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Abstract

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Keywords

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LOWER BOUNDS ON THE KOBAYASHI METRIC NEAR A POINT OF INFINITE TYPE

TRAN VU KHANH

Abstract. Under a potential-theoretical hypothesis named f-property which holds for all pseudoconvex domains of finite type and many examples of infinite type, we give a new method for constructing a family of bumping functions and hence plurisubharmonic peak functions with good estimates. The rate of lower bounds on the Kobayashi metric follows by the estimates of peak functions. The application to the continuous extendibility of proper holomorphic maps is given.

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Key words and phrases: Kobayashi metric, proper holomorphic map, finite and infinite type

Contents

1. Introduction

Let Ω be a C^2, pseudoconvex domain in $\mathbb{C}^n$ and z_o be a boundary point. For a smooth monontonic increasing function $f : [1 + \infty) \to [1, +\infty)$ with $\dfrac{f(t)}{t^{1/2}}$ decreasing, we say that Ω has the f-property at z_o if there exist a neigborhood U of z_o and a family of functions $\{\phi_\delta\}$ such that

(i) functions ϕ_δ are plurisubharmonic, $-1 \le \phi_\delta \le 0$ and C^2 on U;
(ii) $i\partial\bar\partial\phi_\delta \gtrsim f(\delta^{-1})^2 Id$ and $|D\phi_\delta| \lesssim \delta^{-1}$ for any $z \in U \cap \{z \in \Omega : -\delta < r(z) < 0\}$, where r is a C^2-defining function of Ω.

In the joint work with G. Zampieri [KZ10], we show that the f-property implies an f-estimate for the $\bar\partial$-Neumann problem. In another paper [KZ12], we prove that an f-estimate with $\frac{f}{\log}\to\infty$ at ∞ implies that the Bergman metric has a lower bound with the rate $g(t)=\frac{f}{\log}(t^{1-\eta})$ for $\eta>0$. The ideas leading to these results follow by Kohn [Koh02], Catlin [Cat83, Cat87] and McNeal [McN92b]. Combining the above results, we obtain the following:

Theorem 1.1. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with C^∞-smooth boundary and z_o a point in the boundary $b\Omega$. Assume that the f-property holds at z_o with $g(t):=\frac{f(t)}{\log(t)}\nearrow\infty$ as $t\to\infty$. Then for any $\eta>0$ there are a neighborhood V_η of z_o and a constant C_η such that the Bergman metric B of Ω satisfies*

$$B(z,X)\geq C_\eta g(\delta_\Omega^{-1+\eta}(z))|X| \tag{1.1}$$

for any $z\in V_\eta\cap\Omega$ and $X\in T^{1,0}_z\mathbb{C}^n$, where δ_Ω denotes the distance of z to $b\Omega$.

The purpose of this paper is to prove a result similar to Theorem 1.1 for the Kobayashi metric. Let us recall the definition of the Kobayashi metric: Let Ω be a pseudoconvex domain in $\mathbb{C}^n$; the function $K:T^{1,0}\Omega\to\mathbb{R}$ on the holomorphic tangent bundle, given by

$$\begin{aligned} K(z,X)=&\inf\{\alpha>0|\exists\kappa:\Delta\to\Omega\text{ holomorphic with }\kappa(0)=z,\kappa'(0)=\alpha^{-1}X\}\\ =&\inf\{s^{-1}|\exists\kappa:\Delta_s\to\Omega\text{ holomorphic with }\kappa(0)=z,\kappa'(0)=X\},\end{aligned} \tag{1.2}$$

is called the Kobayashi metric of Ω. Here Δ denotes the unit disc and Δ_s the disc in $\mathbb{C}$ centered at 0 with radius s.

Our main result in this paper is the following:

Theorem 1.2. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary $b\Omega$ and z_o be a boundary point. Assume that Ω has the f-property at z_o with f satisfying $\int_t^\infty\frac{da}{af(a)}<\infty$ for some $t>1$, and denote by $(g(t))^{-1}$ this finite integral. Then, there are a neighborhood V of z_o and a constant $C>0$ such that*

$$K(z,X)\geq Cg(\delta_\Omega^{-1}(z))|X| \tag{1.3}$$

for any $z\in V\cap\Omega$ and $X\in T^{1,0}_z\mathbb{C}^n$, where δ_Ω denotes the distance of z to $b\Omega$.

We remark that both Theorem 1.1 and 1.2 may apply to all pseudoconvex domains of finite type and many class of pseudoconvex domains of infinite type (see Corollary 1.3 below). Compared with Theorem 1.1, in Theorem 1.2 we reduce the C^∞-smoothness of the boundary and slightly strengthen the hypothesis if f in the f-property since $\int_t^\infty\frac{da}{af(a)}<\infty$ is stronger than $\lim\limits_{t\to\infty}\frac{f(t)}{\log t}=\infty$. Moreover, we obtain a larger size of the lower bound of the Kobayashi metric in the case $f(t) = t^\epsilon$, i.e., we have $g(t) = t^\epsilon$ in Theorem 1.2 instead of $g(t) = t^{\epsilon-\eta}$ in Theorem 1.1; and the same size of the lower bounded of both metrics in the case $f(t) = \log^{1+\epsilon}$, in this case we have $g(t) = \log^\epsilon(t)$ in two theorems.

