtain

$$\mathcal{L}_0 P_{0,3} + \mathcal{L}_1 P_{0,2} + \mathcal{L}_2 P_{0,1} = 0,$$

which is again a Poisson equation in the variable y. As a result, it is solvable if and only if

$$< \mathcal{L}_1 P_{0,2} + \mathcal{L}_2 P_{0,1} >= 0,$$

which can be further simplified as

$$< \mathcal{L}_2 > P_{0,1} = - < \mathcal{L}_1 P_{0,2} > . \tag{3.12}$$

On the other hand, from (3.9) and (3.12), one can find that

$$\mathcal{L}_0 P_{0,2} = -\mathcal{L}_2 P_{0,0} + < \mathcal{L}_2 P_{0,0} >= -\frac{1}{2}[f^2(y, z) - \bar{f}^2(z)]S^2\frac{\partial^2 P_{0,0}}{\partial S^2},$$

and thus

$$\begin{aligned} P_{0,2} &= -\frac{1}{2}\mathcal{L}_0^{-1}\{[f^2(y, z) - \bar{f}^2(z)]S^2\frac{\partial^2 P_{0,0}}{\partial S^2}\}, \\ &= -\frac{1}{2}[\phi(y, z) + q(S, z, t)]S^2\frac{\partial^2 P_0}{\partial S^2}, \end{aligned} \tag{3.13}$$

where $\mathcal{L}_0\phi(y, z) = f^2(y, z) - \bar{f}^2(z)$, and the function q does not depend on y. By substituting (3.13) into (3.12), we have

$$< \mathcal{L}_2 > P_{0,1} = \frac{1}{2} < \mathcal{L}_1\phi(y, z) > S^2\frac{\partial^2 P_{0,0}}{\partial S^2} = H_1(S, z, t),$$

14

where

$$H_1(S, z, t) = V_3 S^3 \frac{\partial^3 P_{0,0}}{\partial S^3} + V_2 S^2 \frac{\partial^2 P_{0,0}}{\partial S^2},$$

with

$$V_2 = \frac{v}{\sqrt{2}} \rho_1 < f \frac{\partial \phi}{\partial y} >, \quad V_3 = \frac{v}{\sqrt{2}} [2\rho_1 < f \frac{\partial \phi}{\partial y} > - < \Lambda_1 \frac{\partial \phi}{\partial y} >].$$

It should be remarked that V_2 appeared in the above equation is a volatility level correction, and depends on both ρ_1 and γ, whereas the V_3 term is the "skew effect" resulted from the presence of the third derivative and it will vanish if ρ_1 is equal to zero [11].

Now, by taking all the boundary conditions for $P_{0,1}$ into consideration, we find that $P_{0,1}$ satisfies

$$\begin{cases} < \mathscr{L}_2 > P_{0,1} = H_1(S, z, t), \\ P_{0,1}(S, T) = 0, \\ P_{0,1}(L, t) = 0. \end{cases} \quad (3.14)$$

From (3.14), it is clear that $P_{0,1}$ is governed by an inhomogeneous PDE system, where S and t are the only variables involved, and z is treated as a constant and y disappears as a result of taking a statistical average w.r.t the fast factor. Moreover, this PDE system is characterized by the B-S operator $< \mathscr{L}_2 >$ with source terms from the volatility level correction and the "skew effect". This PDE system cannot be solved in a straightforward way as we did for (3.10). In the following, we shall concentrate on solving for (3.14).

The method that will be used to solve for (3.14) is mainly based on the properties of the B-S operator. A simple calculation shows that $L_{BS}[(t-T)H_1] = H_1 - (T-t)L_{BS}[H_1] = H_1$, because

$$\begin{aligned} L_{BS}[H_1] &= L_{BS}[V_3 S^3 \frac{\partial^3 P_{0,0}}{\partial S^3} + V_2 S^2 \frac{\partial^2 P_{0,0}}{\partial S^2}], \\ &= V_3 S^3 \frac{\partial^3}{\partial S^3} L_{BS}[P_{0,0}] + V_2 S^2 \frac{\partial^2}{\partial S^2} L_{BS}[P_{0,0}] = 0. \end{aligned}$$

