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Abstract
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ITERATES OF HOLOMORPHIC SELF-MAPS ON PSEUDOCONVEX DOMAINS OF FINITE AND INFINITE TYPE IN \( \mathbb{C}^n \)

TRAN VU KHANH AND NINH VAN THU

Abstract. Using the lower bounds on the Kobayashi metric established by the first author [16], we prove a Wolff-Denjoy-type theorem for a very large class of pseudoconvex domains in \( \mathbb{C}^n \). This class includes many pseudoconvex domains of finite type and infinite type.

1. Introduction

In 1926, Wolff [22] and Denjoy [9] established their famous theorem on the behavior of iterates of holomorphic self-mappings of the unit disk \( \Delta \) of \( \mathbb{C} \) that do not admit fixed points.

**Theorem** (Wolff-Denjoy [22, 9], 1926). Let \( \phi: \Delta \rightarrow \Delta \) be a holomorphic self-map without fixed points. Then there exists a point \( x \) in the unit circle \( \partial \Delta \) such that the sequence \( \{\phi^k\} \) of iterates of \( \phi \) converges, uniformly on any compact subsets of \( \Delta \), to the constant map taking the value \( x \).

The generalization of this theorem to domains in \( \mathbb{C}^n, n \geq 2 \), is the focus of this paper. This has been done in several cases:

- the unit ball (see [13]);
- strongly convex domains (see [2, 4, 5]);
- strongly pseudoconvex domains (see [3, 14]);
- pseudoconvex domains of strictly finite type in the sense of Range [20] (see [3]);
- pseudoconvex domains of finite type in \( \mathbb{C}^2 \) (see [15, 23]).

The main goal of this paper is to prove a Wolff-Denjoy-type theorem for a general class of bounded pseudoconvex domains in \( \mathbb{C}^n \) that includes many pseudoconvex domains of both finite and infinite type. In particular, we shall prove the following (the definitions are given below).

**Theorem 1.** Let \( \Omega \subset \mathbb{C}^n \) be a bounded, pseudoconvex domain with \( C^2 \)-smooth boundary \( \partial \Omega \).

Assume that

(i) \( \Omega \) has the \( f \)-property with \( f \) satisfying \( \int_1^\infty \frac{\ln \alpha}{af(\alpha)} d\alpha < \infty \); and

(ii) the Kobayashi distance of \( \Omega \) is complete.

If \( \phi: \Omega \rightarrow \Omega \) is a holomorphic self-map such that the sequence of iterates \( \{\phi^k\} \) is compactly divergent, then the sequence \( \{\phi^k\} \) converges, uniformly on a compact set, to a point of the boundary.

We say that a Wolff-Denjoy-type theorem for \( \Omega \) holds if the conclusion of Theorem 1 holds. We will prove Theorem 1 in Section 3 using the (known) estimates of the Kobayashi distance on domains of the \( f \)-property and the work by Abate [2, 3, 4].

We now recall some the definitions of the \( f \)-property (see also [16, 17]) and the Kobayashi distance.

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Key words and phrases. Wolff-Denjoy-type theorem, finite type, infinite type, \( f \)-property, Kobayashi metric, Kobayashi distance.
Definition 1. Let \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) be a smooth, monotonically increasing functions so that \( f(\alpha)\alpha^{-1/2} \) is decreasing. We say that \( \Omega \subset \mathbb{C}^n \) has the \( f \)-property if there exists a family of functions \( \{\psi_\eta\} \) such that

(i) the functions \( \psi_\eta \) are plurisubharmonic, \( |\psi_\eta| \leq 1 \), and \( C^2 \) on \( \Omega \);
(ii) \( i\partial\bar{\partial}\psi_\eta \geq c_1 f(\eta^{-1})^2 \text{Id} \) and \( |D\psi_\eta| \leq c_2 \eta^{-1} \) on \( \{z \in \Omega : 0 < \delta_\Omega(z) < \delta\} \) for some constants \( c_1, c_2 > 0 \), where \( \delta_\Omega(z) \) is the Euclidean distance from \( z \) to the boundary \( \partial \Omega \).