The f-property is a potential-theoretical condition of the boundary where the function f reflects the geometric type. In [Cat87, Theorem 9.2], Catlin proves that every smooth, pseudoconvex domain Ω of finite type m in $\mathbb{C}^n$ has the f-property for $f(t) = t^\epsilon$ with $\epsilon = m^{-n^2m^{n^2}}$. Additionally, there are several cases when Ω is known to have the f-property with $f(t) = t^{1/m}$ where m is the type: strongly pseudoconvex (see [Kha10, Theorem 7.1]), pseudoconvex of finite type in $\mathbb{C}^2$ (see [Cat89, Proposition 2.1]), decoupled or convex in $\mathbb{C}^n$ (see [Kha10, Theorem 7.7], [McN92a, Proposition 3.2])). Furthermore, there is a large class of infinite type pseudoconvex domains that satisfy an f-properties, for example, let Ω be defined by

$$\Omega = \{z \in \mathbb{C}^n : \mathrm{Im} z_n + \sum_{j=1}^{n-1} P_j(z_j) < 0\}, \tag{1.4}$$

where $\Delta P_j(z_j) \gtrsim \dfrac{\exp(-1/|x_j|^\alpha)}{x_j^2}$ or $\dfrac{\exp(-1/|y_j|^\alpha)}{y_j^2}$ for $j = 1, \ldots, n-1$, then the f-property holds with $f(t) = \log^{1/\alpha} t$ (see [KZ10, Proposition 3.1]). This remark immediately leads to the following.

Corollary 1.3. *1) Let Ω be a pseudoconvex domain of finite type m in $\mathbb{C}^n$. Then* (1.3) *holds for $g(t) = t^{\frac{1}{m}}$ if Ω satisfies at least one of the following conditions: Ω is strongly pseudoconvex, or Ω is convex, or $n = 2$, or Ω is decoupled. In general case, we have $g(t) = t^\epsilon$ with $\epsilon = m^{-n^2m^{n^2}}$.*

2) Let Ω be defined by (1.4) *with $\alpha < 1$. Then* (1.3) *holds for $g(t) = \log^{\frac{1}{\alpha}-1} t$.*

An important tool for estimating the Kobayashi metric from below is the following bumping theorem. The bumping functions on pseudoconvex domains of finite type had been constructed by K. Diederich and J.E. Fornaess [DF79] for the real analytic case and S. Cho [Cho92] for smooth case. However, their methods are strictly for domains of finite type since they can be approximated by real analytic domains (or they themselves are real analytic). Here, we give a new method for constructing the bumping functions on pseudoconvex domains of general type that includes many infinite type examples. We will prove that, for any boundary point w on $b\Omega$ which satisfies the f-property, we can find a pseudoconvex hypersurface touching $\bar{\Omega}$ exactly at w from the outside such that the distance from $z \in \Omega$ to the new hypersurface is exactly controlled by the rate in $|z-w|^{-1}$ of the reciprocal of the inverse of g (see Theorem 2.1 in Section 2).

The lower bound of the Kobayashi metric is an important tool in the function theory of several complex variables and has been studied by many authors. In the following, we briefly review some significant, classical results related to this paper.

When Ω is strongly pseudoconvex or else it is pseudoconvex of finite type in $\mathbb{C}^2$ and decoupled or convex in $\mathbb{C}^n$, then the size of the Kobayashi metric has been described by I. Graham [Gra75], D. Catlin [Cat89], G. Herbort [Her92] and L. Lee [Lee08]. In these classes of domains, the Kobayashi metric $K(z,X)$ is asymptotically equivalent to

$$\delta_\Omega^{-1/m}(z)|X^\tau| + \delta_\Omega^{-1}(z)|X^\nu| \qquad \text{as} \qquad z \to b\Omega,$$

where X^τ and X^ν are the tangential and normal components of X and m is the type of the boundary.

For a general pseudoconvex domain in $\mathbb{C}^n$, K. Diederich and J. E. Fornaess [DF79] proved, by using Kohn's algorithm [Koh79], that there is a $\epsilon > 0$ such that $K(z,X) \gtrsim \delta_\Omega(z)^{-\epsilon}|X|$ if $b\Omega$ is real analytic of finite type. By using the method of Catlin in [Cat87, Cat89], S. Cho [Cho92] improved the result of [DF79] for domains which are not necessarily real analytic. However, in the case of infinite type we know very little except from the recent results by S. Lee [Lee01] for the exponentially-flat infinite type.

Among other uses of the lower bound of the Kobayashi metric, we mention the continuous extendibility of proper holomorphic maps to the boundary of a domain of general type. We refer readers to [Hen73, BF78, DF79, Ran78] for this problem on domains of finite type.

Theorem 1.4. *Let Ω and Ω' be pseudoconvex domains. Let $\eta, 0 < \eta \leq 1$ be such that there is a C^2 defining function r of Ω with the property that $-(-r)^\eta$ is strictly plurisubharmonic on Ω. Assume that Ω' has the f-property with f satisfying $\displaystyle\int_t^\infty \frac{(\ln a - \ln t)da}{af(a)} < \infty$ for some $t > 1$, and denote by $(\tilde{f}(t))^{-1}$ this finite integral. Then any proper holomorphic map $\Psi : \Omega \to \Omega'$ can be extended as a general Hölder continuous map $\hat{\Psi} : \bar{\Omega} \to \bar{\Omega}'$ with a rate $\tilde{f}(t^\eta)$, that is,*

$$|\hat{\Psi}(z) - \hat{\Psi}(w)| \leq C\tilde{f}(|z-w|^{-\eta})^{-1}$$

for any $z, w \in \bar{\Omega}$.

