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A critique of the Granger representation theorem

Edgar J. Wilson
University of Wollongong, ewilson@uow.edu.au

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Abstract

The Granger representation theorem states that if a set of nonstationary variables are cointegrated then they can be characterized as generated by an error correction mechanism. This paper uses the continuous time equivalent representation for two variables to demonstrate the relatively large number of restrictions required to represent a cointegrating relationship as an error correction mechanism. It is shown that the restrictions result from placing too much importance on the long run, which excludes interesting and possibly important short run dynamics. This is surprising because these restrictions are at odds with the a-theoretical vector autoregressive approach, which criticises the ad-hoc specification and identification of the Cowles foundation style structural models. The second criticism relates to the justification of using cointegration because economic theories are mostly about long run relationships with little to contribute to modeling short run economic behaviour. It is argued in this paper that cointegration places too much importance on the long run and excludes interesting short run dynamics. After the formal presentation of the conditions for stability of an economic model, an exchange rate and endogenous growth examples are provided. They highlight the importance of short run dynamics in modeling economic behaviour and providing policy prescriptions. It is then shown that applying the cointegrating restrictions eliminates these short run dynamics. It is also possible that many researchers are not aware of the restrictions that this procedure forces on the parameters which are to be estimated. This paper demonstrates these restrictions on the coefficients of economic relationships.

Keywords

critique, Granger, representation, theorem

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E. J. Wilson

School of Economics
University of Wollongong
Northfields Avenue
Wollongong NSW 2500
AUSTRALIA
E-mail: edgar_wilson@uow.edu.au
Phone: +61 (0)2 42213666

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ABSTRACT

The *Granger representation theorem* states that if a set of non-stationary variables are cointegrated then they can be characterized as generated by an error correction mechanism. This paper uses the continuous time equivalent representation for two variables to demonstrate the relatively large number of restrictions required to represent a cointegrating relationship as an error correction mechanism.

It is shown that the restrictions result from placing too much importance on the long run, which excludes interesting and possibly important short run dynamics. This is surprising because these restrictions are at odds with the a-theoretical vector autoregressive approach, which criticises the *ad-hoc* specification and identification of the Cowles foundation style structural models.

The second criticism relates to the justification of using cointegration because economic theories are mostly about long run relationships with little to contribute to modeling short run economic behaviour. It is argued in this paper that cointegration places too much importance on the long run and excludes interesting short run dynamics. After the formal presentation of the conditions for stability of an economic model, an exchange rate and endogenous growth examples are provided. They highlight the importance of short run dynamics in modeling economic behaviour and providing policy prescriptions. It is then shown that applying the cointegrating restrictions eliminates these short run dynamics.

It is also possible that many researchers are not aware of the restrictions that this procedure forces on the parameters which are to be estimated. This paper demonstrates these restrictions on the coefficients of economic relationships.

Keywords: VECM, cointegration, short run, dynamics, exchange rates, endogenous growth.
JEL Classifications: .

I Introduction

The influential *Granger representation theorem* states that if a set of non-stationary variables are cointegrated then they can be characterized as being generated by an error correction mechanism. Consider the simultaneous vector autoregressive (SVAR) system for the $n \times 1$ vector of endogenous variables, $\underline{x}(t)$:

$$\underline{x}(t) = \underline{\phi} + \sum_{i=1}^{k} \Phi_i \underline{x}(t-i) + \sum_{j=0}^{l} \Psi_j \underline{z}(t-j) + \underline{u}(t) \ , \qquad t = 1, 2, ..., T \tag{1}$$

where $\underline{z}(t-j)$ is a vector of stationary exogenous variables, Φ_i and Ψ_j are $n \times n$ and $n \times m$ respectively dimensioned coefficient matrices, $\underline{\phi}$ is a $n \times 1$ vector of intercepts and $\underline{u}(t)$ is $n \times 1$ vector of disturbances with the usual *iid* properties.

According to Granger's representation theorem, (1) has an equivalent vector error correction mechanism (VECM) representation:

$$\Delta \underline{x}(t) = \underline{\phi} + \sum_{i=1}^{k-1} \Gamma_i \Delta \underline{x}(t-i) + \Pi \underline{x}(t-k) + \sum_{j=0}^{l} \Psi_j \underline{z}(t-j) + \underline{\varepsilon}(t) \tag{2}$$

with $\Pi = \sum_{i=1}^{k} \Phi_i - \mathrm{I}$, where I is the identity matrix. The rank of the Π matrix can be determined using Johansen's trace, eigenvalue and model selection criteria and it can be decomposed into $\Pi = \underline{\alpha\beta}'$. The $r \times n$ dimensioned $\underline{\beta}$ matrix gives the $r \times 1$ cointegrating vectors $\underline{\beta}' \underline{x}(t)$, which are stationary, $I(0)$.

The SVAR given by (1) and the VECM in (2) are powerful analytic devices which have had major impacts on how empirical research is conducted.[1] Researchers working with non-stationary time series are required to transform the variables

[1] The (recursively or non-recursively) identified Sim (1980) SVAR procedure uses impulse responses to trace the intertemporal effects of shocks on variables and variance decomposition to analyse the contribution of a shock to one variable on the forecast variance of other variables. There has been surprisingly little criticism of the non-stationary issues inherent with the specification (1), perhaps due to the system being identified to be stable with all characteristic roots lying within the unit circle.

to first differences to avoid the charges of estimating spurious regressions and making incorrect statistical inferences using the estimated standard errors. However, empirical work using distributed lags of variables in first differences:

$$\Delta \underline{x}(t) = \underline{\phi} + \sum_{i=1}^{k-1} \Gamma_i \Delta \underline{x}(t-i) + \sum_{j=0}^{l} \Psi_j \underline{z}(t-j) + \underline{v}(t) \tag{3}$$

have been criticised due to the term, $\Pi \underline{x}(t-k)$ being omitted. Ignoring the long run cointegrating vector, $\underline{\beta}' \underline{x}(t)$ and the error correction, $\underline{\alpha}\left[\underline{\beta}' \underline{x}(t)\right]$ means the SVAR in (3) is mis-specified.[2] The potential seriousness of this, and its effective policing by academics, has resulted in the widespread specification of the VECM (2) in time series research. This paper considers two important consequences of the ubiquitous use the VECM in empirical research.

First, it is shown in Section II that a relatively large number of restrictions are required to represent a cointegrating relationship as being generated by a VECM. Ironically, these restrictions are at odds with the a-theoretical VAR approach which criticises the *ad-hoc* specification and identification of the Cowles foundation style structural models. It is also possible that many researchers are not aware that this procedure forces these restrictions on the parameters which are to be estimated.

The second criticism relates to the justification of using cointegration because economic theories are long run and say little about short run economic behaviour. It is argued in this paper that cointegration places too much importance on the long run and applying the cointegrating restrictions via the VECM excludes interesting short to medium run dynamics, which may have relevance for policy formulation. This is demonstrated theoretically in Section III. Two well known examples, the first with bounded solution and the second with

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[2] This also applies to traditional tests of Granger causality using distributed lags of the variables in first differences.

an unbounded solution, are considered in Sections IV and V. Conclusions are provided in Section VI.

II Restrictions Required by the VECM

Consider (1) with only two endogenous variables, x_1 and x_2 $(n=2)$ having only one lag $(k=1)$ each, and two stationary exogenous variables, z_1 and z_2:

$$\begin{aligned} x_1(t) &= \phi_1 + a_1 x_1(t-1) + a_{12} x_2(t-1) + a_{13} z_1(t) + \varepsilon_1(t) \\ x_2(t) &= \phi_1 + a_{21} x_1(t-1) + a_2 x_2(t-1) + a_{23} z_2(t) + \varepsilon_2(t) \end{aligned} . \tag{4}$$

Transforming (4) into first differences gives:

$$\begin{aligned} \Delta x_1(t) &= \phi_1 + (a_1 - 1) x_1(t-1) + a_{12} x_2(t) + a_{13} z_1(t) + \varepsilon_1(t) \\ \Delta x_2(t) &= \phi_1 + a_{21} x_1(t) - (1 - a_2) x_2(t-1) + a_{23} z_2(t) + \varepsilon_2(t) \end{aligned} . \tag{5}$$

Rather than work with this specification it is preferred to work with the continuous time equivalent of (5). The analysis in continuous time allows a simpler formal presentation of the conditions for stability of the model, which is also consistent with the well known continuous time examples of the Dornbusch exchange rate overshooting model and the endogenous growth models.

