2013

Degree theory for oblique boundary problems

Jiakun Liu
University of Wollongong, jiakunl@uow.edu.au

Publication Details

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au
Degree theory for oblique boundary problems

Abstract
Considering a second order fully nonlinear elliptic operator with a nonlinear oblique boundary condition of the general form.

Keywords
problems, theory, boundary, degree, oblique

Disciplines
Engineering | Science and Technology Studies

Publication Details

This journal article is available at Research Online: http://ro.uow.edu.au/eispapers/2132
DEGREE THEORY FOR OBLIQUE BOUNDARY PROBLEMS

JIAKUN LIU
(joint work with Yanyan Li and Luc Nguyen)

Consider a second order fully nonlinear elliptic operator with a nonlinear oblique boundary condition of the general form,

\[ F[u] = f(\cdot, u, Du, D^2 u), \quad \text{in } \Omega, \]
\[ G[u] = g(\cdot, u, Du), \quad \text{on } \partial \Omega, \]

where \( \Omega \) is a bounded smooth domain in Euclidean \( n \)-space, \( \mathbb{R}^n \), and \( f \in C^3(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n) \) and \( g \in C^4(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \) are real valued functions. Here \( S^n \) denotes the \( (n+1)/2 \)-dimensional linear space of \( n \times n \) real symmetric matrices, and \( Du = (D_i u) \) and \( D^2 u = [D_{ij} u] \) denote the gradient vector and Hessian matrix of the real valued function \( u \).

Let \( (x, z, p, r) \) denote points in \( \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n \). The operator \( F : C^4(\overline{\Omega}) \to C^2(\Omega) \) is uniformly elliptic on some bounded open subset \( O \) of \( C^4(\overline{\Omega}) \), namely there exists a constant \( \lambda > 0 \) such that for all \( u \in O \), \( x \in \Omega \) and \( \xi \in \mathbb{R}^n \) there holds

\[ \frac{\partial f}{\partial r_{ij}}(x, u, Du, D^2 u) \xi_i \xi_j \geq \lambda |\xi|^2. \]

The operator \( G : C^4(\overline{\Omega}) \to C^3(\partial \Omega) \) is uniformly oblique on \( O \), namely there exists a constant \( \chi > 0 \) such that for all \( u \in O \) and \( x \in \partial \Omega \)

\[ \frac{\partial g}{\partial p}(x, u, Du) \cdot \gamma(x) \geq \chi, \]

where \( \gamma(x) \) denotes the outer unit normal of \( \partial \Omega \) at \( x \).

**Theorem 1** ([3]). Let \( O \subset C^4(\overline{\Omega}) \) be a bounded open set with \( \partial O \cap (F,G)^{-1}(0) = \emptyset \), where \( F,G \) are as above. There exists a unique integer-valued degree for \((F,G)\) on \( O \) at 0, which satisfies the following key properties:

- If \( \text{deg}((F,G), O, 0) \neq 0 \), then \( \exists u \in O \) s.t. \( (F[u], G[u]) = 0 \).
- If \( U_1, U_2 \subset O \) and \( (O \setminus U_1 \cup U_2) \cap (F,G)^{-1}(0) = \emptyset \), then \( \text{deg}((F,G), O, 0) = \text{deg}((F,G), U_1, 0) + \text{deg}((F,G), U_2, 0) \).
- (Homotopy invariance) If \( t \mapsto (f_t, g_t) \) is continuous from \([0,1]\) to \( C^4(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^{n \times n}) \times C^4(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \) s.t. \( F_t \) is uniformly elliptic, \( G_t \) is uniformly oblique on \( O \), and \( \partial O \cap (F_t,G_t)^{-1}(0) = \emptyset \) for all \( t \in [0,1] \), then \( \text{deg}((F_t,G_t), O, 0) \) is independent of \( t \).

Moreover, we have the following immediate properties:

*Date: August 10, 2013.*
• Assume \((F,G)[u_0] = 0\) and the Fréchet derivative \((F',G')[u_0]\) is invertible. Then \(\deg((F,G),\mathcal{O},0) = \deg((F',G'),\mathcal{O},0)\), where \(\mathcal{O}\) is a neighborhood of \(u_0\) in \(C^{4,\alpha}(\overline{\Omega})\) which does not contain any other points of \((F,G)^{-1}(0)\).

• Compatibility with Leray-Schauder degree in linear cases: If \(F\) and \(G\) are linear, then \(\deg((F,G),\mathcal{O},0)\) “coincides” with the Leray-Schauder degree for linear operators.