The paper is organized as follows. In section 2, using the f-property, we construct the bumping functions. In Section 3, by the existence of suitable exhaustion functions, we obtain the plurisubharmonic peak functions having the good estimates and hence the lower bound of the Kobayashi metric follows. Modifying the argument by Diederich-Fornaess [DF79] we prove Theorem 1.4 in Section 4.

Throughout this paper, we use $\lesssim$ and $\gtrsim$ to denote inequality up to a positive constant. Moreover, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$, and $\partial\bar\partial$ for $i\partial\bar\partial$.

2. The bumping function

In this section, we construct the family of bumping functions from the f-property.

Theorem 2.1. *Let Ω be pseudoconvex and z_o be a boundary point. Assume that Ω has the f-property at z_o with f satisfying $\displaystyle\int_t^\infty \frac{da}{af(a)} < \infty$ for some $t > 1$, and denote by $(g(t))^{-1}$ this finite integral. Then there is a neighborhood U of z_o and a real C^2 function ρ on $U \times (U \cap b\Omega)$ with the following properties:*

1. *$\rho(w, w) = 0$ for any $w \in (U \cap b\Omega)$.*
2. *$\rho(z, w) \le -G(|z - w|)$ for any $(z, w) \in (U \cap \Omega) \times (U \cap b\Omega)$ where $G(\delta) = (g^*((\gamma\delta)^{-1}))^{-1}$. Here, the supercript * denotes the inverse function and $\gamma > 0$ sufficiently small.*
3. *$\rho(z, \pi(z)) \gtrsim -\delta_\Omega(z)$ for any $z \in U \cap \Omega$ where $\pi(z)$ is the projection of z to the boundary $b\Omega$.*
4. *For each fixed $w \in U \cap b\Omega$, denote $S_w = \{z \in U : \rho(z, w) = 0\}$. One has:*
   (a) *$|D_z\rho(z, w)| \approx 1$ everywhere on S_w.*
   (b) *S_w is pseudoconvex. In fact, one can choose ρ such that S_w is strongly pseudoconvex outside of w.*
   (c) *S_w touches $\bar\Omega$ exactly at w from outside.*

Proof of Theorem 2.1. The proof is divided in four steps. In Step 1, we show the equivalence of the f-property between the pseudoconvex and the pseudoconcave side of a hypersurface. In Step 2, we prove that there exists a single function with self-bounded gradient which has a lower bound $f(r^{-1}(z))$ for the Levi form. In Step 3, we estimate the function G. The properties of bumping function is checked on Step 4.

Step 1. Since the hypersurface defined by each bumping function lies outside the original domain except from one point and the f-property takes place inside the domain, we first show hat the f-property still holds outside the domain.

Without loss of generality, we can assume that the original point z_o belongs to $U \cap b\Omega$. We choose special coordinates $z = (x, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ at z_o. Assume that there is a family of functions ϕ_δ which have properties (i) and (ii) in the first paragraph of Section 1. Define $\tilde\phi_\delta(x, r) = \phi_\delta(x, r - \delta)$ for each $\delta > 0$ and $\tilde U = \{(x, r) : (x, r - \delta) \in U\}$. Then $\tilde\phi_\delta$ is C^2 and $-1 \le \tilde\phi_\delta \le 0$ on $\tilde U$ and also

$$\partial\bar\partial\tilde\phi_\delta(\tilde X, \tilde X)|_{(x,r)} = \partial\bar\partial\phi_\delta(X, X)|_{(x,r-\delta)} \ge 0, \text{ for all } \tilde X \in T^{1,0}_{(x,r)}\mathbb{C}^n \quad \text{and } (x, r) \in \tilde U.$$

where $X|_{(x,r-\delta)} := \tilde{X}|_{(x,r)}$, i.e., $\tilde{\phi}$ is plurisubharmonic on $\tilde{U}$. Moreover, we also have

$$\partial\bar\partial\tilde{\phi}_\delta(\tilde{X},\tilde{X})|_{(x,r)} = \partial\bar\partial\phi_\delta(X,X)|_{(x,r-\delta)} \gtrsim f^2(\delta^{-1})|X|^2|_{(x,r-\delta)} = f^2(\delta^{-1})|\tilde{X}|^2|_{(x,r)}$$

and $|D\tilde{\phi}_\delta| \lesssim \frac{1}{\delta}$ for all $\tilde{X} \in T^{1,0}\mathbb{C}^n$ on $\{z : -\delta < r - \delta < 0\}$ or $\{z : 0 < r < \delta\}$. Therefore, the family $\{\tilde{\phi}_\delta\}$ satisfies the f-property outside Ω. In what follows, we use notation ϕ_δ for $\tilde{\phi}_\delta$.

Step 2. In this step we construct a single function which has self-bounded gradient and has lower bound $f(r^{-1}(z))$ for the Levi form.

