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The structure and ergodic theory of higher-rank graph algebras

A Thesis submitted in fulfillment
of the requirements for the award of the degree

Doctor of Philosophy in Mathematics

from the

University of Wollongong, NSW

by

Peter Lewin, BMath(Hons)

School of Mathematics and Applied Statistics

2011

I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

Peter Lewin

Acknowledgments.

To my supervisor Aidan Sims,

colleagues,

friends,

administration staff,

and of course my amazing wife,

Tracy Lewin.

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Abstract

A higher-rank graph is a higher-dimensional analogue of a directed graph. These were introduced by Kumjian and Pask in 2000 as a combinatorial model for the higher-rank Cuntz-Krieger algebras studied by Robertson and Steger in 1999. Since their inception, studying the properties of higher-rank graph algebras has been a flourishing research area. The most general class of higher-rank graphs currently under investigation is called *finitely aligned*. Raeburn, Sims and Yeend showed in 2004 how to associate to each finitely aligned higher-rank graph Λ a universal C^*-algebra $C^*(\Lambda)$. These algebras share many important properties with the Cuntz-Krieger algebras, in particular they are highly tractable. Hence higher-rank graphs provide an excellent mechanism for constructing examples of tractable C^*-algebras, which accounts for the interest in the area in recent years.

Recently, Shotwell has formulated two conditions – aperiodicity and cofinality – in a finitely aligned higher-rank graph Λ which characterise simplicity of $C^*(\Lambda)$. Working independently of Shotwell, we give an alternate formulation of aperiodicity and cofinality in Λ which characterises simplicity of $C^*(\Lambda)$. The key point of difference is that our formulations of aperiodicity and cofinality are given only in terms of finite paths. Thus our conditions are more easily verified in examples. Moreover, all of our proofs are direct and employ new techniques for studying higher-rank graph algebras. We also show how our cofinality condition simplifies in a number of special cases occurring in the literature; in these cases our results are also new.

One of they key dynamical invariants used in ergodic theory is topological entropy. Here we investigate topological entropy of an endomorphism, known as the noncommutative shift map, of the core of a higher-rank graph algebra. We consider the class of *locally finite higher-rank graphs with no sources*, a subclass of all finitely aligned higher-rank graphs. Recently, Jeong and Park provided estimates for topological entropy of the noncommutative shift map for C^*-algebras associated to infinite directed graphs. In addition, Skalski and Zacharias have computed the topological entropy of the noncommutative shift map for C^*-algebras associated to finite higher-rank graphs. We provide a common generalisation of the work of Jeong and Park, and of Skalski and Zacharias. That is, we provide estimates for topological entropy of the noncommutative shift map for C^*-algebras associated to infinite higher-rank graphs.

CHAPTER 1

Introduction and literature review

1.1. The origin of graph algebras

In 1980, Cuntz and Krieger [**9**] published their seminal paper in which they construct a class of C^*-algebras, now known as *Cuntz-Krieger algebras*, from finite $\{0,1\}$-matrices A. This was achieved by defining a set of relations — which we refer to as the *Cuntz-Krieger relations* — on a set of partial isometries indexed by the same set as A. The construction of Cuntz and Krieger established a link between C^*-algebras and symbolic dynamics.

Shortly after, Enomoto and Watatani [**10**] formalized a link between Cuntz-Krieger algebras and graph theory. A directed graph E consists of a set of vertices E^0, edges E^1 and range and source maps $r, s : E^1 \to E^0$. For $n \in \mathbb{N}$ and $A \in \mathrm{M}_n(\mathbb{N})$, we may construct a finite directed graph E_A, where $E_A^0 = \{v_1, \ldots, v_n\}$ and whenever $a_{ij} \neq 0$ we draw a_{ij} edges with range v_i and source v_j. Conversely, given a finite directed graph E, we can define a $E^0 \times E^0$ matrix A_E by

$$A_E(v,w) = \#\{e \in E^1 : r(e) = v, s(e) = w\}.$$

We call the matrix A_E the *adjacency matrix* of E. These constructions are compatible: That is $E_{A_E} \cong E$ and A_{E_A} and A are conjugate by a permutation matrix. In particular, they encode the same Cuntz-Krieger relations.

Utilizing these constructions, Enomoto and Watatani sought to describe the results of Cuntz and Krieger using notions from graph theory. Enomoto and Watatani reinterpreted the Cuntz-Krieger relations in terms of a directed graph E_A. As in [**9**], there exists a universal C^*-algebra generated by partial isometries, indexed by the edges of a directed graph, which satisfy the Cuntz-Krieger relations. This C^*-algebra is called a *graph algebra*. Enomoto and Watatani showed that for a finite $\{0,1\}$-matrix A the Cuntz-Krieger algebra associated to A is isomorphic to the graph algebra associated to E_A.

The results of Enomoto and Watatani apply to graph algebras associated to finite directed graphs with at most one edge between any two vertices and no fewer than one edge coming in to, and leaving from, each vertex. For such directed graphs, we say that a *Cuntz-Krieger E-family* consists of a set of mutually orthogonal projections $\{P_v : v \in E^0\}$, and a set of partial isometries $\{S_e : e \in E^1\}$ which satisfy

(DG1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$; and

(DG2) $P_v = \sum_{\{e:r(e)=v\}} S_e S_e^*$ for all $v \in E^0$.

Through the work of many authors, for example [**3, 4, 11, 16, 25**], the Cuntz-Krieger relations of [**10**] have been extended to allow the construction of graph algebras for more general classes of directed graphs. There is now a well developed theory of C^*-algebras associated to arbitrary directed graphs. In particular, we are able to associate graph algebras to infinite directed graphs in which some vertices may not emit or receive any edges, or may emit or receive an infinite number of edges. Vertices which do not emit any edges are called sinks, and vertices which do not receive any edges are called sources.

In order to associate graph algebras to arbitrary directed graphs it is necessary to deal with problems which arise from the existence of sources and infinite receivers: If a vertex v is a source, then (DG2) forces $P_v = 0$ and thus not all generators are nonzero; and if v is an infinite receiver, then (DG2) involves in infinite sum of projections that can't converge in norm. These problems were overcome by replacing (DG2) with a relation which agrees with (DG2) when a vertex receives a finite, nonzero number of edges, but still makes sense when a vertex is a source of an infinite receiver. We discuss this point in greater detail a little later on in this chapter.

1.2. Higher-rank graphs and associated C^*-algebras

In this thesis our primary interest surrounds C^*-algebras associated to combinatorial objects called higher-rank graphs, or k-graphs for short. A k-graph is a higher-dimensional analogue of a directed graph. In this section we explain how to construct a k-graph, along with an overview of the history and significant developments in this area of study.

In 1999, Robertson and Steger [**38**] studied a class of higher-rank generalisations of Cuntz-Krieger algebras. The motivation for their work was the example given in [**38**, Section 7] of C^*-algebras associated to boundary actions of affine buildings. Soon after [**38**] appeared, Kumjian and Pask [**23**] introduced the concept of a k-graph as a graph-based model for the algebras of Robertson and Steger. Since this time, the study of k-graphs and their C^*-algebras has generated a lot of interest. One of the main reasons for this is because k-graphs provide a link between operator algebras, dynamical systems, and graph theory.

Formally, a k-graph is a pair (Λ, d), consisting of a small category Λ and a functor $d : \Lambda \to \mathbb{N}^k$, which satisfies a condition called the *factorisation property.* The degree, or shape, $d(\lambda)$ of a path λ is a higher-rank generalisation of length. Note that each $\lambda \in \Lambda$ has a unique range, denoted by $r(\lambda)$, and a unique source, denoted by $s(\lambda)$. To facilitate our discussion we define $\Lambda^n = \{\lambda \in \Lambda : d(\lambda) = n\}$, and for any $X \subset \Lambda$, $v \in \Lambda^0$ we let $vX = \{\lambda \in X : r(\lambda) = v\}$ and $Xv = \{\lambda \in X : s(\lambda) = v\}$. When

$k = 1$, we obtain the collection of all paths in a directed graph. In other words, every 1-graph is the path category of a directed graph. For this reason we commonly refer to a directed graph as a 1-graph. This notation continues the notational conventions used in [**32**].

A k-graph can be visually represented as a directed graph, with the exception that now edges occur in k-different colours, rather than just one. Let e_i is the ith generator of the semigroup $\mathbb{N}^k$. For $i \in \{1, \ldots, k\}$, edges with degree e_i are drawn in the ith colour. In a directed graph, different sequences of edges describe different paths. However in a k-graph this need not be the case. In fact, the role of the factorisation property is to tell us when different compositions of edges in Λ describe the same path.

From this point on we require some terminology relating to different distinguishing properties of k-graphs: We say a k-graph

- is row-finite if each vertex receives only finitely many edges;
- is locally finite if each vertex receives and emits only finitely many edges;
- has no sources if every vertex receives at least one edge of each colour; and
- has no sinks if every vertex emits at least one edge of each colour.

Given a row-finite k-graph Λ with no sources, Kumjian and Pask [**23**] associated a C^*-algebra to Λ, denoted by $C^*(\Lambda)$. By definition, $C^*(\Lambda)$ is the C^*-algebra generated by a collection $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying what Kumjian and Pask called the *Cuntz-Krieger relations*:

(1) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
(2) $s_{\mu\nu} = s_\mu s_\nu$ for all $\mu, \nu \in \Lambda$ such that $s(\mu) = r(\nu)$;
(3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$; and
(4) for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ we have $s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$.

The collection $\{s_\lambda : \lambda \in \Lambda\}$ is called a *Cuntz-Krieger* Λ*-family*, and $C^*(\Lambda)$ is called the higher-rank graph algebra. $C^*(\Lambda)$ is universal in the following sense: Suppose $\{t_\lambda : \lambda \in \Lambda\}$ is also a Cuntz-Krieger Λ-family; then there exists a unique homomorphism π_t of $C^*(\Lambda)$ such that $\pi_t(s_\lambda) = t_\lambda$ for every $\lambda \in \Lambda$. As we mentioned in passing earlier, this formulation provides a graph-based model for the higher-rank Cuntz-Krieger algebras of Robertson and Steger.

Since [**23**], several different generalisations of the Cuntz-Krieger relations have arisen for different classes of higher-rank graphs. The most general of these classes is called *finitely aligned*. Throughout this thesis, all k-graphs are assumed to be finitely aligned unless otherwise stated. To define a finitely aligned k-graph we require some background before we are able to give the definition on page 30. For the remainder of this section we describe how our ability to obtain k-graph algebras from increasingly general classes of k-graphs has developed.

To show that each generating element of $C^*(\Lambda)$ is non-zero, Kumjian and Pask constructed an *infinite path space*, denoted Λ^∞. Critically each vertex v is the range of some $x \in \Lambda^\infty$. Moreover, for each $x \in \Lambda^\infty$, if we attach a finite path to, or remove from, the front of x then we obtain a new element of Λ^∞. This allowed Kumjian and Pask to construct a Cuntz-Krieger Λ-family $\{S_\lambda : \lambda \in \Lambda\} \subset \mathrm{B}\left(\ell^2(\Lambda^\infty)\right)$. Since each vertex is the range of some element of Λ^∞, each $S_v \neq 0$ and then the Cuntz-Krieger relations ensure that each $S_\lambda \neq 0$. The universal property of $C^*(\Lambda)$ implies that there is a homomorphism $\pi_S : C^*(\Lambda) \to \mathrm{B}\left(\ell^2(\Lambda^\infty)\right)$ satisfying $\pi_S(s_\lambda) = S_\lambda$. Thus it follows that the generating elements $\{s_\lambda : \lambda \in \Lambda\}$ of $C^*(\Lambda)$ are also non-zero.

1.2.1. Extending the construction of Kumjian and Pask. Following on from the work of Kumjian and Pask, the Cuntz-Krieger relations have been generalised to k-graphs which may have sources or may not be row-finite (or both). In such examples the Cuntz-Krieger relations can become quite technical. Thus we introduce the relations as required, in particular, the most general version formulated to date appears in Chapter 2.

In 2003, Raeburn, Sims and Yeend [**34**] considered how to associate C^*-algebras to k-graphs with sources. Given a row-finite k-graph Λ, they defined, for every $n \in \mathbb{N}^k$, a subset $\Lambda^{\leq n} \subset \Lambda$ (see (2.2.1) on page 29) which can be regarded as containing all paths which extend as far as possible in direction n. They used this concept to extend the definition of a Cuntz-Krieger Λ-family to k-graphs with sources.

In [**34**, Definition 3.9], Raeburn, Sims, and Yeend defined a k-graph Λ to be *locally convex* if, for all $v \in \Lambda^0$, $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $\lambda \in v\Lambda^{e_i}$ and $\mu \in v\Lambda^{e_j}$ then $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are both non-empty. They then showed that there exists a Cuntz-Krieger Λ-family in which every s_λ is non-zero if and only if Λ is locally convex.

To prove the existence of a non-trivial Cuntz-Krieger Λ-family for a locally convex k-graph Λ, Raeburn, Sims and Yeend developed a substitute for the Λ^∞ representation of [**23**]. They did this by defining a set $\Lambda^{\leq\infty}$ which is to Λ^∞ as $\Lambda^{\leq n}$ is to Λ^n. The key point is that every vertex in a locally convex k-graph is the range of an element of $\Lambda^{\leq\infty}$, so a similar analysis to [**23**] shows that $C^*(\Lambda)$ is non-trivial.

1.2.2. What about Infinite receivers? Shortly after, Raeburn, Sims and Yeend published another paper [**35**] which extended the theory to k-graphs which need not be row-finite. For a k-graph Λ, if we allow $v \in \Lambda^0$ to receive an infinite number of edges of any given degree, then the fourth Cuntz-Krieger relation involves an infinite sum of projections, which cannot converge in norm. Overcoming this problem was a major advancement so we discuss it in some detail. To understand how this development came about, we introduce some history from the 1-graph case.

Let A be a C^*-algebra. Then a *Hilbert bimodule X over A* is a right Hilbert A-module with a left action of A by adjoinable operators. In 1997, Pimsner [**29**] constructed a C^*-algebra $\mathcal{O}_X$ associated to a Hilbert bimodule X. One of the reasons this construction is so interesting for us is that for certain X, the resulting algebra $\mathcal{O}_X$ is a Cuntz-Krieger algebra.

In 1999, Fowler and Raeburn [**18**] used the earlier methods developed in [**17, 26**] to study Pimsner algebras. For a directed graph E, they constructed what they called the Cuntz-Krieger Hilbert bimodule $X(E)$ of E (see [**18**, Example 1.2] for a technical definition). They observed that $X(E)$ and $\mathcal{O}_{X(E)}$ make sense whether or not E has infinite receivers. Next, Fowler and Raeburn proved a uniqueness theorem for C^*-algebras associated to graphs in which every vertex receives infinitely many edges [17, Corollary 4.2]. Their approach was to require (DG2) to hold only at vertices v which receive finitely many edges.

Shortly after, Fowler, Laca and Raeburn [**16**] established a link between $C^*(E)$ and the Cuntz-Krieger algebras of arbitrary infinite matrices defined in [**11**]. They then gave conditions which ensure that the resulting graph algebras are simple, and purely infinite. The key point was that Fowler and Raeburn's idea of only applying (DG2) at vertices which receive only finitely many edges is the right condition for Cuntz-Krieger algebras as well as for the generalisations used in [**18**].

In order to identify suitable Cuntz-Krieger relations for arbitrary k-graphs, Raeburn, Sims and Yeend [**35**] sought to follow an analogous program to that of [**16, 18**]. For this they turned to Fowler's theory of Toeplitz algebras of product systems of Hilbert Bimodules [**13, 14**]. Very briefly, a product system over a countable semigroup S, taking values in a tensor groupoid $\mathcal{G}$ is a family of complex Hilbert Spaces $(E_t)_{t>0}$ tendowed with an associative multiplacation which satisfies the following property:

> For every $s, t > 0$ there is a unitary isomorphism $E_s \otimes E_t \to E_{s+t}$ which maps the elementary tensor $x \otimes y$ to the product xy.

Before we introduce the contribution of [**35**], we give a brief background of the theory of Toeplitz algebras of product systems of Hilbert Bimodules. Our discussion begins with the work of Fowler and Sims in [**15**]. They observed that a k-graph could be viewed as a product system of coordinate 1-graphs over $\mathbb{N}^k$. Then following on from [**15**], Raeburn and Sims showed in [**33**] how to extend the construction $E \mapsto X(E)$ to product systems of graphs over $\mathbb{N}^k$. The aim was to find clues as to how to define the Cuntz-Krieger algebras of arbitrary k-graphs. We discuss the key observations of Raeburn and Sims below.

For a k-graph Λ, viewed as a product system of 1-graphs, recall from [**33**] that a *Toeplitz Λ-family* is a set of partial isometries $\{s_\lambda : \lambda \in \Lambda^{e_i}, 1 \leq i \leq k\}$ which satisfy four technical conditions very similar to the Cuntz-Krieger relations. When

every vertex in Λ receives infinitely many paths of every degree, Raeburn and Sims showed how to model $C^*(\Lambda)$: This is done in terms of a Toeplitz Λ-family which corresponds to a Toeplitz representation of Hilbert bimodules studied by Fowler [**14**]. Raeburn and Sims observed that in general these representations are too large to act like Cuntz-Krieger algebras. Thus Raeburn and Sims restricted their attention to the *Nica covariant representations* introduced by Fowler [**14**]. This required a quasi-lattice ordering on a semigroup as defined by Nica [**28**].

To make this restriction a C^*-algebraic condition, rather than a spatial condition, Raeburn and Sims assumed that $X(\Lambda)$ was *compactly aligned* in the sense of Fowler [**13**]. They then identified a condition on a k-graph, called *finitely aligned*, which ensures that the associated product system is compactly aligned. Their analysis suggested that the family of finitely-aligned k-graphs is the largest family of k-graphs which are likely to have associated Cuntz-Krieger algebras. Using the strong foundation of Fowler [**14**], Raeburn and Sims were also able to reinterpret the abstract concept of "Nica covariance" in terms of Cuntz-Krieger Λ-families.

In [**35**], Raeburn, Sims and Yeend considered the class of finitely aligned k-graphs and showed how to associate to each such k-graph Λ, a Cuntz-Krieger type algebra $C^*(\Lambda)$. In particular, they showed that every row-finite k-graph is finitely aligned, and that when Λ is locally convex, the Cuntz-Krieger relations of [**35**] simplify to the Cuntz-Krieger relations of [**34**]. Thus the results of [**35**] extend the results of their earlier paper on the C^*-algebras of row-finite locally convex k-graphs [**34**]. By using a modified version of the definition of $\Lambda^{\leq\infty}$ used in [**34**], the authors proved that the generators of $C^*(\Lambda)$ are all nonzero.

1.3. Structure theory of higher-rank graph algebras

In [**23**], Kumjian and Pask identified two conditions, which we now call aperiodicity and cofinality, on a row-finite k-graph Λ with no sources. They showed that if Λ is aperiodic, then $C^*(\Lambda)$ is simple if and only if Λ is cofinal. These conditions are analogues of conditions arising in directed graphs. In particular, when $k = 1$, aperiodicity is equivalent to the condition that "every loop has an exit" and thus Kumjian and Pask's simplicity theorem [**23**, Proposition 4.8] contains that of [**25**].

For 1-graphs, aperiodicity and cofinality are intuitive and easy to identify. However when $k > 1$, the aperiodicity and cofinality conditions both involve infinite paths. This is less than ideal because infinite paths, by their very nature, are difficult to work with. Moreover, such complications are compounded by the factorisation property.

In 2007, Robertson and Sims [**36**] defined an alternative aperiodicity condition to the one used by Kumjian and Pask, this time using only finite paths. Robertson and Sims show that the two formulations are equivalent [**36**, Lemma 3.2]. Moreover,

Robertson and Sims show in [**36**, Theorem 3.1] that if Λ is a row-finite k-graph with no sources, then $C^*(\Lambda)$ is simple if and only if Λ aperiodic and cofinal. This improves on [**23**, Proposition 4.8] where aperiodicity and cofinality are only shown to be necessary.

Earlier in this chapter we reviewed the development of the theory of k-graph algebras. Each new generalisation of the Cuntz-Krieger relations raises the question of how to characterise simplicity of the associated k-graph algebras. In each case, the first step is to prove what are known as the "Uniqueness theorems." In this thesis we prove a version of the Cuntz-Krieger Uniqueness Theorem for finitely aligned k-graphs in Chapter 3. We state the theorem below and then make some comments immediately after.

THEOREM (Theorem 3.4.4). *Suppose Λ is an aperiodic, finitely aligned k-graph. Suppose also that π is a representation of $C^*(\Lambda)$ such that $\pi(s_v) \neq 0$ for every $v \in \Lambda^0$. Then π is injective.*

Validity of the Cuntz-Krieger Uniqueness Theorem is a litmus test for any generalisation of Cuntz-Krieger theory; for example, see [**35**, Remarks 2.6]. Slightly different versions of the Cuntz-Krieger Uniqueness Theorem have been proven for row-finite k-graphs with no sources [**23**, Theorem 4.6], row-finite locally convex k-graphs [**34**, Theorem 4.3], and finitely aligned k-graphs [**35**, Theorem 4.5], [**12**, Theorem 7.1]. In each case, the definition of aperiodicity needed to be modified to allow for the increase in generality of the underlying k-graph.

Of course, aperiodicity has consequences beyond the Cuntz-Krieger Uniqueness Theorem. In [**23**], for a row-finite k-graph Λ with no sources, Kumjian and Pask relate aperiodicity to a groupoid $\mathcal{G}_\Lambda$ associated to Λ^∞. They show in [**23**, Proposition 4.5] that $\mathcal{G}_\Lambda$ is *essentially free* if and only if Λ is aperiodic. For locally convex k-graphs, Robertson and Sims show that aperiodicity is equivalent to the assertion that every non-trivial ideal in $C^*(\Lambda)$ contains a vertex projection [**37**, Proposition 4.3]. In [**12**], Farthing, Muhly and Yeend take an inverse semigroup and groupoid approach to k-graph algebras. They provide an aperiodicity condition in [**12**, Proposition 7.2], which is an extension of Kumjian and Pask's result [**23**, Proposition 4.5] to finitely aligned k-graphs.

As we have seen, the Cuntz-Krieger Uniqueness Theorem may also be interpreted as the assertion that if Λ is an aperiodic k-graph, then any non-trivial ideal in $C^*(\Lambda)$ must contain a vertex projection. To use this interpretation to show that $C^*(\Lambda)$ is simple, we identify a condition on Λ which ensures that if an ideal contains one vertex projection, it must contain them all. As in [**23**], this condition is called cofinality.

For row-finite k-graphs with no sources, Kumjian and Pask showed in [**23**, Corollary 3.5] that if Λ is row-finite with no sources, then $C^*(\Lambda)$ is isomorphic to $C^*(\mathcal{G}_\Lambda)$. When $\mathcal{G}_\Lambda$ is essentially free, $C^*(\mathcal{G}_\Lambda)$ is simple if and only if $\mathcal{G}_\Lambda$ is minimal. In [**23**, Definition 4.7], Kumjian and Pask define a cofinality condition on Λ which ensures that $\mathcal{G}_\Lambda$ is minimal. This simplicity theorem for $C^*(\Lambda)$ appears as [**23**, Proposition 4.8].

For a row-finite locally convex k-graph Λ, Robertson and Sims [**37**, Definition 3.1] restate the definition of aperiodicity from [**36**] and the definition of cofinality of [**23**], making adjustments for the increase in generality. In [**37**, Theorem 3.4] they prove that $C^*(\Lambda)$ is simple if and only if Λ is aperiodic and cofinal. This generalises [**36**, Theorem 3.1] to locally convex k-graphs. Note that similar to [**36**], in [**37**], aperiodicity is called *no local periodicity.* In this paper the definitions of aperiodicity and cofinality involve comparing elements of $\Lambda^{\leq\infty}$, which can be quite difficult in general.

Recently Shotwell [**40**] has identified formulations of aperiodicity and cofinality on a k-graph Λ which characterise simplicity in $C^*(\Lambda)$. These conditions are phrased in terms of another set of paths, denoted $\partial\Lambda$. The set $\partial\Lambda$ arose from the analysis of Farthing, Muhly and Yeend [**12**], which used inverse semigroups and groupoids to study k-graph algebras. Typically, $\partial\Lambda$ contains $\Lambda^{\leq\infty}$ as a proper subset. Describing this set is quite technical and so we reserve the details for Chapter 2 and Section 3.2.

In [**12**], Farthing, Muhly and Yeend describe a groupoid $\mathcal{G}_\Lambda$. This groupoid has path space $\mathcal{G}^{(0)}$, homeomorphic to the space of all paths — finite, partially infinite and infinite — in Λ. In [**12**, Proposition 6.10] they show that $\partial\Lambda$ is a nonempty closed invariant subset of $\mathcal{G}^{(0)}$. Then in [**12**, Theorem 6.13] it is shown that $C^*(\mathcal{G}_\Lambda|_{\partial\Lambda}) \cong C^*(\Lambda)$.

In [**40**], Shotwell uses Condition (A) of Farthing, Muhly and Yeend [**12**, Theorem 7.1] as his definition of aperiodicity. He shows that this definition is equivalent to the notion of no local periodicity, and applies this equivalence to show that the Cuntz-Krieger Uniqueness Theorem of [**12**] still holds when the use of aperiodicity is replaced with no local periodicity. Next, Shotwell identifies a new cofinality condition for k-graphs. Again, this condition is phrased in terms of $\partial\Lambda$-paths. Shotwell proves that if Λ is a k-graph, then $C^*(\Lambda)$ is simple if and only if Λ has no local periodicity and is cofinal.

Note that the notation $\partial\Lambda$ was intended by Farthing, Muhly and Yeend to indicate a kind of boundary, and interpretaion in this vein do exist. For example, the boundaries of quasi-lattice ordered semigroups studied by Crisp and Laca [**8**].

Throughout all the research mentioned above, each formulation of cofinality and aperiodicity has involved complicated sets of paths. In particular, we must consider at least all of the infinite paths in the relevant k-graph. However, in the special case of row-finite k-graphs with no sources, a finite-path formulation of aperiodicity

has been provided by Robertson and Sims [**36**, Lemma 3.2]. This formulation was shown to be equivalent to the definition of aperiodicity used by Kumjian and Pask in [**23**]. The significance of this advancement is that under this formulation, identifying aperiodicity in examples becomes a much more manageable task. We build on this idea in this thesis.

In this thesis we extend the current literature concerning the ideal structure of k-graph algebras in three ways. Firstly, we provide new formulations for aperiodicity and cofinality, both phrased only in terms of finite paths. We then check that our formulations are equivalent to several earlier conditions appearing in the literature. Moreover, in the case of cofinality, we show how our definition simplifies in a number of special cases. Secondly, we prove our primary result of Chapter 3, Theorem 3.3.3: The C^*-algebra of a finitely aligned k-graph Λ is simple if and only if Λ is aperiodic and cofinal. Thirdly, our results are proved without reference to the classification of gauge invariant ideals [**43**] and consequently our proofs exhibit a number of new techniques.

1.4. Ergodic theory of higher-rank graph algebras

1.4.1. Background information. A large part of ergodic theory is identifying when two dynamical systems are *conjugate*. This is done through comparing various different *topological invariants*. A topological invariant of a dynamical system is any property which is invariant under homeomorphisms. The best known topological invariant is called *topological entropy*. In its simplest form, entropy is a measure of growth within a dynamical system. In this thesis, we are interested in two forms of entropy: topological entropy of an endomorphism of a C^*-algebra and topological entropy of an action of $\mathbb{N}^k$ on a multi-dimensional dynamical system. The versions of topological entropy we use are technical, so we only give a very broad overview in this chapter and reserve the details for Chapter 4.

Let T be a homeomorphism of a compact, separable topological space X. We call the pair (X, T) a topological dynamical system. In 1965, Adler, Konheim and McAndrew [**1**] defined the topological entropy $h(T)$ of T. This definition involves finite open covers of X, so it is suitable only for compact X.

It wasn't for another 10 years that the definition of Adler, Konheim and McAndrew was extended to the case when X is not compact. This definition is due to Bowen [**6**] and applies when X is a metric space. Let (X, d) be a metric space and $T : X \to X$ be uniformly continuous. Bowen defined a kind of subentropy $h_d(T, K)$ on each compact subset K of X. Then Bowen defined the topological entropy $h_d(T)$ to be the supremum of all the $h_d(T, K)$, where the supremum is taken over all compact subsets K of T [**6**, Definition 2]. Importantly, when X is compact this definition coincides with the definition given in [**1**].

As mentioned above, entropy measures growth. To measure growth we need to establish a concept of size. For a compact topological space, Adler, Konheim and McAndrew use the cardinality of a minimal subcover of an open cover. For a non-compact metric space (X, d) and a uniformly continuous map $T : X \to X$, Bowen uses the cardinality of a maximal (n, ε)-separated set, which is defined as follows: A set $E \subset X$ is an (n, ε)-separated set if for every $x, y \in E$ there exists j such that $0 \leq j \leq n$ and $d\big(T^j(x), T^j(y)\big) > \varepsilon$. Less formally, a (n, ε)-separated set is a set of elements E in X, which diverge from every other element in E within n iterations of T; a maximal (n, ε)-separated set is a (n, ε)-separated set of maximal cardinality. Lemma 1 of [**6**] shows that whenever X is compact, the cardinality of a maximal (n, ε)-separated set is finite for any $\varepsilon > 0$ and $n < \infty$.

To extend the definition of topological entropy from topological spaces to C^*-algebras, one must define a notion of "size" within a C^*-algebra. Early attempts at measuring size involved measuring the trace of finite dimensional subalgebras. This idea ran into problems because it is possible to generate infinite dimensional C^*-algebras from a finite set of elements. This problem was not overcome until 1995 when Voiculescu [**45**] defined the *completely positive approximation* of a C^*-algebra, and used the rank of the approximation as a measure of size. This measure of size is the most common measure presently used.

Voiculescu defines entropy for C^*- and W^*-algebras by approximating arbitrary algebras by finite-dimensional ones. This version of topological entropy is commonly called Voiculescu approximation entropy. From here on in we only consider the C^*-case.

Let A be a unital, nuclear C^*-algebra, let C be a finite dimensional C^*-algebra and let $\phi : C \to A$ and $\psi : A \to C$ be unital and completely positive maps. Then we call (ϕ, ψ, C) a completely positive approximation, or an approximating triple, for A and write $(\phi, \psi, C) \in \mathrm{CPA}(A)$. If, in addition, there exists a finite subset Ω of A and $\varepsilon > 0$ such that $\|\phi \circ \psi(a) - a\| < \varepsilon$ for every $a \in \Omega$, then we write $(\phi, \psi, C) \in \mathrm{CPA}(A, \Omega, \varepsilon)$. We define the rank of a completely positive approximation of A, with respect to Ω and ε, by

$$\mathrm{rcp}_A(\Omega, \varepsilon) := \inf\{\mathrm{rank}(C) : (\phi, \psi, C) \in \mathrm{CPA}(A, \Omega, \varepsilon)\}.$$

Voiculescu [**45**] uses this rcp function to define the topological entropy of an automorphism of an unital, nuclear C^*-algebra. To state the definition we need some notation: for an automorphism Φ of A, a finite subset $\Omega \subset A$ and $\varepsilon > 0$, let

$$\mathrm{ht}(\Phi, \Omega, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log\Big(\mathrm{rcp}_A\Big(\bigcup_{l=0}^{n-1} \Phi^l(\Omega), \varepsilon\Big)\Big);$$

and

$$\mathrm{ht}(\Phi, \Omega) = \sup_{\varepsilon > 0} \mathrm{ht}(\Phi, \Omega, \varepsilon).$$

Voiculescu defines the topological entropy $\mathrm{ht}(\Phi)$ of Φ to be the supremum of the $\mathrm{ht}(\Phi, \Omega)$, taken over all finite subsets Ω of A. Voiculescu also points out that the above definition of topological entropy equally applies when Φ an endomorphism and much of the theory remains unaffected.

Since its inception, Voiculescu entropy has been generalised to cover additional examples. In 1999, Brown [**7**] extended the definition of Voiculescu entropy to include endomorphisms of exact algebras. As part of this, he was able to relax the requirement that ϕ and ψ be unital and instead only required them to be contractive, completely positive maps. In 2004, Pop and Smith [**31**] modify the definition of rcp again and this time only require that ϕ and ψ are completely contractive. They show that under this modification, the definition of approximation entropy coincides with the versions introduced by Voiculescu and Brown. Note that in this final case, Pop and Smith only consider the topological entropy of an automorphism Φ, and exlude the case when Φ is an endomorphism.

1.4.2. Infinite directed graphs. As we saw in Section 1.3, given a locally finite k-graph Λ with no sinks, there exists a universal C^*-algebra $C^*(\Lambda)$ associated to Λ. Moreover, for $p \in \mathbb{N}^k$ we recall that the non-commutative shift map $\Phi_p : C^*(\Lambda) \to C^*(\Lambda)$ is defined by $\Phi_p(x) = \sum_{\lambda \in \Lambda^p} s_\lambda x s_\lambda^*$. In this subsection we consider the case when $k = 1$. Let E be a directed graph. The core $C^*(E)^\gamma$ of $C^*(E)$ is the Φ_1-invariant, AF subalgebra $\overline{\mathrm{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in E, d(\alpha) = d(\beta), r(\alpha) = r(\beta)\}$ of $C^*(E)$. The map Φ_p is an endomorphism when restricted to $C^*(E)^\gamma$ [**5**, page 1], so we may calculate the topological entropy $\mathrm{ht}(\Phi_p|_{C^*(E)^\gamma})$. In general, calculating $\mathrm{ht}(\Phi_p)$ for arbitary directed graphs directly is quite difficult, so several authors have shown how to simplify the calculations in a number of special cases. Below we give an overview of main developments in this area and explain how we extend the current theory.

The first analysis in this area was performed in 2000 by Boca and Goldstein [**5**]. They considered the Cuntz-Krieger algebra $\mathcal{O}_A$ associated to a finite, 0,1-matrix A where each row and column is non-zero. Recall that $\mathcal{O}_A$ is generated by the partial isometries $\{s_j : 1 \leq j \leq |E_A^0|\}$ which satisfy the Cuntz-Krieger relations of [**9**]. Boca and Goldstein were interested in calculating the topological entropy of the unital and completely positive map $\phi_A : \mathcal{O}_A \to \mathcal{O}_A$ defined by $\phi_A(x) = \sum_{j=1}^{|E_A^0|} s_j x s_j^*$. They showed that ϕ_A has topological entropy equal to $\log r(A)$, where $r(A)$ denotes the spectral radius of A.

Now recall that given a 1-graph E, we define the adjacency matrix A_E of E to be a $E^0 \times E^0$ matrix such that

$$A_E(v,w) = \#\{e \in E^1 : r(e) = v, s(e) = w\}.$$

As we have seen, a matrix A of the form studied by Boca and Goldstein is isomorphic to the matrix A_{E_A}. Thus we may reinterpret the results of [**5**] to see that $\mathrm{ht}(\Phi_1) = \mathrm{ht}(\phi_{A_E}) = \log r(A_E)$.

In 2004 Jeong and Park [**19**] considered row-finite directed graphs, with finitely many vertices, which may contain sinks. They calculate the topological entropy of Φ_1, which they denote by ϕ_E. This extends the results of Boca and Goldstein to the case when E may contain sources. For this, Jeong and Park define a map $\psi_E : C^*(E) \to C^*(E)$ by

$$\psi_E(x) = \Phi_1(x) + \sum_{\{v \in E^0 : v \text{ is a sink}\}} p_v x p_v.$$

They then show that $\mathrm{ht}(\Phi_1) = \mathrm{ht}(\psi_E) = \log r(A_E)$.

From 2004 to 2006, Jeong and Park published three papers which extend the above entropy calculations to a family of infinite 1-graphs [**19, 20, 21**]. For technical reasons they considered only locally finite 1-graphs which have no sinks and are *strongly connected*: A k-graph Λ is strongly connected if for all $u, v \in \Lambda^0$ the set $v\Lambda u \neq \emptyset$. That is, there is a path from any vertex u to any other vertex v. To explain the contribution of Jeong and Park we also need to introduce the following definition: For a locally finite 1-graph with no sources and any infinite path $e_1e_2e_3 \ldots \in E$, the map $\sigma : E^\infty \to E^\infty$ defined by $\sigma(e_1e_2e_3\ldots) = e_2e_3\ldots$ is called the *shift map* of E.