Defining $Dx_i(t) = \lim_{\Delta t \to 0} \Delta x_i(t)$ in (5) gives the differential equations:

$$\begin{aligned} Dx_1(t) &= a_{11} x_1(t) + a_{12} x_2(t) + b_1(t) \\ Dx_2(t) &= a_{21} x_1(t) + a_{22} x_2(t) + b_2(t) \end{aligned} \tag{6}$$

with $a_{11} = a_1 - 1$, $a_{22} = a_2 - 1$, $b_1 = \phi_1 + a_{13} z_1 + \varepsilon_1$ and $b_2 = \phi_1 + a_{23} z_2 + \varepsilon_2$.

Without time subscripts (to keep the notation simple) *Granger's representation theorem* normalizes (6) with respect to x_1 for the cointegrating vector, $x_1 - \beta x_2$:

$$
\begin{aligned}
Dx_1 &= \alpha_1\left(x_1 - \beta x_2\right) + b_1 \\
Dx_2 &= \alpha_2\left(x_1 - \beta x_2\right) + b_2
\end{aligned}
\tag{7}
$$

where $\beta > 0$, $\alpha_1 < 0$ and $\alpha_2 > 0$. It is important to note that these requirements for (7) can only be achieved when the characteristic roots of the system (6) are $\lambda_1 < 0$ and $\lambda_2 = 0$. To see this, consider the matrix form of (6):

$$
DX = AX + B \tag{8}
$$

with $DX = \begin{bmatrix} Dx_1 \\ Dx_2 \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. The restricted reduced form solution with the steady state requirement $DX = 0$ imposed is:

$$
\bar{X} = -A^{-1}B \tag{9}
$$

for $|A| \neq 0$. The dynamic adjustments of x_1 and x_2 to respective steady states $\bar{x}_1$ and $\bar{x}_2$ can be determined by solving (8) without imposing $DX = 0$:

$$
\begin{aligned}
DX &= AX + B \\
\therefore X &= (D - A)^{-1}B
\end{aligned}
\tag{10}
$$

provided $|D - A| \neq 0$. Note that pre-multiplying both sides by $(D - A)^{-1}$ integrates DX in (8) to give the solution for X in (10). The integral general solution therefore needs to include a 'constant of integration' term: [3]

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[3] It makes sense when solving a dynamic system to include a growth term as the unknown constant of integration. The compound exponential function characterises dynamic cumulative growth and decay (as experienced in the biological, physical and social sciences). The discrete time compound growth of a principal, *C*, at *r* rate of return over *t* periods given

$$\therefore X = (D - A)^{-1}B + Ce^{\lambda t} \tag{11}$$

for $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and suitably defined parameter λ. The value of λ determines the dynamics of the system and its value can be determined from the homogeneous subset of the differential equations (8):

$$DX_h = AX_h .$$

The general solution for the homogeneous sub-system (11) with $B = 0$ is:

$$\begin{aligned} X_h &= Ce^{\lambda t} \\ \therefore DX_h &= \lambda Ce^{\lambda t} \end{aligned} \tag{12}$$

Now, $DX_h - AX_h = 0$, and substituting using the two terms in (12) gives:

$$\begin{gathered} DX_h - AX_h = 0 \\ \therefore \lambda Ce^{\lambda t} - ACe^{\lambda t} = 0 \\ \therefore (\lambda I_2 - A)Ce^{\lambda t} = 0 \\ \therefore (\lambda I_2 - A)C = \left(A - \lambda I_2\right)C = 0 \end{gathered} \tag{13}$$

Ruling out the trivial solution, $C = 0$, implies that $(A - \lambda I_2)^{-1}$cannot exist. This singularity requires, $\left|A - \lambda I_2\right| = 0$ such that: [4]

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by: $k = C(1+r)^t$ can be represented as the exponential function, $Ce^{\lambda t}$. To see that these are equivalent, equate them and solve for λ as a function of *r*:

$$Ce^{\lambda t} = C(1+r)^t, \quad \therefore e^{\lambda t} = (1+r)^t .$$

Taking Naperian logs (log to the base *e*) denoted, ln, of both sides:

$$\ln(e^{\lambda t}) = \ln(1+r)^t, \quad \therefore \lambda = \ln(1+r)$$

and so: $$Ce^{\lambda t} = Ce^{\ln(1+r)t} = C(1+r)^t .$$

4 We can use (13) to prove that the additional term $Ce^{\lambda t}$ in (11) should disappear when this solution is differentiated to give the structural equations (8), $DX = AX + B$.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 .$$

Solving the determinant gives:

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

$$\therefore \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\therefore \lambda^2 - (trA)\lambda + |A| = 0 \tag{14}$$

where $trA = a_{11} + a_{22}$ and $|A| = a_{11}a_{22} - a_{12}a_{21}$. The solutions of this quadratic equation are the characteristic roots:

$$\therefore \lambda_{1,2} = \frac{trA \pm \sqrt{(trA)^2 - 4|A|}}{2} . \tag{15}$$

If $(trA)^2 > 4|A|$ then there are two distinct real roots, λ_1 and λ_2 which need to include in the general homogeneous solution (11):

$$X_h = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} . \tag{16}$$

The non-homogeneous solution for X is the steady state solution, $\bar{X}$ shown as $\bar{X} = -A^{-1}B$ in (9), plus the homogeneous solution, $X_h = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$ in (16):

$$X = \bar{X} + X_h$$

Proof: Consider solution (11), $X = (D-A)^{-1}B + Ce^{\lambda t}$. Pre-multiplying both sides by $(D-A)$ effectively differentiates X with respect to time:

$$\therefore (D-A)X = (D-A)(D-A)^{-1}B + (D-A)Ce^{\lambda t}$$

$$= B + DCe^{\lambda t} - ACe^{\lambda t} = B + \lambda Ce^{\lambda t} - ACe^{\lambda t} = B + \left(\lambda I_2 - A\right)Ce^{\lambda t}$$

and from (13), $(\lambda I_2 - A)C = 0$ then $(D-A)X = B$, which is the specification of the original structural equations (8), namely $DX = AX + B$. *Q.E.D.*

$$\therefore X = -A^{-1}B + C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

$$\begin{aligned} \therefore x_1(t) &= \bar{x}_1 + c_{11}e^{\lambda_1 t} + c_{21}e^{\lambda_2 t} \\ \therefore x_2(t) &= \bar{x}_2 + c_{12}e^{\lambda_1 t} + c_{22}e^{\lambda_2 t} \end{aligned}. \tag{17}$$

The signs of the characteristic roots, λ_1 and λ_2 in (17) indicate important dynamic properties of the variables.[5] If the characteristic roots are both less than zero then there will be stable solutions for $x_1(t)$ and $x_2(t)$. However, in this case the variables will be stationary and therefore cannot be cointegrated. If both the roots are greater than zero, then $x_1(t)$ and $x_2(t)$ will have unstable solutions. The variables will not be stationary and therefore cannot be cointegrated. When $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = 0$ then (15) shows trivially that $\lambda_1 = \lambda_2 = 0$. Whilst the variables $x_1(t)$ and $x_2(t)$ must be stationary ($a_{11} = a_{22} = 0$) they will be unrelated ($a_{12} = a_{21} = 0$) and therefore cannot be cointegrated.