Lemma 2.2. *Assume that Ω enjoys the f-property at z_o. Then there are a neighborhood U of z_o, a single function $\Phi \in C^2(U \setminus b\Omega)$ such that $\Phi = 0$ on $U \cap \bar\Omega$ and on $U \setminus \bar\Omega$ we have*

(1) $-1 \lesssim \Phi \le 0$

(2) $\partial\bar\partial\Phi(X,\bar X) \ge -C(\dfrac{1}{r}|\partial\bar\partial r(X,\bar X)| + \dfrac{1}{r^2}|Xr|^2) + \dfrac{1}{8}|X\Phi|^2 + cf^2(\dfrac{1}{r})|X|^2$

(3) $|D\Phi| \lesssim \dfrac{1}{r}$.

where C and c are positive constants.

Proof. Let χ be a cut-off function such that $\chi(t) = \begin{cases} 0 & \text{if } t \le \frac{1}{4} \text{ or } t \ge 2, \\ 1 & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$ We also suppose that $|\dot\chi|$, $|\ddot\chi|$ and $\dfrac{\dot\chi^2}{\chi}$ are bounded. Define

$$\Phi(z) := \sum_{j=1}^{\infty} \left(\exp(\phi_{2^{-j}}(z)) - 1\right) \chi(2^j r(z)), \tag{2.1}$$

where $\phi_{2^{-j}}$ is constructed in Step 1. Denote $S_\delta := \{z \in U | 0 < r(z) < \delta\}$. Let $z \in U \setminus \bar\Omega = \bigcup\limits_{j=1}^{\infty} S_{2^{-k}} \setminus S_{2^{-(k+1)}}$, then there is an integer k such that

$$z \in S_{2^{-k}} \setminus S_{2^{-(k+1)}} = \{z \in U : 2^{-k-1} \le r(z) < 2^{-k}\}. \tag{2.2}$$

We notice that $\chi(2^j r(z)) = 0$ if $j < k-1$ or $j > k+1$, and $\chi(2^k r(z)) = 1$ for any $z \in S_{2^{-k}} \setminus S_{2^{-(k+1)}}$. Hence, (2.1) can be rewritten that

$$\Phi(z) = \sum_{j=k-1}^{k+1} \left(\exp(\phi_{2^{-j}}(z)) - 1\right) \chi(2^j r(z)).$$

This proves (1). We observe that

$$\begin{aligned}\partial\bar\partial\left(\left(e^{\phi_{2^{-j}}}-1\right)\chi(2^j r)\right)(X,\bar X) &= \left(\partial\bar\partial\phi_{2^{-j}}(X,\bar X)+|X\phi_{2^{-j}}|^2\right)e^{\phi_{2^{-j}}}\chi\\ &\quad + 2^{j+1}\mathrm{Re}\langle X\phi_{2^{-j}},\overline{Xr}\rangle e^{\phi_{2^{-j}}}\dot\chi\\ &\quad + \left(2^j\partial\bar\partial r(X,\bar X)\dot\chi+2^{2j}|Xr|^2\ddot\chi\right)\left(e^{\phi_{2^{-j}}}-1\right)\\ &\geq \left(\partial\bar\partial\phi_{2^{-j}}(X,\bar X)+\frac{1}{2}|X\phi_{2^{-j}}|^2\right)e^{\phi_{2^{-j}}}\chi\\ &\quad - 2^j|\dot\chi|(1-e^{\phi_{2^{-j}}})|\partial\bar\partial r(X,\bar X)|\\ &\quad - 2^{2j}(|\ddot\chi|(1-e^{\phi_{2^{-j}}})+2\frac{\dot\chi^2}{\chi}e^{\phi_{2^{-j}}})|Xr|^2.\end{aligned} \tag{2.3}$$

Here, we use the Cauchy-Schwartz inequality for the second line of (2.3), that is,

$$\left|2^{j+1}\mathrm{Re}\langle X\phi_{2^{-j}}\overline{Xr}\rangle e^{\phi_{2^{-j}}}\dot\chi\right| \leq \frac{1}{2}|X\phi_{2^{-j}}|^2 e^{\phi_{2^{-j}}}\chi + 2^{2j+1}|Xr|^2\frac{\dot\chi^2}{\chi}.$$

Moreover, we also observe that

$$\begin{aligned}|X\Phi(z)|^2 =& \Big|\sum_{j=k-1}^{k+1} X(\phi_{2^{-j}})e^{\phi_{2^{-j}}}\chi + 2^j(e^{\phi_{2^{-j}}}-1)X(r)\dot\chi(2^j r)\Big|^2\\ \leq& 4\sum_{j=k-1}^{k+1}|X\phi_{2^{-j}}|^2 e^{\phi_{2^{-j}}}\chi(2^j r) + 2^{2k+2}(1-e^{-1})^2|Xr|^2\left(\frac{1}{4}\dot\chi^2(2^{k-1}r)+4\dot\chi^2(2^{k+1}r)\right)\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\begin{aligned}\partial\bar\partial\Phi(X,\bar X) \geq& e^{-1}\partial\bar\partial\phi_{2^{-k}}(X,\bar X)+\frac{1}{8}|X\Phi|^2 - C(2^k|\partial\bar\partial r(X,\bar X)|+2^{2k}|Xr|^2)\\ \geq& cf^2(2^k)|X|^2+\frac{1}{8}|X\Phi|^2 - C(2^k|\partial\bar\partial r(X,\bar X)|+2^{2k}|Xr|^2)\end{aligned} \tag{2.5}$$

for any $z \in S_{2^{-k}} \setminus S_{2^{-(k+1)}}$. From (2.4), we also obtain $|D\Phi| \lesssim 2^k$ $z \in S_{2^{-k}} \setminus S_{2^{-(k+1)}}$ since $|D\phi_\delta| \lesssim \delta^{-1}$ and $|Dr| \lesssim 1$. This completes the proof of (2) and (3).