This is an analytic condition where the function \( f \) reflects the geometric “type” of the boundary. For example, viewing Catlin’s results on pseudoconvex domains of finite type through the lens of the \( f \)-property [6, 7], a domain is of finite type if and only if there exists an \( \epsilon > 0 \) such that the \( t^\epsilon \)-property holds. If domain is convex and of finite type \( m \), then the \( t^1/m \)-property holds [18]. Furthermore, there is a large class of infinite type pseudoconvex domains that satisfy an \( f \)-properties [17, 16]. For example (see [17]), the \( \ln \eta \)-property holds for both the complex ellipsoid of infinite type

\[
\Omega = \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n \exp\left(-\frac{1}{|z_j|^{\alpha_j}}\right) - e^{-1} < 0 \right\}
\]

with \( \alpha := \max_j \{\alpha_j\} \), and the real ellipsoid of infinite type

\[
\tilde{\Omega} = \left\{ z = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n : \sum_{j=1}^n \exp\left(-\frac{1}{|x_j|^{\alpha_j}}\right) + \exp\left(-\frac{1}{|y_j|^{\beta_j}}\right) - e^{-1} < 0 \right\}
\]

with \( \alpha := \max_j \{\min\{\alpha_j, \beta_j\}\} \), where \( \alpha_j, \beta_j > 0 \) for all \( j = 1, 2, \ldots \). The influence of the \( f \)-property on estimates of the Kobayashi metric and distance will be given in Section 2.

On hyperbolic manifolds, completeness of the Kobayashi distance (or \( k \)-completeness for short) is a natural condition. For a bounded domain \( \Omega \subset \mathbb{C}^n \), \( k \)-completeness of means

\( k_\Omega(z_0, z) \to \infty \) as \( z \to \partial \Omega \)

for any point \( z_0 \in \Omega \) where \( k_\Omega(z_0, z) \) is the Kobayashi distance from \( z_0 \) to \( z \). It is well-known that this condition holds for strongly pseudoconvex domains [11], convex domains [19], pseudoconvex domains of finite type in \( \mathbb{C}^2 \) [23], pseudoconvex Reinhardt domains [21], domains enjoying a local holomorphic peak function at any boundary point [12]. We also remark that the domain defined by \( \{1\} \) (resp. \( \{2\} \)) is \( k \)-complete because it is a pseudoconvex Reinhardt domain (resp. convex domain). These remarks immediately lead to the following corollary.

Corollary 2. Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) with smooth boundary \( \partial \Omega \). The Wolff-Denjoy-type theorem for \( \Omega \) holds if \( \Omega \) satisfies at least one of the following settings:

(a) \( \Omega \) is a strongly pseudoconvex domain;
(b) \( \Omega \) is a pseudoconvex domain of finite type and \( n = 2 \);
(c) \( \Omega \) is a convex domain of finite type;
(d) \( \Omega \) is a pseudoconvex Reinhardt domain of finite type;
(e) \( \Omega \) is a pseudoconvex domain of finite type (or of infinite type having the \( f \)-property with \( f(t) \geq \ln^{2+\epsilon}(t) \) for any \( \epsilon > 0 \)) such that \( \Omega \) has a local, continuous, holomorphic peak function at each boundary point, i.e., for any \( x \in \partial \Omega \) there exist a neighborhood \( U \) of \( x \) and a holomorphic function \( p \) on \( \Omega \cap U \), continuous up to \( \Omega \cap U \), and satisfies

\[
p(x) = 1, \quad p(z) < 1, \quad \text{for all } z \in \Omega \cap U \setminus \{x\};
\]

(f) \( \Omega \) is defined by \( \{1\} \) or \( \{2\} \) with \( \alpha < \frac{1}{2} \).
Finally, throughout the paper we use \( \lesssim \) and \( \gtrsim \) to denote inequalities up to a positive multiplicative constant, and \( H(\Omega_1, \Omega_2) \) to denote the set of holomorphic maps from \( \Omega_1 \) to \( \Omega_2 \).

## 2. The Kobayashi metric and distance

We start this section by defining the Kobayashi metric.