It is argued (Enders, 1995, pp. 368-369 and others) that two variables will be cointegrated when one characteristic root is equal to zero, $\lambda_2 = 0$. The VECM requires the other characteristic root must be less than zero, $\lambda_1 < 0$. Examination of (15) shows that $trA < 0$ and $|A| = 0$ must apply. The second condition gives the important relationship:

$$a_{11} = \frac{a_{12}a_{21}}{a_{22}} \tag{18}$$

---

[5] The equivalent discrete time conditions are given by applying the lag operator L to (4) to give $\begin{bmatrix} (1-a_1L) & -a_{12}L \\ -a_{21}L & (1-a_2L) \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}$. The inverse characteristic equation is similarly derived from the singular matrix with zero determinant $(1-a_1L)(1-a_2L) - a_{12}a_{21}L^2 = 0$ and defining $\mu = 1/L$ gives the characteristic equation, $\mu^2 - (a_1 + a_2)\mu + (a_1a_2 - a_{12}a_{21}) = 0$ which has characteristic roots, $\mu_{1,2} = 1/2\left[(a_1 + a_2) \pm \sqrt{(a_1 + a_2)^2 - 4(a_1a_2 - a_{12}a_{21})}\right]$. The benchmark for the discrete time characteristic roots is unity, which is equivalent to the benchmark of zero for the continuous time analogue, ie. $\lambda_{1,2} = \mu_{1,2} - 1$.

and substituting into (6) gives:

$$\begin{aligned} Dx_1 &= \frac{a_{12}a_{21}}{a_{22}}x_1 + a_{12}x_2 + b_1 \\ Dx_2 &= a_{21}x_1 + a_{22}x_2 + b_2 \end{aligned} . \tag{19}$$

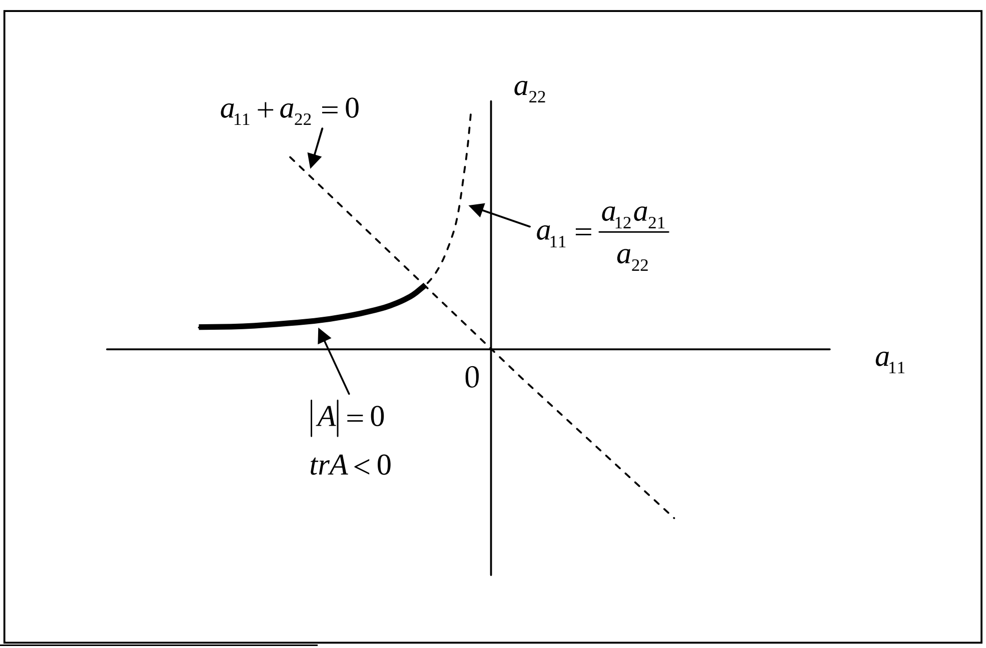
Normalising with respect to x_1 for the cointegrating vector, $x_1 - \beta x_2$, gives the error correction (7):

$$\begin{aligned} Dx_1 &= \alpha_1\left(x_1 - \beta x_2\right) + b_1 \\ Dx_2 &= \alpha_2\left(x_1 - \beta x_2\right) + b_2 \end{aligned}$$

with the parameters for the cointegrating vector, β and error corrections, α_1 and α_2 given by: [6]

$$\beta = \frac{a_{22}}{a_{21}}, \quad \alpha_1 = \frac{a_{12}a_{21}}{a_{22}} \quad \text{and} \quad \alpha_2 = a_{21}. \tag{20}$$

Figure 1



[6] This is equivalent to the discrete time specification (5) and Footnote 4 with the substitutions, $a_{11} = a_1 - 1$ and $a_{22} = a_2 - 1$.

The restrictions imposed by $|A| = 0$ are considerable, for example, the VECM requires $\beta > 0$, $\alpha_1 < 0$ and $\alpha_2 > 0$. Since $\alpha_2 = a_{21}$ then $a_{21} > 0$ and so $a_{22} > 0$, because $\beta > 0$. For $\alpha_1 < 0$ then $a_{21} > 0$ and $a_{22} > 0$ means that $a_{12} < 0$. Figure 1 graphs the hyperbola for given $a_{12}a_{21} < 0$ in (a_{11}, a_{22}) space. The other requirement, $trA = a_{11} + a_{22} = 0$ is graphed by the straight line and so only the points on the thick black line satisfy the joint requirements, $|A| = 0$ and $trA < 0$. Clearly these requirements substantially restrict the possible parameter space (a_{11}, a_{22}). In addition, these restrictions seriously affect the possible short run dynamics of adjustment of the system via the VECM and this will be considered in the next section.

III Possible Dynamic Solutions

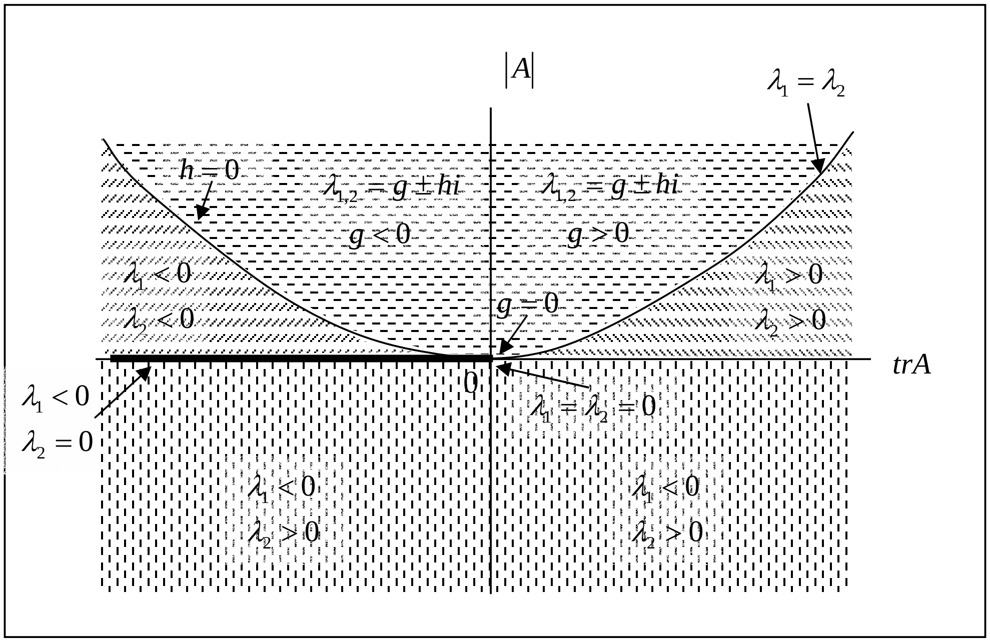
The relationship (15), $\lambda_{1,2} = \frac{1}{2}\left[trA \pm \sqrt{(trA)^2 - 4|A|}\right]$ shows that *Granger's representation theorem* only applies on the manifold, $trA < 0$ and $|A| = 0$, as indicated by the thick black line with label, $\lambda_1 < 0$, $\lambda_2 = 0$ in Figure 2.

When $|A| > 0$ and $trA > 0$, the system is globally unstable with $\lambda_1 > 0$ and $\lambda_2 > 0$. The exponential terms in (17):

$$
\begin{aligned}
x_1(t) &= \bar{x}_1 + c_{11}e^{\lambda_1 t} + c_{21}e^{\lambda_2 t} \\
x_2(t) &= \bar{x}_2 + c_{12}e^{\lambda_1 t} + c_{22}e^{\lambda_2 t}
\end{aligned}
$$

will grow without bound and the general solutions for x_1 and x_2 will diverge exponentially from the steady state values $\bar{x}_1$ and $\bar{x}_2$ over time, *t*.