□

Step 3. We recall that $g(t) = \left(\int_t^\infty \frac{da}{af(a)}\right)^{-1} = \left(\int_0^{t^{-1}} \frac{da}{af(a^{-1})}\right)^{-1}$ for any $t > 1$. Then, it is easy to check that g is increasing, $g \to \infty$ at ∞, and $g \leq f$ on $(1, +\infty)$. We define

$$G(\delta) := g^*\left(((\gamma\delta)^{-1})^{-1}\right.$$

where $\gamma > 0$ is a constant to be chosen later. We also notice that G is an increasing, convex function, and $G(0) = 0$.

Claim: For $\delta > 0$, we have

1. $\dot{G}(\delta) = \gamma G(\delta) f(G^{-1}(\delta))$,
2. $0 \leq \ddot{G}(\delta) \leq \gamma^2 G(\delta) f^2(G^{-1}(\delta))$,
3. $\dfrac{\dot{G}(\delta)}{\delta} \leq \gamma^2 G(\delta) f^2(G^{-1}(\delta))$.

Proof of the Claim. By the definition of G and g, we have

$$g(G(\delta)^{-1}) = (\gamma\delta)^{-1} \quad \text{or} \quad \int_0^{G(\delta)} \frac{da}{af(a^{-1})} = \gamma\delta. \tag{2.6}$$

Taking the derivative with respect to δ in the second equation of (2.6), we prove the first and second claims, that is,

$$\dot{G}(\delta) = \gamma G(\delta) f(G^{-1}(\delta)), \tag{2.7}$$

and

$$\begin{aligned} \ddot{G}(\delta) =& \gamma \dot{G}(\delta) f(G^{-1}(\delta)) - \gamma \dot{G}(\delta) G^{-1}(\delta) \dot{f}(G^{-1}(\delta)) \\ =& \gamma^2 G(\delta) f^2(G^{-1}(\delta)) - \gamma^2 f(G^{-1}(\delta)) \dot{f}(G^{-1}(\delta)). \end{aligned} \tag{2.8}$$

Then (2) follows by $\dfrac{f(t)}{t}$ decreasing and $f(t)$ increasing. Moreover, since $f \geq g$, we have $f(G^{-1}(\delta)) = f(g^*((\gamma\delta)^{-1})) \geq f(f^*((\gamma\delta)^{-1})) = (\gamma\delta)^{-1}$. From (2.7) we then get

$$\frac{\dot{G}(\delta)}{\delta} = \frac{1}{\gamma\delta}\gamma^2 G(\delta) f(G^{-1}(\delta)) \leq \gamma^2 G(\delta) f^2(G^{-1}(\delta)).$$

The proof of the claim is complete.

□

Step 4. We define

$$\rho(z, w) = r(z) + G(|z - w|)\,(-1 + \epsilon\Phi(z))$$

where $\epsilon > 0$ will be chosen later.

Let $S_w = \{z \in U | \rho(z, w) = 0\}$ be a hypersurface defined by $\rho(z, w) = 0$ where w is fixed. We will prove that ρ satisfies the following properties:

(i) $\rho(w, w) = 0$ for any $w \in b\Omega$.
(ii) $\rho(z, w) \leq -G(|z - w|)$ for $z \in U \cap \Omega$ and $w \in U \cap b\Omega$.
(iii) $\rho(z, \pi(z)) \gtrsim -r(z)$ for $z \in U \cap \Omega$, where $\pi(z)$ is the projection of z to the boundary.
(iv) S_w is pseudoconvex.
(v) $|D_z\rho(z, w)| \approx 1$ on S_w.

Now, (i) is obvious. Since Φ is negative and bounded, we first choose ϵ so small that $-2 \leq -1+\epsilon\Phi \leq -1$. For $z \in U \cap \Omega$, we have $r(z) < 0$, and $|r(z)| \geq G(|r(z)|)$, hence (ii) and (iii) follow. To prove (iv), we estimate the Levi form of ρ with respect to $z \in U \setminus \bar{\Omega}$:

