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Keywords
fell, theorem, douady, algebras, dixmier

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A Dixmier–Douady theorem for Fell algebras ☆

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Abstract

We generalise the Dixmier–Douady classification of continuous-trace C^*-algebras to Fell algebras. To do so, we show that C^*-diagonals in Fell algebras are precisely abelian subalgebras with the extension property, and use this to prove that every Fell algebra is Morita equivalent to one containing a diagonal subalgebra. We then use the machinery of twisted groupoid C^*-algebras and equivariant sheaf cohomology to define an analogue of the Dixmier–Douady invariant for Fell algebras, and to prove our classification theorem.
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Keywords: Brauer group; Dixmier–Douady; Extension property; Fell algebra; Groupoid; Sheaf cohomology

1. Introduction

The Dixmier–Douady theorem classifies continuous-trace C^*-algebras with spectrum T up to Morita equivalence by classes in a third cohomology group [17], and the Phillips–Raeburn theorem classifies their $C_0(T)$-automorphisms using classes in the corresponding second cohomology group [35]. The Dixmier–Douady Theorem has been very influential in the study of C^*-dynamical systems (see for example [36]), and has been applied in differential geometry [10],

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in mathematical physics [8,12,30], and in the definition of twisted K-theory [40]. The object of this paper is to extend the Dixmier–Douady theorem to Fell algebras.

A Fell algebra is a C^*-algebra A such that every irreducible representation π_0 of A satisfies Fell's condition: there is a positive $b \in A$ and a neighbourhood U of $[\pi_0]$ in $\widehat{A}$ such that $\pi(b)$ is a rank-one projection whenever $[\pi] \in U$. The spectrum of a Fell algebra is always locally Hausdorff [6, Corollary 3.4], and is Hausdorff if and only if the Fell algebra is a continuous-trace C^*-algebra. The class of Fell algebras coincides with the class of Type I_0 algebras defined by Pedersen in [34, §6.1] as the C^*-algebras generated by their abelian elements (see p. 1546). Fell algebras are the natural building blocks for Type I C^*-algebras: every Type I C^*-algebra has a canonical composition series consisting of Fell algebras [34, Theorem 6.2.6] (by contrast there always exists a composition series consisting of continuous-trace C^*-algebras, but no canonical one).

Let T be a locally compact, Hausdorff space. Given a continuous-trace C^*-algebra A with spectrum identified with T, the Dixmier–Douady invariant $\delta(A)$ belongs to a second sheaf-cohomology group $H^2(T, \mathcal{S})$. The Dixmier–Douady classification of continuous-trace C^*-algebras says that if A and B are continuous-trace C^*-algebras with spectra identified with T, then $\delta(A) = \delta(B)$ if and only if there is an A–B-imprimitivity bimodule whose Rieffel homeomorphism respects the identifications of $\widehat{A}$ and $\widehat{B}$ with T.

If we replace continuous-trace C^*-algebras with Fell algebras, we must deal with locally compact, locally Hausdorff spaces X. There is no difficulty with sheaf cohomology for such spaces, but the definition of our analogue of the Dixmier–Douady invariant $\delta(A) \in H^2(\widehat{A}, \mathcal{S})$ is more involved. We tackle the problem using the machinery of C^*-diagonals and of twisted groupoid C^*-algebras.

A C^*-diagonal consists of a C^*-algebra A and a maximal abelian subalgebra B of A with properties modeled on those of the subalgebra of diagonal matrices in $M_n(\mathbb{C})$ (see Definition 5.2). Diagonals relate to Fell algebras as follows. Consider a Fell algebra A with a generating sequence a_i of pairwise orthogonal abelian elements such that $a := \sum_i \frac{1}{i} a_i$ is strictly positive in A. That is, the hereditary subalgebra generated by a is equal to A. Then $B := \bigoplus_i \overline{a_i A a_i}$ is an abelian subalgebra of A, which we prove is a diagonal. Indeed, Theorem 5.17 shows that every separable Fell algebra A is Morita equivalent to a C^*-algebra C with a diagonal subalgebra D arising in just this fashion. In outline the construction is straightforward. Fix a sequence a_i of abelian elements which generate A and let $\tilde{a}_i = a_i \otimes \Theta_{ii} \in A \otimes \mathcal{K}(l^2(\mathbb{N}))$ for each i. Let C be the smallest hereditary C^*-subalgebra containing all the $\tilde{a}_i$ and let

$$D := \bigoplus_i \overline{\tilde{a}_i (A \otimes \mathcal{K}) \tilde{a}_i} = \bigoplus_i (\overline{a_i A a_i} \otimes \Theta_{ii}).$$

To prove that D is a diagonal, we show in Theorem 5.14 that diagonals in Fell algebras A can be characterised as the abelian subalgebras B which have the extension property relative to A: every pure state of B extends uniquely to a state of A. This extends [27, Theorem 2.2] from continuous-trace C^*-algebras to Fell algebras. Example 5.15 shows that this characterisation does not generalise to bounded-trace C^*-algebras.