**Definition 2.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \), and \( T^{1,0}\Omega \) be its holomorphic tangent bundle. The Kobayashi (pseudo)metric \( K_\Omega : T^{1,0}\Omega \to \mathbb{R} \) is defined by

\[
K_\Omega(z, X) = \inf \{ \alpha > 0 \mid \exists \Psi \in H(\Delta, \Omega) : \Psi(0) = z, \Psi'(0) = \alpha^{-1} X \},
\]

for any \( z \in \Omega \) and \( X \in T^{1,0}\Omega \), where \( \Delta \) be the unit open disk of \( \mathbb{C} \).

In the case that \( \Omega \) is a smoothly pseudoconvex bounded domain of finite type, it is known that there exists \( \epsilon > 0 \) such that the Kobayashi metric \( K_\Omega \) has the lower bound \( \delta_\Omega^{-\epsilon}(z) \) (see [8, 10]), in the sense that,

\[
K_\Omega(z, X) \gtrsim \frac{\| X \|}{\delta_\Omega(z)},
\]

where \( \| X \| \) is the Euclidean length of \( X \). Recently, the first author [16] obtained lower bounds on the Kobayashi metric for a general class of pseudoconvex domains in \( \mathbb{C}^n \), that contains all domains of finite type and many domains of infinite type.

**Theorem 3.** Let \( \Omega \) be a pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary \( \partial \Omega \). Assume that \( \Omega \) has the \( f \)-property with \( f \) satisfying \( \int_a^\infty \frac{\alpha \, d\alpha}{f(\alpha)} < \infty \) for \( s \geq 1 \), and denote by \( (g(s))^{-1} \) this finite integral. Then,

\[
K(z, X) \gtrsim g(\delta_\Omega^{-1}(z))\| X \|,
\]

for any \( z \in \Omega \) and \( X \in T^{1,0}\Omega \).

The Kobayashi (pseudo)distance \( k_\Omega : \Omega \times \Omega \to \mathbb{R}^+ \) on \( \Omega \) is the integrated form of \( K_\Omega \). \( k_\Omega \) is given by

\[
k_\Omega(z, w) = \inf \left\{ \int_a^b K_\Omega(\gamma(t), \dot{\gamma}(t)) \, dt \mid \gamma : [a, b] \to \Omega, \text{piecewise } C^1 \text{-smooth curve}, \gamma(a) = z, \gamma(b) = w \right\}
\]

for any \( z, w \in \Omega \). An essential property of \( k_\Omega \) is that it is a contraction under holomorphic maps, i.e.,

\[
\text{if } \phi \in H(\Omega, \hat{\Omega}) \text{ then } k_\Omega(\phi(z), \phi(w)) \leq k_\Omega(z, w), \text{ for all } z, w \in \Omega.
\]

We need the following lemma from [11].

**Lemma 4.** Let \( \Omega \) be a bounded \( C^2 \)-smooth domain in \( \mathbb{C}^n \) and \( z_0 \in \Omega \). Then there exists a constant \( c_0 > 0 \) depending on \( \Omega \) and \( z_0 \) such that

\[
k_\Omega(z_0, z) \leq c_0 - \frac{1}{2} \ln \delta_\Omega(z)
\]

for any \( z \in \Omega \).

We recall that the curve \( \gamma : [a, b] \to \Omega \) is called a minimizing geodesic with respect to the Kobayashi metric between two points \( z = \gamma(a) \) and \( w = \gamma(b) \) if

\[
k_\Omega(\gamma(s), \gamma(t)) = t - s = \int_s^t K_\Omega(\gamma(\tau), \dot{\gamma}(\tau)) \, d\tau, \quad \text{for any } s, t \in [a, b], s \leq t.
\]

This implies that

\[
K(\gamma(t), \dot{\gamma}(t)) = 1, \quad \text{for any } t \in [a, b].
\]
The relation between the Kobayashi distance \( k_\Omega(z, w) \) and the Euclidean distance \( \delta_\Omega(z, w) \) is contained in the following lemma, itself a generalization of [15, Lemma 36].