In their 2004 paper [**19**], Jeong and Park show that the topological entropy of Φ_1 is bounded below by the topological entropy of the shift map on the one-point compactification of E^∞. For the same family of 1-graphs as in [**19**], the authors compute an upper bound for topological entropy of Φ_1 in [**20, 21**]. However unlike in [**19**], here Jeong and Park no longer consider Φ_1 as a map from $C^*(E)$ to itself, but rather restrict it to the *core*.

With this restriction, Jeong and Park identify a net of increasing sub-algebras of the core. The idea is that each subalgebra is a completely positive approximation of itself. Since we know the rank of each subalgebra in terms of the number of paths in Λ, Jeong and Park were able to provide an upper bound on $\mathrm{ht}(\Phi_1|_{C^*(E)^\gamma})$ in terms of the number of paths in Λ. It turns out that this upper bound is equal to the topological entropy of the shift map σ on E^∞ [**20**, Theorem 4.3].

As part of this thesis, we provide an alternative proof for [**20**, Theorem 4.3] in Section 4.3. We do this because [**20**, Theorem 4.3] relies on [**21**, Lemma 3.7], which is false. We provide a counterexample in Appendix A. It is important to note that the statement of [**20**, Theorem 4.3] is correct and hence remains unchanged.

In general, the results of Jeong and Park provide an upper bound for the topological entropy $\Phi_1|_{C^*(E)^\gamma}$. However in [**21**, Example 3.11], Jeong and Park show that the upper and lower bounds coincide for some specific examples. These examples were originally formulated by Salama [**39**]. In addition, because these examples followed a particular pattern, the authors were able to calculate a precise value for the topological entropy.

1.4.3. Finite higher-rank graphs. Computation of topological entropy has also been undertaken when $k > 1$. To introduce these results we define a higher-dimensional analogue of the adjacency matrix applicable to finite k-graphs. The idea is to define an adjacency matrix for each colour. Suppose Λ is a row-finite k-graph, with no sources, where $|v\Lambda^{e_i}| \in \{0, 1\}$ for each $v \in \Lambda^0$, $i \in \{1, \ldots, k\}$. Then there exists a k-tuple of $\{0, 1\}$-matrices $M = (M_1, \ldots, M_k)$ which satisfy Robertson and Steger's conditions (H0) - (H3) [**23**, Corollary 3.5]. When Λ is aperiodic, Skalski and Zacharias extend the results of Boca and Goldstein to show that the topological entropy of Φ_p is equal to $\log r(M^p)$ [**44**, Corollary 2.5]. This corollary was born out of a series of calculations involving *topological pressure*, a type of generalisation of topological entropy. A precise definition of topological pressure can be found in [**44**].

This result also has another interpretation. When $k = 1$ there is only one subshift of finite type associated to Λ. However when we allow $k > 1$, there are many subshifts associated to Λ, one for each p. Thus we think of the entropy of Φ_p as a sort of directional entropy inside the multi-dimensional shift space associated to Λ.

Computing topological entropy in the context of infinite k-graphs is much more complicated. This is because the associated infinite path space is not compact, and hence the associated k-graph algebras are not unital. For each $p \in \mathbb{N}^k$, only estimates for the topological entropy of Φ_p, restricted to the fixed point subalgebra, have been calculated. These estimates come in the form of the topological entropy of the *p-shift map* on the underlying k-graphs, which is a higher-dimensional analogue of the shift map for directed graphs. The advantage of this is that the p-shift map is commutative and so computing its entropy is possible.

One of the main goals of this thesis is to find a common generalisation of the work of Jeong and Park and the work of Skalski and Zacharias. That is, we compute an upper bound for the topological entropy of the non-commutative shift map Φ_p on the fixed point subalgebra associated to a strongly connected, locally finite k-graph Λ. In particular, we allow for the possibility that Λ has an infinite number of vertices. This extends the results of Jeong and Park to higher-rank graphs and also the results of Skalski and Zacharias from finite to infinite k-graphs. In doing so, we also fill the gap mentioned above in the proof of [**20**, Theorem 4.3].

For a locally finite k-graph with no sinks and $k > 1$, there is an action σ of $\mathbb{N}^k$ on Λ^∞ which maps each p to the p-shift map σ_p. We investigate the entropy $\mathrm{h}(\Lambda, \sigma)$ of this action. This question was first posed by Skalzki and Zacharias. Suppose Λ is a row-finite k-graph with no sources and only a finite number of vertices. Skalski and Zacharias showed in [**44**, Proposition 1.2] that when $k > 1$, we have $\mathrm{h}(\Lambda, \sigma) = 0$. Moreover, in [**44**, Theorem 3.3], Skalski and Zacharias prove the same result for the Voiculescu entropy of the action of $\mathbb{N}^k$ on $C^*(\Lambda)$, given by $p \mapsto \Phi_p$.

In the final section of this thesis we explore the properties of the action σ when Λ is a row-finite k-graph with no sinks and $|\Lambda^0| = \infty$. As in the case of Voiculescu entropy, when we allow Λ to be an infinite k-graph the situation becomes much more complicated than when Λ^0 is finite. This is because when Λ^0 is infinite, when we view the infinite path space Λ^∞ as a metric space we see that Λ^∞ is not compact. Thus we need to extend the definition used by Skalski and Zacharias to the case when Λ^∞ is locally compact.

As we saw above, Bowen [**6**] extends the definition of topological entropy from compact to non-compact spaces. We use his techniques to extend the definition of $\mathrm{h}(\Lambda, \sigma)$ to the case when Λ has an infinite number of vertices. Unlike the case when Λ is finite, we see that $\mathrm{h}(\Lambda, \sigma)$ may be nonzero. We provide conditions on a locally finite k-graph Λ with no sinks which ensure that $\mathrm{h}(\Lambda, \sigma) = 0$. We give two different conditions, one for when $k = 2$ and a stronger condition which applies when $k \geq 3$.

1.5. Content summary

The outline of this thesis is as follows. In Chapter 2 we introduce some standard facts and notation which will be used throughout this thesis. The majority of Chapter 2 introduces the fundamentals of higher-rank graphs and higher-rank subshifts as well as background about the construction of higher-rank graph algebras.

We begin Chapter 3 by providing a more detailed introduction to the structure theory of k-graph algebras in Section 3.1. In Section 3.2 we outline the $\partial\Lambda$-representation. This representation is significant because it ensures that the higher-rank graph algebras with which we are working are non-trivial.

The main body of Chapter 3 begins with Section 3.3. Here we state our new formulations of aperiodicity and cofinality, together with our main result, Theorem 3.3.3: Suppose Λ is a finitely aligned k-graph. Then the C^*-algebra $C^*(\Lambda)$ is simple if and only if Λ is aperiodic and cofinal. Note that this theorem is not proved until the end of the chapter. Section 3.3 is then concluded by showing that our formulation of aperiodicity is equivalent to two formulations appearing previously in the literature.

In Section 3.4 we characterise how aperiodicity relates to the ideal structure of a k-graph algebra. As part of this we prove the Cuntz-Krieger Uniqueness Theorem.

The main result of Section 3.4 is as follows: Suppose Λ is a finitely aligned k-graph. Then Λ is aperiodic if and only if every non-zero ideal in $C^*(\Lambda)$ contains a vertex projection. The proof of our main theorem occurs in Section 3.5. For this proof we show that our cofinality condition ensures that any ideal of $C^*(\Lambda)$ which contains a single vertex projection, must contain all vertex projections. The proof of this result, together with the proof of Theorem 3.3.3, complete Section 3.5.

In Section 3.6 we finish off Chapter 3 by showing how our cofinality condition simplifies in a number of special cases which have appeared elsewhere in the literature. The key point is that in these special cases our cofinality condition is also new.

Chapter 4 begins with a recap of the ergodic theory surrounding k-graph algebras. In Section 4.2 we provide some fundamental results which we require throughout Chapter 4. In particular, in Subsection 4.2.2 we define the topological entropy of an endomorphism of a C^*-algebra.

In Section 4.3 we consider topological entropy in graph algebras arising from infinite directed graphs. The main goal of this section is to prove Theorem 4.3.1. This theorem originally appeared as [**20**, Theorem 4.3], however the proof there is incomplete (See Appendix A). This theorem is significant because it gives us an upper bound for $\mathrm{ht}(\phi_E|_{C^*(E)^\gamma})$ in terms of the entropy of the shift map σ. The point is that the entropy of σ can be computed in terms of the number of paths in E, thus it gives a meaningful combinatorial upper bound for the purely algebraic measure $\mathrm{ht}(\phi_E|_{C^*(E)^\gamma})$.

In Section 4.4, we provide a common generalisation of the work of Jeong and Park and Skalski and Zacharias. This takes the form of three main results. The most demanding of the three is Theorem 4.4.2 which appears in Subsection 4.4.1. This theorem is a higher-rank generalisation of Theorem 4.3.1. That is, given a locally finite higher-rank graph Λ with no sources, we compute an upper bound for the topological entropy of $\Phi_p|_{C^*(\Lambda)^\gamma}$ in terms of the classical entropy of σ_p. Since we are able to compute the classical entropy of σ_p, Theorem 4.4.2 gives a meaningful and intuitive upper bound for $\mathrm{ht}(\Phi_p|_{C^*(\Lambda)^\gamma})$.

The final two main results appear in Subsection 4.4.2. Here we are interested in computing the topological entropy of the action σ of $\mathbb{N}^k$ on Λ^∞. We give conditions on an infinite k-graph Λ which ensure that the topological entropy of σ is 0. This extends [**44**, Proposition 1.1] to cover infinite k-graphs.

The final section of this thesis is Section 4.5. In this section we show how to compute the rcp of a simple, finite dimensional C^*-subalgebra of a nuclear C^*-algebra. The point is that computing the rcp of a C^*-algebra is a crucial step in calculating the topological entropy of an endomorphism of a C^*-algebra. We hope that the results contained in this subsection will be a starting point for the future computation of an exact value for $\mathrm{ht}(\Phi_p|_{C^*(\Lambda)^\gamma})$ where Λ is an infinite k-graph.

1.6. Connections with other work

The analysis of Chapter 3 is an expansion of joint work with my supervisor, Aidan Sims, which has been published as [**27**]. While the results in Chapter 3 are the same as in [**27**], in this thesis the author has the liberty of explaining and describing these results in much more detail.

CHAPTER 2

Background and elementary facts

2.1. Shift spaces, C^*-algebras and completely positive maps

We regard $\mathbb{N}^k$ as a semigroup under addition, and use e_i to denote the ith generator. In this thesis we assume $0 \in \mathbb{N}$. For $m, n \in \mathbb{N}^k$ we write m_i, n_i for the ith coordinates of m and n, $m \vee n$ for their coordinatewise maximum, and $m \wedge n$ for their coordinatewise minimum. For $l \in \mathbb{N}$ we write $\bar{l}$ for $(l, l, \ldots, l) \in \mathbb{N}^k$.

Suppose A is a countable set and $k \in \mathbb{N}$. Then A is a topological space when we endow it with the discrete topology. It then follows that $A^{\mathbb{N}^k}$ is a topological space with the product topology. For each $p \in \mathbb{N}^k$, define a map $\sigma_p : A^{\mathbb{N}^k} \to A^{\mathbb{N}^k}$ by $\big(\sigma_p(x)\big)_c = x_{c+p}$. For any subset $S \subseteq A$, we call the pair $(S^{\mathbb{N}^k}, \sigma_p)$ a *shift space* if σ_p is a continuous, onto and a n-to-1 transformation. $(S^{\mathbb{N}^k}, \sigma_p)$ is also referred to as a *rank-k subshift*.

For $x \in S^{\mathbb{N}^k}$ and $N \subset \mathbb{N}^k$, let $x|_N$ denote the restriction of x by N. A map $f : N \to A$ is called an *allowed block of* $S^{\mathbb{N}^k}$ if there exists $y \in S^{\mathbb{N}^k}$ such that $y|_N = f$. The set of *allowable blocks of* $S^{\mathbb{N}^k}$ *of length* N is denoted $\mathcal{B}_N$.

For a shift space $(S^{\mathbb{N}^k}, \sigma_p)$, the *block entropy* of $(S^{\mathbb{N}}, \sigma_p)$ is the value

$$\mathrm{h}(S^{\mathbb{N}^k}) := \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{B}_{\bar{n}}(S)|.$$

In this thesis we also consider a number of more complicated definitions of entropy in Chapter 4.

In this paper we denote the C^*-algebra of bounded, linear operators on a Hilbert space H by $B(H)$. Let $A \subset B(H)$ be a C^*-algebra such that $1_{B(H)} \notin A$. Define $\widetilde{A} = \{a + \lambda 1_{B(H)} : a \in A, \lambda \in \mathbb{C}\}$. Then $\widetilde{A}$ is the *smallest unital C^*-algebra containing* A.

Suppose A and B are C^*-algebras. The *direct sum of A and B* is $A \oplus B := \{(a, b) : a \in A, b \in B\}$ where multiplication, addition and the $*$-operation are all performed pointwise. Now suppose A and B are C^*-subalgebras of a C^*-algebra C. If $ab = 0$ for all $a \in A$, $b \in B$ then $A + B$ is a C^*-subalgebra of C. Moreover, there is an isomorphism of $A + B$ onto $A \oplus B$ given by $a + b \mapsto (a, b)$. In this case it is common to write $A + B = A \oplus B$.

2.2. Higher-rank graphs and higher-rank subshifts

A k-graph (Λ, d) is a category Λ endowed with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ satisfying $d(\lambda) = m + n$

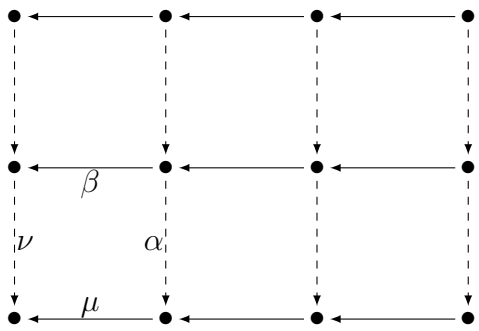
there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We read $d(\lambda)$ as the *degree of* λ and think of d as a generalised length function. For $n \in \mathbb{N}^k$ we write Λ^n for $\{\lambda \in \Lambda : d(\lambda) = n\}$.

The factorisation property implies that for each $\lambda \in \Lambda$ there exist unique elements $r(\lambda), s(\lambda) \in \Lambda^0$ such that $\lambda = r(\lambda)\lambda = \lambda s(\lambda)$, and also that $\Lambda^0 = \{r(\lambda) : \lambda \in \Lambda\} = \{s(\lambda) : \lambda \in \Lambda\}$. We call elements of Λ^0 vertices, and the functions r, s the *range* and *source* maps. Given k-graphs Λ and Γ, a k-graph morphism ϕ from Λ to Γ is a functor which preserves the degree map.

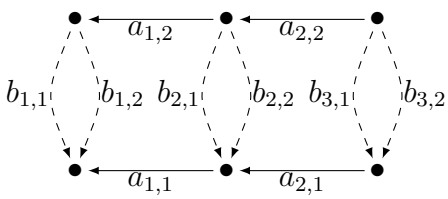
To visualise a k-graph we draw its *skeleton*. The skeleton of a k-graph is the directed graph whose vertices are Λ^0 and edges are $\bigcup_{i\in\{1,\ldots,k\}} \Lambda^{e_i}$ with range and source maps inherited from Λ. To distinguish edges of different degrees we draw them using k different colours. For example, for $k = 2$ we draw edges of degree e_1 using blue or solid lines and edges of degree e_2 using red or dashed lines.

Suppose we have k-graph Λ and let $\lambda \in \Lambda$. Then the factorisation property allows us to write down λ as a sequence of edges $\alpha_j \in \Lambda$ such that $d(\alpha_j) = e_i$ for some $i \in \{1, \ldots, k\}$. The consequence of this is that when we draw a skeleton we can always draw it so that it will appear to be a convex hull of squares for $k = 2$, cubes for $k = 3$ and so on.

The following picture is an example of the skeleton of a k-graph Λ.



Consider the labeled square in the bottom left hand corner of the above skeleton. Let $\lambda = \mu\alpha \in \Lambda$. Then the factorisation property with $m = e_2$ and $n = e_1$ forces $\nu\beta = \lambda = \mu\alpha$. This is an example of a *commuting square*. That is, the labeled square above is a commuting diagram for elements Λ. In the above example the factorisation property is uniquely determined by the skeleton, but this need not be the case. Consider the following skeleton.



To obtain a k-graph from the this skeleton we need to say what the factorisation property is; to do this we specify a collection of commuting squares. In this example there are four possible collections: for each of $i = 1, 2$ we may either have $b_{i,j}a_{i,2} = a_{i,1}b_{i+1,j}$ or $b_{i,j}a_{i,2} = a_{i,1}b_{i+1,3-j}$.

For $n \in \mathbb{N}^k$ define

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n, \text{ and } d(\lambda)_i < n_i \Rightarrow s(\lambda)\Lambda^{e_i} = \emptyset\}. \tag{2.2.1}$$

For $v \in \Lambda^0$ and $X \subset \Lambda$, define $vX = \{\lambda \in X : r(\lambda) = v\}$ and $Xv = \{\lambda \in X : s(\lambda) = v\}$.

We say a k-graph Λ is *row finite* if for any $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ the set $v\Lambda^n$ is finite. In addition, if each $\Lambda^n v$ is also finite we say that Λ is *locally finite*. Λ is said to be *irreducible* if for all $v, w \in \Lambda^0$ we have $v\Lambda w \neq \emptyset$. A vertex $v \in \Lambda^0$ is called a sink if there exists $i \in \{1, \ldots, k\}$ such that $\Lambda^{e_i}v = \emptyset$. Similarly, v is source if $v\Lambda^{e_i} = \emptyset$ for some i.

So far, for a k-graph Λ, we only have a definition for paths of finite degree. Next we define paths of infinite degree. For this we introduce the following example.

Example 2.2.1. *Let $k \in \mathbb{N}$. Define a partial ordering on $\mathbb{N}^k$ by $l \leq m$ if and only if $l_i \leq m_i$ for all i. Fix $n \in (\mathbb{N} \cup \{\infty\})^k$. We define a category $\Omega_{k,n}$ by*

$$\begin{aligned} \mathrm{Obj}(\Omega_{k,n}) &:= \{p \in \mathbb{N}^k : p \leq n\}, \\ \mathrm{Hom}(\Omega_{k,n}) &:= \{(p, q) \in \mathrm{Obj}(\Omega_{k,n}) \times \mathrm{Obj}(\Omega_{k,n}) : p \leq q\}, \\ r(p, q) &:= p, \text{ and } s(p, q) = q. \end{aligned}$$

Then $\Omega_{k,n}$ together with degree map $d(p, q) = q - p$ is a k-graph.

Suppose Λ is a k-graph. For $m \in \mathbb{N}^k$ and $\lambda \in \Lambda^m$ it is shown in [**34**, Examples 2.2 (ii)] that there exists a unique graph morphism $x_\lambda : \Omega_{k,m} \to \Lambda$ defined by $x_\lambda(p, q) = \lambda''$, where $\lambda = \lambda'\lambda''\lambda'''$, $d(\lambda') = p$ and $d(\lambda''') = m - q$. This sets up a bijection between Λ^n and the set of k-graph morphisms $x : \Omega_{k,n} \to \Lambda$.

Observe that the definition of $\Omega_{k,n}$ makes sense even when n is not an element of $\mathbb{N}^k$. Thus we may extend the k-graph morphisms defined above to the case when $n \notin \mathbb{N}^k$. That is, for $n \in (\mathbb{N} \cup \infty)^k$ define Λ^n to be the set of k-graph morphisms $x : \Omega_{k,n} \to \Lambda$ with range map defined by $r(x) = x(0)$ and degree map $d(x) = n$. Just as for $n \in \mathbb{N}^k$, when $n \in (\mathbb{N} \cup \infty)^k$ elements x of Λ^n are determined by the paths $\{x(0, p) : p \leq d(x)\}$. Note that when $n = \infty^k$, the resulting path space Λ^∞ is called the *infinite path space*. Note that $\Omega_{k,\infty}$ is the main example of [**23**, Chapter 2].

For a k-graph Λ with no sinks or sources we show how to realise Λ^∞ as a multi-dimensional shift space. For any $p \in \mathbb{N}^k$ we define a map $\sigma_p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma_p\big(x(m, n)\big) = x(m + p, n + p)$. We call σ_p the *p-shift map* for Λ. Fix $x \in \Lambda^\infty$ and

$p \in \mathbb{N}^k$. Then $\sigma_p(x) \in \Lambda^\infty$. Thus for any $p \in \mathbb{N}^k$ the pair $(\Lambda^\infty, \sigma_p)$ is a k-dimensional shift space.

We now extend the definition of σ_p to graph morphisms $x : \Omega_{k,m} \to \Lambda$ for $m \neq \infty^k$. Note that the following construction has previously appeared as [**12**, Notation 5.2]. Suppose Λ is a k-graph, $q \in (\mathbb{N} \cup \infty)^k$ and $x : \Omega_{k,q} \to \Lambda$ is a k-graph morphism. Then for $p \leq q$ we write $\sigma_p(x)$ for the unique graph morphism $\sigma_p(x) : \Omega_{k,q-p} \to \Lambda$ defined by $\sigma_p(x)(m,n) = x\big(m+p, n+p\big)$ for each pair m, n such that $0 \leq m \leq n \leq q-p$. For $\lambda \in \Lambda x(0)$, we write λx for the unique graph morphism $\lambda x : \Omega_{k,d(x)+d(\lambda)} \to \Lambda$ defined by $(\lambda x)\big(0, d(\lambda)\big) = \lambda$ and $(\lambda x)\big(d(\lambda), d(\lambda) + d(x)\big) = x$.

2.2.1. Cuntz-Krieger families and higher-rank graph algebras. In this subsection we indicate how to associate a C^*-algebra to a higher-rank graph.

Fix a k-graph Λ. For $\mu, \nu \in \Lambda$, we write

$$\mathrm{MCE}(\mu,\nu) := \{\lambda \in \Lambda : d(\lambda) = d(\mu) \vee d(\nu), \lambda\big(0, d(\mu)\big) = \mu, \lambda\big(0, d(\nu)\big) = \nu\} \tag{2.2.2}$$

for the collection of *minimal common extensions* of μ and ν. We say that Λ is *finitely aligned* if $|\mathrm{MCE}(\mu,\nu)| < \infty$ for all $\mu, \nu \in \Lambda$.

Let Λ be a k-graph, $v \in \Lambda^0$ and $E \subset v\Lambda$. We say that E is *exhaustive* if for each $\mu \in v\Lambda$ there exists $\lambda \in E$ such that $\mathrm{MCE}(\mu,\lambda) \neq \emptyset$. If $|E| < \infty$ we say E is *finite exhaustive*. Define $\mathrm{FE}(\Lambda)$ to be the set of finite exhaustive sets of Λ. For each $v \in \Lambda^0$, define $v\,\mathrm{FE}(\Lambda)$ to be the set of finite exhaustive sets whose elements all have range v.

In the following definition, conditions (CK1) to (CK4) are called the *Cuntz-Krieger relations*.

Definition 2.2.2 ([**35**, Definition 2.5]). Let Λ be a finitely aligned k-graph. A *Cuntz-Krieger Λ-family* is a set $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in a C^*-algebra B satisfying:

(CK1) $\{t_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $t_\mu t_\nu = t_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $t_\mu^* t_\nu = \sum_{\mu\alpha=\nu\beta\in\mathrm{MCE}(\mu,\nu)} t_\alpha t_\beta^*$ for every $\mu, \nu \in \Lambda$ and;
(CK4) $\prod_{\lambda\in E}(t_v - t_\lambda t_\lambda^*) = 0$ for every $v \in \Lambda^0$ and $E \in v\,\mathrm{FE}(\Lambda)$.

The projections $\{t_v : v \in \Lambda^0\}$ are commonly referred to as *vertex projections*.

To see that (CK4) above is well defined, we need to check that the order of multiplication is unambiguous. As we shall see below, it turns out that $\{t_\lambda : \lambda \in \Lambda\}$ is a set of mutually orthogonal projections and hence the product in (CK4) is commutative.

Given a k-graph Λ there exists a C^*-algebra $C^*(\Lambda)$ generated by a Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal in the following sense: given any other

Cuntz-Krieger Λ-family $\{t_\lambda : \lambda \in \Lambda\}$ there exists a unique homomorphism π_t such that $\pi_t(s_\lambda) = t_\lambda$ for every $\lambda \in \Lambda$.

The following lemma tells us some useful properties of Cuntz-Krieger Λ-families.

Lemma 2.2.3 ([**35**, Lemma 2.7]). *Let Λ be a finitely aligned k-graph and let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz-Krieger Λ-family. Then*

(i) $t_\mu t_\mu^* t_\nu t_\nu^* = \sum_{\lambda \in \mathrm{MCE}(\mu,\nu)} t_\lambda t_\lambda^*$ *for all* $\mu, \nu \in \Lambda$. *In particular,* $\{t_\lambda t_\lambda^* : \lambda \in \Lambda\}$ *is a family of commuting projections.*

(ii) *For* $\mu, \nu \in \Lambda^{\leq n}$, *we have* $t_\mu^* t_\nu = \delta_{\mu,\nu} t_{s(\mu)}$.

(iii) *If* $E \subset v\Lambda^{\leq n}$ *is finite, then* $t_v \geq \sum_{\lambda \in E} t_\lambda t_\lambda^*$.

(iv) $C^*(\{t_\lambda : \lambda \in \Lambda\}) = \overline{\mathrm{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda\} = \overline{\mathrm{span}}\{t_\mu t_\nu^* : \mu, \nu \in \Lambda, s(\mu) = s(\nu)\}$.

We have written (CK3) and Lemma 2.2.3 (i) in terms of $\mathrm{MCE}(\mu, \nu)$, whereas in [**35**] they are written in terms of a different set, denoted $\Lambda^{\min}(\mu, \nu)$. The two definitions are equivalent because, for fixed $\mu, \nu \in \Lambda$ the map $(\alpha, \beta) \mapsto \mu\alpha$ is a bijection between $\Lambda^{\min}(\mu, \nu)$ and $\mathrm{MCE}(\mu, \nu)$. We will avoid using $\Lambda^{\min}$ throughout this thesis in order to reduce the amount of notation required as much as possible.

Given a k-graph, define the *gauge action* $\gamma : \mathbb{T}^k \to \mathrm{Aut}\big(C^*(\Lambda)\big)$ by $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ where z^m is defined using the multi-index notation $z_1^{m_k} \dots z_k^{m_k} \in \mathbb{T}$. It is well known that the gauge action is a strongly continuous action. The resulting fixed point algebra $C^*(\Lambda)^\gamma = \overline{\mathrm{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu)\}$ is called the *core* of $C^*(\Lambda)$.

CHAPTER 3

Characterising simplicity for higher-rank graph algebras

3.1. Introduction

One of the great advantages of higher-rank graph algebras is that we can characterise complex properties of the C^*-algebra in terms of intuitive combinatorial properties of the higher-rank graph. In other words, we are able to construct and visualise higher-rank graphs with certain combinatorial properties which ensure that the resulting C^*-algebra has certain specific properties. Starting with Kumjian and Pask in [**23**], many authors have characterised simplicity in terms of two conditions, called aperiodicity and cofinality (see Chapter 1 for more details). In this chapter we introduce new notions of aperiodicity and cofinality — given only in terms of finite paths — which are appropriate for finitely aligned k-graphs, and use them to show that a k-graph algebra is simple if and only if the associated k-graph is aperiodic and cofinal.

We begin the chapter by constructing a Cuntz-Krieger Λ-family consisting entirely of non-zero partial isometries in Section 3.2. In Section 3.3 we define aperiodicity and cofinality for k-graphs. We then state the main theorem of this chapter, Theorem 3.3.3: Suppose Λ is a finitely aligned k-graph. Then $C^*(\Lambda)$ is simple if and only if Λ is aperiodic and cofinal.

Simplicity of $C^*(\Lambda)$, for a k-graph Λ, has previously been characterised by Shotwell in [**40**, Theorem 4.5]. Theorem 3.3.3 is an improvement on Shotwell's result because our formulation of aperiodicity and cofinality are phrased entirely in terms of finite paths. This is desirable because previous formulations of aperiodicity and cofinality were defined in terms of a set of generalised paths denoted by $\partial\Lambda$. This is problematic because elements of $\partial\Lambda$ are difficult to identify and work with in practice. Thus any condition phrased in terms of such paths is extremely difficult to verify in examples. See Example 3.2.2 for an example of the complexities involved with identifying elements of $\partial\Lambda$. We finish off Section 3.3 by showing that our definition of aperiodicity is equivalent to two earlier formulations present in the literature.

In order to characterise simplicity our first step is to prove the Cuntz-Krieger Uniqueness Theorem. One way to interpret the Cuntz-Krieger Uniqueness theorem is as the following statement: If a k-graph Λ is aperiodic, then every nonzero ideal in $C^*(\Lambda)$ contains at least one of the generators. This is the subject of Section 3.4.

In this section we prove our version of the Cuntz-Krieger Uniqueness Theorem as part of Theorem 3.4.1.

Once we have established the Cuntz Krieger Uniqueness Theorem there is one further step required to establish a characterisation of simplicity. We must identify a condition on Λ which ensures that if an ideal in $C^*(\Lambda)$ contains a generator, then it must contain all generators. Recently, Shotwell [**40**] has identified such a condition for finitely aligned higher-rank graphs. However this condition is phrased in terms of $\partial\Lambda$ and is potentially quite difficult to check in examples. We present a condition, also called cofinality, which is phrased in terms of finite paths and so is more easily checked in examples. We show in Section 3.5 that our formulation is equivalent to Shotwell's and then prove Theorem 3.3.3. Our cofinality condition is new, even in the special cases of locally convex k-graphs and infinite directed graphs. We provide details of this in Section 3.6.

3.2. The $\partial\Lambda$-representation

Suppose Λ is a k-graph. In this section we show that there exists a Cuntz-Krieger Λ-family consisting of non-zero partial isometries. To do this we construct a Cuntz-Krieger Λ-family on a Hilbert space with basis induced by a collection $\partial\Lambda$ of paths in Λ. This representation is not new and the summary presented here draws on several sources [**12, 35, 41, 42**].

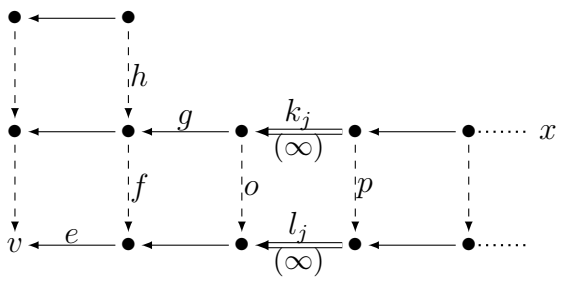
The following definition is based on [**12**, Definition 5.10]. We retain the original meaning but change the notation to be consistent with the rest of this thesis.

Definition 3.2.1. Suppose Λ is a finitely aligned k-graph. We define a set $\partial\Lambda$ of graph morphisms $\{x : \Omega_{k,m} \to \Lambda\}$ by

$$\partial\Lambda = \bigcup_{m\in(\mathbb{N}\cup\{\infty\})^k} \{x : \Omega_{k,m} \to \Lambda \mid \text{for all } n \in \mathbb{N}^k \text{ with } n \leq m \text{ and all } E \in x(n)\,\mathrm{FE}(\Lambda) \text{ there exists } p \leq m-n \text{ such that } x(n, n+p) \in E\}.$$

In [**12, 40**] the elements of $\partial\Lambda$ were called boundary paths. However, the same term was used in to refer to elements of $\Lambda^{\leq\infty}$, for example in [**34, 35, 37**]. In order to avoid confusion we will eschew the the term "boundary path" altogether.

Example 3.2.2. *Consider the following skeleton Λ.*



Then (Λ, d) together with the factorisation property defined by $ok_j = l_j p$ for each j is a 2-graph.

We provide three examples of elements of $\partial\Lambda$.

(1) $efh \in \partial\Lambda$. *For illustrative purposes we check that Definition 3.2.1 is satisfied for $n = (1,1)$ and $n = (1,2)$.*

We first verify Definition 3.2.1 when $n = (1,2)$. In this case we have $(efh)\big((1,2)\big) = s(h)$. Since $s(h)\,\mathrm{FE}(\Lambda) = \{s(h)\}$ and $s(h) = (efh)\big((1,2)\big)$, we see that Definition 3.2.1 is satisfied for $n = (1,2)$.

We now check Definition 3.2.1 when $n = (1,1)$. In this case we have $(efh)\big((1,1)\big) = s(f)$. We claim any element E of $s(f)\,\mathrm{FE}(\Lambda)$ must contain either h or $s(f)$. For if this is not the case, then $\mathrm{MCE}(\mathrm{h}, \eta) = \emptyset$ for every $\eta \in E$, contradicting the fact that $E \in \mathrm{FE}(\Lambda)$. This proves the claim. Since $h = (efh)\big((1,1),(1,2)\big)$ and $s(f) = (efh)\big((1,1)\big)$ then Definition 3.2.1 is satisfied for $n = (1,1)$.

Verification of the Definition 3.2.1 for other values of $n \leq (1,2)$ follows the same process as for $n = (1,1)$.

(2) $efg \in \partial\Lambda$. *This is because every element of $s(g)\,\mathrm{FE}(\Lambda)$ must contain $s(g)$. To prove this we suppose $E \subset s(g)\Lambda$ is an exhaustive set which doesn't contain $s(g)$. Since E is exhaustive, for every $k_j \in s(g)\Lambda^{e_1}$ there exists $\lambda \in E$ such that $\lambda(0, e_1) = k_j$. However $|s(g)\Lambda^{e_1}| = \infty$ and hence we must have $|E| = \infty$. Thus E is not a finite exhaustive set.*

(3) *Fix $j \in \mathbb{N}$, $y \in s(k_j)\Lambda^{\leq\infty}$ and let $x = k_j y$. Then $efgx \in \partial\Lambda$. Fix $n \leq d(efgx)$. Examples (1) and (2) above show that Definition 3.2.1 holds for $n \leq (2,1)$. So suppose $n > (2,1)$. Fix $E \in x(n)\,\mathrm{FE}(\Lambda)$ and $\eta \in E$. Then $\mathrm{MCE}\Big(x\big(n, n + d(\eta)\big), \eta\Big) = \eta$ as required.*

The following lemma is a special case of [**41**, Lemma 4.3.3] with $\mathcal{E} = \mathrm{FE}(\Lambda)$, we do however provide a proof for completeness. We use it to construct a Cuntz-Krieger family consisting of nonzero partial isometries in $\ell^2(\partial\Lambda)$.

Lemma 3.2.3. *Suppose Λ is a finitely aligned k-graph and let $x \in \partial\Lambda$.*

(1) *If $n \in \mathbb{N}^k$ such that $n \leq d(x)$, then $\sigma_n(x) \in \partial\Lambda$.*
(2) *If $\lambda \in \Lambda r(x)$, then $\lambda x \in \partial\Lambda$.*

Proof. **(1)** Fix $m \in \mathbb{N}^k$ such that $m \leq d(x)$, $n \leq d\big(\sigma_m(x)\big)$ and that $E \in \sigma_m(x)(n)\,\mathrm{FE}(\Lambda)$. We show there exists $p \leq d\big(\sigma_m(x)\big) - n$ such that $\sigma_m(x)(n, n+p) \in E$. By definition of $\sigma_m(x)$ we know $n + m \leq d(x)$ and $E \in x(n+m)\,\mathrm{FE}(\Lambda)$. Since $x \in \partial\Lambda$ there exists $p \in d(x) - n - m$ such that $x(n+m, n+m+p) \in E$. Since $x(n+m, n+m+p) = \sigma_m(x)(n, n+p)$ this proves (1).