$$
\begin{aligned}
&\partial_z\bar{\partial}_z\rho(z,w)(X,\bar{X})\\
&\qquad =\partial\bar{\partial}r(X,\bar{X}) + \frac{1}{2}\frac{\dot{G}(|z-w|)}{|z-w|}(-1+\epsilon\Phi(z))|X|^2\\
&\qquad\quad + \frac{1}{4}\left(\frac{\ddot{G}(|z-w|)}{|z-w|^2} - \frac{\dot{G}(|z-w|)}{|z-w|^{3/2}}\right)(-1+\epsilon\Phi(z))|X(|z-w|^2)|^2\\
&\qquad\quad + 2\epsilon\mathrm{Re}\langle XG(|z-w|), X\Phi(z)\rangle + \epsilon G(|z-w|)\partial\bar{\partial}\Phi(z)(X,\bar{X})\\
&\qquad \geq\partial\bar{\partial}r(X,\bar{X}) - \big(\frac{\dot{G}(|z-w|)}{|z-w|} + \frac{1}{2}\ddot{G}(|z-w|) + 16\epsilon\frac{\dot{G}^2(|z-w|)}{G(|z-w|)}\big)|X|^2\\
&\qquad\quad + \epsilon G(|z-w|)\big(\partial\bar{\partial}\Phi(z)(X,\bar{X}) - \frac{1}{16}|X\Phi(z)|^2\big)\\
&\qquad \geq\partial\bar{\partial}r(X,\bar{X}) - \epsilon CG(|z-w|)\left(\frac{1}{r}|\partial\bar{\partial}r(X,\bar{X})| + \frac{1}{r^2}|Xr|^2\right) + \frac{\epsilon}{16}G(|z-w|)|X\Phi|^2\\
&\qquad\quad + G(|z-w|)\left(\epsilon cf^2(\frac{1}{r}) - \frac{\dot{G}(|z-w|)}{|z-w|G(|z-w|)} - \frac{\ddot{G}(|z-w|)}{2G(|z-w|)} - 16\epsilon\frac{\dot{G}^2(|z-w|)}{G^2(|z-w|)}\right)|X|^2\\
&\qquad \geq \underbrace{\partial\bar{\partial}r(X,\bar{X}) - \epsilon CG(|z-w|)\left(\frac{1}{r}|\partial\bar{\partial}r(X,\bar{X})| + \frac{1}{r^2}|Xr|^2\right) + \frac{\epsilon}{16}G(|z-w|)|X\Phi|^2}_{I}\\
&\qquad\quad + \underbrace{G(|z-w|)\left(\epsilon cf^2(\frac{1}{r}) - \gamma^2(\frac{3}{2}+16\epsilon)f^2(G^{-1}(|z-w|))\right)|X|^2}_{II}.
\end{aligned}
\tag{2.9}
$$

Here, the first inequality follows by using Cauchy-Schwartz inequality for the first term in the fourth line; the second inequality follows from Lemma 2.2(ii); and the last follows by Claim.

Now we consider $z \in (S_w \cap U) \setminus \{w\} \subset U \setminus \bar{\Omega}$, that is, $r(z) = G(|z-w|)\big(1-\epsilon\Phi(z)\big)$. By the choice of ϵ, we obtain

$$G(|z-w|) \leq |r(z)| \leq 2G(|z-w|). \tag{2.10}$$

For $X \in T^{1,0}S_w$, that is, $X\rho = 0$. This implies

$$Xr = (1-\epsilon\Phi)XG(|z-w|) - \epsilon G(|z-w|)X\Phi,$$

and hence,

$$|Xr|^2 \le 2\dot{G}^2(|z-w|)|X|^2 + 2\epsilon^2 G^2(|z-w|)|X\Phi|^2. \tag{2.11}$$

Choose ϵ small such that $2\epsilon C \le 1$, thus expression (I) in (2.9) continues as

$$\begin{aligned}(I) \ge& \partial\bar\partial r(X,\bar X) - |\partial\bar\partial r(X,\bar X)| - \frac{1}{G(|z-w|)}|Xr|^2 + \frac{\epsilon}{16}G(|z-w|)|X\Phi|^2\\ \ge& - C_1|X||Xr| - \frac{1}{G(|z-w|)}|Xr|^2 + \frac{\epsilon}{16}G(|z-w|)|X\Phi|^2\\ \ge& - \frac{C_1^2}{4}G(|z-w|)|X|^2 - \frac{2}{G(|z-w|)}|Xr|^2 + \frac{\epsilon}{16}G(|z-w|)|X\Phi|^2\end{aligned} \tag{2.12}$$

Using inequality (2.11), it continues as

$$\begin{aligned}(I) \ge& -\left(\frac{C_1^2}{4}G(|z-w|) + \frac{4\dot{G}^2(|z-w|)}{G(|z-w|)}\right)|X|^2 + (\frac{\epsilon}{16} - 4\epsilon^2)G(|z-w|)|X\Phi|^2\\ \ge& - G(|z-w|)\left(\frac{C_1^2}{4} + 4\gamma^2 f^2(G^{-1}(|z-w|))\right)|X|^2,\end{aligned} \tag{2.13}$$

where choose again ϵ such that $\frac{\epsilon}{16} - 6\epsilon^2 \ge 0$; then the term in left hand side of the first line of (2.13) can be disregarded; the first term follows by Claim.

To estimate the expression (II), we remark again that $\frac{f(t)}{t}$ is decreasing, it implies

$$f(t/2) = (t/2)\frac{f(t/2)}{t/2} \ge (t/2)\frac{f(t)}{t} = f(t)/2.$$

Hence

$$f(\frac{1}{r}) \ge f(\frac{1}{2G(|z-w|)}) \ge \frac{1}{2}f(G^{-1}(|z-w|)).$$

Combining (2.9), (2.13) and above remark

$$\partial\bar\partial\rho(X,\bar X) \ge G(|z-w|)\left(\left(\frac{\epsilon c}{4} - \gamma^2(\frac{11}{2} + 16\epsilon)\right)f^2\left(\frac{1}{G(|z-w|)}\right) - \frac{C_1^2}{4}\right)|X|^2 \tag{2.14}$$

Choosing $\gamma > 0$ small enough and shrinking U, we conclude that

$$\partial\bar\partial\rho(X,X) \gtrsim G(|z-w|)f^2\left(\frac{1}{G(|z-w|)}\right)$$

on S_w for any $X \in T^{1,0}S_w$. The proof of property (iv) is complete.