**Lemma 5.** Let \( \Omega \) be a bounded, pseudoconvex, \( C^2 \)-smooth domain in \( \mathbb{C}^n \) satisfying the \( f \)-property with \( \int_1^\infty \frac{\ln \alpha}{\alpha f(\alpha)} \, d\alpha < \infty \) and \( z_0 \in \Omega \). Then there exists a constant \( c \) only depending on \( z_0 \) and \( \Omega \) such that

\[
\delta_\Omega(z, w) \leq c \int_{e^{2k_\Omega(z, \gamma)}}^{e^{2k_\Omega(z, \gamma)}} \frac{c_0 + \ln \alpha}{\alpha f(\alpha)} \, d\alpha,
\]

for all \( z, w \in \Omega \), where \( \gamma \) is a minimizing geodesic connecting \( z \) to \( w \) and \( c_0 \) is the constant given in Lemma 4. Here, \( k_\Omega(z_0, \gamma) \) is the Kobayashi distance from \( z_0 \) to the curve \( \gamma \).

**Proof.** We may assume that \( z \neq w \). Let \( p \) be a point on \( \gamma \) of minimal distance to \( z_0 \). We can assume that \( p \neq z \) (if not, we interchange \( z \) and \( w \)) and denote by \( \gamma_1 : [0, a] \to \Omega \) the reparametrized piece of \( \gamma \) going from \( p \) to \( z \). By the minimality of \( k_\Omega(z_0, \gamma) = k_\Omega(z_0, p) \) and the triangle inequality we have

\[
k_\Omega(z_0, \gamma_1(t)) \geq k_\Omega(z_0, \gamma) \quad \text{and} \quad k_\Omega(z_0, \gamma_1(t)) \geq k_\Omega(p, \gamma_1(t)) - k_\Omega(z_0, p) = t - k_\Omega(z_0, \gamma)
\]

for any \( t \in [0, a] \). Substituting \( z = \gamma_1(t) \) into the inequality in Lemma 4 it follows

\[
\frac{1}{\delta_\Omega(\gamma_1(t))} \geq e^{2k_\Omega(z_0, \gamma_1(t)) - 2c_0}
\]

for all \( t \in [0, a] \). Since \( \gamma_1 \) is a unit speed curve with respect to \( K_\Omega \) we have

\[
\delta_\Omega(p, z) \leq \int_0^a \|\gamma'_1(t)\|dt
\]

\[
\leq \int_0^a \left( g \left( \frac{1}{\delta_\Omega(\gamma_1(t))} \right) \right)^{-1} K_\Omega(\gamma_1(t), \gamma'_1(t))dt
\]

\[
\leq \int_0^a \left( g \left( e^{2k_\Omega(z_0, \gamma_1(t)) - 2c_0} \right) \right)^{-1} dt.
\]

We now compare \( a \) with \( 2k_\Omega(z_0, \gamma) + c_0 \). In the case \( a > 2k_\Omega(z_0, \gamma) + c_0 \), we split the integral into two parts and use the inequalities (7) and the fact that \( g \) is increasing. We then have

\[
\delta_\Omega(p, z) \leq \int_0^{2k_\Omega(z_0, \gamma) + c_0} \left( g \left( e^{2k_\Omega(z_0, \gamma_1(t)) - 2c_0} \right) \right)^{-1} dt + \int_{2k_\Omega(z_0, \gamma) + c_0}^a \left( g \left( e^{2k_\Omega(z_0, \gamma_1(t)) - 2c_0} \right) \right)^{-1} dt
\]

\[
\leq \int_0^{2k_\Omega(z_0, \gamma) + c_0} \left( g \left( e^{2k_\Omega(z_0, \gamma) - 2c_0} \right) \right)^{-1} dt + \int_{2k_\Omega(z_0, \gamma) + c_0}^a \left( g \left( e^{2t - 2k_\Omega(z_0, \gamma) - 2c_0} \right) \right)^{-1} dt
\]

\[
\leq \frac{2k_\Omega(z_0, \gamma) + c_0}{g \left( e^{2k_\Omega(z_0, \gamma) - 2c_0} \right)} + \int_{e^{2k_\Omega(z_0, \gamma) - 2c_0}}^\infty \frac{d\beta}{\beta g(\beta)}
\]

\[
\leq \left( \frac{c_0 + \ln s}{g(s)} + \int_s^\infty \frac{d\beta}{\beta g(\beta)} \right) \bigg|_{s=e^{2k_\Omega(z_0, \gamma)}}.
\]

By the definition of \( (g(s))^{-1} \) in Theorem 3 and the fact that \( f(\alpha)\alpha^{-1/2} \) decreasing, it follows

\[
\frac{1}{g(s)} \leq \int_{e^{2c_0}}^{\infty} \frac{d\alpha}{\alpha f(\alpha)} = \int_s^\infty \frac{e^{c_0}d\alpha}{\alpha f(\alpha)} \leq \int_s^\infty \frac{e^{c_0}d\alpha}{\alpha^{3/2} f(\alpha)} = \frac{e^{c_0}}{g(s)}.
\]
Let $\Omega$ be a domain in $\mathbb{C}^n$.