(2) Fix $\lambda \in \Lambda r(x)$. To show $\lambda x \in \partial\Lambda$ fix $n \leq d(\lambda x)$, $E \in (\lambda x)(n)\,\mathrm{FE}(\Lambda)$ and let $\lambda' = \big(\lambda x\big)\big(n, n \vee d(\lambda)\big)$. Then Lemma 3.5.2 implies that

$$\mathrm{Ext}(\lambda', E) \in s(\lambda')\,\mathrm{FE}(\Lambda) = \big(\lambda x\big)\big(n \vee d(\lambda)\big)\,\mathrm{FE}(\Lambda) = x\big((n - d(\lambda)) - d(\lambda)\big)\,\mathrm{FE}(\Lambda).$$

In particular, $\mathrm{Ext}(\lambda', E)$ is a finite exhaustive set. Let $n' = \big(n \vee d(\lambda)\big) - d(\lambda)$ and $E' = \mathrm{Ext}(\lambda', E)$; then that $x \in \partial\Lambda$ implies that there exists $p' \leq d(\lambda) - n'$ such that $x(n', n' + p') \in E'$. By definition of E there exists $\mu \in E$ such that $\mathrm{MCE}(\lambda', \mu) = \big(x(n', n' + p'), \beta\big)$ for some $\beta \in \Lambda$. That is $\lambda' x(n', n' + p') = \mu\beta$.

Let $p = d(\mu)$. We show that $\big(\lambda x\big)\big(n, n + p\big) \in E$:

$$\big(\lambda x\big)\big(n, n + p\big) = \Big(\big(\lambda x\big)\big(n, d(\lambda x)\big)\Big)(0, p) = \big(\lambda' x(n', d(x)\big)(0, p). \tag{3.2.1}$$

Since $d\big(\lambda' x(n', n' + p)\big) = d(\mu\beta) \geq p$, equation (3.2.1) yields

$$\big(\lambda x\big)\big(n, n + p\big) = (\mu\beta)(0, p) = \mu \in E. \qquad \blacksquare$$

Example 3.2.4. *Recall from Example 3.2.2 that we saw that $efgx \in \partial\Lambda$. Then Lemma 3.2.3 (1) implies that $\sigma_{(3,0)}(efgx) = px \in \partial\Lambda$. In addition, Lemma 3.2.3 (2) implies that for every $j \in \mathbb{N}$ and any $y \in s(l_j)(\partial\Lambda)$, when we append l_j to the front of y we see that $l_j y \in \partial\Lambda$. In particular, for every $j \in \mathbb{N}$ we have $l_j px \in \partial\Lambda$.*

Definition 3.2.5 (The $\partial\Lambda$-representation). Let Λ be a finitely aligned k-graph and let $\{\zeta_x : x \in \partial\Lambda\}$ denote the standard orthonormal basis for $\ell^2(\partial\Lambda)$. We define a set of operators $\{S_\lambda : \lambda \in \Lambda\}$ by

$$S_\lambda \zeta_x = \begin{cases} \zeta_{\lambda x} & \text{if } s(\lambda) = r(x), \\ 0 & \text{otherwise.} \end{cases}$$

We call the set $\{S_\lambda : \lambda \in \Lambda\}$ the $\partial\Lambda$-representation.

We prove that $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger family, and hence induces a representation $\pi : C^*(\Lambda) \to \ell^2(\partial\Lambda)$. Suppose Λ is a k-graph. In order to check that $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family we need to be able to compute S_λ^* for each $\lambda \in \Lambda$. Fix $\lambda \in \Lambda$. Then for $x, y \in \partial\Lambda$ we have

$$(S_\lambda^* \zeta_x | \zeta_y) = (\zeta_x | S_\lambda \zeta_y) = \begin{cases} (\zeta_x | \zeta_{\lambda y}) & \text{when } s(\lambda) = r(y), \\ (\zeta_x | 0) & \text{otherwise} \end{cases}$$

and hence

$$(S_\lambda^* \zeta_x | \zeta_y) = \begin{cases} 1 & \text{if } y = \sigma_{d(\lambda)}(x) \text{ and } x\big(0, d(\lambda)\big) = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$S_\lambda^* \zeta_x = \begin{cases} \zeta_{\sigma_{d(\lambda)}(x)} & \text{if } x(0, d(\lambda)) = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

It follows by definition that for all $x \in \partial\Lambda$ such that $r(x) = s(\lambda)$ we have

$$S_\lambda^* S_\lambda \zeta_x = S_{s(\lambda)}\zeta_x = \zeta_x. \tag{3.2.2}$$

We now finish off this section by proving that $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family. This lemma is proven in greater generality compared to [**42**, Lemma 4.6]. We provide a proof here for completeness.

LEMMA 3.2.6. *Suppose Λ is a finitely aligned k-graph. Then the set $\{S_\lambda : \lambda \in \Lambda\}$ of Definition 3.2.5 is a Cuntz-Krieger Λ-family.*

PROOF. Throughout this proof $\{\zeta_x : x \in \partial\Lambda\}$ denotes the standard orthonormal basis for $\ell^2(\partial\Lambda)$. We first establish that $\{S_\lambda : \lambda \in \Lambda\}$ is a set of partial isometries. To see that the S_λ are partial isometries, observe that $x \mapsto \lambda x$ is a one-to-one map from $X = \{x \in \partial\Lambda : r(x) = s(\lambda)\}$ to $Y = \{y \in \partial\Lambda : y\big(0, d(\lambda)\big) = \lambda\}$. Thus for each $\lambda \in \Lambda$, the operator S_λ is a partial isometry with range projection onto $\overline{\operatorname{span}}\{\zeta_y : y \in Y\}$ and source projection onto $\overline{\operatorname{span}}\{\zeta_x : x \in X\}$. We now verify (CK1) – (CK4).

(CK1) For $v \in \Lambda^0$, the element $S_v \in B\big(\ell^2(\partial\Lambda)\big)$ is the orthogonal projection onto the subspace $E_v = \overline{\operatorname{span}}\{\zeta_x : r(x) = v\}$. Since $\{E_v : v \in \Lambda^0\}$ is a set of mutually orthogonal subspaces, it follows that $\{S_v : v \in \Lambda^0\}$ is a set of mutually orthogonal projections.

(CK2) Fix $\mu, \nu \in \Lambda$ such that $s(\mu) = r(\nu)$. Then for $x \in \partial\Lambda$

$$S_\mu S_\nu \zeta_x = \begin{cases} S_\mu \zeta_{\nu x} & \text{if } s(\nu) = r(x), \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \zeta_{\mu\nu x} & \text{if } s(\nu) = r(x), s(\mu) = r(\nu), \\ 0 & \text{otherwise} \end{cases} = S_{\mu\nu}\zeta_x.$$

(CK3) Fix $\mu, \nu \in \Lambda$ and let $p = d(\mu) \vee d(\nu)$. We must show that $S_\mu^* S_\nu = \sum_{\mu\alpha=\nu\beta\in\mathrm{MCE}(\mu,\nu)} S_\alpha S_\beta^*$. We begin by establishing the following equality.

$$S_\mu (S_\mu^* S_\nu) S_\nu^* = \sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} S_\lambda S_\lambda^*. \tag{3.2.3}$$

To see this we fix $x \in \partial\Lambda$ and calculate

$$\begin{aligned} S_\mu S_\mu^* S_\nu S_\nu^* \zeta_x &= \begin{cases} S_\mu S_\mu^* \zeta_x & \text{if } x\big(0, d(\nu)\big) = \nu, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \zeta_x & \text{if } x\big(0, d(\nu)\big) = \nu \text{ and } x\big(0, d(\mu)\big) = \mu, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By definition of MCE we have

$$S_\mu S_\mu^* S_\nu S_\nu^* \zeta_x = \begin{cases} \zeta_x & \text{if } x(0,p) \in \mathrm{MCE}(\mu,\nu), \\ 0 & \text{otherwise.} \end{cases}$$

Hence to establish (3.2.3) it suffices to show

$$\sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} S_\lambda S_\lambda^*\zeta_x = \begin{cases}\zeta_x & \text{if } x(0,p)\in\mathrm{MCE}(\mu,\nu),\\ 0 & \text{otherwise.}\end{cases}$$

For each $\lambda\in\mathrm{MCE}(\mu,\nu)$ we have

$$S_\lambda S_\lambda^*\zeta_x = \begin{cases}\zeta_x & \text{if } x(0,p)=\lambda,\\ 0 & \text{otherwise.}\end{cases}$$

Thus

$$\sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} S_\lambda S_\lambda^*\zeta_x = \sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} \begin{cases}\zeta_x & \text{if } x(0,p)=\lambda,\\ 0 & \text{otherwise.}\end{cases}$$

We now observe that $x(0,p)$ can be equal to at most one element of $\mathrm{MCE}(\mu,\nu)$. Hence at most one term in the sum above can be non-zero. The sum contains a non-zero term precisely when $x(0,p)\in\mathrm{MCE}(\mu,\nu)$. Hence

$$\sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} S_\lambda S_\lambda^*\zeta_x = \begin{cases}\zeta_x & \text{if } x(0,p)\in\mathrm{MCE}(\mu,\nu),\\ 0 & \text{otherwise}\end{cases}$$

and we have verified (3.2.3) as claimed.

We now multiply (3.2.3) on the left by S_μ^* and on the right by S_ν to see that

$$S_\mu^*(S_\mu S_\mu^* S_\nu S_\nu^*)S_\nu = S_\mu^*\Big(\sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} S_\lambda S_\lambda^*\Big)S_\nu = \sum_{\mu\alpha=\nu\beta\in\mathrm{MCE}(\mu,\nu)} S_\mu^* S_\mu S_\alpha S_\beta^* S_\nu^* S_\nu$$

Hence (3.2.2) implies

$$S_\mu^*(S_\mu S_\mu^* S_\nu S_\nu^*)S_\nu = \sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} S_\alpha S_\beta^*. \tag{3.2.4}$$

On the other hand, since S_μ and S_ν are partial isometries, we also observe that $S_\mu^*(S_\mu S_\mu^* S_\nu S_\nu^*)S_\nu = (S_\mu^* S_\mu S_\mu^*)(S_\nu S_\nu^* S_\nu) = S_\mu^* S_\nu$. Hence (3.2.4) becomes

$$S_\mu^* S_\nu = \sum_{\lambda\in\mathrm{MCE}(\mu,\nu)} S_\alpha S_\beta^*.$$

(CK4) Fix $v\in\Lambda^0$, $E\in v\,\mathrm{FE}(\Lambda)$ and $x\in v\partial\Lambda$. Applying the definition of $\partial\Lambda$ with $n=0$, we deduce that there exists $p\in\mathbb{N}^k$ such that $x(0,p)\in E$. Then

$$(S_v - S_{x(0,p)}S_{x(0,p)}^*)\zeta_x = S_v\zeta_x - S_{x(0,p)}S_{x(0,p)}^*\zeta_x = \zeta_x - \zeta_x = 0.$$

Thus

$$\prod_{\lambda\in E}(S_v - S_\lambda S_\lambda^*)\zeta_x = \Big(\prod_{\lambda\in E\setminus\{x(0,p)\}}(S_v - S_\lambda S_\lambda^*)\Big)(S_v - S_{x(0,p)}S_{x(0,p)}^*)\zeta_x = 0. \qquad \blacksquare$$

REMARK 3.2.7. Suppose Λ is a finitely aligned k-graph. Farthing, Muhly and Yeend show in [**12**, Lemma 5.15] that each S_v is non-zero. Since $\{S_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family, (CK3) implies $S_\lambda^* S_\lambda = S_{s(\lambda)} \neq 0$ for all $\lambda \in \Lambda$ and hence each $S_\lambda \neq 0$.

3.3. Aperiodicity and cofinality

The main result of this chapter is Theorem 3.3.3 which says that for a finitely aligned k-graph Λ, the associated C^*-algebra $C^*(\Lambda)$ is simple if and only if Λ is both aperiodic and cofinal. We first introduce the notions of aperiodicity and cofinality which we use in this thesis. We will show later that aperiodicity is equivalent to earlier notions used by Shotwell [**40**] and Farthing, Muhly and Yeend [**12**].

DEFINITION 3.3.1. A finitely aligned k-graph Λ is *aperiodic* if for every pair of distinct paths $\alpha, \beta \in \Lambda$ with $s(\alpha) = s(\beta)$ there exists $\tau \in s(\alpha)\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$.

DEFINITION 3.3.2. Suppose Λ is a finitely aligned k-graph. We say that Λ is *cofinal* if for every $v, w \in \Lambda^0$ there exists $E \in w\,\mathrm{FE}(\Lambda)$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in E$.

THEOREM 3.3.3. *Suppose Λ is a finitely aligned k-graph. Then $C^*(\Lambda)$ is simple if and only if Λ is aperiodic and cofinal.*

We defer the proof of Theorem 3.3.3 until Section 3.5 as we first require some supporting results.

In examples, we want to be able to check that a k-graph is aperiodic by considering as few pairs of paths μ, ν as possible. To this end we prove the following lemma

LEMMA 3.3.4. *Let Λ be a k-graph. Then Λ is aperiodic if and only if for all $\mu, \nu \in \Lambda$ such that $s(\mu) = s(\nu)$, $r(\mu) = r(\nu)$, $d(\mu) \neq d(\nu)$ and $d(\mu) \wedge d(\nu) = 0$ there exists $\tau \in s(\mu)\Lambda$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$.*

PROOF. Suppose that for all distinct $\mu, \nu \in \Lambda$ such that $s(\mu) = s(\nu)$, $r(\mu) = r(\nu)$ and $d(\mu) \wedge d(\nu) = 0$ there exists $\tau \in s(\mu)\Lambda$ such that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. Fix distinct $\alpha, \beta \in \Lambda$ such that $s(\alpha) = s(\beta)$. We must find $\tau \in s(\alpha)\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$.

Let $m = d(\alpha) \wedge d(\beta)$. If $\alpha(0, m) \neq \beta(0, m)$ then $\tau = s(\alpha)$ will do. So suppose otherwise. That is, $\alpha(0, m) = \beta(0, m)$. Then by definition of m we have $\big(d(\alpha) - m\big) \wedge \big(d(\beta) - m\big) = 0$. Moreover, since α and β are distinct, if follows that $d(\alpha) \neq d(\beta)$. Thus $d(\alpha) - m \neq d(\beta) - m$ which forces $\alpha\big(m, d(\alpha)\big) \neq$

$\beta\big(m, d(\beta)\big)$. Since $\alpha(m) = \beta(m)$, by assumption there exists $\tau \in s(\alpha)\Lambda$ such that $\mathrm{MCE}\Big(\alpha\big(m, d(\alpha)\big)\tau, \beta\big(m, d(\beta)\big)\tau\Big) = \emptyset$. Hence

$$\mathrm{MCE}(\alpha\tau, \beta\tau) = \big\{\alpha(0, m)\rho : \rho \in \mathrm{MCE}\big(\alpha(m, d(\alpha))\tau, \beta(m, d(\beta))\tau\big)\big\} = \emptyset$$

as required. Since the only if direction is trivial, this completes the proof. ■

In the history of k-graph algebras, one of the main goals has been to extend the Cuntz-Krieger Uniqueness Theorem for directed graphs, to also cover k-graphs. In broad terms, the Cuntz-Krieger Uniqueness Theorem says that given a k-graph Λ and a homomorphism $\pi : C^*(\Lambda) \to B$ such that $\pi(s_v) \neq 0$ for each $v \in \Lambda^0$, then π is injective if Λ is aperiodic. Several authors have identified various aperiodicity conditions which ensure that the Cuntz-Krieger Uniqueness Theorem holds. For the remainder of this section we show that the aperiodicity condition we have identified in Definition 3.3.1 is equivalent to two earlier formulations in the literature. The first such condition is due to Farthing, Muhly and Yeend and appears in [**12**, Theorem 7.1] as Condition (A). We restate [**12**, Theorem 7.1] below.

THEOREM 3.3.5 ([**12**, Theorem 7.1]). *Let (Λ, d) be a finitely aligned k-graph, and suppose that*

for each $v \in \Lambda^0$ there exists $x \in v\partial\Lambda$ such that

(A) *for all $m, n \in \mathbb{N}^k$, $m, n \leq d(x)$ we have $\sigma_m(x) = \sigma_n(x) \Rightarrow m = n$.*

Suppose that $\pi : C^*(\Lambda) \to B$ is a homomorphism such that $\pi(s_v)$ is non-zero for all $v \in \Lambda^0$. Then π is injective.

In a recent preprint, Shotwell has used a property called *no local periodicity* in place of aperiodicity. Shotwell's formulation is given below.

DEFINITION 3.3.6. We say Λ has *no local periodicity* if for every $v \in \Lambda^0$ and for every distinct $m, n \in \mathbb{N}^k$ either

(NLP1) there exists $x \in v(\partial\Lambda)$ such that $d(x) \not\geq m \vee n$; or

(NLP2) $d(x) \geq m \vee n$ for every $x \in v(\partial\Lambda)$, and there exists $y \in v(\partial\Lambda)$ such that $\sigma_m(y) \neq \sigma_n(y)$.

In the following Proposition, we show that aperiodicity is equivalent to no local periodicity, and is also equivalent to Condition (A). Note that the equivalence of no local periodicity with Condition (A) is due to Shotwell [**40**, Proposition 3.7]. We include this result here only for completeness.

PROPOSITION 3.3.7. *Suppose Λ is a finitely aligned k-graph. Then the following are equivalent:*

(1) *Λ is aperiodic;*

(2) Λ *has no local periodicity; and*
(3) Λ *satisfies Condition* (A).

Proof. In [**40**], Shotwell defines aperiodicity to be Condition (A). He then establishes (2) $\Leftrightarrow$ (3) in [**40**, Proposition 3.7]. So it suffices to establish (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (2): Suppose Λ is aperiodic. Fix $v \in \Lambda^0$ and distinct $m, n \in \mathbb{N}^k$. Either for each $x \in v(\partial\Lambda)$ we have $d(x) \geq m \vee n$, or there exists $x \in v(\partial\Lambda)$ such that $d(x) \ngeq m \vee n$. If $d(x) \ngeq m \vee n$ then (NLP1) holds and hence Λ does not have local periodicity m, n at v. Now suppose that $d(x) \geq m \vee n$ for all $x \in v(\partial\Lambda)$. We aim to establish (NLP2).

Since $v(\partial\Lambda) \neq \emptyset$ and each $x \in v(\partial\Lambda)$ has degree greater than $m \vee n$, we have $v\Lambda^{m\vee n} \neq \emptyset$. Fix $\lambda \in v\Lambda^{m\vee n}$. Since $m, n \leq d(\lambda)$ the factorisation property implies that there exist $\mu \in \Lambda^m$, $\nu \in \Lambda^n$, $\alpha \in \Lambda^{(m\vee n)-m}$ and $\beta \in \Lambda^{(m\vee n)-n}$ such that $\lambda = \mu\alpha = \nu\beta$. Note that in particular $d(\alpha) \neq d(\beta)$ and hence $\alpha \neq \beta$.

Since Λ is aperiodic, there exists $\tau \in s(\alpha)\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$. Fix $z \in s(\tau)(\partial\Lambda)$ and let $y := \lambda\tau z \in v(\partial\Lambda)$. Since $\lambda = \mu\alpha = \nu\beta$ we have $\sigma_m(y) = \alpha\tau z$ and $\sigma_n(y) = \beta\tau z$, so it suffices to show that $\alpha\tau z \neq \beta\tau z$. To see this observe that if $\alpha\tau z = \beta\tau z$ then $z\big(0, d(\alpha\tau) \vee d(\beta\tau)\big) \in \mathrm{MCE}(\alpha\tau, \beta\tau)$, contradicting our choice of τ.

(2) $\Rightarrow$ (1): Suppose Λ has no local periodicity. Fix $\alpha, \beta \in \Lambda$ such that $s(\alpha) = s(\beta)$, $r(\alpha) = r(\beta)$, $d(\alpha) \neq d(\beta)$ and $d(\alpha) \wedge d(\beta) = 0$. Lemma 3.3.4 implies that to show Λ is aperiodic it suffices to construct $\tau \in s(\alpha)\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$. Let $D = d(\alpha) \vee d(\beta)$; then $D = d(\alpha) + d(\beta)$ because $d(\alpha) \wedge d(\beta) = 0$. Recall from Definition 3.3.6 that Λ has no local periodicity if for each $v \in \Lambda^0$ and $m, n \in \mathbb{N}^k$ either (NLP1) holds or (NLP2) holds. Thus we consider the following two cases: for $m = d(\alpha)$, $n = d(\beta)$ and $v = r(\alpha)$ in Definition 3.3.6 either

(1) (NLP1) holds; or
(2) (NLP2) holds.

Case (1): Suppose (NLP1) holds. That is, there exists $x \in v(\partial\Lambda)$ such that $d(x) \ngeq D$. We consider the following two subcases: either

(i) for every $n \leq d(x)$ there exist $\eta \in x(n)\Lambda$ such that $n + d(\eta) \geq D$; or
(ii) there exists $n \leq d(x)$ such that for each $\eta \in x(n)\Lambda$ we have $n + d(\eta) \ngeq D$.

Case (1)(i): In this case we claim that there exist $\tau_f \in v\Lambda$ and $i \in \{1, \ldots, k\}$ such that $d(\alpha)_i \neq d(\beta)_i$ and $|s(\tau_f)\Lambda^{e_i}| = \infty$. Let $p = D \wedge d(x)$ and $\lambda = x(0, p)$. We first show that $s(\lambda)\Lambda^{D-p}$ is an infinite, exhaustive set.

Fix $\mu \in s(\lambda)\Lambda$. To see that $s(\lambda)\Lambda^{D-p}$ is exhaustive we must find $\nu \in s(\lambda)\Lambda^{D-p}$ such that $\mathrm{MCE}(\mu, \nu) \neq \emptyset$. If $d(\mu) \geq D - p$ then $\nu = \mu(0, D - p) \in s(\lambda)\Lambda^{D-p}$ and hence $\mathrm{MCE}(\mu, \nu) = \{\mu\}$. So suppose $d(\mu) \ngeq D - p$.

Since $d(\lambda\mu) \leq d(x)$, by assumption there exists $\mu' \in x\big(d(\lambda\mu)\big)\Lambda$ such that $d(\lambda\mu) + d(\mu') \geq D$. Since $d(\lambda\mu) + d(\mu') = d(\lambda) + d(\mu\mu')$ and $d(\lambda) = p$, it follows that

$d(\mu\mu') \geq D-p$. Thus $d(\mu\mu') \geq D-p$. Let $\nu = (\mu\mu')(0, D-p)$. Then $\nu \in s(\lambda)\Lambda^{D-p}$ and $\mathrm{MCE}(\mu,\nu) = \{\mu\mu'\}$. Hence $s(\lambda)\Lambda^{D-p}$ is exhaustive.

We now show that $|s(\lambda)\Lambda^{D-p}| = \infty$. For suppose $|s(\lambda)\Lambda^{D-p}| < \infty$. Then $s(\lambda)\Lambda^{D-p} \in s(\lambda)\,\mathrm{FE}(\Lambda)$. However, since $d(x) \not\geq D$ there is no $q \in \mathbb{N}^k$ such that $x(p, p+q) \in s(\lambda)\Lambda^{D-p}$, contradicting $x \in \partial\Lambda$. Thus $s(\lambda)\Lambda^{D-p}$ is an infinite, exhaustive set.

We are now able to find the required τ_f as claimed at the beginning of this case. Since $D-p < \infty$ there exists $i \in \{1,\dots,k\}$ and a smallest $a < (D-p)_i$ such that $\big|\{\mu\big(ae_i, (a+1)e_i\big) : \mu \in s(\lambda)\Lambda^{D-p}\}\big| = \infty$. Since $a \in \mathbb{N}$ is the smallest such element we have $|\{\mu(0, ae_i) : \mu \in s(\lambda)\Lambda^{D-p}\}| < \infty$. By definition of a, together with the factorisation property, there exists $\eta \in s(\lambda)\Lambda^{D-p}$ such that $|\eta(ae_i)\Lambda^{e_i}| = \infty$. Let $\tau_f = (\lambda\eta)\big(d(\alpha), p + ae_i\big)$. Then $|s(\tau_f)\Lambda^{e_i}| = |\eta(ae_i)\Lambda^{e_i}| = \infty$. We now show that $d(\alpha)_i \neq d(\beta)_i$. First we observe that $\eta \in s(\lambda)\Lambda^{D-p}$ forces $0 < a < d(\eta)_i = (D-p)_i$. In particular, $D_i > p_i \geq 0$. Since $D_i = \max\{d(\alpha)_i, d(\beta)_i\}$ and $d(\alpha) \wedge d(\beta) = 0$, it follows that $d(\alpha)_i \neq d(\beta)_i$.

If $\mathrm{MCE}(\alpha\tau_f, \beta\tau_f) = \emptyset$ we are done. So suppose that $\mathrm{MCE}(\alpha\tau_f, \beta\tau_f) \neq \emptyset$ and let $m = \min\{d(\alpha\tau_f)_i, d(\beta\tau_f)_i\}$. Since $d(\alpha\tau_f)_i \neq d(\beta\tau_f)_i$ the definition of MCE implies $d(\psi)_i > m$ for all $\psi \in \mathrm{MCE}(\alpha\tau_f, \beta\tau_f)$. However $|s(\tau_f)\Lambda^{e_i}| = \infty$ and because Λ is finitely aligned we have $|\,\mathrm{MCE}(\alpha\tau_f, \beta\tau_f)| < \infty$. Hence there exists $\tau_i \in s(\tau_f)\Lambda^{e_i}$ such that $\psi(m, m+e_i) \neq \tau_i$ for all $\psi \in \mathrm{MCE}(\alpha\tau_f, \beta\tau_f)$.

Let $\tau = \tau_f\tau_i$. We show $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$. Suppose for contradiction that $\psi \in \mathrm{MCE}(\alpha\tau, \beta\tau)$. Then by definition of MCE we have $\psi\big(0, d(\alpha\tau_f) \vee d(\beta\tau)\big) \in \mathrm{MCE}(\alpha\tau, \beta\tau_f)$. But now $\psi(m, m+e_i) = \tau_i$, which contradicts our choice of τ_i.

Case (1)(ii): Let $n \in \mathbb{N}^k$ be as in Case (1)(ii) and let $\lambda := x(0,n)$. Then, by assumption, for all $\eta \in x(n)\Lambda$ we have $n + d(\eta) \not\geq D$. Fix $\eta \in x(n)\Lambda^{\leq D-n} \subset x(n)\Lambda$. By hypothesis $d(\lambda\eta) = n + d(\eta) \not\geq D$ and hence $d(\lambda) \neq D-n$. So there exists $i \in \{1,\dots,k\}$ such that $d(\lambda)_i < (D-n)_i$. For each such i, by definition of $\Lambda^{\leq D-n}$ we have $s(\eta)\Lambda^{e_i} = \emptyset$.

We now claim that there exists $i \in \{1,\dots,k\}$ such that $d(\lambda\eta)_i < D_i$ and $d(\alpha)_i \neq d(\beta)_i$. Suppose for contradiction that there exists $i \in \{1,\dots,k\}$ such that $d(\lambda\eta)_i < D_i$ and $d(\alpha)_i = d(\beta)_i$. Since $d(\alpha) \wedge d(\beta) = 0$, we must have $d(\alpha)_i = d(\beta)_i = 0$. Then we have the contradiction $0 \leq d(\lambda\eta)_i < D_i = \big(d(\alpha) \vee d(\beta)\big)_i = 0$.

Fix $i \in \{0,\dots,k\}$ such that $d(\lambda\eta)_i < D_i$ and $d(\alpha)_i \neq d(\beta)_i$. Fix a path $\tau = (\lambda\eta)\big(d(\alpha), d(\lambda\eta)\big)$. Then $d(\alpha\tau)_i \neq d(\beta\tau)_i$. However by construction $s(\tau)\Lambda^{e_i} = s(\lambda\eta)\Lambda^{e_i} = \emptyset$. Hence $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$ as required.

Case (2): Recall that $D = d(\alpha) + d(\beta)$ and suppose (NLP2) holds. That is, for every $x \in v(\partial\Lambda)$ we have $d(x) \geq D$, and there exists $y \in v(\partial\Lambda)$ such that $\sigma_{d(\alpha)}(y) \neq \sigma_{d(\beta)}(y)$. Then there exists $M_{\alpha,\beta} \in \mathbb{N}^k$ such that $M_{\alpha,\beta} + D \leq d(\mu)$ and

$$\big(\sigma_{d(\alpha)}(y)\big)(0, M_{\alpha,\beta}) \neq \big(\sigma_{d(\beta)}(y)\big)(0, M_{\alpha,\beta}).$$

Or equivalently,

$$y\big(d(\alpha), d(\alpha) + M_{\alpha,\beta}\big) \neq y\big(d(\beta), d(\beta) + M_{\alpha,\beta}\big). \tag{3.3.1}$$

Let $\tau = y(0, D + M_{\alpha,\beta})$. Then $d(\alpha) + M_{\alpha,\beta}$ and $d(\beta) + M_{\alpha,\beta}$ are both less than $d(\tau)$. Hence

$$(\alpha\tau)(D, D + M_{\alpha,\beta}) = \tau\big(D - d(\alpha), D - d(\alpha) + M_{\alpha,\beta}\big) = y\big(d(\beta), d(\beta) + M_{\alpha,\beta}\big)$$

and

$$(\beta\tau)(D, D + M_{\alpha,\beta}) = \tau\big(D - d(\beta), D - d(\beta) + M_{\alpha,\beta}\big) = y\big(d(\alpha), d(\alpha) + M_{\alpha,\beta}\big).$$

Then (3.3.1) implies

$$(\alpha\tau)(D, D + M_{\alpha,\beta}) \neq (\beta\tau)(D, D + M_{\alpha,\beta}).$$

Hence for $\psi \in \alpha\tau\Lambda^{(d(\alpha\tau)\vee d(\beta\tau)) - d(\alpha\tau)}$, we have

$$\psi(D, D + M_{\alpha,\beta}) = (\alpha\tau)(D, D + M_{\alpha,\beta}) \neq (\beta\tau)(D, D + M_{\alpha,\beta}).$$

So $\psi \notin \beta\tau\Lambda$ and hence $\psi \notin \mathrm{MCE}(\alpha\tau, \beta\tau)$. Thus $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$. ■

3.4. Ideals in Cuntz-Krieger algebras of aperiodic k-graphs

In this section we describe how aperiodicity of a k-graph Λ relates to the ideal structure of $C^*(\Lambda)$. Specifically, we prove the following theorem, the proof of which occupies the entire section.

THEOREM 3.4.1. *Let (Λ, d) be a finitely aligned k-graph. Then the following are equivalent:*

(1) *Λ is aperiodic.*
(2) *Every non-zero ideal of $C^*(\Lambda)$ contains a vertex projection.*
(3) *The $\partial\Lambda$-representation π_S of $C^*(\Lambda)$ is faithful.*

The implication (1) $\Rightarrow$ (2) is a direct consequence of what is known as the Cuntz-Krieger Uniqueness Theorem, which we prove in Subsection 3.4.1 (see Theorem 3.4.4). The implication (2) $\Rightarrow$ (3) was established in Remark 3.2.7. Thus we begin this section by establishing (3) $\Rightarrow$ (1). We first prove two technical lemmas.

LEMMA 3.4.2. *Let Λ be a finitely aligned k-graph and fix $v \in \Lambda^0$. Suppose that Λ has local periodicity m, n at v. If $x \in v(\partial\Lambda)$ and $i \in \{1, \ldots, k\}$ satisfy $d(x)_i < \infty$, then $m_i = n_i$.*

PROOF. Suppose $x \in v(\partial\Lambda)$ and $i \in \{1, \ldots, k\}$ satisfy $d(x)_i < \infty$. Since $\sigma_m(x) = \sigma_n(x)$ we have $d(x) - m = d(x) - n$. In particular $\big(d(x) - m\big)_i = \big(d(x) - n\big)_i$ and it follows that since $d(x)_i < \infty$ we must have $m_i = n_i$. ■

LEMMA 3.4.3. *Let Λ be a finitely aligned k-graph. Suppose Λ has local periodicity m, n at v. Then there exist $\mu, \nu, \alpha \in \Lambda$ such that $r(\mu) = r(\nu) = v$, $s(\mu) = s(\nu) = r(\alpha)$, $d(\mu) = m$, $d(\nu) = n$ and $\mu\alpha z = \nu\alpha z$ for all $z \in s(\alpha)\partial\Lambda$.*

PROOF. Fix $x \in v(\partial\Lambda)$. By definition of local periodicity m, n at v we know that $m \vee n \leq d(x)$. Let $\mu = x(0, m)$, $\nu = x(0, n)$, $\alpha = x(m, m \vee n)$ and fix $z \in s(\alpha)\partial\Lambda$. Since $(\mu\alpha z)(0, n) = \big(x(0, m \vee n)z\big)(0, n) = x(0, n) = \nu$ and $d(\mu) = m$, we have

$$\mu\alpha z = (\mu\alpha z)(0, n)\sigma_n(\mu\alpha z) = \nu\sigma_n(\mu\alpha z).$$

Then the definition of local periodicity m, n at v implies

$$\mu\alpha z = \nu\sigma_m(\mu\alpha z) = \nu\alpha z. \qquad \blacksquare$$

We now prove (3) $\Rightarrow$ (1) in Theorem 3.4.1. For this we recall from page 31 that the gauge action $\gamma : \mathbb{T}^k \to \mathrm{Aut}\big(C^*(\Lambda)\big)$ is defined, using multi-index notation, by $\gamma_z(s_\lambda) = z^{d(\lambda)}s_\lambda$ where $z^m \in \mathbb{T}$.

PROOF OF THEOREM 3.4.1 $\big((3) \Rightarrow (1)\big)$. For this implication we prove the contrapositive statement. Suppose Λ is not aperiodic. We must show that the $\partial\Lambda$-representation π_S is not faithful. By Proposition 3.3.7, there exists $v \in \Lambda^0$ and distinct $m, n \in \mathbb{N}^k$ such that Λ has local periodicity m, n at $v \in \Lambda^0$. Then Lemma 3.4.3 implies that there exist $\mu, \nu, \alpha \in \Lambda$ such that $r(\mu) = r(\nu)$, $s(\mu) = s(\nu)$, $d(\mu) = m$, $d(\nu) = n$ and $\mu\alpha z = \nu\alpha z$ for all $z \in s(\alpha)(\partial\Lambda)$.

We claim that $a := s_{\mu\alpha}s_{\mu\alpha}^* - s_{\nu\alpha}s_{\mu\alpha}^* \in \ker(\pi_S) \setminus \{0\}$ and hence π_S is not faithful. The remainder of this proof follows the same lines as (iii) $\Rightarrow$ (i) in the proof of [**36**, Proposition 3.5]. We provide the details for completeness.

We first check that $a \neq 0$. Since $n \neq m$, there exists $\omega \in \mathbb{T}^k$ such that $\omega^{n-m} = -1$. Then

$$(\mathrm{id} + \gamma_\omega)(a) = a + \gamma_\omega(a) = (s_{\mu\alpha}s_{\mu\alpha}^* - s_{\nu\alpha}s_{\mu\alpha}^*) + (\omega^0 s_{\mu\alpha}s_{\mu\alpha}^* - \omega^{(n+d(\alpha))-(m+d(\alpha))}s_{\nu\alpha}s_{\mu\alpha}^*).$$

Since $\omega^{(n+d(\alpha))-(m+d(\alpha))} = \omega^{n-m} = -1$ it follows from the equation above that $(\mathrm{id} + \gamma_\omega)(a) = 2s_{\mu\alpha}s_{\mu\alpha}^*$. Recall that Remark 3.2.7 ensures that $s_{\mu\alpha} \neq 0$. Thus it follows that $(\mathrm{id} + \gamma_\omega)(a) \neq 0$ and hence $a \neq 0$.

To complete the proof we show $\pi_S(a) = 0$. Let $\{\zeta_y : y \in \partial\Lambda\}$ be the standard orthonormal basis for $\ell^2(\partial\Lambda)$ and fix $x \in \partial\Lambda$. We calculate

$$\pi_S(a)\zeta_x = (S_{\mu\alpha}S_{\mu\alpha}^* - S_{\nu\alpha}S_{\mu\alpha}^*)\zeta_x = S_{\mu\alpha}S_{\mu\alpha}^*\zeta_x - S_{\nu\alpha}S_{\mu\alpha}^*\zeta_x. \tag{3.4.1}$$

We consider two cases: either $x = \mu\alpha z$ for some $z \in \partial\Lambda$ or $x \neq \mu\alpha z$ for all $z \in \partial\Lambda$. If $x = \mu\alpha z$ for some $z \in \partial\Lambda$ then $x = \nu\alpha z$ by choice of μ, ν and α. Thus (3.4.1) yields $\pi_S(a)\zeta_x = \zeta_x - \zeta_x = 0$. On the other hand, if $x \neq \mu\alpha z$ for all $z \in \partial\Lambda$ then $\pi_S(a)\zeta_x = 0 - 0 = 0$. In both cases $\pi_S(a)\zeta_x = 0$, completing the proof. $\blacksquare$

The remainder of this section is devoted to proving (1) $\Rightarrow$ (2).