Now, we suppose that the solution of (3.14) can be written as $P_{0,1} = V(S,t) - (T-t)H_1$. It is clear that $V(S,t)$ satisfies

$$
\begin{cases}
<\mathscr{L}_2> V = 0, \\
V(S,T) = 0, \\
V(L,t) = (T-t)H_1(L,t).
\end{cases}
\tag{3.15}
$$

The above PDE system can be solved by using the Laplace transform technique, and we find that

$$
V(S,t;z) = (\frac{S}{L})^{\frac{1}{2}-\frac{r}{\bar{f}^2}} \frac{\ln\frac{S}{L}}{2\pi\bar{f}(z)} \int_t^T \frac{H_1(L,z,T+t-\xi)}{(T-\xi)^{\frac{3}{2}}} e^{-\frac{1}{2}(\frac{r}{\sigma}+\frac{\sigma}{2})^2(T-\xi)-\frac{(\ln\frac{S}{L})^2}{2\bar{f}^2}(T-\xi)} d\xi. \tag{3.16}
$$

Therefore,

$$
P_{0,1} = (\frac{S}{L})^{\frac{1}{2}-\frac{r}{\bar{f}^2}} \frac{\ln\frac{S}{L}}{2\pi\bar{f}(z)} \int_t^T \frac{H_1(L,z,T+t-\xi)}{(T-\xi)^{\frac{3}{2}}} e^{-\frac{1}{2}(\frac{r}{\sigma}+\frac{\sigma}{2})^2(T-\xi)-\frac{(\ln\frac{S}{L})^2}{2\bar{f}^2}(T-\xi)} d\xi - (T-t)H_1(S,z,t).
$$

One should notice that y does not appear explicitly in $P_{0,1}$, although it is the correction resulting from the introduction of the fast mean-revision volatility. This is in fact reasonable in an average sense, because when the mean-reversion rate is extremely fast, Y_t will fluctuate around its ergodic mean rapidly, and the volatility will thus oscillate around $\bar{f}(z)$.

3.2.2   The first-order correction with respect to the slow scale

One can clearly observe that z is kept constant in the above derivation for P_0^ϵ, and thus P_0^ϵ can be treated as the solution in the fast mean-revision scenario. In this subsection, we shall find the first-order correction accounting for the randomness of the slow factor Z_t.

Now, by eliminating the order of $\sqrt{\delta}$ in (3.6), it is obvious that

$$
(\frac{1}{\epsilon}\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_1 + \mathscr{L}_2)P_1^\epsilon = -(\mathscr{M}_1 + \frac{1}{\sqrt{\epsilon}}\mathscr{M}_3)P_0^\epsilon. \tag{3.17}
$$

To solve for P_1^ϵ, we expand it in the powers of $\sqrt{\epsilon}$ as

$$P_1^\epsilon = \sum_{n=0}^{+\infty} \epsilon^{\frac{n}{2}} P_{1,n}. \tag{3.18}$$

By substituting (3.18) into (3.17), we find that, at the lowest order $\mathscr{O}(\frac{1}{\epsilon})$, the following equation holds true

$$\mathscr{L}_0 P_{1,0} = 0,$$

which implies that $P_{1,0}$ does not depend on y, i.e., $P_{1,0} = P_{1,0}(S, z, t)$. At the order of $\mathscr{O}(\frac{1}{\sqrt{\epsilon}})$, it is not difficult to find that

$$\mathscr{L}_0 P_{1,1} + \mathscr{L}_1 P_{1,0} = -\mathscr{M}_3 P_{0,0},$$

which can be simplified as

$$\mathscr{L}_0 P_{1,1} = 0, \tag{3.19}$$

because both $P_{0,0}$ and $P_{1,0}$ do not depend on y. Due to the same reason as for $P_{0,0}$, one can deduce from (3.19) that $P_{1,1}$ is also a constant w.r.t y.