Definition 3. Let $\Omega$ be a domain in $\mathbb{C}^n$. Fix $z_0 \in \Omega$, $x \in \partial \Omega$ and $R > 0$. Then the small horosphere $E_{\Omega}(x, R)$ and the big horosphere $F_{\Omega}(x, R)$ of center $x$, pole $z_0$ and radius $R$ are defined by

$$E_{\Omega}(x, R) = \{z \in \Omega: \limsup_{\Omega \ni w \to x} [k_{\Omega}(z, w) - k_{\Omega}(z_0, w)] < \frac{1}{2} \ln R\}$$
From (11) and (12), we obtain

\[ F_{z_0}(x, R) = \{ z \in \Omega : \liminf_{\Omega \ni w \to x} [k_\Omega(z, w) - k_\Omega(z_0, w)] < \frac{1}{2} \ln R \}. \]

**Definition 4.** (see [3, p.185]) A domain \( \Omega \subset \mathbb{C}^n \) is called \( F \)-convex if for every \( x \in \partial \Omega \)

\[ F_{z_0}(x, R) \cap \partial \Omega \subseteq \{x\} \]

holds for every \( R > 0 \) and for every \( z_0 \in \Omega \).

**Remark 1.** The bidisk \( \Delta^2 \) in \( \mathbb{C}^2 \) is not \( F \)-convex. Indeed, since \( d_{\Delta^2}((1/2, 1 - 1/k), (0, 1 - 1/k)) - d_{\Delta^2}((0, 0), (0, 1 - 1/k)) = d_\Delta(1/2, 0) - d_\Delta(0, 1 - 1/k) \to -\infty \) as \( \mathbb{N}^* \ni k \to \infty \), \( (1/2, 1) \in \overline{F_{\Delta^2}((0, 1), R)} \cap \partial(\Delta^2) \) for any \( R > 0 \).

**Remark 2.** If \( \Omega \) is either a strongly pseudoconvex domain in \( \mathbb{C}^n \), or a pseudoconvex domain of finite type in \( \mathbb{C}^2 \), or a pseudoconvex domain of strict finite type in \( \mathbb{C}^n \) then \( \Omega \) is \( F \)-convex (see [2, 3, 23]).

Now, we prove that \( F \)-convexity holds on a larger class of pseudoconvex domains.

**Proposition 7.** Let \( \Omega \) be a domain satisfying the hypotheses of Theorem 7. Then \( \Omega \) is \( F \)-convex.

**Proof.** Let \( R > 0 \) and \( z_0 \in \Omega \). Assume by contradiction that there exists \( y \in \overline{F_{z_0}(x, R)} \cap \partial \Omega \) with \( y \neq x \). Then we can find a sequence \( \{z_n\} \subset \Omega \) with \( z_n \to y \in \partial \Omega \) and a sequence \( \{w_n\} \subset \Omega \) with \( w_n \to x \in \partial \Omega \) such that

\[ k_\Omega(z_n, w_n) - k_\Omega(z_0, w_n) \leq \frac{1}{2} \ln R. \] (11)

Moreover, for each \( n \in \mathbb{N}^* \) there exists a minimizing geodesic \( \gamma_n \) connecting \( z_n \) to \( w_n \). Let \( p_n \) be a point on \( \gamma_n \) of minimal distance \( k_\Omega(z_n, \gamma_n) = k_\Omega(z_0, p_n) \) to \( z_0 \). We consider the following two cases for the sequence \( \{p_n\} \).