Figure 2



When $|A| > 0$ and $trA < 0$ the system is globally stable with $\lambda_1 < 0$ and $\lambda_2 < 0$. The inverse exponential terms will decay to zero so that the time paths of x_1 and x_2 must converge to the steady state values, $\lim_{t \to \infty} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$. Alternatively, $|A| = 0$ means that one root will be zero and the other non-zero. If $trA = 0$ then both roots will be equal to zero, $\lambda = 0$. The relationship (17) will reduce to:

$$\begin{aligned} x_1(t) &= \bar{x}_1 + (c_{11} + c_{21}) \\ x_2(t) &= \bar{x}_2 + (c_{12} + c_{22}) \end{aligned} \qquad (21)$$

so that x_1 and x_2 must always be a constant value away from steady state.

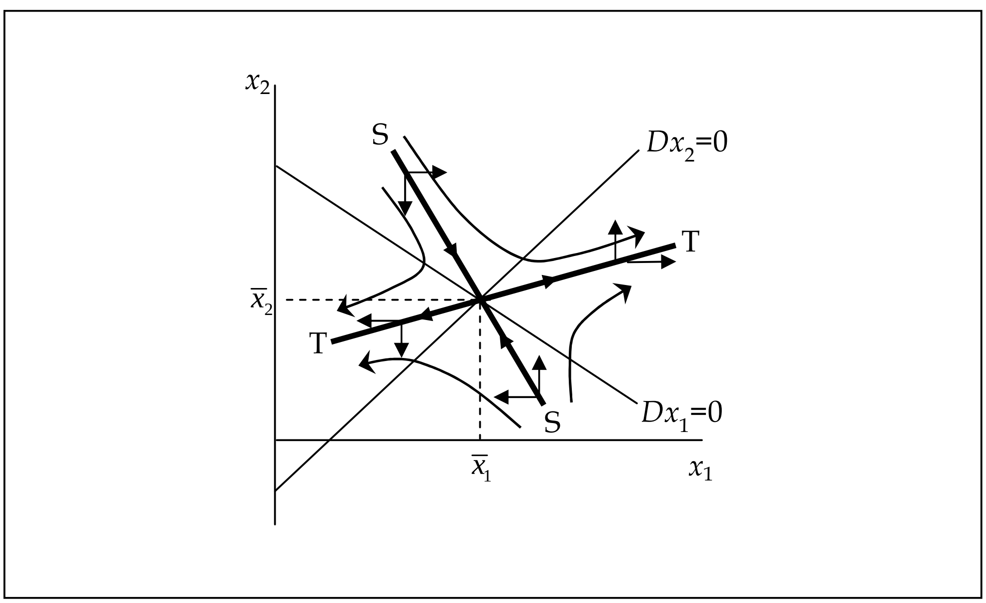
When $|A| < 0$ the characteristic roots must be opposite in sign with either ($\lambda_1 < 0$, $\lambda_2 > 0$) or ($\lambda_1 > 0$, $\lambda_2 < 0$). These values describe a dynamic saddlepath solution with the negative characteristic root characterizing the stable arm and positive root the unstable arm. These conditions are shown as the areas below the horizontal trA axis in Figure 2. Let's consider the saddlepath solution with $\lambda_1 < 0$ and $\lambda_2 > 0$. To achieve this outcome we can keep the required inequalities of the VECM, $a_{11} < 0$, $a_{12} < 0$ and $a_{22} > 0$ but drop the restrictions, $a_{11} = a_{12}a_{21}/a_{22}$ and $a_{21} < 0$. For $a_{21} > 0$ and $\lambda_1 < 0$, the stable arm (22) will be:

$$\therefore x_2(t) = -\frac{\lambda_1 \bar{x}_2 + b_2}{a_{22} - \lambda_1} - \frac{a_{21}}{a_{22} - \lambda_1} x_1(t) . \tag{22}$$

It will have positive slope and is shown as the *SS* schedule in Figure 3. According to (23), $\lambda_2 > 0$ ensures the unstable *TT* arm will have positive slope:

$$\therefore x_2\left(t\right) = -\frac{\lambda_2 \bar{x}_1 + b_1}{a_{12}} + \frac{\lambda_2 - a_{11}}{a_{12}} x_1\left(t\right) . \tag{23}$$

Figure 3



Note that if both of the *SS* and *TT* arms are stable (consistent with $|A| > 0$ and $trA < 0$ causing $\lambda_1 < 0$ and $\lambda_2 < 0$) then the system would be globally stable. There would be an infinite number of possible solution paths and so this under-identification will result in a non-unique solution. Conversely, if both arms are unstable (with $|A| > 0$ and $trA > 0$ giving $\lambda_1 > 0$ and $\lambda_2 > 0$) then the system would be globally unstable. In this case the initial values of x_1 and x_2 on either *SS* or *TT* will cause the variables to diverge from their steady state values. The only solution to the system is for the variables to jump to the steady state values $\bar{x}_1$ and $\bar{x}_2$. Whilst this solution is unique there are no dynamics of adjustment and it is therefore of little interest.

The interesting saddlepath solution (with $|A| < 0$ giving either ($\lambda_1 < 0$, $\lambda_2 > 0$) or ($\lambda_1 > 0$, $\lambda_2 < 0$) provides a locally stable manifold, *SS* and a globally unstable system.[7] The solution requires one of the variables to jump onto the stable arm and then both variables to adjust as the system moves along *SS* to the steady state values $\bar{x}_1$ and $\bar{x}_2$.

Finally if $4|A| > (trA)^2$ then the characteristic roots will be complex numbers:

$$\begin{aligned}\lambda_{1,2} &= \frac{trA \pm \sqrt{(-1)(4|A| - (trA)^2)}}{2} \\ &= \frac{trA \pm i\sqrt{4|A| - (trA)^2}}{2}\end{aligned}.$$

with imaginary part, $i = \sqrt{-1}$. These complex conjugate solutions are defined to include a real part and an imaginary part in linear form, $\lambda_{1,2} = g \pm hi$ where $g = \frac{trA}{2}$ and $h = \frac{\sqrt{4|A| - (trA)^2}}{2}$. Substituting into the general solution (17):

$$\begin{aligned}X &= -A^{-1}B + c_1 e^{(g+hi)t} + c_2 e^{(g-hi)t} \\ &= -A^{-1}B + e^{gt}\left(c_1 e^{hit} + c_2 e^{-hit}\right)\end{aligned} \qquad (24)$$

[7] The stable arm *SS* represents a U shaped part of the saddle which is a ridge where points off the *SS* schedule will fall away from the schedule. The *SS* schedule is said to be locally stable and globally unstable. The *TT* schedule is locally unstable.

and changing from Cartesian co-ordinates (g,k) to polar co-ordinates (θ,k) expressed in trigonometric form, $g = k\cos\theta$, $h = k\sin\theta$, for $k > 0$ gives:[8]

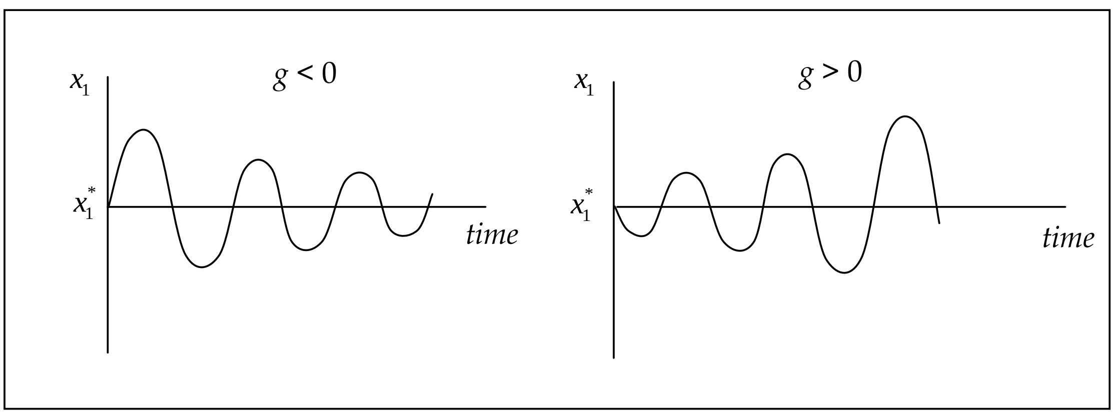
$$g \pm hi = k\left(\cos\theta \pm i\sin\theta\right).$$

Using the Euler equation, $g \pm hi == ke^{\pm i\theta}$ and substituting into (24) gives the complex solution:

$$\begin{aligned} X &= -A^{-1}B + e^{gt}\left[c_1(\cos ht + i\sin ht) + (\cos ht - i\sin ht)\right] \\ &= -A^{-1}B + e^{gt}\left[(c_1 + c_2)\cos ht + (c_1 - c_2)i\sin ht\right] \\ &= -A^{-1}B + e^{gt}\left[d_1\cos ht + d_2 i\sin ht\right] \end{aligned} \tag{25}$$

where the trigonometric term, $d_1\cos ht + d_2 i\sin ht$, specifies the periodic fluctuations of the solution.[9]

Figure 4



8 The angle, θ, is measured in radians, $0 \le \theta \le 2r\pi$, $r \ge 1$, and the trigonometric functions have period, 2π and amplitude, k.