Now, we turn to the order of $\mathscr{O}(1)$. We find that

$$\mathscr{L}_2 P_{1,0} + \mathscr{L}_1 P_{1,1} + \mathscr{L}_0 P_{1,2} + \mathscr{M}_1 P_{0,0} + \mathscr{M}_3 P_{0,1} = 0,$$

which can be simplified as

$$\mathscr{L}_2 P_{1,0} + \mathscr{L}_0 P_{1,2} = -\mathscr{M}_1 P_{0,0}, \tag{3.20}$$

because $\mathscr{L}_1 P_{1,1} = 0$ and $\mathscr{M}_3 P_{0,1} = 0$. By applying the Fredholm alternative theorem to

(3.20), we obtain

$$
\begin{aligned}
< \mathscr{L}_2 > P_{1,0} &= -< \mathscr{M}_1 P_{0,0} >= -< \mathscr{M}_1 > P_{0,0}, \\
&= g(z) < \lambda_2(y,z) > \frac{\partial P_{0,0}}{\partial z} - \rho_2 g(z)\bar{f}(z)S\frac{\partial^2 P_{0,0}}{\partial z \partial S}, \\
&\triangleq H_2(S,z,t).
\end{aligned}
$$

Therefore, with the boundary conditions taken into consideration, the PDE system for $P_{1,0}$ can be found as

$$
\begin{cases}
< \mathscr{L}_2 > P_{1,0} = H_2(S,z,t), \\
P_{1,0}(S,z,T) = 0, \\
P_{1,0}(L,z,t) = 0.
\end{cases}
\tag{3.21}
$$

It can be shown that H_2 is also a solution of the B-S equation, and thus we have

$$
L_{BS}[(t-T)H_2] = H_2 - (T-t)L_{BS}[H_2] = H_2.
$$

Therefore, (3.21) can be solved in a same way as we did for (3.14), and $P_{1,0}$ can be finally derived as

$$
P_{1,0} = \left(\frac{S}{L}\right)^{\frac{1}{2}-\frac{r}{\bar{f}^2}}\frac{\ln\frac{S}{L}}{2\pi\bar{f}(z)}\int_t^T\frac{H_2(L,z,T+t-\xi)}{(T-\xi)^{\frac{3}{2}}}e^{-\frac{1}{2}(\frac{r}{\sigma}+\frac{\sigma}{2})^2(T-\xi)-\frac{(\ln\frac{S}{L})^2}{2\bar{f}^2}(T-\xi)}d\xi-(T-t)H_2(S,z,t).
$$

We remark that $P_{1,0}$ derived in this subsection is the correction from the slow volatility factor. Therefore, $P_{0,0} + \sqrt{\delta}P_{1,0}$ can be viewed as the price of the down-and-out option under a single but slowly varying volatility.

By combing the solutions derived from the fast and slow time scales, the closed-form

approximation for the down-and-out binary option price can be expressed as

$$
\begin{aligned}
\tilde{P}(S, y, z, t) &= e^{-r(T-t)}[N(d_1) - N(d_2)e^{-(\frac{2r}{\bar{f}^2(z)}-1)\ln(S/L)}] \\
&+ (\frac{S}{L})^{\frac{1}{2}-\frac{r}{\bar{f}^2}} \frac{\ln\frac{S}{L}}{2\pi\bar{f}(z)} \int_t^T \frac{\sqrt{\epsilon}H_1(L, z, T+t-\xi) + \sqrt{\delta}H_2(L, z, T+t-\xi)}{(T-\xi)^{\frac{3}{2}}} \\
&\quad \cdot\ e^{-\frac{1}{2}(\frac{r}{\sigma}+\frac{\sigma}{2})^2(T-\xi)-\frac{(\ln\frac{S}{L})^2}{2\bar{f}^2}(T-\xi)}d\xi \\
&- (T-t)[\sqrt{\delta}H_2(S, z, t) + \sqrt{\epsilon}H_1(S, z, t)], \quad\quad\quad (3.22)
\end{aligned}
$$

and the corresponding CDS price can then be derived straightforwardly via (3.3).