For any $z \in (S_w \cap U) \setminus \{w\}$, we have

$$\begin{aligned} |D\left(G(|z-w|)(-1+\epsilon\Phi(z))\right)| &\leq \dot{G}(|z-w|) + \epsilon G(|z-w|)|D\Phi(z)| \\ &\leq \gamma G(|z-w|) f(G^{-1}(|z-w|)) + \epsilon \frac{G(|z-w|)}{r(z)} \\ &\leq \gamma + \epsilon \end{aligned} \tag{2.15}$$

where, the second inequality follows from Claim (1) and Lemma 2.2.(3), the third inequality follows from the hypothesis that $f(t) \leq t$ and (2.10). Since $|Dr| \approx 1$, then for ϵ and γ small enough, we obtain $|D\rho| \approx 1$. That is the proof of property (v). The proof of Theorem 2.1 is complete.

□

3. Proof of Theorem 1.2

The proof of Theorem 1.2 follows immediately from Theorem 3.1 and 3.3 below. Theorem 3.1 consists in the construction of plurisubharmonic peak functions with good estimates. This is a consequence of the construction of bumping functions in last section. More precisely, we obtain the following

Theorem 3.1. *Assume that there exists a family of bumping functions on a local path V of the boundary as in the conclusion of Theorem 2.1. Fix $0 < \eta < 1$; then for any $w \in V \cap b\Omega$ there is a plurisuhharmonic function ψ_w on $V \setminus \{w\}$ verifying*

(1) $|\psi_w(z) - \psi_w(z')| \lesssim |z - z'|^\eta$
(2) $\psi_w(z) \lesssim -G^\eta(|z-w|)$
(3) $\psi_{\pi(z)}(z) \gtrsim -\delta_\Omega^\eta(z)$

for any z and z' in $V \cap \bar{\Omega}$.

Remark 3.2. The construction of the plurisubharmonic peak functions on a pseudoconvex domain of finite type in $\mathbb{C}^2$ and a convex domain of finite type in $\mathbb{C}^n$ has been obtained by J. E. Fornaess and N. Sibony in [FS89].

Proof of Theorem 3.1. Using the argument in Section 3 and 4 of Diederich-Fornaess's paper [DF79], we obtain that for any $\eta > 0$, there exist an open neighborhood $V \subset U$, and a constant $L > 0$, such that

$$\psi_w(z) = -\left(-\rho(z,w)e^{L|z-w|^2}\right)^\eta,$$

is a plurisubharmonic function on $V \cap \{z \in U : \rho(z,w) < 0\}$, where $\rho(\cdot, w)$ is a bumping function was constructed in Theorem 2.1. By the properties of ρ in Theorem 2.1, it is easily to check that ψ_w satisfies (1), (2) and (3). That is the proof of Theorem 3.1.

□

Now, we prove the lower bound for the Kobayashi metric by using the plurisubharmonic peak function. We state the theorem in a more general setting and the proof is adapted from the argument given by Diederich-Fornaess [DF79].

Theorem 3.3. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, z_o be a given boundary point, F_1 and F_2 are postive functions such that F_1 is increasing and convex. Assume that there is a neighborhood V of z_o such that for each $w \in V \cap b\Omega$, there is a plurisubharmonic function ψ_w such that*

i) $\psi_w(z) \le -F_1(|z-w|)$
ii) $\psi_{\pi(z)}(z) \ge -F_2(\delta_\Omega(z))$

for $z \in U \cap \Omega$.
Then

$$K_\Omega(z,X) \ge (F_1^*(F_2(\delta_\Omega(z)))^{-1}|X|$$

for all $z \in V\cap\Omega$, $X \in T_z^{1,0}\mathbb{C}^n$.

Proof. We fix now a point $z \in V \cap \Omega$, put $w = \pi(z)$ and assume that $\kappa = (\kappa_1, \dots, \kappa_n) : \overline{\Delta} \to \Omega$ is a holomorphic map of the closed unit disc into Ω with $\kappa(0) = z$.

By applying the mean value inequality to the subharmonic function $\psi_w(\kappa(t))$ on $\overline{\Delta}$ we get

$$\psi_w(z) = \psi_w(\kappa(0)) \le \int_0^1 \psi_w \circ \kappa(e^{i2\pi\theta})d\theta$$

The hypothesis (ii) gives

$$F_2(\delta(z))) \ge \int_0^1 -\psi_w \circ \kappa(e^{i2\pi\theta})d\theta \tag{3.1}$$

We now use the hypothesis (i) of ψ_w,

$$\begin{aligned}\int_0^1 -\psi_w \circ f(e^{i2\pi\theta})d\theta &= \int_0^1 \big(-\psi_w\circ\kappa(e^{i2\pi\theta}) - F_1(|\kappa_j(e^{i2\pi\theta}) - w_j|)\big)d\theta \\ &\quad + \int_0^1 F_1\big(|\kappa(e^{i2\pi\theta}) - w|\big)d\theta \\ &\ge \int_0^1 F_1\big(|\kappa(e^{i2\pi\theta}) - w|\big)d\theta.\end{aligned} \tag{3.2}$$