**Case 1.** There exists a subsequence \( \{p_{n_k}\} \) of the sequence \( \{p_n\} \) such that \( p_{n_k} \to p_0 \in \Omega \) as \( k \to \infty \).

\[ k_\Omega(w_{n_k}, z_{n_k}) \geq k_\Omega(w_{n_k}, p_{n_k}) + k_\Omega(p_{n_k}, z_{n_k}) \]
\[ \geq k_\Omega(w_{n_k}, z_0) - k_\Omega(z_0, p_{n_k}) + k_\Omega(p_{n_k}, z_{n_k}). \] (12)

From (11) and (12), we obtain

\[ k_\Omega(p_{n_k}, z_{n_k}) \leq k_\Omega(w_{n_k}, z_{n_k}) - k_\Omega(w_{n_k}, z_0) + k_\Omega(z_0, p_{n_k}) \leq \frac{1}{2} \ln R + k_\Omega(z_0, p_{n_k}) \lesssim 1. \]

This is a contradiction since \( \Omega \) is \( k \)-complete.

**Case 2.** Otherwise, \( p_n \to \partial \Omega \) as \( n \to \infty \). By Lemma 5 there are constants \( c \) and \( c_0 \) only depending on \( z_0 \) such that

\[ \delta_\Omega(w_n, z_n) \leq c \int_{\alpha = 2\delta_\Omega(z_0, \gamma_n)} \frac{c_0 + \ln \alpha}{\alpha f(\alpha)} d\alpha. \] (13)

On the other hand, \( \delta_\Omega(w_n, z_n) \gtrsim 1 \) since \( x \neq y \). Thus, the inequality (13) implies that

\[ k_\Omega(z_0, \gamma_n) = k_\Omega(z_0, p_n) \lesssim 1. \] (14)

Therefore,

\[ k_\Omega(z_n, w_n) \geq k_\Omega(z_n, p_n) + k_\Omega(p_n, w_n) \]
\[ \geq k_\Omega(z_0, z_n) + k_\Omega(z_0, w_n) - 2k_\Omega(z_0, p_n). \] (15)

Combining with (11) and (14), we get

\[ k_\Omega(z_0, z_n) \leq k_\Omega(z_n, w_n) - k_\Omega(z_0, w_n) + 2k_\Omega(z_0, p_n) \lesssim \ln R + 1. \]

This is a contradiction since \( z_n \to y \in \partial \Omega \) and hence the proof is complete.

The following theorem is a generalization of Theorem 3.1 in [3].
Lemma 9. Let $\Omega$ be a domain satisfying the hypothesis in Theorem \ref{thm:1} and fix $z_0 \in \Omega$. Let $\phi \in H(\Omega, \Omega)$ such that $\{\phi^k\}$ is compactly divergent. Then there is a point $x \in \partial \Omega$ such that for all $R > 0$ and for all $m \in \mathbb{N}$

$$
\phi^m(E_{z_0}(x, R)) \subset F_{z_0}(x, R).
$$

Proof. Since $\{\phi^k\}$ is compactly divergent and $\Omega$ is $k$-complete,

$$
\lim_{k \to +\infty} k_\Omega(z_0, \phi^k(z_0)) = \infty.
$$

For every $\nu \in \mathbb{N}$, let $k_\nu$ be the largest integer $k$ satisfying $k_\Omega(z_0, \phi^k(z_0)) \leq \nu$; then

$$
k_\Omega(z_0, \phi^{k_\nu}(z_0)) \leq \nu < k_\Omega(z_0, \phi^{k_\nu+m}(z_0)) \forall \nu \in \mathbb{N}, \forall m > 0. \tag{16}
$$

Again, since $\{\phi^k\}$ is compactly divergent, up to a subsequence, we can assume that

$$
\phi^{k_\nu}(z_0) \to x \in \partial \Omega.
$$

Fix an integer $m \in \mathbb{N}$. Without loss of generality we may assume that $\phi^{k_\nu}(\phi^m(z_0)) \to y \in \partial \Omega$. Using Corollary \ref{cor:6} and the fact that

$$
k_\Omega(\phi^{k_\nu}(\phi^m(z_0)), \phi^{k_\nu}(z_0)) \leq k_\Omega(\phi^m(z_0), z_0) \quad \text{(by (5))}
$$

it must hold that $x = y$.