9 The period of the cycle is $2\pi/h$, so that higher values of $h = 1/2\left[\sqrt{4|A| - (trA)^2}\right]$ increase the frequency, that is, shorten the time the cycle repeats itself.

The sign of g in (25) determines the stable (g < 0) and unstable (g > 0) dynamic paths. These cases can be viewed in Figure 4 as dynamic oscillations along the stable saddlepath *SS* for $g < 0$ and unstable oscillations along the unstable path *TT* for $g > 0$. The complex solution occurs when $(trA)^2 - 4|A| < 0$, which descibes all points vertically above the parabola in Figure 2.

To summarise, of all the possible outcomes in $(trA, |A|)$ space, *Granger's representation theorem* substantially restricts the possible parameter space for the VECM to the $(trA < 0, |A| = 0)$ manifold, as indicated by the thick black line (labeled as $\lambda_1 < 0$, $\lambda_2 = 0$) in Figure 2.

IV Example of Overshooting Exchange Rate

The seminal model of Dornbusch (1976) demonstrates that exchange rates may overshoot (even with continuous asset market clearing and rational expectations). This is brought about by different relative speeds of adjustment between asset and goods markets. Domestic and foreign assets are assumed to be perfect substitutes reflecting perfect international capital mobility (and relative fast adjustment) whilst real output is assumed fixed so that domestic prices are required to adjust (relatively slowly).

The model comprises interest rate parity, the demand for money (asset market) and a Phillips curve (goods market).[10] Interest rate parity ($\dot{s} = Dx_1^e$) is given by:

$$\dot{s} = i - i^* \tag{26}$$

where s is defined as the domestic price of foreign exchange for the small open economy. The demand for money (with money supply m assumed exogenous) is:

$$m = p + k\overline{y} - \theta i \qquad k > 0, \theta > 0 . \tag{27}$$

Subtracting the long run equilibrium in the money market, $\overline{m} = \overline{p} + k\overline{y} - \theta i$, from (27) gives, $(p - \overline{p}) - (m - \overline{m}) = \theta(i - i^*)$ and substituting for $(i = i^*)$ using (26) derives the first equation of motion:

$$\dot{s} = \frac{1}{\theta}\left[(p - \overline{p}) - (m - \overline{m})\right] \tag{28}$$

An increase in price *p* increases the domestic interest rate, *i* which appreciates the spot exchange rate, *s* and increases the expectation of a depreciation of the forward rate $\dot{s}$. An unexpected increase in the money supply, *m* has the opposite effect by depreciating the spot rate and increasing the expected future appreciation.

The final equation is the Phillips curve:

$$\dot{p} = \gamma\left[\delta + \varphi(s - p) - \overline{y}\right] \qquad \gamma > 0, \varphi > 0 \tag{29}$$

where excess aggregate demand, $\delta + \varphi(s - p)$, over the fixed supply, $\overline{y}$, will be inflationary.[11]

In long run equilibrium, $0 = \gamma\left[\overline{\delta} + \varphi(\overline{s} - \overline{p}) - \overline{y}\right]$ and subtracting from (29) gives the second required equation of motion:

$$\dot{p} = \gamma\varphi(s - \overline{s}) - \gamma\varphi(p - \overline{p}) + \gamma(\delta - \overline{\delta}) \tag{30}$$

[10] This demonstration differs from the original model in order to make clearer the differences with the requirements of the cointegrating VECM specification.

[11] This differs from the Mundell-Fleming model, where *AD* affects real output, because here only movements in the price level, *p* equilibrates the goods market.

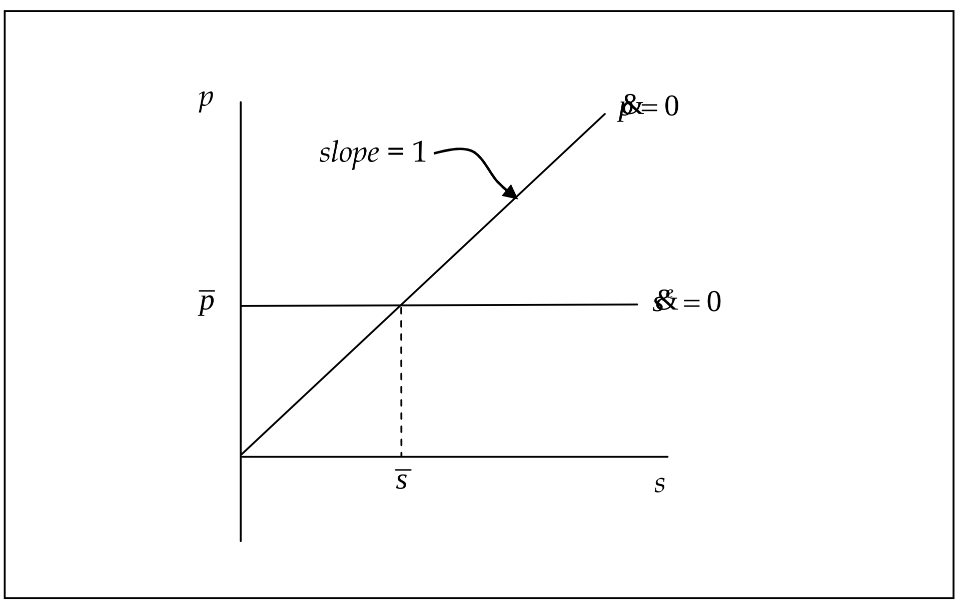
Equations (28) and (30) comprise the dynamic continuous time system:

$$\begin{bmatrix} \dot{s} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\theta} \\ \gamma\varphi & -\gamma\varphi \end{bmatrix} \begin{bmatrix} s-\bar{s} \\ p-\bar{p} \end{bmatrix} + \begin{bmatrix} \frac{-1}{\theta}\left(m-\bar{m}\right) \\ \gamma\left(\delta-\bar{\delta}\right) \end{bmatrix}. \tag{31}$$

The stability of the system is determined by $|A| = -\frac{\gamma\varphi}{\theta} < 0$ for $\theta > 0, \gamma > 0, \varphi > 0$, which means there is a saddlepath solution.

Steady state occurs when $\dot{s} = 0$ which means that $p = \bar{p}$, when there are no unexpected changes in the money supply, $m = \bar{m}$. For $\dot{p} = 0$ then $p = s$ when $\bar{p} = \bar{s}$ and there are no unexpected changes in autonomous *AD*, $\delta = \bar{\delta}$. These steady states are graphed in Figure 5.