**Sketch of proof:** Consider the operator \(S : C^{2,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega)\) such that 
\[
F(u) = (\triangle u, (\gamma_i Du + u)|_{\partial\Omega}), \quad \gamma : \text{unit outer normal}
\]
and the boundary operator \(T : C^{3,\alpha}(\partial\Omega) \to C^{1,\alpha}(\partial\Omega)\) s.t. 
\[
u \mapsto \triangle_T u - u,
\]
where \(\triangle_T\) denotes the tangential Laplacian over \(\partial\Omega\). Define

\[
\tilde{F}(u) = F(u) = (\triangle u, (\gamma_i Du + u)|_{\partial\Omega}), \quad \gamma : \text{unit outer normal}
\]
\[
\tilde{G}(u) = G(u) = (b_i(x, u, Du)\triangle_T(D_i u) + H_i(x, u, Du, D^2 u)|_{\partial\Omega}),
\]
where \(a_s = \frac{\partial F}{\partial u_s}(x, u, Du, D^2 u)\) and \(b_i = \frac{\partial G}{\partial u_i}(x, u, Du)\).

For a constant \(N > 0\), define a linear operator 
\[
L^N : C^{4,\alpha}(\overline{\Omega}) \to C^{\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega) \times C^{1,\alpha}(\partial\Omega)
\]
\[
\begin{align*}
L^N_{(1)} w &= a_s(x, u, Du, D^2 u)D_s w + C_s(x, u, Du, D^2 u), \\
L^N_{(2)} w &= (a_s(x, u, Du, D^2 u)D_s w)|_{\partial\Omega}, \\
L^N_{(3)} w &= (b_i(x, u, Du, D^2 u)|_{\partial\Omega}).
\end{align*}
\]

Then we split the operators \((\tilde{F}^N, \tilde{G}^N) = (F^N, G^N)\) maps \(C^{4,\alpha}(\overline{\Omega})\) into \(C^{1,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega) \times C^{2,\alpha}(\partial\Omega)\) and

\[
\begin{align*}
R_{(1)}^u [u] &= N a_s(x, u, Du, D^2 u)D_s w + C_s(x, u, Du, D^2 u, D^3 u), \\
R_{(2)}^u [u] &= E_s(x, u, Du, D^2 u)|_{\partial\Omega}, \\
R_{(3)}^u [u] &= N b_i(x, u, Du)D_i u + Nu + H_i(x, u, Du, D^2 u)|_{\partial\Omega}.
\end{align*}
\]

A main technical estimate is the following
Theorem 2 ([3]). Let \( a_{st} \in C^{2,\alpha}(\Omega) \), symmetric, and there exists a constant \( \lambda > 0 \) such that \( a_{st}(x)\xi_i \xi_j \geq \lambda |\xi|^2, \forall \xi \in \mathbb{R}^n \) and \( \forall x \in \Omega \). Let \( b_i \in C^{3,\alpha}(\partial \Omega) \), and there exists a constant \( \chi > 0 \) such that \( b_i(x) \gamma_i(x) \geq \chi \), \( \forall x \in \partial \Omega \).

Then there exists a constant \( N_0 \), depending only on \( \|a_{st}\|_{C^{2,\alpha}}, \|b_i\|_{C^{3,\alpha}}, n, \lambda, \chi \) such that for all \( N > N_0 \), \( L^N \) is a bijection. Furthermore, \( L^N \) depends continuously on \( a_{st}, b_i \) with respect to the corresponding topologies.

Having the above theorem, by [1, Theorem 7.3], \( (L^u,N)^{-1} \) maps \( C^{1,\alpha}(\Omega) \times C^{2,\alpha}(\partial \Omega) \times C^{2,\alpha}(\partial \Omega) \) into \( C^{5,\alpha}(\Omega) \), and its norm as a linear map between these spaces is bounded by a constant depends only on \( \|a_{st}\|_{C^{2,\alpha}}, \|b_i\|_{C^{3,\alpha}}, \lambda \) and \( \chi \). It follows that
\[
\begin{align*}
\text{u \mapsto (L}^u,N)^{-1}R^u,N[u] 
\end{align*}
\]
is a compact operator from \( O \) to \( C^{4,\alpha}(\Omega) \).

Moreover, \( (F,G)^0[u] = 0 \) is the same as \( u + (L^u,N)^{-1}R^u,N[u] = 0 \), i.e.,
\[
\partial O \cap (Id + (L^u,N)^{-1}R^u,N)^{-1}(0) = \partial O \cap (F,G)^{-1}(0) = \emptyset.
\]
Therefore, we can define the degree of \( (F,G) \) as the Leray-Schauder degree of the map \( u \mapsto u + (L^u,N)^{-1}R^u,N[u] \). More precisely we have the following definition.

Definition 1. Let \( F,G \) be operators as above, and \( O \subset C^{4,\alpha}(\Omega) \) is a bounded open set with \( \partial O \cap (F,G)^{-1}(0) = \emptyset \). We define a degree of \( (F,G) \) on \( O \) at \( 0 \) by
\[
\deg ((F,G),O,0) = \deg_{L.S.}(Id + (L^u,N)^{-1}R^u,N,O,0),
\]
where \( N > N_0, N_0 \) is the constant in Theorem 2.

We remark that as in [2], this definition of the degree is independent of \( N > N_0 \) according to the homotopy invariance of the Leray-Schauder degree. And the degree satisfies our desired properties in Theorem 1.

References


Institute for Mathematics and its Applications, School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, AUSTRALIA.

E-mail address: jiaikunl@uow.edu.au