Set $w_\nu = \phi^{k_\nu}(z_0)$. Then $w_\nu \to x$ and $\phi^m(w_\nu) = \phi^{k_\nu}(\phi^m(z_0)) \to x$. From (16), we also have for $m \geq 0$

$$
\limsup_{\nu \to +\infty} [k_\Omega(z_0, w_\nu) - k_\Omega(z_0, \phi^m(w_\nu))] \leq 0. \tag{17}
$$

Now, fix $m > 0$, $R > 0$ and take $z \in E_{z_0}(x, R)$. Then

$$
\liminf_{\Omega \ni w \to x} [k_\Omega(\phi^m(z), w) - k_\Omega(z_0, w)]
\leq \liminf_{\nu \to +\infty} [k_\Omega(\phi^m(z), \phi^m(w_\nu)) - k_\Omega(z_0, \phi^m(w_\nu))]
\leq \liminf_{\nu \to +\infty} [k_\Omega(z, w_\nu) - k_\Omega(z_0, \phi^m(w_\nu))]
\leq \liminf_{\nu \to +\infty} [k_\Omega(z, w_\nu) - k_\Omega(z_0, w_\nu)]
\leq \limsup_{\nu \to +\infty} [k_\Omega(z_0, w_\nu) - k_\Omega(z_0, \phi^m(w_\nu))]
\leq \limsup_{\nu \to +\infty} [k_\Omega(z, w_\nu) - k_\Omega(z_0, w_\nu)]
\leq \limsup_{\Omega \ni w \to x} k_\Omega(z, w) - k_\Omega(z_0, w)]
< \frac{1}{2} \ln R,
$$

that is $\phi^m(z) \in F_{z_0}(x, R)$. Here, the first inequality follows by $\phi^m(w_\nu) \to x$, the second follows by (5), the fourth follows by (17), and the last one follows from the fact that $z \in E_{z_0}(x, R)$. 

\qed

Lemma 9. Let $\Omega$ be a $F$-convex domain in $\mathbb{C}^n$. Then for any $x, y \in \partial \Omega$ with $x \neq y$ and for any $R > 0$, we have $\lim_{a \to y} E_a(x, R) = \Omega$, i.e., for each $z \in \Omega$, there exists a number $\epsilon > 0$ such that $z \in E_a(x, R)$ for all $a \in \Omega$ with $|a - y| < \epsilon$.

Proof. Suppose that for some $z \in \Omega$ such that there exists a sequence $\{a_n\} \subset \Omega$ with $a_n \to y$ and $z \notin E_{a_n}(x, R)$. Then we have

$$
\limsup_{w \to x} [k_\Omega(z, w) - k_\Omega(a_n, w)] \geq \frac{1}{2} \ln R.
$$
This implies that
\[ \liminf_{w \to z} [k_{\Omega}(a_n, w) - k_{\Omega}(z, w)] \leq \frac{1}{2} \ln \frac{1}{R}. \]
Thus, \( a_n \in \overline{F}(x, 1/R) \), for all \( n = 1, 2, \cdots \). Therefore, \( y \in \overline{F}(x, 1/R) \cap \partial \Omega = \{x\} \), which is absurd, and the proof is complete.

Now we are ready to prove our main result.

**Proof of Theorem 1.** First we fix a point \( z_0 \in \Omega \). By Proposition 8 there is a point \( x \in \partial \Omega \) such that for all \( R > 0 \) and for all \( m \in \mathbb{N} \)
\[ \phi^m(E_{z_0}(x, R)) \subset F_{z_0}(x, R). \]
We need to show that for any \( z \in \Omega \)
\[ \phi^m(z) \to x \quad \text{as} \quad m \to +\infty. \]
Indeed, let \( \psi(z) \) be a limit point of \( \{\phi^m(z)\} \). Since \( \{\phi^m\} \) is compactly divergent, \( \psi(z) \in \partial \Omega \). By Lemma 9 for any \( R > 0 \) there is \( a \in \Omega \) such that \( z \in E_a(x, R) \). By Proposition 8, \( \phi^m(z) \in F_a(x, R) \) for every \( m \in \mathbb{N}^* \). Therefore,
\[ \psi(z) \in \partial \Omega \cap \overline{F}_{a}(x, R) = \{x\} \]
by Proposition 7 thus the proof is complete.

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**References**


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