Figure 5



Solving for the stable saddlepath *SS* in terms of the exchange rate gives:[12]

[12] The quantitative solution can be obtained by integrating (31) to give the general solution (17):

$$s = \overline{s} + \frac{1}{\lambda_1 \theta}\left(p - \overline{p} \right) - \frac{1}{\lambda_1 \theta}\left(m - \overline{m} \right) \tag{32}$$

The value of the stable characteristic root is given by:

$$\lambda_1 = \frac{-\gamma\varphi - \sqrt{\left(\gamma\varphi\right)^2 - 4\left(\frac{-\gamma\varphi}{\theta}\right)}}{2}. \tag{33}$$

Completing the square and simplifying gives, $\lambda_1 = -\gamma\varphi$, which for $\gamma > 0$ and $\varphi > 0$ ensures $\lambda_1 < 0$. The equation for the saddlepath is therefore:

$$s = \overline{s} - \frac{1}{\gamma\varphi\theta}\left(p - \overline{p} \right) + \frac{1}{\gamma\varphi\theta}\left(m - \overline{m} \right) \tag{34}$$

and rearranging gives the saddlepath in terms of price, as shown in Figure 6:

$$p = \overline{p} - \gamma\varphi\theta\left(s - \overline{s} \right) - \gamma\varphi\theta\left(m - \overline{m} \right). \tag{35}$$

$$\begin{aligned} s - \overline{s} &= c_{11} e^{\lambda_1 t} + c_{21} e^{\lambda_2 t} \\ p - \overline{p} &= c_{12} e^{\lambda_1 t} + c_{22} e^{\lambda_2 t} \end{aligned}.$$

Selecting the stable arm, $\lambda_1 < 0$, setting $t = 0$ and differentiating gives, $\dot{s} = c_{11}\lambda_1$ and $\dot{p} = c_{12}\lambda_1$. Equating with (31) eliminates $\dot{s}$ and $\dot{p}$ to give:

$$c_{11}\lambda_1 = \frac{1}{\theta}\left(p - \overline{p} \right) - \frac{1}{\theta}\left(m - \overline{m} \right) \text{ and } c_{12}\lambda_1 = \gamma\varphi\left(s - \overline{s} \right) - \gamma\varphi\left(p - \overline{p} \right) + \gamma\left(\delta - \overline{\delta} \right).$$

Eliminating c_{11} and c_{12} by setting $t = 0$ for the stable arm, $s - \overline{s} = c_{11} e^{\lambda_1 t}$ and $p - \overline{p} = c_{12} e^{\lambda_1 t}$ gives, $s = \overline{s} + c_{11}$ and $p = \overline{p} + c_{12}$. Substituting gives:

$$\begin{aligned} \dot{s} = \lambda_1\left(s - \overline{s} \right) &= \frac{1}{\theta}\left(p - \overline{p} \right) - \frac{1}{\theta}\left(m - \overline{m} \right) \\ \dot{p} = \lambda_1\left(p - \overline{p} \right) &= \gamma\varphi\left(s - \overline{s} \right) - \gamma\varphi\left(p - \overline{p} \right) + \gamma\left(\delta - \overline{\delta} \right). \end{aligned}$$

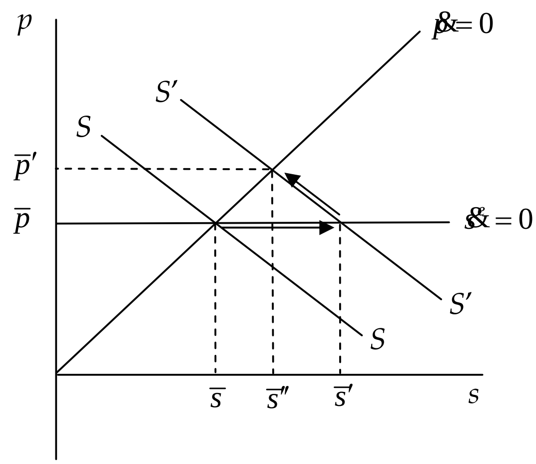
Solving the first equation for the exchange rate gives:

$$s = \overline{s} + \frac{1}{\lambda_1 \theta}\left(p - \overline{p} \right) - \frac{1}{\lambda_1 \theta}\left(m - \overline{m} \right) \tag{32}$$

An unexpected increase in the money supply, m will lower the domestic interest rate, i and depreciate the exchange rate, s. The SS saddlepath schedule will shift to the right on impact to $S'S'$, as shown in Figure 6. Since goods prices will be fixed on impact and because $\partial\overline{s} = \partial m$, the size of the rightward shift is according to (34):

$$\partial s = \partial\overline{s} + \frac{1}{\gamma\varphi\theta}\partial m = \left(1 + \frac{1}{\gamma\varphi\theta}\right)\partial m \tag{36}$$

Figure 6



The spot exchange rate will therefore need to depreciate all the way to point $\overline{s}'$, where $\overline{s} - \overline{s}' = \left(1 + \frac{1}{\gamma\varphi\theta}\right)\partial m$. As prices rise over time, the interest rate will increase, appreciating the exchange rate back to $\overline{s}''$. The subsequent increase in prices to $\overline{p}'$ necessitates the exchange rate to overshoot in order to factor in the future appreciation. The slope of the saddlepath (35) is $-\gamma\varphi\theta$, so lower values of these parameters will flatten the saddlepath schedule and increase the required overshooting. According to (27), $m = p + k\overline{y} - \theta i$, as θ falls, the interest rate will need to fall by more to equilibrate the money market and so the exchange rate will depreciate by more. Lower values of γ and φ in (29), $\dot{p} = \gamma\left[\delta + \varphi(s - p) - \overline{y}\right]$

mean that prices do not have to increase by as much in order to offset the depreciation and equilibrate the goods market.

Now compare this with the Granger representation of the VECM (7):

$$\begin{aligned} Dx_1(t) &= \alpha_1\left[x_1(t)-\beta x_2(t)\right] \\ Dx_2(t) &= \alpha_2\left[x_1(t)-\beta x_2(t)\right] \end{aligned}.$$

In this example, the VECM requires the first equation in relationship (31) to include the real exchange rate:

$$\dot{s} = \frac{-1}{\theta}(s-p)+\frac{1}{\theta}\left[(\bar{s}-\bar{p})-(m-\bar{m})\right] \tag{31′}$$

and the second equation in (31) needs to be rearranged to show the presence of the real exchange rate :

$$\dot{p} = \gamma\varphi(s-p)-\left[\gamma\varphi(\bar{s}-\bar{p})-\gamma\left(\delta-\bar{\delta}\right)\right] \tag{31}$$

Comparing (31′) and (31) with (7) shows the real exchange rate, $e-p$ is the cointegrating vector. So $\beta=1$ and the error corrections, $\alpha_1=\dfrac{-1}{\theta}$ and $\alpha_2=\gamma\varphi$ have the required signs, $\alpha_1<1$ and $\alpha_2>0$ for $\theta>0$, $\gamma>0$ and $\varphi>0$.[13] Putting into matrix form:

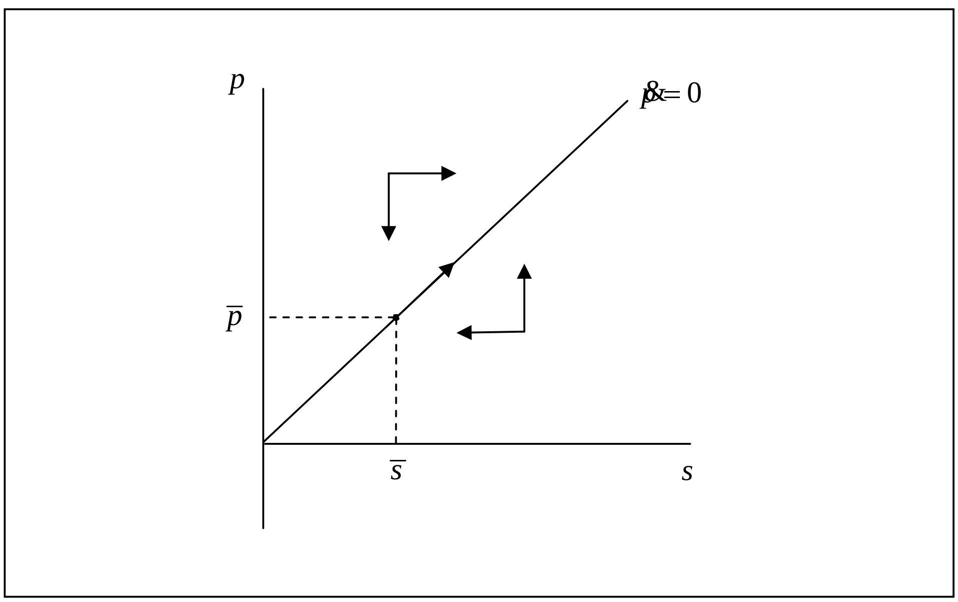
$$\begin{bmatrix}\dot{s}\\ \dot{p}\end{bmatrix}=\begin{bmatrix}\dfrac{-1}{\theta} & \dfrac{1}{\theta}\\ \gamma\varphi & -\gamma\varphi\end{bmatrix}\begin{bmatrix}s\\ p\end{bmatrix}+\begin{bmatrix}\dfrac{-1}{\theta}\left[(\bar{s}-\bar{p})-(m-\bar{m})\right]\\ -\gamma\varphi(\bar{s}-\bar{p})+\gamma\left(\delta-\bar{\delta}\right)\end{bmatrix} \tag{31′}$$