3.4.1. The Cuntz-Krieger Uniqueness Theorem. We begin by stating our version of the Cuntz-Krieger Uniqueness Theorem.

THEOREM 3.4.4 (The Cuntz-Krieger Uniqueness Theorem). *Suppose Λ is an aperiodic, finitely aligned k-graph. Suppose also that π is a representation of $C^*(\Lambda)$ such that $\pi(s_v) \neq 0$ for every $v \in \Lambda^0$. Then π is injective.*

The goal of this subsection is a proof of Theorem 3.4.4. For this we first require a series of technical results.

LEMMA 3.4.5. *Suppose Λ be a finitely aligned k-graph and let $\alpha, \beta \in \Lambda$ such that* $\mathrm{MCE}(\alpha, \beta) = \emptyset$. *Then for all $\tau \in s(\alpha)\Lambda$ we have* $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$.

PROOF. Suppose for contradiction that there exists a path $\tau \in s(\alpha)\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) \neq \emptyset$. Let $\lambda \in \mathrm{MCE}(\alpha\tau, \beta\tau)$. Then $\lambda\big(0, d(\alpha) \vee d(\beta)\big) \in \mathrm{MCE}(\alpha, \beta)$, contradicting the definition of α and β. ■

In the following lemma, we bootstrap the definition of aperiodicity, up from a condition on a pair of distinct paths, to a condition on an arbitrary finite subset H of Λv.

LEMMA 3.4.6. *Suppose Λ is an aperiodic, finitely aligned k-graph. Fix $v \in \Lambda^0$ and let H be is a finite subset of Λv. Then there exists $\tau \in v\Lambda$ such that* $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$ *for all distinct $\alpha, \beta \in H$.*

PROOF. We prove this result by induction on H. For a base case we consider $|H| = 2$. In this case the claim of Lemma 3.4.6 is precisely that Λ is aperiodic. Thus this case is trivially true.

Now suppose $p \geq 2$ and that whenever $H \subseteq \Lambda v$ has at most p elements, there exists $\tau \in v\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$ for all distinct $\alpha, \beta \in H$. Fix $H \subseteq \Lambda v$ with $|H| = p + 1$ and fix $\alpha \in H$. List $H \setminus \{\alpha\}$ as $\{\beta_1, \ldots, \beta_k\}$. Then the inductive hypothesis implies that there exists $\tau_\alpha \in v\Lambda$ such that $\mathrm{MCE}(\beta_i\tau_\alpha, \beta_j\tau_\alpha) = \emptyset$ for all distinct $i, j \leq k$.

Since $\alpha \neq \beta_1$, we have $\alpha\tau_\alpha \neq \beta_1\tau_\alpha$. Then the definition of aperiodicity implies that there exists $\tau_1 \in s(\tau_\alpha)\Lambda$ such that $\mathrm{MCE}(\alpha\tau_\alpha\tau_1, \beta_1\tau_\alpha\tau_1) = \emptyset$. An identical argument on $\alpha\tau_\alpha\tau_1$ and $\beta_2\tau_\alpha\tau_1$ shows that there exists $\tau_2 \in s(\tau_1)\Lambda$ satisfying $\mathrm{MCE}(\alpha\tau_\alpha\tau_1\tau_2, \beta_2\tau_\alpha\tau_1\tau_2) = \emptyset$. By repeating this process we obtain paths $\tau_1, \ldots, \tau_p$ such that τ_1 is as above, and for each $1 < i \leq p$ we have $\tau_i \in s(\tau_{i-1})\Lambda$ which satisfy

$$\mathrm{MCE}(\alpha\tau_\alpha\tau_1 \ldots \tau_i, \beta_i\tau_\alpha\tau_1 \ldots \tau_i) = \emptyset.$$

Let $\tau = \tau_\alpha\tau_1 \ldots \tau_p \in v\Lambda$ and fix distinct $\mu, \nu \in H$. We complete the proof by showing $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. Either $\mu, \nu \in H \setminus \{\alpha\}$ or one of μ, ν is equal to α. If $\mu, \nu \in H \setminus \{\alpha\}$ then $\mu = \beta_i$ and $\nu = \beta_j$ for distinct $i, j \leq p$. Thus by definition

of τ_α we have $\mathrm{MCE}(\mu\tau_\alpha, \nu\tau_\alpha) = \emptyset$. Since $\tau_\alpha = \tau\big(0, d(\tau_\alpha)\big)$, Lemma 3.4.5 implies $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. Now suppose one of μ, ν is equal to α. Without loss of generality we suppose $\mu = \alpha$. Then there exists $i \leq p$ such that $\nu = \beta_i$. By construction we have $\mathrm{MCE}(\mu\tau_\alpha\tau_1 \dots \tau_i, \nu\tau_\alpha\tau_1 \dots \tau_i) = \emptyset$. Since $\tau_\alpha\tau_1 \dots \tau_i = \tau\big(0, d(\tau_\alpha\tau_1 \dots \tau_i)\big)$, Lemma 3.4.5 implies that $\mathrm{MCE}(\mu\tau, \nu\tau) = \emptyset$. ■

Our proof of the Cuntz-Krieger Uniqueness Theorem follows the strategy of Raeburn, Sims and Yeend in [**35**, Subsection 4.2]. Our main job is to show how to substitute the use of Condition (B), from [**35**, Theorem 4.5], with our definition of aperiodicity. The following lemma will allow us to do this.

Lemma 3.4.7. *Let $\{t_\lambda : \lambda \in \Lambda\}$ be a Cuntz-Krieger Λ-family such that $t_v \neq 0$ for each $v \in \Lambda^0$. Fix $v \in \Lambda^0$ and let H be a finite subset of Λv. Fix $N \in \mathbb{N}^k$ and a linear combination $a = \sum_{\mu,\nu\in H} a_{\mu,\nu} t_\mu t_\nu^*$ such that whenever $a_{\mu,\nu} \neq 0$ we have $\mu \in \Lambda^{\leq N}$. Let $a_0 = \sum_{\mu,\nu\in H, d(\mu)=d(\nu)} a_{\mu,\nu} t_\mu t_\nu^*$. Then $\|a_0\| \leq \|a\|$.*

Remark 3.4.8. Fix $n \in \mathbb{N}^k$. If $\alpha, \beta \in \Lambda$ satisfy $s(\alpha) = s(\beta)$, $d(\alpha) = d(\beta)$ and $\alpha \in \Lambda^{\leq n}$, then it follows that $\beta \in \Lambda^{\leq n}$. To see this first observe that if $d(\alpha) = n$ then the result is trivial. So suppose $d(\alpha) < n$. Then for each $i \in \{1, \dots, k\}$ such that $d(\beta)_i < n_i$ we have $d(\alpha) < n_i$ and hence the definition of $\Lambda^{\leq n}$ forces $s(\alpha)\Lambda^{e_i} = \emptyset$. Then

$$d(\beta) < n_i \Rightarrow d(\alpha) < n_i \Rightarrow s(\beta)\Lambda^{e_i} = s(\alpha)\Lambda^{e_i} = \emptyset.$$

Thus $\beta \in \Lambda^{\leq n}$ as claimed.

In the following proofs we index matrices by a finite set rather than by a string of integers. We do this in the following way: Suppose S is a countable set, then we list S as $\{s_1, \dots, s_{|S|}\}$ and define $M_S(\mathbb{C})$ to be $M_{|S|}(\mathbb{C})$.

Proof of Lemma 3.4.7. By Lemma 3.4.6, there exists $\tau \in v\Lambda$ such that $\mathrm{MCE}(\alpha\tau, \beta\tau) = \emptyset$ for all distinct $\alpha, \beta \in H$. For each $n \leq N$, let

$$Q_n = \sum_{\rho\in H\cap\Lambda^{\leq N}\cap\Lambda^n} t_{\rho\tau} t_{\rho\tau}^*$$

and define $Q : C^*(\Lambda) \to C^*(\Lambda)$ by

$$Q(b) = \sum_{n\leq N} Q_n b Q_n.$$

For $n \leq N$ and distinct $\rho, \rho' \in H \cap \Lambda^{\leq N}$, Lemma 2.2.3 (ii) implies that $t_{\rho\tau} t_{\rho\tau}^* t_{\rho'\tau} t_{\rho'\tau}^* = 0$. Hence each Q_n is a projection and distinct Q_n are mutually orthogonal. Thus $Q\big(C^*(\Lambda)\big) = \bigoplus_{n\leq N} Q_n C^*(\Lambda) Q_n$. We claim that Q is norm decreasing. Fix $b \in C^*(\Lambda)$ and calculate

$$\|Q(b)\| = \Big\| \bigoplus_{n\leq N} Q_n b Q_n \Big\| = \max_{n\leq N}\{\|Q_n b Q_n\|\} \leq \max_{n\leq N}\{\|b\|\} = \|b\|.$$

Next we show that $\|Q(a_0)\| = \|a_0\|$. Let $F = \{\alpha, \beta \in H : d(\alpha) = d(\beta), \alpha \in \Lambda^{\leq N}\}$ and $G = \operatorname{span}\{t_\alpha t_\beta^* : \alpha, \beta \in F\}$. Recall that by definition we have $a_0 \in G$. Let the $\theta_{\alpha,\beta}$ denote the standard matrix units of $M_{H\cap\Lambda^{\leq N}\cap\Lambda^n}(\mathbb{C})$. We claim that there is an isomorphism of G onto $\bigoplus_{n\leq N} M_{H\cap\Lambda^{\leq N}\cap\Lambda^n}(\mathbb{C})$ which takes $t_\alpha t_\beta^*$ to $\theta_{\alpha,\beta}$.

For each $n \leq N$ let $F_n = \{t_\alpha t_\beta^* : \alpha, \beta \in F \cap \Lambda^n\}$. We show that F_n is a set of matrix units. Fix $n \leq N$ and $t_\alpha t_\beta^*, t_\mu t_\nu^* \in F_n$. Then $(t_\mu t_\nu^*)^* = t_\nu t_\mu^*$ and Lemma 2.2.3 (ii) implies $t_\alpha(t_\beta^* t_\mu)t_\nu^* = \delta_{\beta,\mu} t_\alpha t_\nu^*$. To see that each element $t_\alpha t_\beta^*$ of F_n is nonzero we calculate

$$\|t_\alpha t_\beta^*\|^2 = \|(t_\alpha t_\beta^*)^* t_\alpha t_\beta^*\| = \|t_\alpha t_{s(\alpha)} t_\alpha\| = \|t_\alpha t_\alpha^*\| = \|t_\alpha^*\|^2 = \|t_\alpha\|^2. \tag{3.4.2}$$

We now observe from (CK3) that $t_\lambda^* t_\lambda = t_{s(\lambda)}$ for all $\lambda \in \Lambda$. Since $t_u \neq 0$ for each $u \in \Lambda^0$ it follows that each $t_\lambda \neq 0$. Then (3.4.2) implies each $t_\alpha t_\beta^* \in F_n$ is nonzero. Hence F_n is a set of non-zero matrix units.

Since $\Theta_{\alpha,\beta}$ are the standard matrix units of $M_{H\cap\Lambda^{\leq N}\cap\Lambda^n}(\mathbb{C})$, [**32**, Proposition A.5] implies there exists an injective homomorphism ϕ of $M_{H\cap\Lambda^{\leq N}\cap\Lambda^n}(\mathbb{C})$ into $C^*(\Lambda)$ such that $\phi(\theta_{\alpha,\beta}) = t_\alpha t_\beta^*$. Since the range of a C^*-homomorphism is a C^*-algebra, ϕ defines an isomorphism of $M_{H\cap\Lambda^{\leq N}\cap\Lambda^n}(\mathbb{C})$ onto its range $G_n := \operatorname{span} F_n$.

We now show that G is the direct sum of the G_n: For distinct $m, n \leq N$ fix $t_\alpha t_\beta^* \in F_m$ and $t_\mu t_\nu^* \in F_n$. Remark 3.4.8 implies $\beta \in \Lambda^{\leq N}$. Since $m \neq n$ we have $|\beta| \neq |\mu|$, so Lemma 2.2.3 (ii) implies $t_\alpha t_\beta^* t_\mu t_\nu^* = 0$ and hence $G_m G_n = 0$. Thus

$$G = \bigoplus_{n\leq N} G_n \cong \bigoplus_{n\leq N} M_{H\cap\Lambda^{\leq N}\cap\Lambda^n}(\mathbb{C}).$$

Fix $\alpha, \beta \in H$ such that $a_{\alpha,\beta} \neq 0$. By hypothesis $\alpha \in \Lambda^{\leq N}$. Hence

$$\begin{aligned} Q(t_\alpha t_\beta^*) &= \sum_{n\leq N} Q_n(t_\alpha t_\beta^*)Q_n \\ &= \sum_{n\leq N} \Big(\sum_{\rho\in H\cap\Lambda^{\leq N}\cap\Lambda^n} t_{\rho\tau}t_{\rho\tau}^*\Big)(t_\alpha t_\beta^*)\Big(\sum_{\rho\in H\cap\Lambda^{\leq N}\cap\Lambda^n} t_{\rho\tau}t_{\rho\tau}^*\Big) \\ &= \sum_{n\leq N} \Big(\sum_{\rho,\rho'\in H\cap\Lambda^{\leq N}\cap\Lambda^n} t_{\rho\tau}t_\tau^*(t_\rho^* t_\alpha)(t_\beta^* t_{\rho'})t_\tau t_{\rho'\tau}^*\Big) \end{aligned} \tag{3.4.3}$$

Since $\alpha, \beta \in \Lambda^{\leq N}$ and each $\rho, \rho' \in \Lambda^{\leq N}$, Lemma 2.2.3 (ii) implies a term in the sum above is nonzero if and only if $\rho = \alpha$ and $\rho' = \beta$. Hence (3.4.3) simplifies to

$$\begin{aligned} Q(t_\alpha t_\beta^*) &= \sum_{\rho'\in H\cap\Lambda^{\leq N}\cap\Lambda^{d(\alpha)}} t_{\alpha\tau}(t_{\beta\tau}^* t_{\rho'\tau})t_{\rho'\tau}^* \\ &= t_{\alpha\tau}t_{\beta\tau}^*. \end{aligned} \tag{3.4.4}$$

Moreover, for $t_\alpha t_\beta^*, t_\mu t_\nu^* \in \bigcup_{n\leq N} F_n$,

$$Q(t_\alpha t_\beta^*)Q(t_\mu t_\nu^*) = t_{\alpha\tau}t_{\beta\tau}^* t_{\mu\tau}t_{\nu\tau}^* = \delta_{\beta,\mu} t_{\alpha\tau}t_{\nu\tau}^*.$$

In other words, the set $\{Q(t_\alpha t_\beta) : \alpha, \beta \in F\}$ is also a set of non-zero matrix units. Thus Q maps non-zero matrix units to non-zero matrix units and hence the map $t_\alpha t_\beta^* \mapsto t_{\alpha\tau} t_{\beta\tau}^*$ defines an isomorphism between G and $Q(G)$. Thus $Q|_G$ is an isomorphism of G onto $Q(G)$ as claimed. Hence $\|Q(a_0)\| = \|a_0\|$.

We now show that $Q(a_0) = Q(a)$. To do this it suffices to show $Q(t_\alpha t_\beta^*) = 0$ whenever $d(\alpha) \neq d(\beta)$. Suppose $d(\alpha) \neq d(\beta)$. Recall from (3.4.4) that

$$Q(t_\alpha t_\beta^*) = \sum_{\rho' \in H \cap \Lambda^{\leq N} \cap \Lambda^{d(\alpha)}} t_{\alpha\tau}(t_{\beta\tau}^* t_{\rho'\tau}) t_{\rho'\tau}^*.$$

Since $d(\alpha) \neq d(\beta)$, for each $\rho' \in H \cap \Lambda^{\leq N} \cap \Lambda^{d(\alpha)}$ we have that $d(\beta) \neq d(\rho')$ and hence $\beta \neq \rho'$. Since $\beta \in H$, for every $\rho' \in H \cap \Lambda^{\leq N} \cap \Lambda^{d(\alpha)}$ the definition of τ forces $\mathrm{MCE}(\beta\tau, \rho'\tau) = \emptyset$. Then (CK3) implies $t_{\beta\tau}^* t_{\rho'\tau} = 0$ and hence $Q(t_\alpha t_\beta^*) = 0$.

To complete the proof we calculate $\|a_0\| = \|Q(a_0)\| = \|Q(a)\| \leq \|a\|$. ■

To prove Theorem 3.4.4 we also require the following technical lemma.

Lemma 3.4.9. *Suppose A is a C^*-algebra and X is a dense subset of A. Let Φ be a norm-decreasing linear map from A to A and let $\pi : A \to B$ be a homomorphism of A such that $\big\|\pi\big(\Phi(x)\big)\big\| \leq \|\pi(x)\|$ for all $x \in X$. Then*

$$\big\|\pi\big(\Phi(a)\big)\big\| \leq \|\pi(a)\| \text{ for all } a \in A.$$

Proof. Fix $a \in A$ and $\varepsilon > 0$. Since X is dense in A, there exists $x \in X$ such that $\|x - a\| < \frac{\varepsilon}{2}$. We calculate

$$\big\|\pi\big(\Phi(a)\big)\big\| = \big\|\pi\big(\Phi(a)\big) - \pi\big(\Phi(x)\big) + \pi\big(\Phi(x)\big)\big\| \leq \big\|\pi\big(\Phi(a-x)\big)\big\| + \big\|\pi\big(\Phi(x)\big)\big\|.$$

Since π is a homomorphism between C^*-algebras, it is norm decreasing. Since Φ is also norm-decreasing we have $\big\|\pi\big(\Phi(a-x)\big)\big\| < \frac{\varepsilon}{2}$. Since $\big\|\pi\big(\Phi(x)\big)\big\| \leq \|\pi(x)\|$ by hypothesis, we calculate

$$\big\|\pi\big(\Phi(a)\big)\big\| \leq \frac{\varepsilon}{2} + \|\pi(x)\| = \frac{\varepsilon}{2} + \|\pi(x) - \pi(a) + \pi(a)\|.$$

Then

$$\big\|\pi\big(\Phi(a)\big)\big\| \leq \frac{\varepsilon}{2} + \|\pi(x-a)\| + \|\pi(a)\|.$$

Since π is norm decreasing we have that

$$\big\|\pi\big(\Phi(a)\big)\big\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \|\pi(a)\| = \|\pi(a)\| + \varepsilon. \tag{3.4.5}$$

Since (3.4.5) holds for all $\varepsilon > 0$, it follows that $\big\|\pi\big(\Phi(a)\big)\big\| \leq \|\pi(a)\|$. ■

We now recall the following proposition of Raeburn, Sims and Yeend [**35**]. In the following proposition, the function $\Phi : C^*(\Lambda) \to C^*(\Lambda)^\gamma$ is defined by $\Phi(a) = \int_{\mathbb{T}^k} \gamma_z(a) dz$.

PROPOSITION 3.4.10 ([**35**, Proposition 4.1]). *Let (Λ, d) be a finitely aligned k-graph. Suppose that π is a homomorphism of $C^*(\Lambda)$ such that $\pi(s_v) \neq 0$ for all $v \in \Lambda^0$ and*

$$\big\|\pi\big(\Phi(a)\big)\big\| \leq \|\pi(a)\| \text{ for all } a \in C^*(\Lambda).$$

Then π is injective.

We now have all the results which we require to prove Theorem 3.4.4.

PROOF OF THEOREM 3.4.4. Let $\Phi : C^*(\Lambda) \to C^*(\Lambda)^\gamma$ be the map defined by $\Phi(a) = \int_{\mathbb{T}^k} \gamma_z(a)dz$. Proposition 3.4.10 shows that in order to prove Theorem 3.4.4, it suffices to show $\big\|\pi\big(\Phi(a)\big)\big\| \leq \|\pi(a)\|$ for all $a \in C^*(\Lambda)$. Lemma 3.4.9 implies that we need only prove that $\|\pi(\Phi(a))\| \leq \|\pi(a)\|$ for a in a dense subset of $C^*(\Lambda)$. Recall from Lemma 2.2.3 (iv) that $\operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$ is a dense subset of $C^*(\Lambda)$.

Fix $b \in \operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$. Since b is a finite linear combination, there exists a finite set $H \subseteq \Lambda$ and scalars $b_{\mu,\nu}$ for each pair $\mu, \nu \in H$ such that $b = \sum_{\mu,\nu \in H} b_{\mu,\nu} s_\mu s_\nu^*$. Recall that by definition of Φ we have that $\Phi(s_\mu s_\nu^*) = \delta_{d(\mu),d(\nu)} s_\mu s_\nu^*$. Let

$$\|b_0\| = \Big\| \sum_{\mu,\nu \in H, d(\mu)=d(\nu)} b_{\mu,\nu} \pi(s_\mu s_\nu^*) \Big\|.$$

Then

$$\|b_0\| = \Big\| \pi\Big(\Phi\big(\sum_{\mu,\nu \in H} b_{\mu,\nu} s_\mu s_\nu^*\big)\Big)\Big\|.$$

Hence to prove the theorem it is enough to verify that

$$\|\pi(b_0)\| \leq \|\pi(b)\|. \tag{3.4.6}$$

Using detailed notation introduced throughout [**35**, Section 3], Raeburn, Sims and Yeend define a projection P_v as the sum of a specific set of range projections of v in [**35**, page 226]. They then show in [**35**, Lemma 4.10 (1)] that $P_v b$ is in the span of elements of the form $s_{\mu\mu'\lambda_{\mu\mu'}} s^*_{\nu\mu'\lambda_{\mu\mu'}}$. Moreover, in the last paragraph of the proof of [**35**, Proposition 4.7], the authors show that there exists $N \in \mathbb{N}^k$ such that each $\mu\mu'\lambda_{\mu\mu'}$ and $\nu\mu'\lambda_{\mu\mu'}$ are elements of $\Lambda^{\leq N}$. Then Lemma 3.4.7 with $a = P_v b$ implies that $\|P_v\pi(b_0) \leq P_v\pi(b)\|$. Moreover, [**35**, Equation (4.4)] implies $\|P_v\pi(b_0)\| = \|\pi(b_0)\|$. Thus we have

$$\big\|\pi(b_0)\big\| = \big\|P_v\pi(b_0)\big\| \leq \|P_v\pi(b)\| \leq \big\|\pi(b)\big\|$$

and hence we have verified (3.4.6). ■

REMARK. In [**35**, Subsection 4.2], Raeburn, Sims and Yeend show that another condition, called Condition (B), is sufficient to prove a version of the Cuntz-Krieger Uniqueness Theorem. The authors perform detailed analysis of the core and their results are critical to the proof of Theorem 3.4.4. In fact, to prove Theorem 3.4.4

we have invoked the main technical results of [**35**, Subsection 4.2], replacing the use of Condition (B) with our definition of aperiodicity.

Proof of Theorem 3.4.1. **(1) ⇒ (2):** Proceed by contrapositive: Let I be an ideal of $C^*(\Lambda)$ and suppose $s_v \notin I$ for every $v \in \Lambda^0$. Let $\pi : C^*(\Lambda) \to C^*(\Lambda)/I$ be the quotient homomorphism. Since each $s_v \notin I$, we have $\pi(s_v) \neq 0$ for all $v \in \Lambda^0$. Theorem 3.4.4 therefore implies that π is injective and hence $I = \ker(\pi) = \{0\}$.

(2) ⇒ (3): Suppose every non-zero ideal of $C^*(\Lambda)$ contains a vertex projection. For each $v \in \Lambda^0$ we have $\pi_S(s_v) = S_v$ which is the projection onto $\overline{\operatorname{span}}\{\zeta_x : x \in v(\partial\Lambda)\}$. Remark 3.2.7 ensures that this subspace is non-zero, so $\pi_S(s_v) \neq 0$. Hence $\ker(\pi_S)$ contains no vertex projection and it follows that $\ker(\pi_S) = 0$. So π_S is faithful.

(3) ⇒ (1): Proved on page 44. ■

3.5. Applications of cofinality

In this section we characterise cofinality of Λ in terms of the ideal structure of $C^*(\Lambda)$. Together with Theorem 3.4.1 this will allow us to easily write down a proof of the main result of this chapter: Theorem 3.3.3. The first result of this section is Theorem 3.5.1. In this theorem we show that the cofinality condition we defined in Definition 3.3.2 is equivalent to an earlier condition used by Shotwell [**40**]. We then show that our cofinality condition ensures that any ideal which contains one vertex projection, must contain every vertex projection. The point is that our condition, unlike earlier formulations, is phrased using only finite paths and hence is more easily verified in examples.

Theorem 3.5.1. *Suppose Λ is a finitely aligned k-graph. Then the following are equivalent.*

(1) *Λ is cofinal;*
(2) *For each $v \in \Lambda^0$ and each $x \in \partial\Lambda$ there exists $n \leq d(x)$ such that $v\Lambda x(n) \neq \emptyset$;*
(3) *If I is an ideal of $C^*(\Lambda)$ which satisfies $s_v \in I$ for some $v \in \Lambda^0$, then $I = C^*(\Lambda)$; and*
(4) *If I is an ideal of $C^*(\Lambda)$ and $I \cap C^*(\Lambda)^\gamma \neq \{0\}$, then $I = C^*(\Lambda)$.*

In [**40**, Proposition 4.3], Shotwell proves (2) ⇔ (3). This implication will ensure that cofinality in Λ, together with aperiodicity, characterises simplicity of $C^*(\Lambda)$. As part of this thesis we have proved (2) ⇔ (3) using a different method to Shotwell. The key point is that our proof is direct. In particular, we avoid relying on the classification of gauge invariant ideals provided by Sims [**43**].

To prove Theorem 3.5.1 we require some additional results. First recall the following background from [**35**]. Suppose Λ is a k-graph. Let $\lambda \in \Lambda$ and $E \subseteq r(\lambda)\Lambda$. We define

$$\operatorname{Ext}(\lambda, E) = \bigcup_{\mu \in E} \big\{\nu\big(d(\lambda), d(\lambda) \vee d(\mu)\big) : \nu \in \operatorname{MCE}(\lambda, \mu)\big\}.$$

We call $\operatorname{Ext}(\lambda, E)$ the collection of *extensions of* λ *by* E.

Lemma 3.5.2 ([**35**, Lemma C.5]). *Let* (Λ, d) *be a finitely aligned* k*-graph, let* $v \in \Lambda^0$*,* $\lambda \in v\Lambda$ *and suppose* $E \in v\operatorname{FE}(\Lambda)$*. Then* $\operatorname{Ext}(\lambda, E) \in s(\lambda)\operatorname{FE}(\Lambda)$*.*

Theorem 3.5.1 also requires the following technical lemma.

Lemma 3.5.3. *Suppose* $K \subseteq \Lambda^0$ *is a nonempty set such that the following two conditions hold.*

(MT1) *If* $u \in K$ *and* $E \in u\operatorname{FE}(\Lambda)$ *then there exists* $\alpha \in E$ *such that* $s(\alpha) \in K$*; and*

(MT2) *if* $u \in K$ *and* $v\Lambda u \neq \emptyset$ *then* $v \in K$*.*

Then there exists $x \in \partial\Lambda$ *such that* $x(n) \in K$ *for every* $n \leq d(x)$*.*

Conditions (MT1) and (MT2) have previously appeared as two *maximal tail* conditions in the thesis [**41**, Proposition 5.5.3].

Proof of Lemma 3.5.3. In this proof we use several techniques from the proof of [**42**, Lemma 4.7]. As in [**42**, Lemma 4.7], define a function $P : (\mathbb{N}\backslash\{0\})^2 \to \mathbb{N}\backslash\{0\}$ by

$$P(m, n) := \frac{(m + n - 1)(m + n - 2)}{2} + m.$$

Then P is the position function for the following ordering of $\mathbb{N}^2 \setminus \{0\}$.

$$\begin{array}{cccc}
(1,1) & (2,1) & (3,1) & \cdots \\
(1,2) & (2,2) & (3,2) & \cdots \\
(1,3) & (2,3) & (3,3) & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

In other words, if $P(m, n) = l$, then (m, n) is the lth term in the listing

$$\{(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \ldots\}$$

of $\mathbb{N}^2$. Since each (m, n) occurs only once in the ordering, for each $l \in \mathbb{N} \setminus \{0\}$ we may define $(i_l, j_l) \in (\mathbb{N} \setminus \{0\})^2$ be the unique pair such that $P(i_l, j_l) = l$.

Fix $v \in K$. We claim there exists a sequence of paths $(\lambda_l)_{l=1}^{\infty} \subset v\Lambda$ and for each $l \in \mathbb{N} \setminus \{0\}$ an enumeration $s(\lambda_l)\,\mathrm{FE}(\Lambda) = \{E_{l,j} : j \geq 1\}$ of the finite exhaust sets with range $s(\lambda_l)$ such that

(i) $\lambda_{l+1}\big(0, d(\lambda_l)\big) = \lambda_l$ for every $l \geq 1$; and
(ii) $\lambda_{l+1}\big(d(\lambda_{i_l}), d(\lambda_{l+1})\big) \in E_{i_l,j_l}\Lambda K$ for all $l \geq 1$.

To prove the claim we proceed by induction on l. First consider $l = 0$ and define $\lambda_{l+1} = \lambda_l = v$. Since Λ is countable, the set $v\Lambda$ is countable and hence $v\,\mathrm{FE}(\Lambda)$ is countable. Fix a listing $\{E_{1,j} : j \geq 1\}$ of $v\,\mathrm{FE}(\Lambda)$. Then for $l = 0$, conditions (i) and (ii) are trivially satisfied because $0 < 1$.

For our inductive step we first fix $l = L \geq 1$. Suppose $(\lambda_n)_{n=1}^{L} \subset v\Lambda$ is a sequence of paths and for each $n < L$ the set $\{E_{n,j} : j \geq 1\}$ is an enumeration of $s(\lambda_n)\,\mathrm{FE}(\Lambda)$. Suppose also that for each $n < L$, the path λ_n together with the enumeration $\{E_{n,j} : j \geq 1\}$ satisfies conditions (i) and (ii). That is, for each $n < L$ there exist paths $\lambda_1, \ldots, \lambda_n \in v\Lambda$ and an enumeration $\{E_{n,j} : j \geq 1\}$ of $s(\lambda_n)\,\mathrm{FE}(\Lambda)$ such that conditions (i) and (ii) hold.

Fix an enumeration $\{E_{L,j} : j \geq 1\}$ of $s(\lambda_L)\,\mathrm{FE}(\Lambda)$. To prove our claim it suffices to construct $\lambda_{L+1} \in v\Lambda$ such that λ_{L+1} satisfies conditions (i) and (ii). For this we recall that by definition $P(i_L, j_L) = L$. In particular, i_L is the horizontal co-ordinate of the Lth term in the diagonal listing above. In particular $i_L \leq L$, so our inductive hypothesis ensures that the listing $\{E_{i_L,j} : j \geq 1\}$ of $s(\lambda_{i_L})\,\mathrm{FE}(\Lambda)$ has already been chosen and satisfies (i) and (ii). In particular, E_{i_L,j_L} is one of the sets in the listing $\{E_{i_L,j} : j \geq 1\}$ of $s(\lambda_{i_L})\,\mathrm{FE}(\Lambda)$.

Let $\mu = \lambda_L\big(d(\lambda_{i_L}), d(\lambda_L)\big)$ and $E = \mathrm{Ext}(\mu, E_{i_L,j_L})$. Lemma 3.5.2 implies $E \in s(\mu)\,\mathrm{FE}(\Lambda)$. Since $i_L < L$, part (i) of the inductive hypothesis implies tells us that $\lambda_L\big(0, d(\Lambda_{i_L})\big) = \lambda_{i_L}$. Hence $\lambda_L = \lambda_{i_L}\mu$. We now see from part (ii) of the inductive hypothesis that $\mu \in E_{i_L,j_L}\Lambda K$. Applying (MT1) from the statement of the lemma to $s(\mu)$ and E, we see that there exists $\alpha \in E$ such that $s(\alpha) \in K$. By definition of E there exists $\nu \in E_{i_L,j_L}$ and $\beta \in \Lambda$ such that $\mu\alpha = \nu\beta \in \mathrm{MCE}(\mu, \nu)$. Let $\lambda_{L+1} = \lambda_L\alpha$. Then λ_{L+1} trivially satisfies (i). The definition of α, together with the the fact that $\mu \in E_{i_L,j_L}\Lambda K$, implies

$$\lambda_{L+1}\big(d(\lambda_{i_L}), d(\lambda_{L+1})\big) = \mu\alpha \in E_{i_L,j_L}\Lambda K.$$

Hence λ_{L+1} satisfies (ii). This proves our earlier claim.

We use the above claim to construct an element of $\partial\Lambda$ as in the lemma. Define $m \in (\mathbb{N} \cup \{\infty\})^k$ by $m_i = \sup\{d(\lambda_l)_i : l \geq 1\}$. Recall that the definition of $(\lambda_n)_{n=1}^{\infty}$ ensures that for each $n < \infty$ we have $\lambda_n = \lambda_{n+1}\big(0, d(\lambda_n)\big)$. Thus there exists $\lambda \in v\Lambda^m$ such that $\lambda_n = \lambda\big(0, d(\lambda_n)\big)$ for every $n \leq \infty$. Example 2.2.1 and the following discussion shows that there exists a bijection between Λ^m and the graph

morphisms $x : \Omega_{k,m} \to \Lambda$. Thus we may define a graph morphism $x : \Omega_{k,m} \to \Lambda$ by $x\big(0, d(\lambda_l)\big) = \lambda_l$ for each l.

We claim that $x \in \partial\Lambda$. Fix $n \leq d(x)$ and $F \in x(n)\,\mathrm{FE}(\Lambda)$. We must show that there exist $p \leq d(x) - n$ such that $x(n, n+p) \in F$. Fix l such that $d(\lambda_l) \geq n$, let $\rho = x\big(n, d(\lambda_l)\big)$, and let $E = \mathrm{Ext}(\rho, F)$. Then Lemma 3.5.2 implies $E \in s(\lambda_l)\,\mathrm{FE}(\Lambda)$. Since $s(\lambda_l)\,\mathrm{FE}(\Lambda) = \{E_{l,j} : j \geq 1\}$ we have $E = E_{l,j}$ for some $j \geq 1$. Since $i_k < k$ for all $k > l$, there exists $k > l$ such that $(l, j) = (i_k, j_k)$. In particular, $E = E_{i_k,j_k} = E_{l,j}$. We then calculate $x\big(d(\lambda_l), d(\lambda_{k+1})\big) = x\big(d(\lambda_{i_k}), d(\lambda_{k+1})\big)$ and hence condition (ii) implies $x\big(d(\lambda_l), d(\lambda_{k+1})\big) \in E_{i_k,j_k}\Lambda K = E\Lambda K$. Thus there exists $\mu \in E$ such that $x\big(d(\lambda_l), d(\lambda_l) + d(\mu)\big) = \mu$. By definition of Ext we have that

$$E = \bigcup_{\phi \in F} \big\{\nu\big(d(\rho), d(\rho) \vee d(\phi)\big) : \nu \in \mathrm{MCE}(\rho, \phi)\big\}.$$

Since $F \in \mathrm{FE}(\Lambda)$ there exists $\phi \in F$, $\nu \in \Lambda$ such that $\phi\nu = \rho\mu$. Then the factorisation property implies that

$$x\big(n, n + d(\phi)\big) = (\rho\mu)\big(0, d(\phi)\big) = (\phi\nu)\big(0, d(\phi)\big) = \phi.$$

Thus we have found $\mu \in F$ such that $x\big(n, n + d(\mu)\big) = \mu$ and hence $x \in \partial\Lambda$.

To complete the proof we show that $x(n) \in K$ for every $n \leq d(x)$. By definition of x, for each $n \leq d(x)$ there exists l such that $d(\lambda_l) \geq n$. Since $x(n)\Lambda s(\lambda_l) \neq \emptyset$ and $s(\lambda_l) \in K$, hypothesis (MT2) forces $x(n) \in K$ as required. ■

Recall the following background from [**35**, Section 3]. Suppose Λ is a k-graph and E is a finite subset of Λ. In [**35**, Lemma 3.2], Raeburn, Sims, and Yeend show that there exists a finite set $F \subset \Lambda$ which contains E and satisfies

(3.5.1) $\lambda, \mu, \sigma, \tau \in F$, $d(\lambda) = d(\mu)$, $d(\sigma) = s(\tau)$, $s((\lambda) = s(\mu)$, and $s(\sigma) = s(\tau)$ imply $\{\lambda\alpha, \tau\beta : \mu\alpha = \sigma\beta \in \mathrm{MCE}(\mu, \sigma)\} \subset F$.