Using the Jensen inequality for the increasing, convex function F_1, we get

$$F_1(|\kappa_j'(0)|) \le F_1\Big(\int_0^1 |\kappa_j(e^{i2\pi\theta}) - w_j|d\theta\Big) \le \int_0^1 F_1\big(|\kappa_j(e^{i2\pi\theta}) - w_j|\big)d\theta,$$

for $j = 1, \dots, n$. Combining the above inequality with (3.1) and (3.2), we obtain

$$F_1(|\kappa_j'(0)|) \le F_2(\delta_\Omega(z)).$$

An immediate consequence of this is

$$|\kappa_j'(0)| \leq F_1^*(F_2(\delta_\Omega(z)).$$

By the definition of $K(z,X)$, we must have for all $X \in T^{1,0}\mathbb{C}^n$

$$K(z,X) \geq (F_1^*(F_2(\delta_\Omega(z)))^{-1}|X|.$$

□

4. Application to proper holomorphic maps

Let Ω_1 and Ω_2 be bounded domains in $\mathbb{C}^n$ with smooth boundary. Assume that Ω_2 is pseudoconvex of finite type at every boundary point. It is well-known that there is $\alpha > 0$, such that every proper holomorphic map $\Psi : \Omega_1 \to \Omega_2$ is Hölder continuous of order α, in particular, Ψ extends continuosly to $\bar{\Omega}_1$. In this section, we prove a similar result for domains of infinite type. For this purpose we give a suitable estimate for generalized Hölder regularity.

Let f be an increasing function such that $\lim\limits_{t\to+\infty} f(t) = +\infty$. For $\Omega \subset \mathbb{C}^n$, define the f-Hölder space on $\bar{\Omega}$ by

$$\Lambda^f(\bar{\Omega}) = \{u : \|u\|_\infty + \sup_{z,z+h\in\overline{\Omega}} f(|h|^{-1}) \cdot |u(z+h) - u(z)| < \infty\}$$

and set

$$\|u\|_f = \|u\|_\infty + \sup_{z,z+h\in\overline{\Omega}} f(|h|^{-1}) \cdot |u(z+h) - u(z)|.$$

Note that the f-Hölder space include the standard Hölder space $\Lambda_\alpha(\bar{\Omega})$ by taking $f(t) = t^\alpha$ (so that $f(|h|^{-1}) = |h|^{-\alpha}$) with $0 < \alpha < 1$.

Before proving Theorem 1.4, we need a generalization of the Hardy-Littlewood Lemma.

Lemma 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $\delta_\Omega(x)$ denote the distance function from x to the boundary of Ω. Let $G : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $\frac{G(\delta)}{\delta}$ is decreasing and $\int_0^d \frac{G(\delta)}{\delta} d\delta < \infty$ for $d > 0$ small enough. Let $u \in C^1(\Omega)$ satisfy*

$$|\nabla u(x)| \lesssim \frac{G(\delta_\Omega(x))}{\delta_\Omega(x)} \quad \text{for every } x \in \Omega. \tag{4.1}$$

Then $u \in \Lambda^f(\bar{\Omega})$ where $f(d^{-1}) = \Big(\int_0^d \frac{G(\delta)}{\delta} d\delta\Big)^{-1}$.

The proof of this theorem can be found in [Kha12].

Remark 4.2. If $G(t) = t^\alpha$, Lemma 4.1 is the classical Hardy-Littlewood Lemma for a domain of finite type.

Proof of Theorem 1.4. Using Theorem 1.2 for Ω', the Schwarz-Pick lemma for the Kobayashi metric, and the upper bound of the Kobayashi metric, we obtain the following estimate

$$g\left(\delta_{\Omega'}^{-1}(\Psi(z))\right)|\Psi'(z)X| \lesssim K_{\Omega'}(\Psi(z), \Psi'(z)X) \le K_\Omega(z, X) \lesssim \delta_\Omega^{-1}(z)|X| \tag{4.2}$$

for any $z \in \Omega$ and $X \in T^{1,0}\mathbb{C}^n$. Moreover, by the fact that $-(-r)^\eta$ is strictly plurisubharmonic on Ω, one has $\delta_{\Omega'}(\Psi(z)) \lesssim \delta_\Omega^\eta(z)$ for any $z \in \Omega$ (Lemma 8 in [DF79]). Therefore,

$$|\Psi'(z)X| \lesssim \delta_\Omega^{-1}(z) g^{-1}(\delta_\Omega^{-\eta}(z))|X|$$

for any $z \in \Omega$ and $X \in T^{1,0}\mathbb{C}^n$. Using Lemma 4.1, Ψ can be extended to a h-Hölder continuous map $\hat{\Psi} : \bar{\Omega} \to \bar{\Omega}'$ with the rate $h(t)$ defined by

$$\begin{aligned}(h(t))^{-1} :=& \int_0^{t^{-1}} \frac{d\delta}{\delta g(\delta^{-\eta})} = \frac{1}{\eta}\int_{t^\eta}^\infty \frac{db}{bg(b)} \\ =& \frac{1}{\eta}\int_{t\eta}^\infty \frac{1}{b}\left(\int_b^\infty \frac{da}{af(a)}\right)db = \frac{1}{\eta}\int_{t^\eta}^\infty \frac{1}{af(a)}\left(\int_{t^\eta}^a \frac{db}{b}\right)da \\ =& \frac{1}{\eta}\int_{t^\eta}^\infty \frac{\ln a - \ln t^\eta}{af(a)}da = \frac{1}{\eta}(\tilde{f}(t^\eta))^{-1}.\end{aligned} \tag{4.3}$$

The proof of Theorem 1.4 is complete.

□

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