---

[13] Note that these values are consistent with the requirements in (21), $\beta=-a_{22}/a_{21}=\gamma\varphi/\gamma\varphi=1$, $\alpha_1=a_{12}a_{21}/a_{22}=\gamma\varphi/-\gamma\varphi\theta=-1/\theta$ and $\alpha_2=a_{21}=\gamma\varphi$.

shows that $|A| = 0$ since the matrix of coefficients are linearly dependent. From (15), one root will be zero, $\lambda_2 = 0$, and the other non-zero, $\lambda_1 = \frac{trA + \sqrt{(trA)^2}}{2} = trA = -\left[\frac{1}{\theta} + \gamma\varphi\right] < 0$. The system will therefore be globally stable and this can be verified by differentiating the first equation in (31′) with respect to s to give $\frac{\partial \dot{s}}{\partial s} = \frac{-1}{\theta} < 0$ and differentiating the second equation in (31′) with respect to p to give, $\frac{\partial \dot{p}}{\partial p} = -\gamma\varphi < 0$, for $\theta > 0, \gamma > 0, \varphi > 0$. Figure 7 demonstrates the global stability with no unique long run steady state values of $\bar{x}_1$ and $\bar{x}_2$, because any positions on the long run cointegrating vector, $\dot{p} = 0$ are possible.[14]

Figure 7



The VECM specification only explains the long run monotonic movement from of the exchange rate depreciating from $\bar{s}$ and the price level increasing from $\bar{p}$ along the $\dot{p} = 0$ cointegrating vector.

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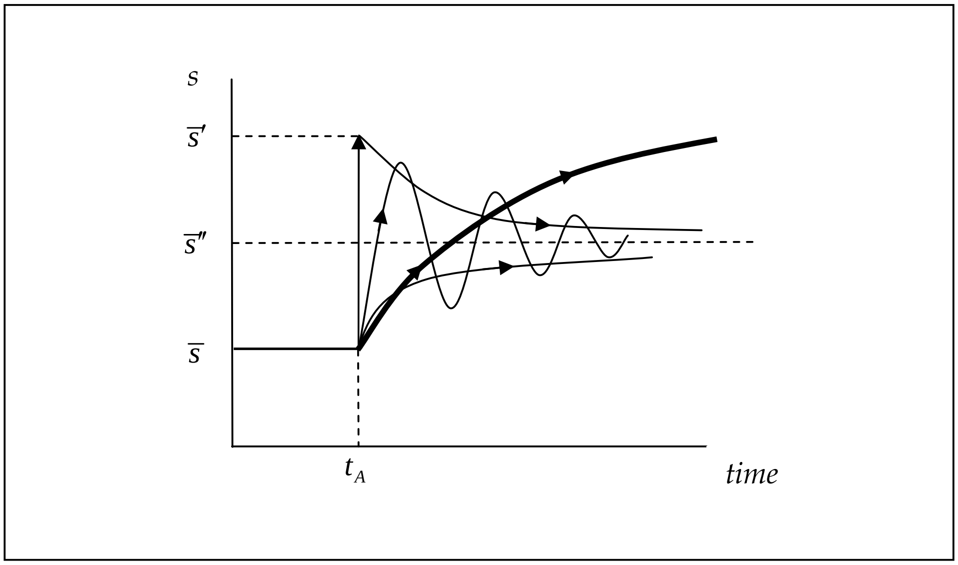
[14] If the system is shocked off the cointegrating locus then the VECM ensures monotonic movement back to it.

Note also that complex (oscillatory) solutions are not possible for the VECM since $|A| = 0$ cannot satisfy the requirement complex conjugate solution requirement, $4|A| > (trA)^2$. Figure 8 shows the only possible VECM solution is the (thick lined) unbounded monotonic locus.[15]

Figure 8



The nature of equilibrium and the dynamic paths by which an economy moves from one equilibrium to another are important. Counter intuitively, the concept of global instability is an important and desirable property. For policy makers and others, the dynamic paths of adjustment are at least as important as the changing equilibrium.

So what is the solution for the dilemma of the acceptance of *Granger's representation theorem* and the widespread use of the VECM in modeling and empirical estimation? The VECM in (7) for this example:

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15 The nonlinear path is due to the exponential, $e^{\lambda_1 t}$ effect for $\lambda_1 < 0$ in the adjustment (32).

$$\begin{bmatrix} \dot{s} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \frac{-1}{\theta} & \frac{1}{\theta} \\ \gamma\varphi & -\gamma\varphi \end{bmatrix} \begin{bmatrix} s \\ p \end{bmatrix} + \begin{bmatrix} \frac{-1}{\theta}\left[(\bar{s}-\bar{p})-(m-\bar{m})\right] \\ -\gamma\varphi(\bar{s}-\bar{p})+\gamma\left(\delta-\bar{\delta}\right) \end{bmatrix} \qquad (31')$$

compares with the saddlepath solution:

$$\begin{bmatrix} \dot{s} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\theta} \\ \gamma\varphi & -\gamma\varphi \end{bmatrix} \begin{bmatrix} s-\bar{s} \\ p-\bar{p} \end{bmatrix} + \begin{bmatrix} \frac{-1}{\theta}(m-\bar{m}) \\ \gamma\left(\delta-\bar{\delta}\right) \end{bmatrix}. \qquad (31)$$

To omit the long run cointegrating vector in (31) means the SVAR will be mis-specified. However, re-arranging (31):

$$\begin{bmatrix} \dot{s} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\theta} \\ \gamma\varphi & -\gamma\varphi \end{bmatrix} \begin{bmatrix} s \\ p \end{bmatrix} + \begin{bmatrix} \frac{-1}{\theta}\left[\bar{p}+(m-\bar{m})\right] \\ -\gamma\varphi(\bar{s}-\bar{p})+\gamma\left(\delta-\bar{\delta}\right) \end{bmatrix}$$

shows that it includes the long run real exchange rate relationship, $s-p$ in the second equation:

$$\dot{p} = \gamma\varphi(s-p)-\gamma\varphi(\bar{s}-\bar{p})+\gamma\left(\delta-\bar{\delta}\right)$$

So the cointegrating vector (with a constant) is included in the $\dot{p}$ equation and importantly, $\beta=1$ and the error correction, $\alpha_2=\gamma\varphi$ are the same as for the VECM. In fact, including the error correction in the first equation would misspecify the relationship. The stable saddlepath *SS* given by (31) can be considered as a short run 'cointegrating vector' which reflects the interest rate parity condition (16), $\dot{s}=i-i^*$ which links to prices via (27), $m=p+k\bar{y}-\theta i$. This moves the variables s and p to the steady state on the long run cointegrating vector, $\dot{p}=0$. The cointegrating vector in the second equation of (31) is therefore binding on the first equation in the long run steady state. In comparison, the cointegrating relationship provides an unbounded solution path. The next brief example shows that a saddlepath solution can also meet this requirement if necessary.

V Example of Endogenous Growth

This short example considers an endogenous growth model with increasing returns to scale. Costs of adjustment means that Tobin's q is the adjustment variable and capital accumulates as the marginal valuation of capital, relative to its replacement cost, is greater than unity. This derives the unstable saddlepath TT as the endogenous growth path (unlike the stable saddlepath SS of the previous example). Growth can therefore be consistent with unbounded capital accumulation, k^{8} and real output, y in the long run.

INCLUDE MATHS HERE

VI Conclusion

The *Granger representation theorem* states that a set of non-stationary cointegrated variables can be characterized by an error correction mechanism. The VECM is a powerful analytic device which has been universally adopted by many empirical researchers. The analysis of VARs without the VECM being included are criticized as being misspecified.

It is demonstrated in Section II that a relatively large number of restrictions are required to represent a cointegrating relationship as being generated by a VECM. The presence of these restrictions on the parameters to be estimated does not appear to be well known or understood by applied researchers. This is possibly due to the preference for the reduced form VAR approach over the frequently criticised the *ad-hoc* specification and identification of structural models.