Remark 3.5.4. In the construction of the set F in the proof of [**35**, Lemma 3.2], we see that every element $\mu \in F$ has the form $\mu = \nu\nu'$ for some $\nu \in E$. To see this, notice that F is defined to be the final element in the sequence of sets $E_0, \ldots, E_{|N|}$, and by the definition of each E_i, every element in each E_i has an initial segment from E_0, and each element of E_0 has an initial segment from E.

For any set $E \subset \Lambda$, Raeburn, Sims, and Yeend defined in [**35**, Definition 3.3] the set ΠE to be the smallest set containing E which satisfies (3.5.1). In particular, ΠE is a subset of the set F as above. Thus every element $\mu \in \Pi E$ also has the form $\mu = \nu\nu'$ for some $\nu \in E$.

For the following remark and proof, recall from [**35**, Lemma 3.2] that for a set $E \subseteq \Lambda$, Raeburn, Sims and Yeend define $M^s_{\Pi E}$ to be $\overline{\mathrm{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Pi E, d(\mu) = d(\nu)\}$.

Remark. Theorem 3.5.1 has previously appeared as [**27**, Theorem 5.1]. The following proof follows the same format as the proof [**27**, Theorem 5.1] with the following exception. For the proof of the implication (3) $\Rightarrow$ (4), it was claimed in [**27**] that $M^s_{\Pi E} = \bigoplus_{v\in\Lambda^0} M^s_{(\Pi E)v}$. However this claim is false. This error has been corrected in the proof below.

Proof of Theorem 3.5.1. **(1) $\Rightarrow$ (2):** Suppose Λ is cofinal and fix $v \in \Lambda^0$ and $x \in \partial\Lambda$. Applying cofinality with $w = x(0)$ there exists $E \in x(0)\,\mathrm{FE}(\Lambda)$ such that $v\Lambda s(\alpha) \neq \emptyset$ for all $\alpha \in E$. Since $x \in \partial\Lambda$ there exists $\beta \in E$ such that $x\big(0, d(\beta)\big) = \beta$. Let $n = d(\beta)$. Then $x(n) = s(\beta)$ and hence $v\Lambda x(n) \neq \emptyset$ as required.

(2) $\Rightarrow$ (1): We prove the contrapositive statement. Suppose Λ is not cofinal. That is, suppose there exists $v, w \in \Lambda^0$ such that for each $E \in w\,\mathrm{FE}(\Lambda)$ there exists $\alpha \in E$ such that $v\Lambda s(\alpha) = \emptyset$. We must find $x \in \partial\Lambda$ such that for all $n \leq d(x)$ we have $v\Lambda x(n) = \emptyset$.

Let

$$K := \{u \in \Lambda^0 : \text{for each } E \in u\,\mathrm{FE}(\Lambda) \text{ there exists } \alpha \in E \text{ such that } v\Lambda s(\alpha) = \emptyset\}.$$

We show that K is nonempty and satisfies (MT1) and (MT2), and then appeal to Lemma 3.5.3. The set K is nonempty because $w \in K$. To check (MT1) and (MT2) we fix $u \in K$ and $E \in u\,\mathrm{FE}(\Lambda)$.

To see K satisfies (MT1) we must find $\alpha \in E$ such that $s(\alpha) \in K$. Suppose for contradiction $s(\alpha) \notin K$ for every $\alpha \in E$. Then for every $\alpha \in E$ there exists $F_\alpha \in s(\alpha)\,\mathrm{FE}(\Lambda)$ such that $v\Lambda s(\rho) \neq \emptyset$ for all $\rho \in F_\alpha$. Let $F = \{\alpha \in E : s(\alpha) \notin F_\alpha\} \subseteq E$ and $G = (E \setminus F) \cup \big(\bigcup_{\alpha\in F} \alpha F_\alpha\big)$. Then [**42**, Definition 5.2] and [**42**, Lemma 5.3], with $\mathcal{E} = \mathrm{FE}(\Lambda)$ imply that $G \in u\,\mathrm{FE}(\Lambda)$. However by construction of G we have $v\Lambda s(\lambda) \neq \emptyset$ for every $\lambda \in G$, contradicting $u \in K$. Hence K satisfies (MT1).

To check that K satisfies (MT2), suppose $v \in \Lambda^0$ and $v\Lambda u \neq \emptyset$. Fix $E \in v\,\mathrm{FE}(\Lambda)$, $\lambda \in v\Lambda u$ and let $F = \mathrm{Ext}(\lambda, E)$. Then Lemma 3.5.2 implies $F \in s(\lambda)\,\mathrm{FE}(\Lambda) = u\,\mathrm{FE}(\Lambda)$. Since $u \in K$ there exists $\alpha \in F$ such that $v\Lambda s(\alpha) = \emptyset$. By definition of F there exists $\mu \in E$, $\nu \in \mathrm{MCE}(\lambda, \mu)$ such that $\nu\big(d(\lambda), d(\lambda) \vee d(\mu)\big) = \alpha$. Hence $\nu\big(d(\mu), d(\lambda) \vee d(\mu)\big) \in s(\mu)\Lambda s(\alpha)$. Since $s(\mu)\Lambda s(\alpha) \neq \emptyset$ and $v\Lambda s(\alpha) = \emptyset$ we also have $v\Lambda s(\mu) = \emptyset$. Hence K satisfies (MT2).

Since K satisfies (MT1) and (MT2), Lemma 3.5.3 implies there exists $x \in \partial\Lambda$ such that $x(n) \in K$ for every $n \leq d(x)$. Since $\{x(n)\} \in x(n)\,\mathrm{FE}(\Lambda)$ for every $n \leq d(x)$, the definition of K then implies $v\Lambda x(n) = \emptyset$ for all n as required.

(2) $\Rightarrow$ (3): Fix an ideal I of $C^*(\Lambda)$ with $s_w \in I$ for some $w \in \Lambda^0$. Let $H = \{w : s_w \in I\}$. We show $\Lambda^0 \setminus H$ satisfies (MT1) and (MT2). Once we have done this we apply Lemma 3.5.3 to see $H = \Lambda^0$.

To check (MT1) we claim that if $v \in \Lambda^0 \setminus H$, then for $F \in v\,\mathrm{FE}(\Lambda)$ there exists $\lambda \in F$ such that $s(\lambda) \in \Lambda^0 \setminus H$. Fix $F \in v\,\mathrm{FE}(\Lambda)$ and suppose for contradiction that $s(\nu) \in H$ for all $\nu \in F$.

Since each $s(\nu) \in H$, by definition of H, each $s_{s(\nu)} \in I$. For every $\mu \in \Pi F$, Remark 3.5.4 implies $\mu = \nu\nu'$ for some $\nu \in F$. Then (CK2) implies that for every $\mu \in \Pi F$ we have $s_{s(\mu)} = s^*_{\nu'} s_{s(\nu)} s_{\nu'} = s^*_{\nu'} s_{\nu'} \in I$. Moreover, each $s_\mu s^*_\mu = s_\mu s_{s(\mu)} s^*_\mu \in I$ also. Then [**35**, Proposition 3.5] implies

$$s_v = \prod_{\mu \in v\Pi F} (s_v - s_\mu s^*_\mu) + \sum_{\mu \in v\Pi F} \Big(s_\mu s^*_\mu \prod_{\nu\nu' \in \Pi F, d(\nu')>0} (s_\nu s^*_\nu - s_{\nu\nu'} s^*_{\nu\nu'}) \Big). \tag{3.5.2}$$

Since F is exhaustive, ΠF is also exhaustive, and since $|F| < \infty$, [**35**, Lemma 3.2] implies $|\Pi F| < \infty$. Hence (CK4) implies $\Pi_{\mu \in v\Pi F}(s_v - s_\mu s^*_\mu) = 0$. Thus (3.5.2) becomes

$$s_v = \sum_{\mu \in v\Pi F} \Big(s_\mu s^*_\mu \prod_{\nu\nu' \in \Pi F, d(\nu')>0} (s_\nu s^*_\nu - s_{\nu\nu'} s^*_{\nu\nu'}) \Big).$$

Since the $s_\mu s^*_\mu, s_\nu s^*_\nu$ and $s_{\nu\nu'} s^*_{\nu\nu'}$ are elements of I, we have $s_v \in I$. However this contradicts that $v \in \Lambda^0 \setminus H$. Thus $\Lambda^0 \setminus H$ satisfies (MT1).

For (MT2) suppose $u \in \Lambda^0 \setminus H$ and $\lambda \in v\Lambda u$. Suppose for contradiction that $v \in H$. Then $s_u = s_{s(\lambda)} = s^*_\lambda s_v s_\lambda \in I$, contradicting $u \in \Lambda^0 \setminus H$. So $\Lambda^0 \setminus H$ satisfies (MT2). We now show $H = \Lambda^0$.

Suppose for contradiction that $H \neq \Lambda^0$. We apply Lemma 3.5.3 with $K = \Lambda^0 \setminus H$ to see that there exists $x \in \partial\Lambda$ such that $x(n) \in \Lambda^0 \setminus H$ for every $n \leq d(x)$. Theorem 3.5.1 (2) implies that there exists $n \leq d(x)$ such that $w\Lambda x(n) \neq \emptyset$. Fix $\mu \in w\Lambda x(n)$. Since $s_w \in I$, (CK2) any (CK3) imply that $s_{x(n)} = s^*_\mu s_w s_\mu \in I$. This contradicts that $x(n) \in \Lambda^0 \setminus H$ for every $n \leq d(x)$. Thus we must have $H = \Lambda^0$ as claimed. Hence for each $\mu \in \Lambda$ we have $s^*_\mu s_\mu = s_{s(\mu)} \in I$. Then $s_\mu = s_\mu s^*_\mu s_\mu \in I$. So Lemma 2.2.3 implies I contains all the generators of $C^*(\Lambda)$ and hence $I = C^*(\Lambda)$.

(3) $\Rightarrow$ (2): We prove the contrapositive statement. That is, we suppose that there exists $v \in \Lambda^0$, $x \in \partial\Lambda$ such that for each $n \leq d(x)$ we have $v\Lambda x(n) = \emptyset$, and prove that there exists an ideal $I \subsetneq C^*(\Lambda)$ which contains a vertex projection.

Define $T \subset \partial\Lambda$ to be the set $\{y \in \partial\Lambda : \sigma_m(y) = \sigma_n(x) \text{ for some } m, n \in \mathbb{N}^k\}$. Fix $y \in T$. That is, there exists $m, n \in \mathbb{N}^k$ such that $\sigma_m(y) = \sigma_n(x)$. Now suppose $\lambda \in \Lambda r(y)$ and $l \leq d(y)$. We claim λy and $\sigma_l(y)$ are elements of T. To see $\lambda y \in T$ we calculate $\sigma_{m+d(\lambda)}(\lambda y) = \sigma_m(y) = \sigma_n(x)$. Since $m, l \leq d(y)$ we have $m \vee l \leq d(y)$. In addition, $\sigma^m(y) = \sigma^n(x)$ implies $(m \vee l) - m \leq d\big(\sigma^n(x)\big)$. Hence we calculate

$$\sigma^{(m\vee l)-l}\big(\sigma^l(y)\big) = \sigma^{m\vee l}(y) = \sigma^{(m\vee l)-m}\big(\sigma^m(y)\big) = \sigma^{(m\vee l)-m}\big(\sigma^n(x)\big) = \sigma^{(m\vee l)-m+n}(xxxxxxx).$$

Thus we see that $\sigma_l(y) \in T$ as claimed.

Let $W = \overline{\mathrm{span}}\{\zeta_y : y \in T\} \subseteq \ell^2(\partial\Lambda)$. Then our above claim, together with Definition 3.2.5, shows that for each $\lambda \in \Lambda$ we have $S_\lambda W \subseteq W$ and $S^*_\lambda W \subseteq W$.

Lemma 3.2.6 shows that the $\{S_\lambda : \lambda \in \Lambda\}$ form a Cuntz-Krieger Λ-family, and since restriction to an invariant subspace is a homomorphism, so too does $\{S_\lambda|_W : \lambda \in \Lambda\}$.

By the universal property of $C^*(\Lambda)$ there exists a representation π of $C^*(\Lambda)$ onto $B(W)$, the bounded linear operators on W, defined by $\pi(s_\lambda) = S_\lambda$. We show that $I = \ker(\pi)$ is the required ideal. Since $\pi(s_{r(x)})\zeta_x = S_{r(x)}\zeta_x = \zeta_x \neq 0$ we see that $\pi(s_{r(x)}) \notin I$ and hence $I \neq C^*(\Lambda)$. To complete the proof of this implication it suffices to show $s_v \in I$. Fix $y \in T$ so that ζ_y is a typical basis element of W. Since $y \in T$, there exists $m, n \in \mathbb{N}^k$ such that $\sigma_m(y) = \sigma_n(x)$. Hence $y(0, m) \in r(y)\Lambda x(n) \neq \emptyset$. Since $v\Lambda x(n) = \emptyset$ we conclude that $r(y) \neq v$. Hence $\pi(s_v)\zeta_y = 0$. Since our choice of y was arbitrary it follows that $s_v \in I$.

(3) $\Rightarrow$ (4): To prove this implication we recall from the proof of [**35**, Theorem 3.1] that $C^*(\Lambda)^\gamma = \overline{\bigcup_{E\subset\Lambda \text{ finite}} M^s_{\Pi E}}$. Moreover [**35**, Lemma 3.2] implies that each $M^s_{\Pi E}$ is a finite-dimensional C^*-algebra. We also use the following notation: For $E \subset \Lambda$, denote by $s(E)$ the set $\{v \in \Lambda^0 : v = s(\lambda) \text{ for some } \lambda \in E\}$.

Suppose I is an ideal of $C^*(\Lambda)$ such that $I \cap C^*(\Lambda)^\gamma \neq \{0\}$. Since $C^*(\Lambda)^\gamma = \overline{\bigcup_{E\subset\Lambda \text{ finite}} M^s_{\Pi E}}$, there exists a finite $E \subset \Lambda$ such that $I \cap M^s_{\Pi E} \neq \{0\}$. For $v \in s(\Pi E)$ and $n \in \mathbb{N}^k$, let

$$T^{\Pi E}(n, v) = \{\nu \in v\Lambda : \text{there exists } \alpha \in \Pi E \cap \Lambda^n \text{ such that } \alpha\nu \in \Pi E\}$$

as in [**35**, Notation 3.12]. For each $E \subset \Lambda$, $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, as in [**35**, Definition 3.8] we let

$$\Theta(s)^{(\Pi E)v\cap\Lambda^n}_{\mu,\nu} = s_\mu s^*_\mu \Big(\prod_{\substack{\mu\theta\in(\Pi E)v\cap\Lambda^n \\ d(\theta)>0}} (s_\mu s^*_\mu - s_{\mu\theta} s^*_{\mu\theta}) \Big) s_\mu s^*_\nu.$$

We also let

$$M^s_{\Pi E}(n, v) := \overline{\operatorname{span}}\Big\{\Theta(s)^{(\Pi E)v\cap\Lambda^n}_{\mu,\nu} : \mu, \nu \in (\Pi E)v \cap \Lambda^n\Big\} = M^s_{(\Pi E)v\cap\Lambda^n}.$$

Then [**35**, Proposition 3.9] implies that each $M^s_{\Pi E}(n, v)$ is spanned by the matrix units $\Theta(s)^{(\Pi E)v\cap\Lambda^n}_{\mu,\nu}$ and that the $M^s_{\Pi E}(n, v)$ are mutually orthogonal. Thus

$$M^s_{\Pi E} = \bigoplus_{\substack{v\in s(\Pi E) \\ n\in\mathbb{N}^k}} M^s_{\Pi E}(n, v).$$

Moreover, [**35**, Proposition 3.13] and the uniqueness of $M_X(\mathbb{C})$ implies that

$$M^s_{\Pi E}(n, v) \cong \begin{cases} M_{(\Pi E)v\cap\Lambda^n}(\mathbb{C}) & \text{when } T^{\Pi E}(n, v) \notin \mathrm{FE}(\Lambda), \\ \{0\} & \text{otherwise.} \end{cases}$$

Since $I \cap C^*(\Lambda)^\gamma \neq \{0\}$ we have that $I \cap M^s_{\Pi E}(n, v) \neq \{0\}$ for some E, n, v. Since $M^s_{\Pi E}(n, v) \cong M_{(\Pi E)v\cap\Lambda^n}(\mathbb{C})$ is simple, it then follows that $M^s_{\Pi E}(n, v) \subseteq I$. Fix

$\lambda \in (\Pi E)v \cap \Lambda^n$. Then [**35**, Lemma 3.15] implies that there exists $\psi \in s(\lambda)\Lambda$ such that

$$s_{\lambda\psi}s_{\lambda\psi}^* = s_{\lambda\psi}s_{\lambda\psi}^*s_\lambda\Big(\prod_{\theta\in T^{\Pi E}(n,v)}(s_v - s_\theta s_\theta^*)\Big)s_\lambda^* \in I.$$

Hence $s_{s(\psi)} = s_{\lambda\psi}^*(s_{\lambda\psi}s_{\lambda\psi}^*)s_{\lambda\psi}^* \in I$ as required.

(4) $\Rightarrow$ (3): This implication is trivial because $s_v \in C^*(\Lambda)^\gamma$ for every $v \in \Lambda^0$. ■

To complete this section we prove Theorem 3.3.3. This proof is a straightforward application of Theorem 3.4.1 together with Theorem 3.5.1.

Proof of Theorem 3.3.3. ($\Rightarrow$). Suppose $C^*(\Lambda)$ is simple. Then (3) $\Rightarrow$ (1) of Theorem 3.5.1 implies Λ is cofinal and (2) $\Rightarrow$ (1) of Theorem 3.4.1 implies Λ is aperiodic.

($\Leftarrow$). Suppose Λ is aperiodic and cofinal. Fix a nonzero ideal I of $C^*(\Lambda)$. Then (1) $\Rightarrow$ (2) of Theorem 3.4.1 implies I contains a vertex projection. It then follows from (1) $\Rightarrow$ (3) of Theorem 3.5.1 that $I = C^*(\Lambda)$. Hence $C^*(\Lambda)$ is simple. ■

3.6. Interpretation of cofinality in special cases

As the theory of higher-rank graph algebras developed from row-finite graphs with no sources [**23**] to the finitely-aligned k-graphs we study here, much of the focus has been on proving versions of the Cuntz-Krieger Uniqueness Theorem. In particular, several different formulations of aperiodicity have been used. However, prior to the research contained in this chapter, the definition of cofinality has largely remained the same. Here we explore how our new definition of cofinality simplifies in a number of special cases.

3.6.1. Row-finite locally convex higher-rank graphs. We use the following notation in this subsection: For $n \in \mathbb{N}^k$ write $|n|$ for $\sum_{i=1}^k n_i$. The following lemma shows how Definition 3.3.2 simplifies in the special case of row-finite locally convex k-graphs.

Proposition 3.6.1. *Suppose Λ is a row-finite, locally convex k-graph. Then Λ is cofinal if and only if for every $v, w \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for each $\alpha \in w\Lambda^{\leq n}$.*

Remark 3.6.2. If Λ has no sources, then each $\Lambda^{\leq n} = \Lambda^n$. In this case Proposition 3.6.1 implies Λ is cofinal if and only if for every $v, w \in \Lambda^0$ there exists $n \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for each $\alpha \in w\Lambda^n$.

To prove Proposition 3.6.1 we use the following result.

Lemma 3.6.3. *Suppose Λ is a locally convex k-graph and $\lambda \in \Lambda^{\leq m+n}$ for some $m, n \in \mathbb{N}^k$. Then $\mu = \lambda\big(0, m \wedge d(\lambda)\big)$ and $\nu = \lambda\big(m \wedge d(\lambda), d(\lambda)\big)$ are the unique paths such that $\mu \in \Lambda^{\leq m}$, $\nu \in \Lambda^{\leq n}$ and $\lambda = \mu\nu$.*

Proof. We first establish the existence of $\alpha \in \Lambda^{\leq m}$ and $\beta \in \Lambda^{\leq n}$ such that $\lambda = \alpha\beta$. For this we proceed by induction on $|n|$. For a base case consider $|n| = 1$. In this case we are precisely in the case of [**34**, Lemma 3.12]. So suppose the lemma holds for $|n| = l \geq 1$. We now consider $|n| = l + 1$.

Since $|n| \geq 1$ there exists $j \in \{1, \ldots, k\}$ such that $n_j \geq 1$. Then [**34**, Lemma 3.12] implies $\lambda = \lambda'\lambda''$ where $\lambda' \in \Lambda^{\leq m+n-e_j}$ and $\lambda'' \in \Lambda^{\leq e_j}$. Since $|d(\lambda')| = l$, the inductive hypothesis implies there exist $\mu \in \Lambda^{\leq m}$ and $\lambda''' \in \Lambda^{\leq n-e_j}$ such that $\lambda' = \mu\lambda'''$. Then [**34**, Lemma 3.6] implies $\lambda'''\lambda'' \in \Lambda^{\leq n}$. Let $\nu = \lambda'''\lambda''$. Then $\lambda = \mu\nu$ as required.

To complete the proof we show that μ, ν are unique and defined as in the statement of the lemma. To this we end first note that if $p \leq d(\lambda)$ satisfies $\lambda(0, p) \in \Lambda^{\leq m}$, then $p = m \wedge d(\lambda)$. Suppose for contradiction that $p_i < \big(m \wedge d(\lambda)\big)_i$. Then by definition of $\Lambda^{\leq m}$ we must have $\lambda(p)\Lambda^{e_i} = \emptyset$. But $p_i < m_i$ so $p + e_i \leq d(\lambda)$ and we are left with the contradiction $\lambda(p, p + e_i) \in \lambda(p)\Lambda^{e_i}$. Thus we must have $p = m \wedge d(\lambda)$ and hence $\mu = \lambda\big(0, m \wedge d(\lambda)\big)$. The factorisation property then forces $\nu = \lambda\big(m \wedge d(\lambda), d(\lambda)\big)$. ■

We now prove Proposition 3.6.1.

Proof of Proposition 3.6.1. ($\Rightarrow$) Suppose Λ is cofinal and fix $v, w \in \Lambda^0$. Then the definition of cofinality says there exists $E \in w\,\mathrm{FE}(\Lambda)$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in E$. We must show there exists $n \in \mathbb{N}^k$ such that for every $\beta \in w\Lambda^{\leq n}$ we have $v\Lambda s(\beta) \neq \emptyset$. Consider $n := \bigvee_{\alpha \in E} d(\alpha)$ and fix $\beta \in w\Lambda^{\leq n}$. Since E is exhaustive there exists $\alpha \in E$, $\mu \in s(\alpha)\Lambda^{(d(\alpha)\vee d(\beta))-d(\alpha)}$ and $\nu \in s(\beta)\Lambda^{(d(\alpha)\vee d(\beta))-d(\beta)}$ such that $\alpha\mu = \beta\nu \in \mathrm{MCE}(\alpha, \beta)$. Since $\beta \in \Lambda^{\leq n}$ the definition of n and $\Lambda^{\leq}$ ensures that $d(\alpha) \leq d(\beta)$. Thus $\alpha\mu = \beta$, and in particular $s(\mu) = \nu = s(\beta)$. Thus $v\Lambda s(\alpha)\mu \subseteq v\Lambda s(\beta)$. Since the cofinality condition ensures $v\Lambda s(\alpha) \neq \emptyset$, it follows that $v\Lambda s(\beta) \neq \emptyset$ as required.

($\Leftarrow$) Fix $w \in \Lambda^0$. It suffices to show $w\Lambda^{\leq n} \in w\,\mathrm{FE}(\Lambda)$. Since Λ is row finite we have $|w\Lambda^{\leq n}| < \infty$. So to complete the proof we show $w\Lambda^{\leq n}$ is exhaustive. Fix $\alpha \in w\Lambda$. We must show that there exists $\mu \in w\Lambda^{\leq n}$ such that $\mathrm{MCE}(\mu, \alpha) \neq \emptyset$. Fix $\beta \in s(\alpha)\Lambda^{(n\vee d(\alpha))-d(\alpha)}$. Then [**34**, Lemma 3.9] implies $\alpha\beta \in w\Lambda^{\leq n\vee d(\alpha)}$. Hence Lemma 3.6.3 with $\lambda = \alpha\beta$ implies there exists $\mu \in \Lambda^{\leq n}$ and $\nu \in \Lambda^{\leq (n\vee d(\alpha))-n}$ such that $\alpha\beta = \mu\nu$. In particular, $\mu \in \Lambda^{\leq n}$ satisfies $\mathrm{MCE}(\mu, \alpha) \neq \emptyset$. ■

3.6.2. Directed graphs. To characterise cofinality in arbitrary directed graphs we require the following notation: Suppose E is an arbitrary directed graph. Then for each $n \in \mathbb{N}$ we define

$$X_n := \big\{\lambda \in E^* : d(\lambda) \leq n, |\lambda(m)E^1| < \infty \text{ for every } m < d(\lambda) \\ \text{and if } d(\lambda) < n \text{ then } |s(\lambda)E^1| \in \{0, \infty\}\big\}.$$

Our final result of this chapter characterises simplicity in an arbitrary directed graph.

PROPOSITION 3.6.4. *Suppose E is a directed graph. Then E is cofinal if and only if for every $v, w \in E^0$ there exists $n \in \mathbb{N}$ such that $vE^*s(\alpha) \neq \emptyset$ for each $\alpha \in wX_n$.*

PROOF. ($\Rightarrow$) Suppose Λ is cofinal and fix $v, w \in E^0$. Then the definition of cofinality tells us there exists $F \in w\,\mathrm{FE}(\Lambda)$ such that for each $\alpha \in F$ we have $vE^*s(\alpha) \neq \emptyset$. Let $n = \max\{|\alpha| : \alpha \in F\}$ and consider wX_n. We must show that $vE^*s(\mu) \neq \emptyset$ for all $\mu \in wX_n$.

Fix $\mu \in wX_n$. Since F is finite there exists a shortest $\nu \in F$ such that $\mathrm{MCE}(\mu, \nu) \neq \emptyset$. We claim $|\nu(n)E^1| < \infty$ for all $n < |\nu|$. To see this, suppose for contradiction that there exists $n < |\nu|$ such that $|\nu(n)E^1| = \infty$. Then since F is finite, there exists $e \in \nu(n)E^1$ such that $\nu(0, n)e$ is not an initial segment of any element of F. Thus $\mathrm{MCE}(\nu(0, n)e, \alpha) = \emptyset$ for every $\alpha \in F$. However, this contradicts that F is exhaustive. So $|\nu(n)E^1| < \infty$ for every $n < |\nu|$.

Next we claim $|\mu| \geq |\nu|$. Again we prove this by contradiction: Suppose $|\mu| < |\nu|$. By definition of n we have $|\mu| < |\nu| < n$. Then the definition of wX_n forces $|s(\mu)E^1| \in \{0, \infty\}$. However, $\mathrm{MCE}(\mu, \nu) \neq \emptyset$ and hence $|s(\mu)E^1| \neq 0$, forcing $|s(\mu)E^1| = \infty$. But F is finite, so there exists $f \in s(\mu)E^1$ such that $\mathrm{MCE}(\mu f, \alpha) = \emptyset$ for all $\alpha \in F$, contradicting that F is exhaustive. Thus $|\mu| \geq |\nu|$.

By definition of MCE we have $\mathrm{MCE}(\mu, \nu) = \mu = \nu\mu(|\nu|, |\mu|)$. Since $vE^*s(\nu) \neq \emptyset$ it follows immediately that $vE^*s(\mu) \neq \emptyset$.

($\Leftarrow$) Fix $v, w \in E^0$. By hypothesis there exists $n \in \mathbb{N}$ such that $vE^*s(\alpha) \neq \emptyset$ for every $\alpha \in wX_n$. To show E is cofinal it suffices to show $wX_n \in w\,\mathrm{FE}(\Lambda)$. By definition of X_n we have $|X_n| < \infty$. Thus it remains to show wX_n is exhaustive.

Fix $\lambda \in wE^*$. Then $\lambda(0) = w = \beta(0)$ for all $\beta \in wX_n$. Hence $\{p \in \mathbb{N} : \lambda(0, p) = \beta(0, p)$ for some $\beta \in wX_n\} \neq \emptyset$. Let $m = \max\{p : \lambda(0, p) = \beta(0, p)$ for some $\beta \in wX_n\}$. Then $m \leq n \wedge d(\lambda)$. To complete the proof we consider two cases: $m = n$ or $m < n$.

First suppose that $m = n$. Then by definition of m there exists $\beta \in wX_n$ such that $\lambda(0, n) = \beta(0, n) \in X_n$. Since $\beta \in X_n$ it follows by definition of X_n that $\beta(0, n) = \beta$. Since $\mathrm{MCE}(\lambda(0, n), \lambda) \neq \emptyset$ we have that $\mathrm{MCE}(\beta, \lambda) \neq \emptyset$. Suppose $m < n$. Then the definition of m tells us that there exists $\beta \in wX_n$ such that $\beta(0, m) = \lambda(0, m)$ and there is no $\beta' \in wX_n$ such that $\beta'(0, m + 1) = \lambda(0, m + 1)$. Moreover, since $m < n$, the definition of X_n tells us that this is only possible if $d(\beta) = m$. It then follows that $\mathrm{MCE}(\beta, \lambda) \neq \emptyset$. ■

CHAPTER 4

Entropy

4.1. Introduction

In this chapter we study the ergodic theory of shift spaces and C^*-algebras associated to locally finite k-graphs with no sinks or sources. One aspect of research in these areas involves studying the properties of dynamical invariants, the most famous of which is entropy. Entropy is a measure of growth within a dynamical system. In this chapter we consider several different measures of entropy.

A key starting point for this chapter is a classical definition of entropy, which we refer to as *block entropy* [**22**, Observation 7.2.10]. Applying the definition of block entropy to $(\Lambda^\infty, \sigma_p)$ gives the formula $\mathrm{h}_p(\Lambda) = \limsup_{n\to\infty} \frac{1}{n} \log |v\Lambda^{np}|$. Let ${}^t\Lambda$ be the *transpose of* Λ obtained by reversing the direction of each edge. Then $\mathrm{h}_p({}^t\Lambda) = \limsup_{n\to\infty} \frac{1}{n} \log |\Lambda^{np}v|$. When $k = 1$ it is common to write h_b for h_1. In this chapter we prove results relating to shift spaces and k-graph algebras associated to k-graphs for $k = 1$ and for $k > 1$.

Let E be a locally finite directed graph. Here we allow for the possibility that E may may contain an infinite number of vertices. In 2005, Jeong and Park [**21**] gave an upper bound for the topological entropy of the non-commutative shift map $\phi_E : C^*(E) \to C^*(E)$ in terms of the block entropy of the underlying shift space. Voiculescu's approximation entropy [**45**] is used and the results of Jeong and Park extend earlier results pertaining to finite directed graphs [**5, 7, 30, 45**] to infinite directed graphs.

Let Λ be a locally finite k-graph with only a finite number of vertices. In 2008, Skalski and Zacharias [**44**] computed, for each $p \in \mathbb{N}^k$, the topological entropy of the non-commutative shift map $\Phi_p : C^*(\Lambda) \to C^*(\Lambda)$. They also showed that the topological entropy of the $\mathbb{N}^k$ action $p \mapsto \sigma_p$ on the infinite path space Λ^∞, is zero. In this paper we develop a common generalisation of the results of Jeong and Park, and of Skalski and Zacharias to cover a class of infinite k-graphs.

We prove two main results. Suppose Λ is a locally finite k-graph and $p \in \mathbb{N}^k$. For our first main result we build on the techniques of Jeong and Park to give an upper bound for the topological entry of Φ_p in terms of the block entropy of the infinite-path space. For our second main result we give conditions on Λ which ensure that the topological entropy of the map $p \mapsto \sigma_p$, when viewed as an action of $\mathbb{N}^k$ on Λ^∞, is zero.

The outline of this chapter is as follows. In Section 4.2 we provide the notation and basic facts required to read the remainder of the chapter. Section 4.3 considers the topological entropy of 1-graph algebras; we observe and correct an error in the literature pertaining to the calculation of entropy in the C^*-algebras of infinite directed graphs. In Section 4.4 we turn our attention back to locally finite k-graphs with no sinks or sources. More specifically, in Subsection 4.4.1 we analyze the topological entropy of the non-commutative shift map. Then in Subsection 4.4.2 we consider the topological entropy of the action $\sigma : p \mapsto \sigma_p$ on the infinite-path space.

In order to calculate topological entropy of an endomorphism of a C^*-algebra A, one needs to be able to identify completely positive approximations of subsets of A. To achieve this, we compute what is known as the *rank of a completely positive approximation*: a measure of the size of a completely positive approximation. This step is quite difficult, especially when A is non-unital. The consequence of this is that often we are only able to compute upper and lower bounds for the topological entropy. However, for some examples we are able to compute exact values. It is this problem which we consider in the final section of this thesis: Section 4.5

4.2. Preliminaries

Recall from Chapter 2 that a k-graph Λ is irreducible, or strongly connected, if for every $u, v \in \Lambda^0$ we have that $u\Lambda v \neq \emptyset$. That is, there is a path from any vertex to any other vertex.

We now introduce some preliminary results which we require later on in this chapter.

Lemma 4.2.1. *Suppose B_1 and B_2 are finite dimensional C^*-algebras and let A_i be a maximal abelian C^*-subalgebra of B_i for $i = 1, 2$. Then $A_1 \oplus A_2$ is a maximal abelian C^*-subalgebra of $B_1 \oplus B_2$.*

Proof. By definition of the direct sum, $A_1 \oplus A_2$ is abelian. We must show that $A_1 \oplus A_2$ is maximal. Fix an abelian C^*-subalgebra C of $B_1 \oplus B_2$ such that $A_1 \oplus A_2 \subseteq C$. Then C is an algebra of ordered pairs (c_1, c_2) where $c_1 \in B_1, c_2 \in B_2$. We must show that $C = A_1 \oplus A_2$.

Fix $(c_1, c_2) \in C$. Then $c_1 \in B_1$ and $c_2 \in B_2$. Also fix $a_1 \in A_1$ and $a_2 \in A_2$. Then $(a_1, 0), (0, a_2) \in C$. Since C is abelian we have $(c_1, c_2)(a_1, 0) = (c_1a_1, 0) = (a_1c_1, 0)$, which implies $c_1a_1 = a_1c_1$. So if $(c_1, c_2) \in C$, then c_1 commutes with every $a \in A_1$. However, A_1 is a maximal abelian C^*-subalgebra of B_1, forcing $c_1 \in A_1$. A similar argument using c_2 and a_2 shows that $c_2 \in A_2$. Hence $(c_1, c_2) \in A_1 \oplus A_2$ implying that $C \subseteq A_1 \oplus A_2$. ■

Let A be a finite-dimensional C^*-algebra A. In this thesis we define $\operatorname{rank}(A)$ to be the dimension of a maximal abelian subalgebra of A. Lemma 4.2.1 and Lemma 4.2.2 together with the well known result of Example 4.2.3 show that rank is well defined.

LEMMA 4.2.2. *Suppose that B_1 and B_2 are finite-dimensional C^*-algebras. Then* $\operatorname{rank}(B_1 \oplus B_2) = \operatorname{rank} B_1 + \operatorname{rank} B_2$.