From the above formula, it can be clearly observed that the fast factor y does not involve explicitly. This is indeed reasonable, and can be explained from a modelling point of view as follows. It is known that in the so-called fast mean-reverting volatility scenario, both the volatility and the underlying price might fluctuate considerably, but the changes of the volatility are not as significant as those of the underlying prices [13]. Therefore, the volatility can be relatively viewed as a constant until the major fluctuation occurs, resulting in the statistical average of all possible paths of Y_t, rather than its sport level, appearing in the current formula. However, not surprisingly, the slow factor z is explicitly involved. This is because as a slowly varying process, Z_t should be "frozen" at its initial value z in the limit sense as $\delta \rightarrow 0$.

3.3 Accuracy of the approximation

In this section, we shall briefly examine the accuracy of the current approximation.

To begin with, we introduce a higher order approximation $\hat{P}$ for P, i.e.,

$$
\hat{P}(S, y, z, t) = P_{0,0} + \sqrt{\epsilon}P_{0,1} + \epsilon P_{0,2} + \epsilon^{\frac{3}{2}}P_{0,3} + \sqrt{\delta}(P_{1,0} + \sqrt{\epsilon}P_{2,0} + \epsilon P_{3,0}).
$$

In addition, we define the residual between $\hat{P}$ and P as $R = \hat{P} - P$. Based on a similar

analysis as outlined in [11], it is found that R can be expressed in an expectation form as

$$
\begin{aligned}
R &= \epsilon E^Q\{[e^{-r(T-t)}G_1(S_T, Y_T, Z_T, T) - \int_t^T e^{-r(s-t)}R_1(S_s, Y_s, Z_s, s)ds]I_{\{\min_{0\leq t^*\leq t} S_{t^*}\geq L\}}|S_t, Y_t, Z_t\} \\
&+ \sqrt{\epsilon\delta}E^Q\{[e^{-r(T-t)}G_2(S_T, Y_T, Z_T, T) - \int_t^T e^{-r(s-t)}R_2(S_s, Y_s, Z_s, s)ds]I_{\{\min_{0\leq t^*\leq t} S_{t^*}\geq L\}}|S_t, Y_t, Z_t\} \\
&+ \delta E^Q\{[-\int_t^T e^{-r(s-t)}R_3(S_s, Y_s, Z_s, s)ds]I_{\{\min_{0\leq t^*\leq t} S_{t^*}\geq L\}}|S_t, Y_t, Z_t\},
\end{aligned}
$$

where

$$
R_1 = \mathscr{L}_1 P_{0,3} + \mathscr{L}_2 P_{0,2}, \; R_2 = \mathscr{L}_1 P_{3,0} + \mathscr{M}_2 P_{0,2},
$$

$$
R_3 = \mathscr{M}_1 P_{1,0} + \mathscr{M}_2 P_{0,0} + \mathscr{M}_3 P_{2,0},
$$

$$
G_1 = P_{0,2} + \sqrt{\epsilon}P_{0,3}, \; G_2 = P_{2,0} + \sqrt{\epsilon}P_{3,0},
$$

and E^Q is the expectation under the risk-neutral measure Q. Using the expression of $P_{0,0}$ and the technique as used in [12], one can show that the expectations appeared in the above expression are bounded by some constants which may depend on S, y, z and t. The analysis is lengthy but trivial, and is thus omitted. Finally, according to the triangle inequality, we have $P - \tilde{P} = \mathcal{O}(\epsilon, \delta, \sqrt{\epsilon\delta})$.

4 Conclusion

In this paper, we consider the pricing of the CDS with the price of the reference asset modelled by a geometric Brownian motion with a multi-scale stochastic volatility. By establishing an equivalence relationship between the CDS and the down-and-out binary option written on the same reference asset, a closed-form approximation for the CDS price is obtained with the perturbation method. It is interesting to notice that the leading-order term of the current formula is the price of a down-and-out binary option with an effective volatility "frozen" at the spot level of the slowly varying factor. The accuracy of

the formula is also briefly discussed to ensure the safe use of the formula in the trading practise.

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