There is also justification for using cointegration because economic theories are put forward as being long run in nature with little to contribute to the understanding of short run economic behaviour. It is argued in this paper that cointegration places too much importance on the long run and applying the cointegrating restrictions via the VECM excludes interesting short to medium run dynamics, which may have relevance for policy formulation.

This is demonstrated theoretically in Section III using a continuous time analogue. It is shown that the VECM means the system is globally stable which significantly restricts the allowable parameter space and the possible short run dynamics of adjustment of the system via the VECM. Other possible globally stable and unstable outcomes are detailed, including saddlepath and complex oscillatory solutions.

Two well known examples, which are not restricted to the VECM outcomes, are considered, the first being Dornbusch's bounded exchange rate overshooting solution in Section IV. The second example of an endogenous growth model with unbounded solution is briefly considered in Section V.

These examples show that including the VECM in all equations of motion may misspecify the dynamic relationships. The stable saddlepath can be considered as a short run 'cointegrating vector' which reflects short run parity conditions. Including the cointegrating vector in the other equation is therefore binding on the first equation in the long run steady state.

This paper claims the nature of equilibrium and the dynamic paths by which an economy moves from one equilibrium to another are important. Counter intuitively, the concept of global instability is an important and desirable property. This contrasts with many economic models which have the property of global stability, consistent with discrete time models where comparative static analysis jumps the variables from an old equilibrium to a new equilibrium. The actual path of adjustment is usually not specified and the centre of focus is on the net changes in the endogenous variables required to achieve the new equilibrium. There is also no specification of the time required for the adjustment except to say it will take so many periods. These periods are discrete in terms of logical time, not chronological time, and therefore say little about the relative speeds of adjustment of the variables. For policy makers and others, the dynamic paths of adjustment are at least as important as the changing equilibrium.

The other main point is the extensive use of pre-testing time series in the form of tests for stationarity which are driving the research process. It has been shown that these tests are not robust under the presence of structural change. Whilst the ARDL procedure allows mixed stationary and non-stationary processes the VECM is enforced on all equations. More flexible estimation procedures are required which will allow a greater range of possible dynamic specifications.

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Appendix

Consider two endogenous variables, x_1 and x_2 $(n=2)$ having only one lag $(k=1)$ each:

$$\begin{aligned} x_1(t) &= a_1 x_1(t-1) + a_{12} x_2(t-1) + a_{13} z_1(t) + \varepsilon_1(t) \\ x_2(t) &= a_{21} x_1(t-1) + a_2 x_2(t-1) + a_{23} z_2(t) + \varepsilon_2(t) \end{aligned} \tag{4}$$

where z_1 and z_2 are stationary exogenous variables. *Granger's representation theorem* normalizes (4) with respect to $x_1(t)$ for the cointegrating vector, $x_1(t-1) - \beta x_2(t-1)$, to give the equivalent error correction:

$$\begin{aligned} \Delta x_1(t) &= \alpha_1 \left[x_1(t-1) - \beta x_2(t-1) \right] + a_{13} z_1(t) + \varepsilon_1(t) \\ \Delta x_2(t) &= \alpha_2 \left[x_1(t-1) - \beta x_2(t-1) \right] + a_{23} z_2(t) + \varepsilon_2(t) \end{aligned} \tag{5}$$

with $\beta > 0$, $\alpha_1 < 0$ and $\alpha_2 > 0$.

Now consider the restrictions the cointegrating vector places on the VECM by applying the lag operator L to (4):

$$\begin{aligned} (1 - a_1 L) x_1(t) - a_{12} L x_2(t) &= a_{13} z_1(t) + \varepsilon_1(t) \\ -a_{21} L x_2(t) + (1 - a_2 L) x_2(t) &= a_{23} z_2(t) \varepsilon_2(t) \end{aligned}$$

$$\therefore \begin{bmatrix} (1 - a_1 L) & -a_{12} L \\ -a_{21} L & (1 - a_2 L) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} a_{13} z_1(t) \\ a_{23} z_2(t) \end{bmatrix} + \begin{bmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{bmatrix}. \tag{6}$$

The inverse characteristic equation is derived from the singular matrix with zero determinant:

$$(1 - a_1 L)(1 - a_2 L) - a_{12} a_{21} L^2 = 0$$

and defining $\mu = \frac{1}{L}$ gives the characteristic equation:

$$\mu^2 - (a_1 + a_2)\mu + (a_1 a_2 - a_{12} a_{21}) = 0$$

which has characteristic roots:

$$\mu_{1,2} = \frac{(a_1 + a_2) \pm \sqrt{(a_1 + a_2)^2 - 4(a_1 a_2 - a_{12} a_{21})}}{2}. \tag{7}$$

The values of the roots indicate important properties of the variables and their possible relationships for (4). If the characteristic roots are both less than unity (ie. lie within the unit circle) then there will be stable solutions for $x_1(t)$ and $x_2(t)$. However, the variables will be stationary and therefore cannot be cointegrated.

If the roots are both greater than unity, then the solutions for $x_1(t)$ and $x_2(t)$ will be unstable. The variables will not be stationary in first difference and therefore cannot be cointegrated.

Substituting $a_{12} = a_{21} = 0$ and $a_1 = a_2 = 1$ in (7) shows $\mu_1 = \mu_2 = 1$, so that the roots will be equal to unity and the variables $x_1(t)$ and $x_2(t)$ must be first difference stationary. However, they will be unrelated and therefore not cointegrated.

It is argued (Enders, 1995, pp. 368-369 and others) that for the variables to be cointegrated then one characteristic root must be equal to unity. We will consider this latter point by letting $\mu_2 = 1$ and solving for (4):

$$x_1(t) = \frac{(1 - a_2 L)\varepsilon_1 + a_{12}\varepsilon_2}{(1 - \mu_1 L)(1 - L)}.$$

Multiplying both sides by $(1-L)$ gives:

$$(1-L)x_1(t)=\frac{(1-a_2L)\varepsilon_1+a_{12}\varepsilon_2}{(1-\mu_1L)}$$

so that $\Delta x_1(t)=(1-L)x_1(t)$ will be stationary only for $|\mu_1|<1$.

Using (7) to solve for $\mu_2=1$:

$$\mu_2=1=\frac{(a_1+a_2)+\sqrt{(a_1+a_2)^2-4(a_1a_2-a_{12}a_{21})}}{2}$$

gives the important relationship:

$$a_1=\frac{(1-a_2)-a_{12}a_{21}}{1-a_2}. \tag{8}$$

Transforming (4) into first differences:

$$\begin{aligned}\Delta x_1(t)&=(a_1-1)x_1(t-1)+a_{12}x_2(t)+a_{13}z_1(t)+\varepsilon_1(t)\\ \Delta x_2(t)&=a_{21}x_1(t)-(1-a_2)x_2(t-1)+a_{23}z_2(t)+\varepsilon_2(t)\end{aligned} \tag{9}$$

and substituting (8), where $a_1-1=-\frac{a_{12}a_{21}}{1-a_2}$, into (9) gives:

$$\begin{aligned}\Delta x_1(t)&=-\frac{a_{12}a_{21}}{1-a_2}x_1(t-1)+a_{12}x_2(t)+a_{13}z_1(t)+\varepsilon_1(t)\\ \Delta x_2(t)&=a_{21}x_1(t)-(1-a_2)x_2(t-1)+a_{23}z_2(t)+\varepsilon_2(t)\end{aligned}. \tag{10}$$

Normalising with respect to $x_1(t)$ for the cointegrating vector: $x_1(t-1)-\beta x_2(t-1)$, gives the error correction:

$$\begin{aligned} \Delta x_1(t) &= \alpha_1\left[x_1(t-1)-\beta x_2(t-1)\right]+\varepsilon_1(t) \\ \Delta x_2(t) &= \alpha_2\left[x_1(t-1)-\beta x_2(t-1)\right]+\varepsilon_2(t) \end{aligned} \tag{5}$$

with the parameters for the cointegrating vector, β and error corrections, α_1 and α_2:

$$\beta=\frac{1-a_2}{a_{21}}, \quad \alpha_1=-\frac{a_{12}a_{21}}{1-a_2} \quad \text{and } \alpha_2=a_{21}. \tag{11}$$