PROOF. Suppose B_1 and B_2 have maximal abelian subalgebras A_1 and A_2 respectively. Then Lemma 4.2.1 implies $A_1 \oplus A_2$ is a maximal abelian subalgebra of $B_1 \oplus B_2$. Thus by definition of rank and dim we have

$$\operatorname{rank}(B_1 \oplus B_2) = \dim(A_1 \oplus A_2) = \dim A_1 + \dim A_2 = \operatorname{rank} B_1 + \operatorname{rank} B_2. \qquad \blacksquare$$

EXAMPLE 4.2.3. *Fix $n \in \mathbb{N}$ and consider $M_n(\mathbb{C})$. We show that* $\operatorname{rank} M_n(\mathbb{C}) = n$. *Let A be the set of diagonal matrices of $M_n(\mathbb{C})$. Then A is an abelian C^*-subalgebra of $M_n(\mathbb{C})$ with dimension n. We claim that any other C^*-subalgebra of $M_n(\mathbb{C})$ which properly contains A is not commutative, thus proving that* $\operatorname{rank} M_n(\mathbb{C}) = n$. *Fix $b \in M_n(\mathbb{C}) \setminus A$. Then there exist distinct $k, l \leq n$ such that $b_{kl} \neq 0$. Define $c \in A$ by $c_{ij} \neq 0$ if and only if $i = j = k$, in which case $c_{kk} = 1$. We calculate $(bc)_{kl} = \sum_{x=1}^{n} b_{kx} c_{xl} = 0$ and $(cb)_{kl} = \sum_{x=1}^{n} c_{kx} b_{xl} = b_{kl} \neq 0$. So b does not commute with A. Hence A is a maximal abelian subalgebra of $M_n(\mathbb{C})$.*

For the remainder of this subsection, we introduce a measure of entropy for rank-k subshifts of the higher-rank shift space Λ^∞. In order to motivate this we first consider the situation of directed graphs. Suppose Λ is an irreducible, locally finite directed graph with no sinks. Jeong and Park propose a definition of the *block entropy of* Λ: they apply the definition of block entropy given for a general symbolic dynamical system [**22**, Observation 7.2.10] to the rank-1 shift space Λ^∞. This value is denoted $h_b(\Lambda)$ and is defined to be $\limsup_{n\to\infty} \frac{1}{n} \log |\Lambda^n v|$. A priori it is not clear that $h_b(\Lambda)$ is well defined since it would appear that the value of the entropy depends on the choice of vertex v. However, the following lemma shows that the value of the block entropy is independent of the choice vertex v. The lemma is proved for k-graphs since we work in this setting later in the chapter.

LEMMA 4.2.4. *Suppose Λ is an irreducible, locally finite k-graph with no sinks. Then for each $p \in \mathbb{N}^k$ and $u, v \in \Lambda^0$ we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log |\Lambda^{np} u| = \limsup_{n\to\infty} \frac{1}{n} \log |\Lambda^{np} v|$$

To prove Lemma 4.2.4 we use the following technical lemmas.

LEMMA 4.2.5. *Suppose (a_n) and (b_n) are sequences such that $(a_n) \to L$ and (b_n) is bounded. Then $\limsup_{n\to\infty} a_n b_n$ exists and is equal to $L \limsup_{n\to\infty} b_n$.*

Proving Lemma 4.2.5 is possible with only a basic understanding of undergraduate calculus. Nevertheless we provide a proof for completeness.

Proof of Lemma 4.2.5. Since (b_n) is bounded, by definition of the supremum, $(\sup_{m\geq n} b_m)_{n=1}^{\infty}$ is a non-increasing sequence which is bounded below. Hence $\limsup_{n\to\infty} b_n$ exists. Let $S = \limsup_{n\to\infty} b_n$. We must show $\limsup_{n\to\infty} a_n b_n = LS$.

Fix $\varepsilon > 0$. By definition of S there exists $N_1 \in \mathbb{N}$ such that $p > N_1$ implies $|\sup_{q\geq p}\{b_q\} - S| < \varepsilon$. Moreover, by definition of L there exists $N_2 \in \mathbb{N}$ such that $p > N_2$ implies $|a_p - L| \leq \frac{\varepsilon}{S}$. Let $N = \max\{N_1, N_2\}$. Then $m > N$ implies

$$(L - \frac{\varepsilon}{S})b_m \leq a_m b_m \leq (L + \frac{\varepsilon}{S})b_m.$$

Hence

$$\sup_{m\geq n}\big((L - \frac{\varepsilon}{S})b_m\big) \leq \sup_{m\geq n} a_m b_m \leq \sup_{m\geq n}\big((L + \frac{\varepsilon}{S})b_m\big).$$

Then we take limits to obtain

$$\limsup_{n\to\infty}(L - \frac{\varepsilon}{S})b_n \leq \limsup_{n\to\infty} a_n b_n \leq \limsup_{n\to\infty}(L + \frac{\varepsilon}{S})b_n.$$

But by definition we have $\limsup_{n\to\infty} b_n = S$, hence

$$(L - \frac{\varepsilon}{S})S \leq \limsup_{n\to\infty} a_n b_n \leq (L + \frac{\varepsilon}{S})S,$$

and consequently

$$(LS - \varepsilon) \leq \limsup_{n\to\infty} a_n b_n \leq (LS + \varepsilon).$$

Thus $\limsup_{n\to\infty} a_n b_n = LS$, as required. ■

Lemma 4.2.6. *Suppose Λ is a locally finite k-graph with no sinks. Fix $m, p \in \mathbb{N}^k$ and $v \in \Lambda^0$. Then*

$$\limsup_{n\to\infty} \frac{1}{n}\log|\Lambda^{m+np}v| = \limsup_{n\to\infty} \frac{1}{n}\log|\Lambda^{np}v| \tag{4.2.1}$$

Proof. Since Λ has no sinks, for every $n \in \mathbb{N}$ we have $1 \leq |\Lambda^{np}v| \leq |\Lambda^{m+np}v|$. Thus $0 \leq \log|\Lambda^{np}v| \leq \log|\Lambda^{m+np}v|$. To complete the proof we must show the reverse inequality. Consider the following two cases:

(1) The sequence $(\frac{1}{n}\log|\Lambda^{np}v|)$ is unbounded; and
(2) the sequence $(\frac{1}{n}\log|\Lambda^{np}v|)$ is bounded.

Case (1): Since $(\frac{1}{n}\log|\Lambda^{np}v|)$ is unbounded, $\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v| = \infty$. Since $|\Lambda^{np}v| \leq |\Lambda^{m+np}v|$ for every $n \in \mathbb{N}$, it follows that $(\frac{1}{n}\log|\Lambda^{m+np}v|)$ is also unbounded and hence $\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{m+np}v| = \infty$ as required.

Case (2): Suppose the sequence $(\frac{1}{n}\log|\Lambda^{np}v|)$ is bounded. We first suppose that $m = m'p$ for some $m' \in \mathbb{N}$. For $n \in \mathbb{N}$, let $a_n = \frac{m'+n}{n}$ and $b_n = \frac{1}{m'+n}\log|\Lambda^{(m'+n)p}v|$. Then $a_n \to 1$ and (b_n) is bounded by hypothesis. Hence Lemma 4.2.5 implies

$$\limsup_{n\to\infty} 1 \cdot b_n = \limsup_{n\to\infty} a_n b_n = \limsup_{n\to\infty} \frac{1}{n}\log|\Lambda^{m+np}v|.$$

Since $\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v| = \limsup_{n\to\infty}\frac{1}{m'+n}\log|\Lambda^{(m'+n)p}v|$, the equality above establishes (4.2.1).

Now suppose $m \neq \mathbb{N}p$. Let $m'' \in \mathbb{N}$ be the unique integer such that $m''p < m < (m''+1)p$. Then $|\Lambda^{(m''+n)p}v| \leq |\Lambda^{(m+n)p}v| \leq |\Lambda^{(m''+1+n)p}v|$ and hence

$$\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{(m''+n)p}v| \leq \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{m+np}v| \leq \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{(m''+1+n)p}v|.$$

The argument for the case when $m = m'p$ shows that

$$\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{(m''+n)p}v| = \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v| = \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{(m''+1+n)p}v|.$$

Hence the squeeze rule implies

$$\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{m+np}v| = \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v|. \qquad \blacksquare$$

With Lemma 4.2.6 in hand, we now return to proving Lemma 4.2.4.

Proof of Lemma 4.2.4. Fix $p \in \mathbb{N}^k$ and $u, v \in \Lambda^0$. Since Λ is irreducible and locally finite there exist $m, m' \in \mathbb{N}^k$ such that $0 < |v\Lambda^m u| < \infty$ and $0 < |u\Lambda^{m'}v| < \infty$. Then Lemma 4.2.6 implies

$$\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v| = \limsup_{n\to\infty}\frac{1}{n}|\Lambda^{m+np}v|$$

and now the factorisation property implies

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v| &\geq \limsup_{n\to\infty}\frac{1}{n}\log(|\Lambda^{np}u||u\Lambda^m v|)\\ &= \limsup_{n\to\infty}\frac{1}{n}(\log|\Lambda^{np}u| + \log|u\Lambda^m v|)\\ &= \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}u|.\end{aligned}$$

An identical argument, replacing the use of m with m', yields

$$\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}u| \geq \limsup_{n\to\infty}\frac{1}{n}\log\big(|\Lambda^{np}v||v\Lambda^{m'}u|\big) \geq \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v|.$$

Thus

$$\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v| \geq \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}u| \geq \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v|$$

which forces $\limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}v| = \limsup_{n\to\infty}\frac{1}{n}\log|\Lambda^{np}u|$ as required. $\blacksquare$

For an irreducible, locally finite k-graph Λ with no sinks we now define the *p-block entropy of* ${}^t\Lambda$ to be

$$\mathrm{h}_p({}^t\Lambda) = \limsup_{n\to\infty} \frac{1}{n} \log |\Lambda^{np} v| \text{ for any } v \in \Lambda^0. \tag{4.2.2}$$

Lemma 4.2.4 shows that the value of $\mathrm{h}_p({}^t\Lambda)$ does not depend on the choice of vertex v, so $\mathrm{h}_p({}^t\Lambda)$ is well defined.

The value $\mathrm{h}_p({}^t\Lambda)$ can be interpreted to be a directional measure of entropy of the shift space ${}^t\Lambda^\infty$. This is because each element x of ${}^t\Lambda^\infty$ is determined by the sequence $x(0,p)x(p,2p)x(2p,3p)\ldots$. Thus we may view x as an element of the 1-dimensional shift with alphabet ${}^t\Lambda^p$. Recall the definition of block entropy given on page 27. Here $k=1$ and hence the entropy $h_p({}^t\Lambda^\infty)$ is just the usual block entropy of the 1-dimensional shift space ${}^t\Lambda^\infty$.

4.2.1. Locally finite higher-rank graph algebras. In Section 4.4 we consider the class of locally-finite k-graphs which have no sinks or sources. In this special case [**35**, Proposition B.1] shows that the Cuntz-Krieger relations of Definition 2.2.2 simplify to the conditions given below.

Definition 4.2.7 ([**23**, Definition 1.5]). Let Λ be a locally finite k-graph with no sources. Then a *Cuntz-Krieger Λ-family* is a set $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

(CK1) $\{s_v : v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $s_\mu s_\nu = s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $s_\mu^* s_\mu = s_{s(\mu)}$; and
(CK4) $s_v = \sum_{\lambda \in v\Lambda^m} s_\lambda s_\lambda^*$ for every $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

Recall the following lemma of Kumjian and Pask.

Lemma 4.2.8 ([**23**, Lemma 3.1]). *Let Λ be a row-finite k-graph with no sources. Then for every $\mu, \nu \in \Lambda$ and $q \geq d(\mu) \vee d(\nu)$ we have*

$$s_\mu^* s_\nu = \sum_{\substack{\mu\alpha = \nu\beta \\ d(\mu\alpha) = q}} s_\alpha s_\beta^*.$$

Hence,

$$C^*(\{s_\lambda : \lambda \in \Lambda\}) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\} = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda, s(\mu) = s(\nu)\}.$$

Remark. When $q = d(\mu) \vee d(\nu)$ in Lemma 4.2.8 then $\mu\alpha, \nu\beta \in \mathrm{MCE}(\mu, \nu)$.

Fix a k-graph Λ and $m \in \mathbb{N}^k$. Recall from Subsection 2.2.1 the C^*-subalgebra of $C^*(\Lambda)$ defined by

$$C^*(\Lambda)^\gamma := \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu), d(\mu) = d(\nu)\};$$

This subalgebra is called the *core* of $C^*(\Lambda)$. Also recall from [**23**] the following important C^*-subalgebras of the core: for $m \in \mathbb{N}^k$, define $\mathcal{F}_m$ to be

$$\overline{\operatorname{span}}\{s_\mu s_\nu^* : d(\mu) = d(\nu) = m \text{ and } s(\mu) = s(\nu)\}$$

and for $v \in \Lambda^0$, define $\mathcal{F}_m(v)$ by

$$\overline{\operatorname{span}}\{\{s_\mu s_\nu^* : d(\mu) = d(\nu) = m \text{ and } s(\mu) = s(\nu) = v\}.$$

We now recall another lemma of Kumjian and Pask.

LEMMA 4.2.9 ([**23**, Lemma 3.2]). *Let Λ be a row-finite k-graph with no sources. For $m \in \mathbb{N}^k$, $v \in \Lambda^0$ there exist isomorphisms*

$$\mathcal{F}(v) \cong \mathcal{K}\big(\ell^2(\{\lambda \in \Lambda^m : s(\lambda) = v\})\big)$$

and $\mathcal{F}_m \cong \bigoplus_{v\in\Lambda^0} \mathcal{F}_m(v)$. Moreover, the C^*-algebras $\mathcal{F}_m$, $m \in \mathbb{N}^k$, form a directed system under inclusion , and $\mathcal{F}_\Lambda = \overline{\cup \mathcal{F}_m}$ is an AF C^*-algebra.

REMARK. In this paper we refer to $\mathcal{F}_\Lambda$ as $C^*(\Lambda)^\gamma$. By definition, $C^*(\Lambda)^\gamma$ is equal to the fixed point algebra $\{a \in C^*(\Lambda) : \gamma_z(a) = a \text{ for all } z \in \mathbb{T}\}$ for the gauge action γ of Subsection 2.2.1. That $\mathcal{F}_\Lambda$ and $C^*(\Lambda)^\gamma$ are one and the same is proved in [**23**, Lemma 3.3].

For a locally finite k-graph Λ and for each $p \in \mathbb{N}^k$, define a map $\Phi_p : C^*(\Lambda) \to C^*(\Lambda)$ by

$$\Phi_p(x) = \sum_{\lambda\in\Lambda^p} s_\lambda x s_\lambda^*.$$

Φ_p is a completely positive map [**44**, p. 274], which we call the *noncommutative p-shift map*. When E is a directed graph, the map Φ_1 is often denoted by ϕ_E.

4.2.2. Topological entropy of a completely positive map. In his seminal paper, Voiculescu [**45**] introduces a non-commutative extension of the classical topological entropy for C^*- and W^*-algebras. In particular, in [**45**, Section 4], Voiculescu defines the topological entropy of an automorphism Φ of an unital and nuclear C^*-algebra. Subsequently, Brown [**7**, Definition 1.1] generalises Voiculescu's definitions to include non-unital C^*-algebras. Brown also shows in [**7**, Remark 1.5] that the definition of topological entropy could equally apply when Φ is an endomorphism. The special case we are interested in is when Φ is a completely positive map. To understand the definition of topological entropy for a completely positive map we introduce some notation and basic facts.

For a set S, we denote the collection of all finite subsets of S by $\mathrm{FS}(S)$. Suppose A is a C^*-algebra, C is a finite dimensional C^*-algebra and the maps $\phi : C \to A$, $\psi : A \to C$ are both completely positive. Then we call (ϕ, ψ, C) a completely positive approximation, or an approximating triple, for A and write $(\phi, \psi, C) \in \mathrm{CPA}(A)$. If,

in addition, there exists $\Omega \in \mathrm{FS}(A)$ and $\varepsilon > 0$ such that $\|\phi \circ \psi(a) - a\| < \varepsilon$ for every $a \in \Omega$, then we write $(\phi, \psi, C) \in \mathrm{CPA}(A, \Omega, \varepsilon)$.

Suppose A is a nuclear C^*-algebra. Then for each $\Omega \in \mathrm{FS}(A)$ and $\varepsilon > 0$ it follows that $\mathrm{CPA}(A, \Omega, \varepsilon) \neq \emptyset$. Thus we may define

$$\mathrm{rcp}_A(\Omega, \varepsilon) := \inf\{\mathrm{rank}(C) : (\phi, \psi, C) \in \mathrm{CPA}(A, \Omega, \varepsilon)\}.$$

When it is clear from the context what A is, we leave it off our notation.

For an endomorphism $\Phi : A \to A$ let

$$\begin{aligned} \mathrm{ht}(\Phi, \Omega, \varepsilon) &= \limsup_{n\to\infty} \frac{1}{n} \log\Big(\mathrm{rcp}_A\Big(\bigcup_{l=0}^{n-1} \Phi^l(\Omega), \varepsilon\Big)\Big); \\ \mathrm{ht}(\Phi, \Omega) &= \sup_{\varepsilon>0} \mathrm{ht}(\Phi, \Omega, \varepsilon); \text{ and} \\ \mathrm{ht}(\Phi) &= \sup_{\Omega\in\mathrm{FS}(A)} \mathrm{ht}(\Phi, \Omega). \end{aligned}$$

We call $\mathrm{ht}(\Phi)$ the *topological entropy* of Φ.

Example 4.2.10. *Let Λ be an irreducible, locally finite k-graph with no sources, fix $p \in \mathbb{N}^k$ and consider the non-commutative p-shift map Φ_p. Since $\Phi_p|_{C^*(\Lambda)^\gamma}$ is an endomorphism and since $C^*(\Lambda^\gamma)$ is nuclear we may write down the topological entropy of $\Phi_p|_{C^*(\Lambda)^\gamma}$. Thus*

$$\mathrm{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) = \sup_{\Omega\in\mathrm{FS}(C^*(\Lambda)^\gamma)} \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{n} \log\Big(\mathrm{rcp}_{C^*(\Lambda)}\Big(\bigcup_{l=0}^{n-1} \Phi_p^l(\Omega), \varepsilon\Big)\Big). \tag{4.2.3}$$

In order to compute entropy, we need to simplify expressions like (4.2.3) as much as possible. The following proposition of Brown, ofton refered to as the Kolmogorov-Sinai property, helps us to do this by allowing us to take a supremum over a net of a C^*-algebra A, rather than over all finite subsets of A.

Proposition 4.2.11 ([**7**, Proposition 2.6]). *If $A \subset B(H)$ and $\{\omega_i\}_{i\in I} \subset \mathrm{FS}(A)$ is a net (partially ordered by inclusion) such that* $\mathrm{span}\bigcup_{i\in I, n\in\mathbb{N}} \Phi^n(\omega_i)$ *is dense in A then*

$$\mathrm{ht}(\Phi) = \sup_{i\in I} \mathrm{ht}(\Phi, \omega_i).$$

The following theorem is also a commonly used result in ergodic theory. We use it to extend completely positive approximations of subspaces of a C^*-algebra A to completely positive approximations of A.

Theorem 4.2.12 (Arveson's Extension Theorem [**2**, Theorem 1.2.3]). *Let S be a norm-closed self-adjoint linear subspace of a unital C^*-algebra B, which contains the identity of B, and let H be a Hilbert space. Then for every completely positive linear map $\phi : S \to B(H)$, there is a completely positive linear map $\Phi : B \to B(H)$ such that $\Phi|_S = \phi$.*

The following Corollary of Arveson's Extension Theorem uses [**7**, Lemma 1.7] and is proved by Brown in [**7**, page 353]. It says that when S is a C^*-algebra we may drop the requirement that S contains the unit of B.

COROLLARY 4.2.13. *Let B be a unital C^*-algebra and suppose S is a nonunital C^*-subalgebra of B. Then for every contractive completely positive linear map $\phi : S \to C$ where $C = B(H)$ or $\dim(C) < \infty$, there is a unital, completely positive linear map $\Phi : B \to C$ such that $\Phi|_S = \phi$.*

We use the following estimate several times throughout this chapter. It says that any C^*-algebra can be approximated by itself.

LEMMA 4.2.14. *Suppose A is a C^*-algebra. Then for every finite subset F of A such that $C^*(F)$ is finite dimensional, and for every $\varepsilon > 0$, we have*

$$\mathrm{rcp}_A(F, \varepsilon) \leq \operatorname{rank} C^*(F).$$

PROOF. Fix $F \in FS(A)$ such that $C^*(F)$ is finite dimensional, and $\varepsilon > 0$. To prove the lemma we first let I be the identity on $C^*(F)$. Then I is a completely positive map and Corollary 4.2.13 implies there exists a completely positive map $\widetilde{I} : \widetilde{A} \to C^*(F)$. Hence $\widetilde{I}|_A : A \to C^*(F)$ is also completely positive. Let $\iota : C^*(F) \to A$ be the inclusion map. Then $\big(\widetilde{I}|_A, \iota, C^*(F)\big) \in \mathrm{CPA}\,\big(A\big)$. Moreover, observe that for each $a \in A$ we have $\|\widetilde{I}|_A\big(\iota(a)\big) - a\| = 0 < \varepsilon$. Thus $\big(\widetilde{I}|_A, \iota, C^*(F)\big) \in \mathrm{CPA}\,\big(A, F, \varepsilon\big)$. Hence by definition of rcp we see that $\mathrm{rcp}_A(F, \varepsilon) \leq \operatorname{rank} C^*(F)$ as required. ■

4.3. Directed graphs

Given a row-finite directed graph E with $|E^0| < \infty$ and edge matrix A_E, the topological entropy of $\phi_E|_{C^*(E)^\gamma}$ is well understood. Indeed, many different authors, working with different classes of row-finite directed graphs, have calculated $\mathrm{ht}(\phi_E|_{C^*(E)^\gamma})$ to be $\log r(A_E)$ where $r(A_E)$ is the spectral radius of A_E [**5, 19, 30**]. In 2005, Jeong and Park [**20**] were interested in extending earlier results, including those mentioned above, to the case when E may have an infinite number of vertices. One of their main aims was the following theorem.

THEOREM 4.3.1 ([**20**, Theorem 4.3]). *Let E be a locally finite, irreducible, infinite directed graph. Then*

$$\mathrm{ht}(\phi_E|_{C^*(E)^\gamma}) \leq \mathrm{h}_b({}^tE)$$

In its original form, the claim of [**20**, Theorem 4.3] appeared as $\mathrm{ht}(\phi_E|_{\mathcal{A}_E}) \leq \mathrm{h}_b({}^tE)$, which looks slightly different from what we have above. The difference in notation is due to the fact that Jeong and Park use $\mathcal{A}_E$ to denote $C^*(E)^\gamma$.

Unfortunately, the proof provided for Theorem 4.3.1 in [**20**] is incorrect. In particular, it relies on [**21**, Lemma 3.7] which is false. (See the counterexample

provided in Appendix A.) Specifically, [**21**, Lemma 3.7] is used in [**21**, Proposition 3.9] to extend a set of matrix units for $\operatorname{span}\omega(n_0, v)$ to a set of matrix units for $\operatorname{span}\big\{\bigcup_{n\leq n_0}\omega(n_0, v)\big\}$. In turn, [**21**, Proposition 3.9] is then invoked in the proof of [**20**, Theorem 4.3]. We correct their approach in this section by proving Theorem 4.3.1 directly. [**21**, Proposition 3.9] is a corollary of Theorem 4.3.1, so our proof fills the gap in the proof of [**20**, Theorem 4.3].

In this section we are dealing exclusively with row-finite directed graphs with no sources. Thus the Cuntz-Krieger relations defined in Definition 4.2.7 simplify. We state the relevant Cuntz-Krieger relations below.

Let E be a row-finite, directed graph with no sources. A *Cuntz-Krieger E-family* $\{S, P\}$ is a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{S_e : e \in E^1\}$ of partial isometries, in a C^*-algebra B, such that

(DG1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$; and
(DG2) $P_v = \sum_{\{e\in E^1 : r(e)=v\}} S_e S_e^*$.

The conditions (DG1) and (DG2) are called the *Cuntz-Krieger relations*; (DG2) is often referred to as the *Cuntz-Krieger relation at v*.

Remark ([**32**, Corollary 1.14]). Recall that if E has no sinks or sources, then by repeatedly applying the Cuntz-Krieger relations to paths of E we obtain the following two results:

(1) for $\lambda \in E^*$ we have $S_\lambda^* S_\lambda = P_{s(\lambda)}$; and
(2) for each $v \in \Lambda^0$ and any $n \in \mathbb{N}$ we have $P_v = \sum_{\{\mu\in E^n : r(\mu)=v\}} S_\mu S_\mu^*$.

Often when we say we are applying (DG1) or (DG2), what we are actually using are the two equalities above.

If $\{S, P\}$ is a Cuntz-Krieger E-family, then for $\mu, \nu \in E^*$, we use Lemma 4.2.8 to compute $S_\mu^* S_\nu$. Note that for directed graphs, Lemma 4.2.8 simplifies to the following corollary.

Corollary 4.3.2 ([**32**, Corollary 1.14 (2)]). *Suppose that E is a row-finite directed graph and $\{S, P\}$ is a Cuntz-Krieger E-family in a C^*-algebra B. Then*

$$S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu\mu' \text{ for some } \mu' \in E^*, \\ S_{\nu'} & \text{if } \nu = \mu\nu' \text{ for some } \nu' \in E^*, \\ 0 & \text{otherwise.} \end{cases}$$

For this section we use the following definitions. Note that for any directed graph E, the set of vertices E^0 is countable. Thus we may choose an enumeration $\{v_j : j \in \mathbb{N}\}$ of E^0. When this has been done we refer to E as an *enumerated directed graph*.

DEFINITIONS 4.3.3. For an enumerated directed graph E and $(m,n) \in \mathbb{N} \times \mathbb{N}$, we define

(1) $\omega_E(m,n) = \{s_\alpha s_\beta^* : \alpha, \beta \in E^m v_j \text{ for some } j \leq n\}$;
(2) $\Omega_E(m,n) = \bigcup_{i \leq m} \omega_E(i,n)$;
(3) $C_E(m,n) = \operatorname{span} \Omega_E(m,n)$; and
(4) $(P_E)_{m,n} = \sum_{j \leq n} \sum_{\lambda \in E^m v_j} s_\lambda s_\lambda^*$

When it is clear from the context what the directed graph E is, we drop the subscript E from our notation.

REMARK 4.3.4. Suppose E is an enumerated directed graph and fix $(m,n) \in \mathbb{N} \times \mathbb{N}$. We claim $C(m,n) = C^*\big(\Omega(m,n)\big)$. To see this it suffices to show $C(m,n)$ is closed under multiplication. Fix $\mu, \nu \in \Omega(m,n)$. If $|\mu| = |\nu|$ then $s_\mu s_\mu^* s_\nu s_\nu^* = \delta_{\mu,\nu} s_\mu s_\mu^* \in C(m,n)$. On the other hand, suppose $|\mu| \neq |\nu|$; without loss of generality we may assume $|\mu| > |\nu|$. Then $s_\mu s_\mu^* s_\nu s_\nu^* = s_\nu s_\nu^* s_\mu s_\mu^* = \delta_{\mu(0,|\nu|),\nu} s_\mu s_\mu^* \in C(m,n)$. Hence $C(m,n)$ is closed under multiplication.

We use Definitions 4.3.3 to help us simplify the calculation of topological entropy. The point is that for an enumerated directed graph E, the $\Omega(m,n)$ defines a net of $C^*(E)^\gamma$, the union of which is dense in $C^*(E)^\gamma$. Proposition 4.2.11 then tells us we need only compute the topological entropy on each $\Omega(m,n)$. This technique has been used previously. For example, by Jeong and Park in the proofs of [**20**, Theorem 4.3] and [**21**, Proposition 3.9].

LEMMA 4.3.5. *Suppose E is an enumerated directed graph. Then*

$$\operatorname{ht}(\phi_E|_{C^*(E)^\gamma}) = \sup_{(m,n) \in \mathbb{N} \times \mathbb{N}} \operatorname{ht}\big(\phi_E|_{C^*(E)^\gamma}, \Omega(m,n)\big).$$

PROOF. Consider $\big(\Omega(m,n)\big)_{(m,n) \in \mathbb{N} \times \mathbb{N}}$ where $\mathbb{N} \times \mathbb{N}$ has partial order $(m,n) \leq (m',n')$ if and only if $m \leq m'$ and $n \leq n'$. Then $\big(\Omega(m,n)\big)_{(m,n)}$ is a net and the linear span of $\bigcup_{(m,n)} \Omega(m,n)$ is dense in $C^*(E)^\gamma$. Moreover, Definitions 4.3.3 ensure that $(m,n) \leq (m',n')$ implies $\Omega(m,n) \subseteq \Omega(m',n')$. The final result follows from Proposition 4.2.11. ■

Suppose E is an enumerated directed graph and $(m,n) \in \mathbb{N} \times \mathbb{N}$. To see that $P_{m,n}$ is a projection, recall that $\{s_\lambda s_\lambda^* : \lambda \in E^m\}$ is a family of mutually orthogonal projections. Since $P_{m,n}$ is a sum of mutually orthogonal projections [**32**, Proposition A.2] implies that $P_{m,n}$ is a projection.

Suppose E is a locally finite, enumerated directed graph with no sinks or sources. To compute an upper bound for the entropy $\operatorname{ht}(\phi_E|_{C^*(\Lambda^\gamma)})$ in Theorem 4.3.1 we need to calculate an upper bound for the rank of $C(m,n)$ for any given $(m,n) \in \mathbb{N} \times \mathbb{N}$. We perform this calculation in Proposition 4.3.9. As a prerequisite for Proposition 4.3.9, we must decompose $C(m,n)$ into a direct sum of matrix algebras. This decomposition is a critical result of the following theorem.

THEOREM 4.3.6. *Suppose E is a locally finite, enumerated directed graph with no sources and let $(m,n) \in \mathbb{N} \times \mathbb{N}$. Then for each $i \leq m$,*

(1) $\prod_{k=i+1}^{m}(1-P_{k,n}) \operatorname{span} \omega(i,n)$ *is a C^*-subalgebra of $C(m,n)$;*
(2) *the projection $\prod_{k=i+1}^{m}(1-P_{k,n})$ commutes with $C(i,n)$; and*
(3) $C(m,n) = \bigoplus_{i=0}^{m} \big(\prod_{k=i+1}^{m}(1-P_{k,n}) \operatorname{span} \omega(i,n)\big)$.

REMARK. In Theorem 4.3.6 we adopt the convention that for any $m \in \mathbb{N}$, the product $\prod_{k=m+1}^{m}(1-P_{k,n})$ is the identity, and more generally that any empty product is the identity.

To prove Theorem 4.3.6 we require the following technical lemma.

LEMMA 4.3.7. *Suppose E is a locally-finite, enumerated directed graph with no sources and $(m,n) \in \mathbb{N} \times \mathbb{N}$. Then*

(1) $P_{m,n}$ *commutes with $C(m,n)$;*
(2) $P_{m,n}C(m,n) = \operatorname{span} \omega(m,n)$*; and*
(3) $(1-P_{m,n})C(m,n) = (1-P_{m,n})C(m-1,n)$.

PROOF. **(1):** Fix $\alpha, \beta \in E^*$ with $|\alpha| = |\beta| \leq m$ and $s(\alpha) = s(\beta) = v_j$ for some $j \leq n$. Since $P_{m,n}s_\alpha s_\beta^* = P_{m,n}s_\alpha s_{s(\beta)}s_\beta^*$, the Cuntz-Krieger relation at $s(\beta)$ implies that

$$P_{m,n}s_\alpha s_\beta^* = P_{m,n}s_\alpha\Big(\sum_{\rho \in s(\beta)E^{m-|\beta|}} s_\rho s_\rho^*\Big)s_\beta^* = \sum_{l \leq n} \sum_{\substack{\lambda \in E^m v_l, \\ \rho \in s(\beta)E^{m-|\beta|}}} s_\lambda s_\lambda^* s_{\alpha\rho} s_{\beta\rho}^*. \tag{4.3.1}$$

Since E is locally finite and each $|\lambda| = |\alpha\rho|$, Corollary 4.3.2 implies that $s_\lambda^* s_{\alpha\rho} = 0$ whenever $\lambda \neq \alpha\rho$ in (4.3.1) above. Thus

$$P_{m,n}s_\alpha s_\beta^* = \sum_{l \leq n} \sum_{\{\rho \in E : \alpha\rho \in E^m v_l\}} s_{\alpha\rho} s_{\beta\rho}^*. \tag{4.3.2}$$

Also, by a symmetric argument we have that

$$s_\alpha s_\beta^* P_{m,n} = \sum_{l \leq n} \sum_{\{\rho \in E : \beta\rho \in E^m v_l\}} s_{\alpha\rho} s_{\beta\rho}^*.$$

Since $s(\alpha) = s(\beta)$ and $|\alpha| = |\beta|$, we see that for each $l \leq n$ we have

$$\{\rho \in E : \alpha\rho \in E^m v_l\} = \{\rho \in E : \beta\rho \in E^m v_l\}.$$

Hence $P_{m,n}s_\alpha s_\beta^* = s_\alpha s_\beta^* P_{m,n}$, as required.

(2): Fix α, β as in part (1). Then (4.3.2) implies $P_{m,n}s_\alpha s_\beta^* \in \operatorname{span} \omega(m,n)$. Thus $P_{m,n}C(m,n) \subseteq \operatorname{span} \omega(m,n)$. For the reverse inclusion fix $j \leq n$ and $\mu, \nu \in E^m v_j$. Then by definition we have $s_\mu s_\nu^* \in C(m,n)$. Since $|\mu| = |\nu| = m$, equation (4.3.2) implies $P_{m,n}s_\mu s_\nu^* = s_\mu s_\nu^*$ and hence $s_\mu s_\nu^* \in P_{m,n}C(m,n)$. Thus $\operatorname{span} \omega(m,n) \subseteq P_{m,n}C(m,n)$ as required.

(3): Fix $\mu, \nu \in E^*$ such that $|\mu| = |\nu| \leq m$ and $s(\mu) = s(\nu) = v_j$ for some $j \leq n$. We consider two cases: either $|\mu| < m$ or $|\mu| = m$. Suppose $|\mu| < m$. Then trivially $s_\mu s_\nu^* \in C(m-1, n)$ and hence $(1 - P_{m,n}) s_\mu s_\nu^* \in (1 - P_{m,n}) C(m-1, n)$. Now suppose that $|\mu| = m$. Recall that by definition $P_{m,n} s_\mu s_\nu^* = s_\mu s_\nu^*$. Thus $(1 - P_{m,n}) s_\mu s_\nu^* = s_\mu s_\nu^* - s_\mu s_\nu^* = 0 \in (1 - P_{m,n}) C(m-1, n)$. Hence $(1 - P_{m,n}) C(i, n) \subseteq (1 - P_{m,n}) C(m-1, n)$. Since the reverse inclusion is trivial, this completes the proof. ■

Remark 4.3.8. For any $(m, n) \in \mathbb{N} \times \mathbb{N}$, Lemma 4.3.7 implies that multiplication by $(1 - P_{m,n})$ is a C^*-homomorphism of $C(m, n)$ onto $C(m-1, n)$.

Proof of Theorem 4.3.6. We prove the theorem by induction on m. Suppose for a base case that $m = 0$. Since the empty product is equal to the identity by convention, (1) and (2) are trivially true. It remains to show $C(0, n) = \operatorname{span} \omega(0, n)$, however this is true by definition. Now fix $m \in \mathbb{N}$ and suppose, as an inductive hypothesis, that for every $i \leq l < m$,

(1) $\prod_{k=i+1}^{l} (1 - P_{k,n}) \operatorname{span} \omega(i, n)$ is a C^*-subalgebra of $C(l, n)$;
(2) the projection $\prod_{k=i+1}^{l} (1 - P_{k,n})$ commutes with $C(i, n)$; and
(3) $C(l, n) = \bigoplus_{i=0}^{l} \Big(\prod_{k=i+1}^{l} (1 - P_{k,n}) \operatorname{span} \omega(i, n) \Big)$.

To prove the theorem we must establish (1) – (3) for $l = m$.

We have

$$C(m, n) = P_{m,n} C(m, n) + (1 - P_{m,n}) C(m, n).$$

Since $P_{m,n}(1 - P_{m,n}) = 0$ and since, by Lemma 4.3.7, $P_{m,n}$ commutes with $C(m, n)$, we see that

$$C(m, n) = P_{m,n} C(m, n) \oplus (1 - P_{m,n}) C(m, n).$$

Then Lemma 4.3.7 (3) implies

$$C(m, n) = P_{m,n} C(m, n) \oplus (1 - P_{m,n}) C(m-1, n).$$

Next we apply Lemma 4.3.7 (2) to see that

$$C(m, n) = \operatorname{span} \omega(m, n) \oplus (1 - P_{m,n}) C(m-1, n).$$

Then the third part of the inductive hypothesis implies

$$C(m, n) = \operatorname{span} \omega(m, n) \oplus (1 - P_{m,n}) \Big(\bigoplus_{i=0}^{m-1} \prod_{k=i+1}^{m-1} (1 - P_{k,n}) \operatorname{span} \omega(i, n) \Big). \tag{4.3.3}$$

The first part of the inductive hypothesis says that each $\prod_{k=i+1}^{m-1} (1 - P_{k,n}) \operatorname{span} \omega(i, n)$ is a C^*-subalgebra of $C(m-1, n)$. By Remark 4.3.8, multiplication by $(1 - P_{m,n})$ is

a homomorphism of $C(m,n)$ onto $C(m-1,n)$. Thus $(1-P_{m,n})\big(\bigoplus_{i=0}^{m-1}\prod_{k=i+1}^{m-1}(1-P_{k,n})\operatorname{span}\omega(i,n)\big)$ is a C^*-subalgebra of $C(m-1,n)$. Further, multiplication by $(1-P_{m,n})$ maps orthogonal subalgebras to orthogonal subalgebras. Hence

(4.3.4)
$$(1-P_{m,n})\Big(\bigoplus_{i=0}^{m-1}\prod_{k=i+1}^{m-1}(1-P_{k,n})\operatorname{span}\omega(i,n)\Big)=\bigoplus_{i=0}^{m-1}\prod_{k=i+1}^{m}(1-P_{k,n})\operatorname{span}\omega(i,n).$$

Thus $\bigoplus_{i=0}^{m-1}\prod_{k=i+1}^{m}(1-P_{k,n})\operatorname{span}\omega(i,n)$ is a C^*-subalgebra of $C(m-1,n)$. Since $C(m-1,n)$ is a C^*-subalgebra of $C(m,n)$, this proves the first claim of the theorem. Next we consider the third claim of the theorem. By combining (4.3.4) with (4.3.3) we obtain

$$C(m,n)=\operatorname{span}\omega(m,n)\oplus\Big(\bigoplus_{i=0}^{m-1}\prod_{k=i+1}^{m}(1-P_{k,n})\operatorname{span}\omega(i,n)\Big).$$

Since the empty product is equal to the identity by convention, it follows that

$$C(m,n)=\bigoplus_{i=0}^{m}\prod_{k=i+1}^{m}(1-P_{k,n})\operatorname{span}\omega(i,n). \tag{4.3.5}$$

This proves the third claim of the theorem.

It remains to prove the second claim. By definition we have that

$$\prod_{k=i+1}^{m}(1-P_{k,n})C(i,n)=(1-P_{m,n})\prod_{k=i+1}^{m-1}(1-P_{k,n})C(i,n).$$

The second part of the inductive hypothesis, with $l=m-1$, tells us that $\prod_{k=i+1}^{m-1}(1-P_{k,n})$ commutes with $C(i,n)$. Hence for every $x\in C(i,n)$ we have

$$\prod_{k=i+1}^{m}(1-P_{k,n})x=(1-P_{m,n})x\prod_{k=i+1}^{m-1}(1-P_{k,n}). \tag{4.3.6}$$

Since $C(i,n)\subset C(m,n)$, Lemma 4.3.7 (1) implies $(1-P_{m,n})$ commutes with $C(i,n)$. Thus continuing from (4.3.6) we see that

$$\prod_{k=i+1}^{m}(1-P_{k,n})C(i,n)=C(i,n)\prod_{k=i+1}^{m}(1-P_{k,n}).\qquad\blacksquare$$

For any $m,n\in\mathbb{N}$, Theorem 4.3.6 allows us to decompose $C(m,n)$ into orthogonal subalgebras. However, our ultimate goal is to calculate the rank of $C(m,n)$ in terms of paths in the underlying directed graph E. By applying results of Kumjian and Pask [**23**], we calculate the rank of each orthogonal subalgebra of $C(m,n)$. We then apply Theorem 4.3.6 to obtain an upper bound for $\operatorname{rank}C(m,n)$. It is this upper bound, calculated in the following proposition, which we use to prove Theorem 4.3.1.

PROPOSITION 4.3.9. *Suppose E is a locally finite, enumerated directed graph with no sources and that $(m,n) \in \mathbb{N} \times \mathbb{N}$. Then* $\operatorname{rank}\big(C(m,n)\big) \leq \sum_{i\leq m}\sum_{j\leq n}|E^i v_j|$.

PROOF. From Theorem 4.3.6 we know that

$$C(m,n) = \bigoplus_{i\leq m}\Big(\prod_{k=i+1}^{m}(1-P_{k,n})\operatorname{span}\omega(i,n)\Big).$$

Thus

$$\operatorname{rank} C(m,n) = \operatorname{rank}\Big(\bigoplus_{i\leq m}\Big(\prod_{k=i+1}^{m}(1-P_{k,n})\operatorname{span}\omega(i,n)\Big)\Big)$$

and then Lemma 4.2.2 implies

$$\operatorname{rank} C(m,n) = \sum_{i\leq m}\operatorname{rank}\Big(\prod_{k=i+1}^{m}(1-P_{k,n})\operatorname{span}\omega(i,n)\Big). \tag{4.3.7}$$

Recall from Subsection 4.2.1 that $\mathcal{F}_i(v) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : |\mu| = |\nu| = i, s(\mu) = s(\nu) = v\}$. Then for distinct $j_1, j_2 \leq n$ we have $\mathcal{F}_i(v_{j_1})\mathcal{F}_i(v_{j_2}) = 0$. Since we have that $\bigcup_{j\leq n}\mathcal{F}_i(j) = \operatorname{span}\omega(i,n)$, [**32**, Corollary A.11] implies $\operatorname{span}\omega(i,n) = \bigoplus_{j\leq n}\mathcal{F}_i(v_j)$. Thus, continuing from (4.3.7), we have

$$\operatorname{rank} C(m,n) = \sum_{i\leq m}\operatorname{rank}\Big(\prod_{k=i+1}^{m}(1-P_{k,n})\bigoplus_{j\leq n}\mathcal{F}_i(v_j)\Big). \tag{4.3.8}$$

Theorem 4.3.6 (2) implies that for each $i \leq m$, the projection $\prod_{k=i+1}^{m}(1-P_{k,n})$ commutes with $\operatorname{span}\omega(i,n)$. Thus multiplication by $\prod_{k=i+1}^{m}(1-P_{k,n})$ is a homomorphism of $\operatorname{span}\omega(i,n)$. Since $\operatorname{span}\omega(i,n) = \bigoplus_{j\leq n}\mathcal{F}_i(v_j)$ and because homomorphisms map orthogonal subalgebras to orthogonal subalgebras, (4.3.8) becomes

$$\begin{aligned}\operatorname{rank} C(m,n) &= \sum_{i\leq m}\operatorname{rank}\bigoplus_{j\leq n}\Big(\prod_{k=i+1}^{m}(1-P_{k,n})\mathcal{F}_i(v_j)\Big)\\ &= \sum_{i\leq m}\sum_{j\leq n}\operatorname{rank}\Big(\prod_{k=i+1}^{m}(1-P_{k,n})\mathcal{F}_i(v_j)\Big).\end{aligned}$$

Since each $\mathcal{F}_i(v_j)$ is simple and multiplication by $\prod_{k=i+1}^{m}(1-P_{k,n})$ is a homomorphism, it follows that each $\prod_{k=i+1}^{m}(1-P_{k,n})\mathcal{F}_i(v_j)$ is either isomorphic to $\mathcal{F}_i(v_j)$ or equal to 0. Hence

$$\operatorname{rank} C(m,n) \leq \sum_{i\leq m}\sum_{j\leq n}\operatorname{rank}\mathcal{F}_i(v_j).$$

Then [**32**, Equation (3.8)] implies that for each $i \leq m$ and $j \leq n$ that $\operatorname{rank}\mathcal{F}_i(v_j) = |E^i v_j|$. Thus $\operatorname{rank} C(m,n) \leq \sum_{i\leq m}\sum_{j\leq n}|E^i v_j|$. ■

We are now able to prove our main result of this chapter: Theorem 4.3.1.

Proof of Theorem 4.3.1. To begin we observe that since E is an irreducible directed graph, E has no sinks or sources. To see this note that if v is a sink, then for every $u \in E^0$ such that $u \neq v$ we have that $uEv = \emptyset$, contradicting the fact that E is irreducible. Now suppose v is a source, then for every $u \in E^0$ such that $u \neq v$ we have that $vEu = \emptyset$, again contradicting the fact that E is irreducible. Thus E must have no sinks or sources.

We continue by following the original argument of Jeong and Park, contained in [**20**, Proof of Theorem 4.3]. Fix $(m,n) \in \mathbb{N} \times \mathbb{N}$ and $s_\alpha s_\beta^* \in \Omega(m,n)$. Then for any $L \in \mathbb{N}$, by definition of ϕ_E and $C(m+L,n)$, for every $l \in \mathbb{N}$ such that $0 \leq l \leq L-1$ we have that $\phi_E^l(s_\alpha s_\beta^*) \in C(m+l,n) \subset C(m+L,n)$. Hence

$$\operatorname{rcp}\Big(\bigcup_{l=0}^{L-1} \phi_E^l\big(\Omega(m,n)\big), \delta\Big) \leq \operatorname{rcp}\big(C(m+L,n), \delta\big).$$

Since $C(m+L,n)$ is finite dimensional, Lemma 4.2.14 implies

$$\operatorname{rcp}\Big(\bigcup_{l=0}^{L-1} \phi_E^l\big(\Omega(m,n)\big), \delta\Big) \leq \operatorname{rank}\big(C(m+L,n)\big).$$

For every $v \in \Lambda^0$, $m' \in \mathbb{N}$, let $C(m',v) := \operatorname{span}\{s_\alpha s_\beta^* : \alpha, \beta \in E^i v \text{ for some } i \leq m'\}$. Then $C(m+L,n) = \bigoplus_{j \leq n} C(m+L, v_j)$ and hence we see that

$$\operatorname{rcp}\Big(\bigcup_{l=0}^{L-1} \phi_E^l\big(\Omega(m,n)\big), \delta\Big) \leq n \max_{j \leq n}\big\{\operatorname{rank}\big(C(m+L,v_j)\big)\big\}. \tag{4.3.9}$$

It is at this point that our proof begins to take a different course to [**20**, Proof of Theorem 4.3]. Since E is irreducible, for every $j \leq n$ there exists $p_j \in \mathbb{N}$ such that $v_j E^{p_j} v_1 \neq \emptyset$. Let $p = \max_{j \leq n}\{p_j\}$. Since E has no sinks, for each $j \leq n$ we have

$$E^{m+L+p} v_1 = E^{m+L+p_j} v_j \Lambda^{p-p_j} v_1 = E^{m+L} v_j \Lambda^p v_1.$$

Then it follows that $|E^{m+L} v_j| \leq |E^{m+L+p} v_1|$. Hence we see that

$$\operatorname{rank}\big(C(m+L,v_j)\big) \leq \operatorname{rank}\big(C(m+L+p_j,v_1)\big) \leq \operatorname{rank}\big(C(m+L+p,v_1)\big).$$

Thus from (4.3.9) we obtain the inequality

$$\operatorname{rcp}\Big(\bigcup_{l=0}^{L-1} \phi_E^l\big(\Omega(m,n)\big), \delta\Big) \leq n \operatorname{rank}\big(C(m+L+p,v_1)\big).$$

Then for any $L \in \mathbb{N}$,

$$\frac{1}{L}\log\Big(\operatorname{rcp}\Big(\bigcup_{l=0}^{L-1} \phi_E^l\big(\Omega(m,n)\big), \delta\Big)\Big) \leq \frac{1}{L}\log\Big(n \operatorname{rank}\big(C(m+L+p,v_1)\big)\Big).$$

So it follows that

$$\limsup_{L\to\infty} \frac{1}{L}\log\Big(\operatorname{rcp}\Big(\bigcup_{l=0}^{L-1} \phi_E^l\big(\Omega(m,n)\big), \delta\Big)\Big) \leq \limsup_{L\to\infty} \frac{1}{L}\log\Big(n \operatorname{rank}\big(C(m+L+p,v_1)\big)\Big).$$

Since the equation above is true for all $\delta > 0$, the supremum of left hand side over $\delta > 0$ of the equation above is equal to $\operatorname{ht}\big(\phi_E|_{C^*(E)^\gamma}, \Omega(m,n)\big)$. Thus

$$\operatorname{ht}\big(\phi_E|_{C^*(E)^\gamma}, \Omega(m,n)\big) \leq \limsup_{L\to\infty} \frac{1}{L} \log\Big(n \operatorname{rank}\big(C(m+L+p, v_1)\big)\Big). \tag{4.3.10}$$

Then Proposition 4.3.9, together with (4.3.10) forces

$$\operatorname{ht}\big(\phi_E|_{C^*(E)^\gamma}, \Omega(m,n)\big) \leq \limsup_{L\to\infty} \frac{1}{L} \log\Big(n \sum_{i\leq m+L+p} |E^i v_1|\Big).$$

We now calculate

$$\begin{aligned}\operatorname{ht}\big(\phi_E|_{C^*(E)^\gamma}, \Omega(m,n)\big) &\leq \limsup_{L\to\infty} \frac{1}{L}\Big(\log n + \log\Big(\sum_{i\leq m+L+p} |E^i v_1|\Big)\Big)\\ &= \limsup_{L\to\infty} \frac{1}{L} \log\Big(\sum_{i\leq m+L+p} |E^i v_1|\Big).\end{aligned}$$

Since E has no sinks, for every $i \leq m+L+p$ we see that $|E^i v_1| \leq |E^{m+L+p} v_1|$. Thus

$$\begin{aligned}\operatorname{ht}\big(\phi_E|_{C^*(E)^\gamma}, \Omega(m,n)\big) &\leq \limsup_{L\to\infty} \frac{1}{L} \log(m+L+p+1)|E^{m+L+p} v_1|\\ &= \limsup_{L\to\infty} \frac{1}{L}\big(\log(m+L+p+1) + \log|E^{m+L+p} v_1|\big)\\ &= \limsup_{L\to\infty} \frac{1}{L} \log|E^{m+L+p} v_1|.\end{aligned}$$

Lemma 4.2.6 implies $\limsup_{L\to\infty} \frac{1}{L}\log|E^{m+L+p}v_1| = \limsup_{L\to\infty} \frac{1}{L}\log|E^L v_1|$ and hence

$$\operatorname{ht}\big(\phi_E|_{C^*(E)^\gamma}, \Omega(m,n)\big) \leq \limsup_{L\to\infty} \frac{1}{L} \log|E^L v_1| = \mathrm{h}_b(E).$$

Since each $|E^L v_1|$ is independent of each $(m,n) \in \mathbb{N}\times\mathbb{N}$, it follows from Lemma 4.3.5 that $\operatorname{ht}(\phi_E|_{C^*(E)^\gamma}) \leq \mathrm{h}_b(E)$. ■

4.4. Higher-rank graphs

4.4.1. Restriction of completely positive maps to the core. In this subsection we extend the main result of Section 4.3, Theorem 4.3.1, from locally finite, irreducible directed graphs to locally finite, irreducible higher-rank graphs.

Definitions 4.4.1. Suppose Λ is a k-graph. We define $\mathrm{FS}(\Lambda)^\gamma = \bigcup_{n\in\mathbb{N}^k} \mathrm{FS}(\Lambda^n)$ and for $F \subset \Lambda$ let $\Omega(F) = \{s_\mu s_\nu^* : \mu, \nu \in F\}$.

We now state the main theorem of this section, Theorem 4.4.2. This theorem provides an upper bound for the topological entropy of $\Phi_p|_{C^*(\Lambda)^\gamma}$ in terms of the number of paths in Λ. It gives us a relationship between topological entropy in the core and block entropy in the rank-k subshifts of ${}^t\Lambda^\infty$.

THEOREM 4.4.2. *Suppose Λ is a locally finite, irreducible k-graph with no sinks or sources, and $p \in \mathbb{N}^k$. Then*

$$\mathrm{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) \leq \mathrm{h}_p({}^t\Lambda). \tag{4.4.1}$$

REMARK 4.4.3. When Λ is a locally finite k-graph with only a finite number of vertices and no sources, there exist finite edge matrices $M_1, \ldots, M_k$ for Λ. Then [**44**, Theorem 2.4] with $f = 0$ shows that for each $p \in \mathbb{N}^k$ we have equality in Theorem 4.4.2. Moreover, when we let $M = M_1 \cdots M_k$, [**44**, Corollary 2.5] tells us that $\mathrm{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) = \log r(M^p)$ where $r(M^p)$ denotes the spectral radius of M^p.

To prove Theorem 4.4.2 we use the following technical lemmas.

LEMMA 4.4.4. *Suppose Λ is a locally finite k-graph, $n \in \mathbb{N}^k$ and $G \in \mathrm{FS}(\Lambda^n)$. Then* $\operatorname{span}\Omega(G)$ *is a C^*-subalgebra of $C^*(\Lambda)^\gamma$, so* $\operatorname{span}\Omega(G) = C^*\big(\Omega(G)\big)$. *In addition,* $\operatorname{rank}\big(\operatorname{span}\Omega(G)\big) = |G|$.

PROOF. Since G is finite, $\operatorname{span}\Omega(G)$ is closed. Fix paths $\alpha, \beta, \mu, \nu \in G$. Then $s_\mu s_\nu^* s_\alpha s_\beta^* = \delta_{\nu,\alpha} s_\mu s_\beta^* \in \Omega(G)$ and $(s_\mu s_\nu^*)^* = s_\nu s_\mu^* \in \Omega(G)$. Thus $\operatorname{span}\Omega(G)$ is closed under multiplication and inclusion. Since $\operatorname{span}\Omega(G) \subset C^*(\Lambda)^\gamma$ it follows that $\operatorname{span}\Omega(G)$ is a C^*-subalgebra of $C^*(\Lambda)^\gamma$.

To prove the second claim of the lemma we use the following notation: for a subset $F \subset \Lambda$, define $s(F) = \{s(\lambda) : \lambda \in F\} \subseteq \Lambda^0$. Since $\operatorname{span}\Omega(G)$ is a C^*-subalgebra of $\mathcal{F}_n$, Lemma 4.2.9 implies that $\operatorname{span}\Omega(G) = \bigoplus_{v \in s(G)} \operatorname{span}\Omega(Gv)$. Furthermore, for each $v \in s(G)$ we have that $\operatorname{span}\Omega(Gv)$ is a C^*-subalgebra of $\mathcal{F}_m(v)$. Then Lemma 4.2.9 implies that the set $\{s_\mu s_\nu^* : \mu, \nu \in Gv\}$ is a complete set of matrix units for $\operatorname{span}\Omega(Gv)$. Thus Lemma 4.2.2 implies

$$\operatorname{rank}\big(\operatorname{span}\Omega(G)\big) = \sum_{v \in s(G)} \operatorname{rank}\big(\operatorname{span}\Omega(Gv)\big) = \sum_{v \in s(G)} |Gv| = |G|. \qquad \blacksquare$$

LEMMA 4.4.5. *Suppose Λ is a k-graph. Then the relation $\leq$ on $\mathrm{FS}(\Lambda)^\gamma$ defined by*

$$I \leq J \text{ if there exists } q, q' \in \mathbb{N}^k \text{ such that } q \leq q', I \subseteq \Lambda^q, J \subseteq \Lambda^{q'} \text{ and } I\Lambda^{q'-q} \subseteq J$$

is a partial order.

PROOF. We must show $\leq$ is reflexive, antisymmetric, and transitive. To see $\leq$ is reflexive, let $I \in \mathrm{FS}(\Lambda)^\gamma$. Then by definition of $\mathrm{FS}(\Lambda)^\gamma$ there exists $q \in \mathbb{N}^k$ such that $I \subseteq \Lambda^q$ and $I\Lambda^0 = I$.

Now suppose $I, J \in \mathrm{FS}(\Lambda)^\gamma$ such that $I \leq J$ and $J \leq I$. Then there exists $q, q' \in \mathbb{N}^k$ such that $q \leq q'$, $I \subseteq \Lambda^q$, $J \subseteq \Lambda^{q'}$ and $I\Lambda^{q'-q} \subseteq J$; and $q' \leq q$, $J \subseteq \Lambda^{q'}$, $I \subseteq \Lambda^q$, and $J\Lambda^{q-q'} \subseteq I$. Since $q \leq q' \leq q$ we have $q = q'$. Then $I = I\Lambda^0 \subset J = J\Lambda^0 \subset I$ and hence $\leq$ is antisymmetric.

For the final part of the proof suppose $I, J, K \in \mathrm{FS}(\Lambda)^\gamma$ such that $I \leq J$ and $J \leq K$. Then there exists q, q', q'' such that $q \leq q' \leq q''$, $I \subseteq \Lambda^q$, $J \subseteq \Lambda^{q'}$ and $I\Lambda^{q'-q} \subseteq J$; and $J \subseteq \Lambda^{q'}$, $K \subseteq \Lambda^{q''}$ and $J\Lambda^{q''-q'} \subseteq K$. To see that $\leq$ is transitive it suffices to show $I\Lambda^{q''-q} \subseteq K$. For this we observe that the factorisation property implies $\Lambda^{q''-q} = \Lambda^{q'-q}\Lambda^{q''-q'}$ and hence $I\Lambda^{q''-q} \subseteq I\Lambda^{q'-q}\Lambda^{q''-q'} \subseteq J\Lambda^{q''-q'} \subseteq K$. ■

We now prove Theorem 4.4.2.

Proof of Theorem 4.4.2. If $|\Lambda^0| < \infty$ then Remark 4.4.3 gives equality in (4.4.1). Thus we consider the case then $|\Lambda^0| = \infty$. The first step in our proof is to apply Proposition 4.2.11. To do this we claim that $\{\operatorname{span}\Omega(I)\}_{I\in\mathrm{FS}(\Lambda)^\gamma}$ is a net, partially ordered by inclusion, where the indexing set $\mathrm{FS}(\Lambda)^\gamma$ has partial order $\leq$

$$I \leq J \text{ if there exists } q, q' \in \mathbb{N}^k \text{ such that } q \leq q', I \subseteq \Lambda^q, J \subseteq \Lambda^{q'} \text{ and } I\Lambda^{q'-q} \subseteq J$$

given by Lemma 4.4.5. Fix $I, J \in \mathrm{FS}(\Lambda)^\gamma$ such that $I \leq J$. Then there exists $q, q' \in \mathbb{N}^k$ such that $q \leq q'$, $I \subseteq \Lambda^q$, $J \subseteq \Lambda^{q'}$ and $I\Lambda^{q'-q} \subseteq J$. We must show $\operatorname{span}\Omega(I) \subset \operatorname{span}\Omega(J)$. To see this we fix $\mu, \nu \in I$. Then (CK4) implies that

$$s_\mu s_\nu^* = s_\mu\Big(\sum_{\lambda\in s(\mu)\Lambda^{q'-q}} s_\lambda s_\lambda^*\Big)s_\nu = \sum_{\lambda\in s(\mu)\Lambda^{q'-q}} s_{\mu\lambda}s_{\nu\lambda}^*.$$

Since $I\Lambda^{q'-q} \subseteq J$, each $\mu\lambda, \nu\lambda \in J$ and hence $s_\mu s_\nu^* \in \operatorname{span}\Omega(J)$. Thus $\operatorname{span}\Omega(I) \subseteq \Omega(J)$ as required.

Lemma 4.2.8 ensures $\bigcup_{F\in\mathrm{FS}(\Lambda)^\gamma}\operatorname{span}\Omega(F)$ is dense in $C^*(\Lambda)^\gamma$. Then since $l = 0 \in \mathbb{N}$, it follows immediately that $\bigcup_{F\in\mathrm{FS}(\Lambda)^\gamma, l\in\mathbb{N}}\Phi_p^l\big(\operatorname{span}\Omega(F)\big)$ is also dense in $C^*(\Lambda)^\gamma$. We now apply Proposition 4.2.11 with $\Phi = \Phi_p|_{C^*(\Lambda)^\gamma}$, $I = \mathrm{FS}(\Lambda)^\gamma$ and with each $\omega_i = \Omega(F)$ to see that

$$\begin{aligned}\operatorname{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) &= \sup_{F\in\mathrm{FS}(\Lambda)^\gamma}\operatorname{ht}\big(\Phi_p|_{C^*(\Lambda)^\gamma}, \Omega(F)\big)\\ &= \sup_{n\in\mathbb{N}^k, F\in\mathrm{FS}(\Lambda^n)}\operatorname{ht}\big(\Phi_p|_{C^*(\Lambda)^\gamma}, \Omega(F)\big). \qquad (4.4.2)\end{aligned}$$

Fix $m \in \mathbb{N}^k$ and $L \in \mathbb{N}$. For every $F \in \mathrm{FS}(\Lambda^m)$ and $\varepsilon > 0$ we claim there exists $G_{F,L} \in \mathrm{FS}(\Lambda^{m+Lp})$ such that

$$\operatorname{rcp}_{C^*(\Lambda)^\gamma}\Big(\bigcup_{l=0}^{L-1}\Phi_p^l\big(\Omega(F)\big), \varepsilon\Big) \leq \operatorname{rank}\big(\operatorname{span}\Omega(G_{F,L})\big). \qquad (4.4.3)$$

For each $\mu, \nu \in F$ such that $s(\mu) = s(\nu)$ and each $l \leq L$ we have

$$\Phi_p^l(s_\mu s_\nu^*) = \sum_{\lambda\in\Lambda^{lp}} s_\lambda s_\mu s_\nu^* s_\lambda^* = \sum_{\lambda\in\Lambda^{lp}r(\mu)} s_{\lambda\mu}s_{\lambda\nu}^*.$$

For each $l \leq L$ let $F_l = \{\lambda\eta : \eta \in F, \lambda \in \Lambda^{lp}r(\mu)\} \subseteq \Lambda^{m+lp}$. Then $\lambda\mu, \lambda\nu \in F_l$ for each $\lambda \in \Lambda^{lp}r(\mu)$. Let $G_{F,L} = \bigcup_{l\leq L} F_l\Lambda^{(L-l)p} \subset \Lambda^{m+Lp}$. Since $G_{F,L}$ is finite,

$\operatorname{span}\Omega(G_{F,L})$ is finite dimensional. Thus Lemma 4.2.14 implies that for every $\varepsilon > 0$ we have

$$\operatorname{rcp}_{C^*(\Lambda)^\gamma}\Big(\Omega(G_{F,L}),\varepsilon\Big) \leq \operatorname{rank}\big(\operatorname{span}\Omega(G_{F,L})\big).$$

So to establish (4.4.3) it suffices to show that $C^*\Big(\bigcup_{l=0}^{L-1}\Phi_p^l\big(\Omega(F)\big)\Big) \subseteq \operatorname{span}\Omega(G_{F,L})$.

By definition, each $F_l \leq G_{F,L}$ and hence (CK4) implies

$$\Phi_p^l(s_\mu s_\nu^*) = \sum_{\lambda\in\Lambda^{lp}r(\mu)} s_{\lambda\mu}\Big(\sum_{\rho\in s(\mu)\Lambda^{(L-l)p}} s_\rho s_\rho^*\Big)s_{\lambda\nu}^* = \sum_{\lambda\in\Lambda^{lp}r(\mu)}\sum_{\rho\in s(\mu)\Lambda^{(L-l)p}} s_{\lambda\mu\rho}s_{\lambda\nu\rho}^*.$$

Since each $\lambda\mu\rho, \lambda\nu\rho \in F_l\Lambda^{(L-l)p} \subset G_{F,L}$ it follows that $\Phi_p^l(s_\mu s_\nu^*) \in \operatorname{span}\Omega(G_{F,L})$. For every $\kappa \in F_l\Lambda^{(L-l)p}$ we calculate $d(\kappa) = (m+lp)+(L-l)p = m+Lp$. Thus $G_{F,L} \in \operatorname{FS}(\Lambda^{m+Lp})$ satisfies $\bigcup_{l=0}^{L-1}\Phi_p^l\big(\Omega(F)\big) \subset \operatorname{span}\Omega(G_{F,L})$. Then

$$C^*\Big(\bigcup_{l=0}^{L-1}\Phi_p^l\big(\Omega(F)\big)\Big) \subseteq C^*\big(\operatorname{span}\Omega(G_{F,L})\big).$$

Lemma 4.4.4 implies that $\operatorname{span}\Omega(G_{F,L}) = C^*\big(\Omega(G_{F,L})\big)$. Thus it follows that

$$C^*\Big(\bigcup_{l=0}^{L-1}\Phi_p^l\big(\Omega(F)\big)\Big) \subseteq \operatorname{span}\Omega(G_{F,L})$$

as required to establish (4.4.3). This proves our earlier claim.

We now calculate $\operatorname{rank}\big(\operatorname{span}\Omega(G_{F,L})\big)$. Since $G_{F,L} \in \operatorname{FS}(\Lambda^{m+Lp})$, Lemma 4.4.4 implies that $\operatorname{rank}\big(\operatorname{span}\Omega(G_{F,L})\big) = |G_{F,L}|$. Hence, for every $\varepsilon > 0$, we continue from (4.4.3) to see that

$$\operatorname{rcp}_{C^*(\Lambda)^\gamma}\Big(\bigcup_{l=0}^{L-1}\Phi_p^l\big(\Omega(F)\big),\varepsilon\Big) \leq |G_{F,L}|. \tag{4.4.4}$$

Recall from (4.4.2) that

$$\operatorname{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) = \sup_{n\in\mathbb{N}^k, F\in\operatorname{FS}(\Lambda^n)} \operatorname{ht}\big(\Phi_p|_{C^*(\Lambda)^\gamma}, \Omega(F)\big).$$

Then by definition of the topological entropy of a completely positive map we have

$$\operatorname{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) = \sup_{n\in\mathbb{N}^k, F\in\operatorname{FS}(\Lambda^n)}\sup_{\varepsilon>0}\limsup_{L\to\infty}\frac{1}{L}\log\Big(\operatorname{rcp}_{C^*(\Lambda)^\gamma}\big(\bigcup_{l=0}^{L-1}\Phi_p^l(\Omega(F)),\varepsilon\big)\Big).$$

Since (4.4.4) holds for all $\varepsilon > 0$, all $n \in \mathbb{N}^k$, and all $F \in \operatorname{FS}(\Lambda^n)$, we obtain

$$\operatorname{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) \leq \sup_{n\in\mathbb{N}^k, F\in\operatorname{FS}(\Lambda^n)}\limsup_{L\to\infty}\frac{1}{L}\log|G_{F,L}|. \tag{4.4.5}$$

Fix $v \in \Lambda^0$. We claim that for every $F \in \operatorname{FS}(\Lambda^n)$ there exists $N \in \mathbb{N}^k$ such that $|G_{F,L}| \leq |\Lambda^{N+n+Lp}v|$. For each $\theta \in G_{F,L}$, since Λ is irreducible there exists a finite path θ' such that $\theta\theta' \in \Lambda^{d(\theta')+n+Lp}v$. Define $N = \max_{\theta\in G_{F,L}}\{d(\theta')\}$ and $G_{F,L,v} =$

$\{\theta''\theta\theta' \in \Lambda^{N+n+Lp}v : \theta'' \in \Lambda^{N-d(\theta')}r(\theta)\}$. Then $|G_{F,L}| \leq |G_{F,L,v}| \leq |\Lambda^{N+n+Lp}v|$. Hence (4.4.5) becomes

$$\begin{aligned}\operatorname{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) &\leq \sup_{n\in\mathbb{N}^k, F\in\operatorname{FS}(\Lambda^n)} \limsup_{L\to\infty} \frac{1}{L}\log|\Lambda^{N+n+Lp}v| \\ &= \sup_{n\in\mathbb{N}^k} \limsup_{L\to\infty} \frac{1}{L}\log|\Lambda^{N+n+Lp}v|.\end{aligned}$$

Then Lemma 4.2.6 implies that

$$\operatorname{ht}(\Phi_p|_{C^*(\Lambda)^\gamma}) \leq \sup_{n\in\mathbb{N}^k} \limsup_{L\to\infty} \frac{1}{L}\log|\Lambda^{Lp}v| = \limsup_{L\to\infty} \frac{1}{L}\log|\Lambda^{Lp}v|. \qquad \blacksquare$$

4.4.2. Topological entropy of the shift map. In 1971, Bowen introduced the notion of an (ε, n)-separated subset E of a locally compact shift space (X, σ) [**6**, Section 1]. This definition was used to extend the existing definition of topological entropy, due to Adler, Konheim and McAndrew [**1**], from compact shift spaces to locally compact shift spaces. In this section we apply Bowen's techniques to the definition of topological entropy for an action of $\mathbb{N}^k$ on a compact metric space to define the topological entropy of an action of $\mathbb{N}^k$ on a locally compact metric space. We show that the definition we obtain coincides with the existing definition for compact spaces. We conclude the section by giving conditions which force the topological entropy of the shift map, when realised as an action of $\mathbb{N}^k$ on the locally compact metric space Λ^∞, to be zero.

Bowen's definition requires the use of a metric. The metric we use is defined by Skalski and Zacharias [**44**] and built on earlier work by Kumjian and Pask [**23**]. We reproduce the details here. Let Λ be a row-finite k-graph. For each $\lambda \in \Lambda$, Kumjian and Pask [**24**, Equation (3)] define the cylinder set $Z(\lambda)$ of λ to be $\{x \in \Lambda^\infty : x\big(0, d(\lambda)\big) = \lambda\}$. Each cylinder set is nonempty and the collection of all cylinder sets is a basis of compact open sets for a topology on the space Λ^∞. In particular, for $v \in \Lambda^0$ we have that $Z(v)$ is equal to the compact subspace $v\Lambda^\infty$. It then follows that for $K \in \operatorname{FS}(\Lambda^0)$ the finite union $K\Lambda^\infty$ is compact. Skalski and Zacharias [**44**, Subsection 1.1] define a metric δ on Λ^∞ which induces this topology. To define this metric recall from Chapter 2 that for every $k, n \in \mathbb{N}$, we define $\bar{n} = \{n, \ldots, n\} \in \mathbb{N}^k$. The metric is as follows: for $x \in \Lambda^\infty$ define $\delta(x, x) = 0$ and for distinct $x, y \in \Lambda^\infty$ define $\delta(x, y) = \frac{1}{a+1}$ where $a = \min\{n \in \mathbb{N} : x(0, \bar{n}) \neq y(0, \bar{n})\}$.

We now have all the background required to apply Bowen's method in our context. Let Λ be a locally finite k-graph and for every $x, y \in \Lambda^\infty$, and $n \in \mathbb{N}^k$ define $\delta_n(x, y) = \max\{\delta\big(\sigma_m(x), \sigma_m(y)\big) : 0 \leq m \leq n\}$. Suppose $\varepsilon > 0$, $n > 0$, $K \in \operatorname{FS}(\Lambda^0)$ and $E \subset K\Lambda^\infty$. We say E is an (ε, n)-*separated subset* of $K\Lambda^\infty$, with respect to σ, if for all distinct $x, y \in E$ we have $\delta_n(x, y) > \varepsilon$. We commonly refer to such a

set as simply an (ε, n)-separated subset of $K\Lambda^\infty$. We also define the set of *maximal (ε, n)-separated subsets of $K\Lambda^\infty$* to be the set

$$s_{K\Lambda^\infty}(\varepsilon, n) := \max\{|E| : E \text{ is an } (\varepsilon, n)\text{-separated subset of } K\Lambda^\infty\}.$$

Remark 4.4.6. Suppose Λ is a locally finite k-graph with no sources. Fix $\varepsilon > 0$, $n > 0$ and $K, L \in \Lambda^0$ such that $K \subseteq L$. We claim that $s_{K\Lambda^\infty}(\varepsilon, n) \leq s_{L\Lambda^\infty}(\varepsilon, n)$. Let E be an (ε, n)-separated subset of $K\Lambda^\infty$. Then by definition, for every $x, y \in E$ we have $\delta_n(x, y) > \varepsilon$. Since $E \subseteq K\Lambda^\infty \subseteq L\Lambda^\infty$ it follows that E is an (ε, n)-separated subset of $L\Lambda^\infty$. Thus $s_{K\Lambda^\infty}(\varepsilon, n) \leq s_{L\Lambda^\infty}(\varepsilon, n)$ as claimed. Moreover, we have equality precisely when $K = L$.

For a finite k-graph and the standard $\mathbb{N}^k$-action $\sigma : n \mapsto \sigma_n$ on Λ^∞ by shifts, Skalski and Zacharias define on [**44**, Page 272] the topological entropy of σ:

$$\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma) := \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n^k} \log s_{\Lambda^\infty}(\varepsilon, \bar{n}). \tag{4.4.6}$$

In this section we allow for the possibility that Λ^0 is not finite, and hence that Λ^∞ is not compact. Thus (4.4.6) does not apply. To overcome this problem we apply Bowen's method [**6**, Definition 2]. For this we take the supremum, over finite subsets K of Λ^0, of $\mathrm{h}_{\mathrm{sep}}(K\Lambda, \sigma)$ to obtain the following definition.

Definition 4.4.7. Suppose Λ is a locally finite k-graph with no sinks or sources. Let

$$\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma) = \sup_{\substack{K \subset \Lambda^0 \\ |K| < \infty}} \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n^k} \log s_{K\Lambda^\infty}(\varepsilon, \bar{n}).$$

The number $\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma)$, which may be infinite, is called the *topological entropy of σ on Λ^∞*.

Remark. Let Λ be a locally finite k-graph with $|\Lambda^0| < \infty$. Then Remark 4.4.6 implies that $s_{K\Lambda^\infty}(\varepsilon, \bar{n}) \leq s_{\Lambda^\infty}(\varepsilon, \bar{n})$ for every $K \subseteq \Lambda^0$. Hence

$$\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma) = \sup_{\varepsilon > 0} \limsup_{n \to \infty} \frac{1}{n^k} \log s_{\Lambda^\infty}(\varepsilon, \bar{n}).$$

Thus our definition of $\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma)$ agrees with (4.4.6) in this case.

Suppose Λ is a locally finite k-graph with no sinks. Fix a finite set $K \subset \Lambda^0$, $n \in \mathbb{N}$, $\varepsilon_1 > \varepsilon_2 > 0$ and an $(\varepsilon_1, \bar{n})$-separated subset E of $K\Lambda^\infty$. Fix $x, y \in E$. Then by definition of E we have $\delta_n(x, y) > \varepsilon_1 > \varepsilon_2$. Hence E is also an $(\varepsilon_2, \bar{n})$-separated subset of $K\Lambda^\infty$.

For every $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that $\frac{1}{l+2} < \varepsilon$. Hence the preceding paragraph implies that for $n \in \mathbb{N}^k$ we have $s_{K\Lambda^\infty}(\varepsilon, \bar{n}) < s_{K\Lambda^\infty}\big(\frac{1}{l+2}, \bar{n}\big)$. Thus we may express Definition 4.4.7 as

$$\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma) = \sup_{\substack{K \subset \Lambda^0 \\ |K| < \infty}} \lim_{l \to \infty} \limsup_{n \to \infty} \frac{1}{n^k} \log s_{K\Lambda^\infty}\Big(\frac{1}{l+2}, \bar{n}\Big). \tag{4.4.7}$$

The following lemma enables us to calculate $\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma)$ in terms of the number of paths in the underlying k-graph Λ.

Lemma 4.4.8. *Suppose Λ is a locally finite k-graph with no sinks. Then*

$$\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma) = \sup_{\substack{K \subset \Lambda^0 \\ |K| < \infty}} \lim_{l \to \infty} \limsup_{n \to \infty} \frac{1}{n^k} \log |K\Lambda^{\overline{l+n}}|.$$

Proof. Fix $l, n \in \mathbb{N}$. From (4.4.7) we see that to prove the lemma it suffices to show $s_{K\Lambda^\infty}\big(\frac{1}{l+2}, \bar{n}\big) = |K\Lambda^{\overline{l+n}}|$. For each $\lambda \in K\Lambda^{\overline{l+n}}$ we fix $x_\lambda \in K\Lambda^\infty$ such that $x(0, \overline{l+n}) = \lambda$ and let $E = \{x_\lambda : \lambda \in K\Lambda^{\overline{l+n}}\}$. Since $|E| = |K\Lambda^{\overline{l+n}}|$, it suffices to show that $|E| = \max\{|F| : F \text{ is an } \big(\frac{1}{l+2}, \bar{n}\big)\text{-separated subset of } K\Lambda^\infty\}$.

We first show that E is an $\big(\frac{1}{l+2}, \bar{n}\big)$-separated subset of $K\Lambda^\infty$. To this end fix distinct $\mu, \nu \in K\Lambda^\infty$. We must show $\delta_n(x_\mu, x_\nu) > \frac{1}{l+2}$. Recall that by definition we have

$$\delta_n(x_\mu, x_\nu) = \max\big\{\delta\big(\sigma_m(x_\mu), \sigma_m(x_\nu)\big) : 0 \le m \le \bar{n}\big\}. \tag{4.4.8}$$

For each $m \le \bar{n}$, the definition of δ says that

$$\delta\big(\sigma_m(x_\mu), \sigma_m(x_\nu)\big) = \begin{cases} 0 & \text{if } \sigma_m(x_\mu) = \sigma_m(x_\nu), \\ \frac{1}{a_m+1} & \text{if } \sigma_m(x_\mu) \ne \sigma_m(x_\nu). \end{cases}$$

where $a_m = \min\{N \in \mathbb{N} : \sigma_m(x_\mu)(0, \overline{N}) \ne \sigma_m(x_\nu)(0, \overline{N})\}$. Since $x_\mu(0, \overline{l+n}) = \mu \ne \nu = x_\nu(0, \overline{l+n})$ there exists $M \in \mathbb{N}^k$ such that $M \le \bar{n}$ and $\sigma_M(x_\mu)(0, \bar{l}) \ne \sigma_M(x_\nu)(0, \bar{l})$. Hence $0 < a_M \le l$. Then from (4.4.8) we see that

$$\delta_n(x_\mu, x_\nu) \ge \frac{1}{a_M + 1} \ge \frac{1}{l+1} > \frac{1}{l+2}.$$

Showing that E is an $\big(\frac{1}{l+2}, \bar{n}\big)$-separated subset of $K\Lambda^\infty$.

We now show $|E| = \max\{|F| : F \text{ is an } \big(\frac{1}{l+2}, \bar{n}\big)\text{-separated subset of } K\Lambda^\infty\}$. Suppose for contradiction that F is an $\big(\frac{1}{l+2}, \bar{n}\big)$-separated subset of $K\Lambda^\infty$ such that $|F| > |K\Lambda^{\overline{l+n}}|$. Then there exist distinct $x, y \in F$ such that $x(0, \overline{l+n}) = y(0, \overline{l+n})$. Thus for each $m \le \bar{n}$ we have $\sigma_m(x) = \sigma_m(y)$ and hence $a_m \ge l+1$. Then for each $m \le \bar{n}$ it follows that $\delta\big(\sigma_m(x), \sigma_m(y)\big) \le \frac{1}{l+2}$. Hence $\delta_n(x, y) \le \frac{1}{l+2}$, contradicting that F is an $(\frac{1}{l+2}, \bar{n})$-separated subset of $K\Lambda^\infty$. Thus we have now shown that $s_{K\Lambda^\infty}\big(\frac{1}{l+2}, \bar{n}\big) = |K\Lambda^{\overline{l+n}}|$, completing the proof. ■

In order to introduce the main results of this subsection, we require the following definition.

Definition 4.4.9. For a k-graph Λ and $v \in \Lambda^0$ we let $\mathrm{val}(v) = \sum_{i\in\{1,\dots,k\}} |v\Lambda^{e_i}|$ and call $\mathrm{val}(v)$ the *valency of* v.

There are two main results of this subsection. The first proposition tells us that for a k-graph with $k \geq 2$ and no sources, if there exists a global upper bound on the valency of each vertex, then the topological entropy of σ equals 0.

Proposition 4.4.10. *Suppose Λ is a locally finite k-graph with $k \geq 2$ and no sources. Also suppose there exists $N \in \mathbb{N}$ with the property that for each $v \in \Lambda^0$ and $i \in \{1,\dots,k\}$ we have $|v\Lambda^{e_i}| \leq N$. Then $\mathrm{h}_{\mathrm{sep}}(\Lambda,\sigma) = 0$.*

If $k \geq 3$ then we can say more: That is, we just need a global upper bound on $\mathrm{val}(v)/\mathrm{val}(u)$ when $u\Lambda^{e_i}v \neq \emptyset$ for each i.

Proposition 4.4.11. *Suppose Λ is a locally finite k-graph with $k \geq 3$ and no sources. Also suppose there exists $N \in \mathbb{N}$ such that, for each $u, v \in \Lambda^0$ and $i \in \{1,\dots,k\}$ such that $u\Lambda^{e_i}v \neq \emptyset$, we have $|v\Lambda^{e_i}| \leq N|u\Lambda^{e_i}|$. Then $\mathrm{h}_{\mathrm{sep}}(\Lambda,\sigma) = 0$.*

In order to prove Proposition 4.4.10 and Proposition 4.4.11 we require the following lemma.

Lemma 4.4.12. *Suppose Λ is a k-graph with no sources. Then for every $v \in \Lambda^0$, non-zero $n \in \mathbb{N}^k$ and $i \in \{1,\dots,k\}$ we have $|v\Lambda^{n+e_i}| \leq |v\Lambda^n| \cdot \max_{\lambda\in v\Lambda^n} |s(\lambda)\Lambda^{e_i}|$.*

Proof. Fix $v \in \Lambda^0$, non-zero $n \in \mathbb{N}^k$, and $i \in \{1,\dots,k\}$. Then the factorisation property implies $|v\Lambda^{n+e_i}| = \sum_{\lambda\in v\Lambda^n} |\lambda\Lambda^{e_i}|$. Thus we obtain the inequality

$$|v\Lambda^{n+e_i}| = \sum_{\lambda\in v\Lambda^n} |\lambda\Lambda^{e_i}| = \sum_{\lambda\in v\Lambda^n} |s(\lambda)\Lambda^{e_i}| \leq |v\Lambda^n| \cdot \max_{\lambda\in v\Lambda^n} |s(\lambda)\Lambda^{e_i}|. \qquad \blacksquare$$

We now prove the first proposition of this subsection.

Proof of Proposition 4.4.10. Fix $p \in \mathbb{N}^k$ and finite $K \subseteq \Lambda^0$. We claim that for every $n \in \mathbb{N}^k$ and every $v \in K$ we have $|v\Lambda^n| \leq N^{|n|}$. We proceed by induction.

For a base case consider $n = 0$. Then $|v\Lambda^0| = |v| = 1 = N^0$. Now suppose that $|v\Lambda^m| \leq N^{|m|}$ for some $m \in \mathbb{N}^k$. To prove our claim we must show that $|v\Lambda^{m+e_i}| \leq N^{|m+e_i|}$ for every $i \in \{1,\dots,k\}$.

Fix $i \in \{1,\dots,k\}$. Then Lemma 4.4.12 implies

$$|v\Lambda^{m+e_i}| \leq |v\Lambda^m| \cdot \max_{\lambda\in v\Lambda^m} |s(\lambda)\Lambda^{e_i}|.$$

By hypothesis we have that $\max_{\lambda\in v\Lambda^m} |s(\lambda)\Lambda^{e_i}| \leq N$ and hence $|v\Lambda^{m+e_i}| \leq |v\Lambda^m|N$. Then the inductive hypothesis implies $|v\Lambda^{m+e_i}| \leq N^{|m|+1} \leq N^{|m+e_i|}$. This completes the proof our claim.

For each $l, n \in N$ we now observe that $|K\Lambda^{\overline{l+n}}| = \sum_{u\in K} |u\Lambda^{\overline{l+n}}|$. Then the above claim shows that $|K\Lambda^{\overline{l+n}}| \leq \sum_{u\in K} N^{|\overline{l+n}|}$ and hence $|K\Lambda^{\overline{l+n}}| \leq |K|N^{k(l+n)}$. Thus

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n^k} \log |K\Lambda^{\overline{l+n}}| &\leq \limsup_{n\to\infty} \frac{1}{n^k} \log(|K|N^{k(l+n)}) \\ &= \limsup_{n\to\infty} \frac{1}{n^k} \big(\log |K| + k(l+n) \log N \big).\end{aligned}$$

Since $k \geq 2$ we have that $\frac{n}{n^k} \to 0$. Then the algebra of limits implies that $\limsup_{n\to\infty} \frac{1}{n^k} \log |K\Lambda^{\overline{l+n}}| = 0$ and hence Lemma 4.4.8 implies $\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma) = 0$. ■

To prove Proposition 4.4.11 we use the following lemma.

Lemma 4.4.13. *Suppose Λ is a locally finite k-graph with $k \geq 3$ and no sources. Also suppose there exists $N \in \mathbb{N}$ such that, for each $u, v \in \Lambda^0$ and $i \in \{1, \ldots, k\}$ such that $u\Lambda^{e_i}v \neq \emptyset$, we have $|v\Lambda^{e_i}| \leq N|u\Lambda^{e_i}|$. Then for every $j \in \{1, \ldots, k\}$ and every $\lambda \in \Lambda$ we have that $|s(\lambda)\Lambda^{e_j}| \leq N^{|d(\lambda)|} \max_{h\in\{1,\ldots,k\}}\{r(\lambda)\Lambda^{e_h}\}$.*

Proof. We proceed by induction on $|d(\lambda)|$. For a base case suppose $|d(\lambda)| = 0$. That is, $\lambda \in \Lambda^0$. Fix $j \in \{1, \ldots, k\}$. Then by definition we have $s(\lambda) = r(\lambda)$ and hence $|s(\lambda)\Lambda^{e_j}| \leq \max_{h\in\{1,\ldots,k\}}\{s(\lambda)\Lambda^{e_h}\} = \max_{h\in\{1,\ldots,k\}}\{r(\lambda)\Lambda^{e_h}\}$.

Suppose the lemma holds for $|d(\lambda)| = n$. That is, there exists $m \in \mathbb{N}^k$ such that $d(\lambda) = m$ and $|m| = n$. To complete the proof we must show the lemma holds for arbitrary $|d(\lambda)| = n + 1$. That is, fix $j \in \{1, \ldots, k\}$ and let $d(\lambda) = m + e_j$. Then by the hypothesis of the lemma we have that $|s(\lambda)\Lambda^{e_j}| \leq N|\lambda(n)\Lambda^{e_j}|$. Then the inductive hypothesis implies

$$|s(\lambda)\Lambda^{e_j}| \leq N\big(N^{|n|} \max_{h\in\{1,\ldots,k\}}\{r(\lambda)\Lambda^{e_h}\}\big) = N^{|d(\lambda)|} \max_{h\in\{1,\ldots,k\}}\{r(\lambda)\Lambda^{e_h}\}$$

Since our choice of j was arbitary, this completes the proof. ■

We are now in a position to prove Proposition 4.4.11

Proof of Proposition 4.4.11. We first observe that for $k = 2$ then the hypothesis of Proposition 4.4.10 are satisfied and hence the result follows trivially. Suppose $k \geq 3$. Fix $p \in \mathbb{N}^k$ and finite $K \subset \Lambda^0$. Let $a = \max_{v\in K, i\in\{1,\ldots,k\}} |v\Lambda^{e_i}|$. We claim that for every $m \in \mathbb{N}^k$ and every $v \in K$ we have $|v\Lambda^m| \leq a^{|m|}N^{\frac{(|m|-1)|m|}{2}}$. Fix $v \in K$. We prove the claim by induction on $|m|$. For a base case suppose $|m| = 0$. Then we have the equality $|v| = 1 = a^0N^0$.

Suppose the claim holds for some $|m| > 0$. Fix $j \in \{1, \ldots, k\}$ To complete the proof of our claim it suffices to show $|v\Lambda^{m+e_j}| \leq a^{|m|+1}N^{\frac{|m|(|m|+1|)}{2}}$. To see this we first observe that Lemma 4.4.12 implies

$$|v\Lambda^{m+e_j}| \leq |v\Lambda^m| \cdot \max_{\lambda\in v\Lambda^m} \{|s(\lambda)\Lambda^{e_i}|\}.$$

Next we notice that Lemma 4.4.13 implies each $|s(\lambda)\Lambda^{e_i}| \leq N^{|m|}a$. Thus

$$|v\Lambda^{m+e_j}| \leq |v\Lambda^m|N^{|m|}a.$$

Then the inductive hypothesis implies

$$|v\Lambda^{m+e_j}| \leq a^{|m|}N^{\frac{(|m|-1)|m|}{2}}N^{|m|}a = a^{|m|+1}N^{\frac{|m|(|m|+1)}{2}}.$$

This completes the proof of our claim.

For each $l, n \in N$, we have that $|K\Lambda^{\overline{l+n}}| = \sum_{v\in K}|v\Lambda^{\overline{l+n}}|$. The above claim implies that each $|v\Lambda^{\overline{l+n}}| \leq a^{k(l+n)}N^{\frac{1}{2}k(l+n)(k(l+n)+1)}$. Hence we have that $|K\Lambda^{\overline{l+n}}| \leq |K|a^{k(l+n)}N^{\frac{1}{2}k(l+n)(k(l+n)+1)}$. Thus

$$\limsup_{n\to\infty}\frac{1}{n^k}\log|K\Lambda^{\overline{l+n}}| \leq \limsup_{n\to\infty}\frac{1}{n^k}\log(|K|a^{k(l+n)}N^{\frac{1}{2}(k(l+n)+1)^2}). \tag{4.4.9}$$

Since

$$\log(|K|a^{k(l+n)}N^{\frac{1}{2}(k(l+n)+1)^2}) = \log|K| + k(l+n)\log a + \frac{1}{2}(k(l+n)+1)^2\log N,$$

the algebra of limits implies that (4.4.9) becomes

$$\limsup_{n\to\infty}\frac{1}{n^k}\log|K\Lambda^{\overline{l+n}}| \leq \limsup_{n\to\infty}\frac{(k(l+n)+1)^2}{2n^k}\log N.$$

Since $k \geq 3$ we have $\frac{(k(l+n)+1)^2}{2n^k} \to 0$. Then Lemma 4.4.8 implies $\mathrm{h}_{\mathrm{sep}}(\Lambda, \sigma) = 0$. ■

4.5. Completely-positive approximation rank of a C^*-algebra

In the final section of this thesis our focus turns to calculating the completely-positive approximation rank of a C^*-algebra. We prove Proposition 4.5.1: if Ω is a finite subset of A such that $C^*(\Omega)$ is a simple, finite dimensional C^*-algebra, then for any complete set of matrix units E for $C^*(\Omega)$, and for small enough $\varepsilon > 0$, the completely positive ε-approximation rank $\mathrm{rcp}(E, \varepsilon)$ coincides with the rank of $C^*(\Omega)$ as a matrix algebra.

We have seen that when Λ is an infinite k-graph, $C^*(\Lambda)$ is non-unital and we have been unable to calculate the precise entropy $\mathrm{ht}(\Phi_p|_{C^*(\Lambda)^\gamma})$ where Φ_p is the p-shift map. As a first step towards achieving this, at least in some useful examples, we prove the following proposition which calculates completely-positive approximations of simple C^*-subalgebras of nuclear C^*-algebras. In particular, simple C^*-subalgebras of $C^*(\Lambda)^\gamma$.

Proposition 4.5.1. *Suppose A is a nuclear C^*-algebra, $\Omega \in \mathrm{FS}(A)$ and suppose that $C^*(\Omega)$ is a simple, finite dimensional C^*-subalgebra of A; say $\dim\big(C^*(\Omega)\big) = n^2$. Let E be a complete set of matrix units for $C^*(\Omega)$. Then for all $\varepsilon \leq \frac{1}{4n^2}$ we have*

$$\mathrm{rcp}_A(E, \varepsilon) = \mathrm{rank}\, C^*(\Omega).$$

PROOF. Since $C^*(\Omega)$ is simple, we see that $\operatorname{rank} C^*(\Omega) = n$. Then Lemma 4.2.14 implies $\operatorname{rcp}_A(E,\varepsilon) \leq n$ for all $E \in \operatorname{FS}\big(C^*(\Omega)\big)$ and $\varepsilon > 0$.

We now prove that $\operatorname{rcp}_A(E,\varepsilon) \geq n$ for every $\varepsilon < \frac{1}{4n^2}$. Suppose B is another C^*-algebra such that $\operatorname{rank} B < n$. Fix $\varepsilon < \frac{1}{4n^2}$, completely positive maps $\theta : A \to B$, $\psi : B \to A$ such that $\big(\psi, \theta, C^*(\Omega)\big) \in \operatorname{CPA}(A)$ and recall that $E = \{E_{ij} : i,j = 1,\dots,n\}$ is a complete set of matrix units for $C^*(\Omega)$. We show that $(\psi,\theta,B) \notin \operatorname{CPA}(A,E,\varepsilon)$, which will complete the proof. Suppose for contradiction that ψ and θ are completely positive maps such that $(\psi,\theta,B) \in \operatorname{CPA}(A,E,\varepsilon)$. That is, for each $E_{ij} \in E$ we have $\|\psi\big(\theta(E_{ij})\big) - E_{ij}\| < \varepsilon$.

We first observe that $\dim C^*(\Omega) = n^2$ implies $|E| = n^2$. Since $\operatorname{rank} B < n$, it follows that $\dim B < n^2$ and hence $|E| > \dim B$. Hence there exists $E_{kl} \in E$ and scalars $\{a_{ij} : (i,j) \neq (k,l)\}$ such that $\theta(E_{kl}) = \sum_{(i,j)\neq(k,l)} a_{ij}\theta(E_{ij})$. Since ψ is linear it follows that

$$\psi\big(\theta(E_{kl})\big) = \psi\Big(\sum_{(i,j)\neq(k,l)} a_{ij}\theta(E_{ij})\Big) = \sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big). \tag{4.5.1}$$

Consider $E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll} \in C^*(\Omega)$. We calculate

$$\begin{aligned}\big\|E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll}\big\| &= \big\|E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll} - E_{kl} + E_{kl}\big\| \\ &= \Big\|E_{kk}\Big(\psi\big(\theta(E_{kl})\big) - E_{kl}\Big)E_{ll} + E_{kl}\Big\|.\end{aligned}$$

Then the reverse triangle inequality implies

$$\begin{aligned}\big\|E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll}\big\| &\geq \Big|\big\|E_{kk}\big\|\big\|\psi\big(\theta(E_{kl})\big) - E_{kl}\big\|\big\|E_{ll}\big\| - \|E_{kl}\|\Big| \\ &= \Big|\big\|\psi\big(\theta(E_{kl})\big) - E_{kl}\big\| - 1\Big|. \end{aligned}\tag{4.5.2}$$

By definition of E we have $\big\|\psi\big(\theta(E_{kl})\big) - E_{kl}\big\| < \varepsilon$. Hence (4.5.2) becomes

$$\big\|E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll}\big\| \geq 1 - \big\|\psi\big(\theta(E_{kl})\big) - E_{kl}\big\| \geq 1 - \varepsilon > 1 - \frac{1}{4n^2}. \tag{4.5.3}$$

On the other hand, (4.5.1) implies that

$$E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll} = \sum_{(i,j)\neq(k,l)} a_{ij}E_{kk}\psi\big(\theta(E_{ij})\big)E_{ll}.$$

Since we have $E_{jj}E_{ll} = 0$ for each $j \neq l$, it follows that

$$E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll} = \sum_{(i,j)\neq(k,l)} a_{ij}E_{kk}\Big(\psi\big(\theta(E_{ij})\big) - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\Big)E_{ll}.$$

Hence

$$\begin{aligned}\big\|E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll}\big\| &= \Big\|\sum_{(i,j)\neq(k,l)} a_{ij}E_{kk}\Big(\psi\big(\theta(E_{ij})\big) - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\Big)E_{ll}\Big\| \\ &\leq \|E_{kk}\|\Big\|\sum_{(i,j)\neq(k,l)} a_{ij}\Big(\psi\big(\theta(E_{ij})\big) - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\Big)\Big\|\|E_{ll}\| \\ &\leq \Big\|\sum_{(i,j)\neq(k,l)} a_{ij}\Big(\psi\big(\theta(E_{ij})\big) - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\Big)\Big\|.\end{aligned}$$

Then the triangle inequality implies

$$\big\|E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll}\big\| \leq \sum_{(i,j)\neq(k,l)} |a_{ij}|\big\|\psi\big(\theta(E_{ij})\big) - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\big\|. \tag{4.5.4}$$

For each pair (i,j) such that $i,j \leq n$ we have

$$\begin{aligned}\big\|\psi\big(\theta(E_{ij})\big) - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\big\| &= \Big\|\Big(\psi\big(\theta(E_{ij})\big) - E_{ij}\Big) + \Big(E_{ij} - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\Big)\Big\| \\ &\leq \big\|\psi\big(\theta(E_{ij})\big) - E_{ij}\big\| + \big\|E_{ij} - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\big\| \\ &\leq \big\|\psi\big(\theta(E_{ij})\big) - E_{ij}\big\| + \|E_{ii}\|\big\|E_{ij} - \psi\big(\theta(E_{ij})\big)\big\|\big\|E_{jj}\big\|.\end{aligned}$$

Since $(\psi,\theta,B) \in \mathrm{CPA}(A,E,\varepsilon)$ and $\varepsilon < \frac{1}{4n^2}$, we have for each (i,j) that

$$\big\|\psi\big(\theta(E_{ij})\big) - E_{ii}\psi\big(\theta(E_{ij})\big)E_{jj}\big\| < \varepsilon + \varepsilon < \frac{1}{4n^2} + \frac{1}{4n^2} = \frac{1}{2n^2}.$$

Hence (4.5.4) implies

$$\big\|E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll}\big\| \leq \sum_{(i,j)\neq(k,l)} |a_{ij}|\frac{1}{2n^2} = \frac{1}{2n^2}\sum_{(i,j)\neq(k,l)} |a_{ij}|.$$

Then the above inequality, together with (4.5.3) implies

$$1 - \frac{1}{4n^2} \leq \big\|E_{kk}\psi\big(\theta(E_{kl})\big)E_{ll}\big\| \leq \frac{1}{2n^2}\sum_{(i,j)\neq(k,l)} |a_{ij}|. \tag{4.5.5}$$

We claim (4.5.5) is a contradiction. To see this we calculate an upper bound for $\sum_{(i,j)\neq(k,l)} |a_{ij}|$.

Recall that $\big\|\psi\big(\theta(E_{kl})\big) - E_{kl}\big\| < \varepsilon < \frac{1}{4n^2}$. If A is a matrix then

$$\|A\| = \|E_{1i}\|\|A\|\|E_{j1}\| \geq \|E_{1i}AE_{j1}\| = |a_{ij}|.$$

Thus the absolute value of each entry in the matrix $\psi\big(\theta(E_{kl})\big) - E_{kl}$ must be less than $\frac{1}{4n^2}$. Fix $(i_0,j_0) \neq (k,l)$ such that $|a_{i_0j_0}| = \max\{|a_{ij}| : (i,j) \neq (k,l)\}$. Since $(i_0,j_0) \neq (k,l)$,

$$\frac{1}{4n^2} > \Big|\Big(\sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big) - E_{kl}\Big)_{i_0j_0}\Big| = \Big|\sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big)_{i_0j_0}\Big|. \tag{4.5.6}$$

However by hypothesis we have $\big\|\psi\big(\theta(E_{ij})\big) - E_{ij}\big\| < \frac{1}{4n^2}$ for each $(i,j) \neq (k,l)$. Thus for each $(i,j) \notin \{(k,l),(i_0,j_0)\}$,

$$\big|\psi\big(\theta(E_{ij})\big)_{i_0j_0}\big| = \Big|\Big(\psi\big(\theta(E_{ij})\big) - E_{ij}\Big)_{i_0j_0}\Big| < \frac{1}{4n^2}.$$

Hence $\psi\big(\theta(E_{ij})\big)_{i_0j_0} \in \big\{b \in \mathbb{C} : |b| < \frac{1}{4n^2}\big\}$. Similarly, for (i_0,j_0) we have

$$\Big|\Big(\psi\big(\theta(E_{i_0j_0})\big) - E_{i_0j_0}\Big)_{i_0j_0}\Big| < \frac{1}{4n^2}$$

which implies $\psi\big(\theta(E_{i_0j_0})\big)_{i_0j_0} \in \big\{b \in \mathbb{C} : |b-1| < \frac{1}{4n^2}\big\}$. Thus we calculate

$$\begin{aligned}\Big|\Big(\sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big)\Big)_{i_0j_0}\Big| = \Big|a_{i_0j_0}\psi\big(\theta(E_{i_0j_0})\big)_{i_0j_0} \\ + \sum_{(i,j)\notin\{(k,l),(i_0,j_0)\}} a_{ij}\psi\big(\theta(E_{ij})\big)_{i_0j_0}\Big|,\end{aligned}$$

whence the reverse triangle inequality implies

$$\begin{aligned}\Big|\Big(\sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big)\Big)_{i_0j_0}\Big| &\geq \Big|\big|a_{i_0j_0}\psi\big(\theta(E_{i_0j_0})\big)_{i_0j_0}\big| \\ &\qquad - \Big|\sum_{(i,j)\notin\{(k,l),(i_0,j_0)\}} a_{ij}\psi\big(\theta(E_{ij})\big)_{i_0j_0}\Big|\Big| \\ &\geq \big|a_{i_0j_0}\psi\big(\theta(E_{i_0j_0})\big)_{i_0j_0}\big| \\ &\qquad - \Big|\sum_{(i,j)\notin\{(k,l),(i_0,j_0)\}} a_{ij}\psi\big(\theta(E_{ij})\big)_{i_0j_0}\Big|.\end{aligned}$$

Next we apply the triangle inequality to the second summand to obtain

$$\begin{aligned}\Big|\Big(\sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big)\Big)_{i_0j_0}\Big| \geq \big|a_{i_0j_0}\psi\big(\theta(E_{i_0j_0})\big)_{i_0j_0}\big| \\ - \sum_{(i,j)\notin\{(k,l),(i_0,j_0)\}} |a_{ij}|\big|\psi\big(\theta(E_{ij})\big)_{i_0j_0}\big|.\end{aligned}$$

Since $a_{i_0j_0}$ was chosen such that $|a_{i_0j_0}| = \max\{|a_{ij}| : (i,j) \neq (k,l)\}$, we may replace each $|a_{ij}|$ with $|a_{i_0j_0}|$ to obtain the inequality

$$\begin{aligned}\Big|\Big(\sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big)\Big)_{i_0j_0}\Big| &\geq \big|a_{i_0j_0}\psi\big(\theta(E_{i_0j_0})\big)_{i_0j_0}\big| \\ &\qquad - \sum_{(i,j)\notin\{(k,l),(i_0,j_0)\}} |a_{i_0j_0}|\big|\psi\big(\theta(E_{ij})\big)_{i_0j_0}\big| \\ &= |a_{i_0j_0}|\Big(\big|\psi\big(\theta(E_{i_0j_0})\big)_{i_0j_0}\big| \\ &\qquad - \sum_{(i,j)\notin\{(k,l),(i_0,j_0)\}} \big|\psi\big(\theta(E_{ij})\big)_{i_0j_0}\big|\Big).\end{aligned} \tag{4.5.7}$$

Since $\psi\big(\theta(E_{i_0j_0})\big)_{i_0j_0} > 1 - \frac{1}{4n^2}$ and each $\psi\big(\theta(E_{ij})\big)_{i_0j_0} < \frac{1}{4n^2}$, the inequality (4.5.7) becomes

$$\begin{aligned}\Big|\Big(\sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big)\Big)_{i_0j_0}\Big| &> |a_{i_0j_0}|\Big|(1-\frac{1}{4n^2}) - \sum_{(i,j)\notin\{(k,l),(i_0,j_0)\}}\frac{1}{4n^2}\Big| \\ &\geq |a_{i_0j_0}||1-\frac{1}{4n^2}-\frac{1}{4}| \\ &= |a_{i_0j_0}|\frac{3n^2-1}{4n^2}. \end{aligned} \tag{4.5.8}$$

By (4.5.6), we have $\frac{1}{4n^2} > \Big|\Big(\sum_{(i,j)\neq(k,l)} a_{ij}\psi\big(\theta(E_{ij})\big)\Big)_{i_0j_0}\Big|$, and this combined with (4.5.8) shows that $|a_{i_0j_0}| < \frac{1}{3n^2-1}$. Substituting $|a_{i_0j_0}| < \frac{1}{3n^2-1}$ into (4.5.5) yields

$$1-\frac{1}{4n^2} \leq \frac{1}{2n^2}\sum_{(i,j)\neq(k,l)}|a_{ij}| < \frac{1}{2(3n^2-1)}. \tag{4.5.9}$$

In particular, when $n=1$ we have the contradiction $\frac{3}{4} = 1-\frac{1}{4} \leq 1-\frac{1}{4n^2} < \frac{1}{2(3n^2-1)} < \frac{1}{2(3-1)} = \frac{1}{4}$.

We have now proved that $(\psi,\theta,B) \in \mathrm{CPA}(A,E,\varepsilon)$ implies $\operatorname{rank} B \geq n$ whenever $\varepsilon < \frac{1}{4n^2}$ as required. This result, together with the inequality $\mathrm{rcp}(E,\varepsilon) \leq n$ for all $\varepsilon > 0$, completes the proof. ■

APPENDIX A

A counter example to a lemma of Jeong and Park

A.1. Linear independence in the core

In this appendix we provide a counter example to [**21**, Lemma 3.7], which is used to prove a key result of [**20**], namely [**20**, Theorem 4.3]. Note that for a locally finite directed graph E with no sources, Jeong and Park use a slightly different definition of a Cuntz-Krieger E-family. The effect of changing between the two formulations is that the direction of all the edges in E are reversed. More importantly, the graph algebras generated by the two different Cuntz-Krieger E-families are isomorphic. In order to maintain consistency throughout this thesis we continue to use the Cuntz-Krieger relations defined in Section 4.3.

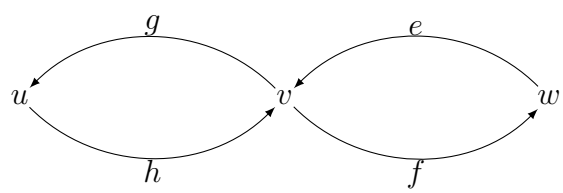
We restate [**21**, Lemma 3.7] for convenience: Let v be a vertex of an irreducible graph E with at least two vertices, and let $n \geq 1$. Then the set

$$\omega(n, v) = \{s_\alpha s_\beta^* : s(\alpha) = s(\beta) = v, |\alpha| = |\beta| \leq n\} \tag{A.1.1}$$

is linearly independent.

We claim that the following is a counterexample to the above statement.

Example A.1.1. *Consider the directed graph E given below.*



We claim $\omega(2, v)$ is linearly dependent. From equation (A.1.1),

$$\omega(2, v) = \{p_v, s_f s_f^*, s_g s_g^*, s_f s_g^*, s_g s_f^*, s_{ef} s_{ef}^*, s_{ef} s_{hg}^*, s_{hg} s_{hg}^*, s_{hg} s_{ef}^*\}.$$

Then (DG2) from page 70 implies

$$p_v = \sum_{\lambda \in E^2 v} s_\lambda s_\lambda^* = s_{ef} s_{ef}^* + s_{hg} s_{hg}^*.$$

Hence $\omega(2, v)$ is ***not*** *linearly independent.*

The issue in the proof of [**21**, Lemma 3.7] arises in the inductive step, which is paragraph two of the proof. There it is claimed that if x is a non-trivial linear

combination of elements in $\omega(n-1,v)$ and is equal 0, then for every edge $\rho \in E^1$ we have

$$s_\rho^* x s_\rho = \sum_{\alpha=\rho\alpha',\beta=\rho\beta'} \lambda_{\alpha\beta} s_\rho^* s_\alpha s_\beta^* s_\rho = \sum_{|\alpha'|=|\beta'|\leq n-1} \lambda_{(\rho\alpha')(\rho\beta')} s_{\alpha'} s_{\beta'}^*. \tag{A.1.2}$$

However the first equality does not hold in general. In particular, it is the case when $d(\alpha)=0$ that was ignored in [**21**]. This equality implicitly asserts that $s_\rho^* s_\alpha s_\beta^* s_\rho = 0$ unless $\alpha = \rho\alpha'$ and $\beta = \rho\beta'$. However if $\alpha = \beta = s(\rho)$, then $s_\rho^* s_\alpha s_\beta^* s_\rho = s_\rho^* s_{s(\rho)} s_\rho = p_{r(\rho)}$. In particular, in Example A.1.1 let $x = p_v - s_{ef}s_{ef}^* - s_{hg}s_{hg}^* = 0$ and $\rho = e$. Then $s_e^* x s_e = s_e^* p_v s_e - s_e^* s_{ef} s_{ef}^* s_e - s_e^* s_{hg} s_{hg}^* s_e = s_e^* s_e - s_f s_f^*$, whereas (A.1.2) claims that $s_e^* x s_e = -s_f s_f^*$.

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