C^*-diagonals arise naturally from topological twists: exact sequences of groupoids

$$\Gamma^{(0)} \to \Gamma^{(0)} \times \mathbb{T} \to \Gamma \to R$$

(just $\Gamma \to R$ for short) such that Γ is a $\mathbb{T}$-groupoid and R is a principal étale groupoid with unit space $\Gamma^{(0)}$ (see p. 1562). The associated twisted groupoid C^*-algebra $C^*_{\mathrm{r}}(\Gamma; R)$ is a completion

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of the space of continuous $\mathbb{T}$-equivariant functions on Γ and contains a subalgebra isomorphic to $C_0(\Gamma^{(0)})$. Moreover, the pair $(C^*_{\mathrm{r}}(\Gamma; R), C_0(\Gamma^{(0)}))$ is a C^*-diagonal. Kumjian showed in [27] that every diagonal pair arises in this way: given a diagonal pair (A, B) there exist a topological twist $\Gamma \to R$ and an isomorphism $\phi : A \to C^*_{\mathrm{r}}(\Gamma; R)$ such that $\phi(B) = C_0(\Gamma^{(0)})$. Together with the results outlined in the preceding paragraph, this implies that each Fell algebra is Morita equivalent to a twisted groupoid C^*-algebra $C^*_{\mathrm{r}}(\Gamma; R)$.

Given a principal étale groupoid R, an isomorphism of twists over R is an isomorphism of exact sequences which identifies ends. The isomorphism classes of topological twists over R form a group $\mathrm{Tw}(R)$ called the twist group [26]. It was shown in [28] how the twist group fits into a long exact sequence of equivariant-sheaf cohomology. In particular, the boundary map ∂^1 in this long exact sequence determines a homomorphism from the twist group to the second equivariant-cohomology group $H^2(R, \mathcal{S})$. We use this construction to define an analogue of the Dixmier–Douady invariant for a Fell algebra A. Given a Fell algebra A with spectrum X, choose any twist $\Gamma \to R$ such that A is Morita equivalent to $C^*_{\mathrm{r}}(\Gamma; R)$. Applying ∂^1 to the class of Γ in the twist group of R yields an element $\partial^1([\underline{\Gamma}])$ of $H^2(R, \mathcal{S})$. We show that the local homeomorphism $\psi : \Gamma^{(0)} \to X$ obtained from the state-extension property yields an isomorphism π^*_ψ from the usual sheaf-cohomology group $H^2(X, \mathcal{S})$ to the equivariant-sheaf cohomology group $H^2(R, \mathcal{S})$. We then show that the class $\delta(A) = (\pi^*_\psi)^{-1}(\partial^1([\underline{\Gamma}])) \in H^2(X, \mathcal{S})$ does not depend on our choice of twist $\Gamma \to R$, and regard $\delta(A)$ as an analogue of the Dixmier–Douady invariant for A.

This paves the way for our main result, Theorem 7.13: Fell algebras A_1 and A_2 are Morita equivalent if and only if there is a homeomorphism between their spectra such that the induced isomorphism $H^2(\widehat{A}_1, \mathcal{S}) \cong H^2(\widehat{A}_2, \mathcal{S})$ carries $\delta(A_1)$ to $\delta(A_2)$. The invariant is exhausted in the sense that each element of $H^2(X, \mathcal{S})$ can be realised as $\delta(A)$ for some Fell algebra A with spectrum X (Proposition 7.16).

A motivating example was a generalisation of Green's theorem for free and proper transformation groups to free transformation groups (G, X) where X is a Cartan G-space. Our Corollary 4.6 gives a Morita equivalence between the transformation-group C^*-algebra $C_0(X) \rtimes G$ and the C^*-algebra of the equivalence relation induced by a local homeomorphism from a Hausdorff space Y to the (not necessarily Hausdorff) quotient space $G\backslash X$. This result and its construction are prototypes for our later investigations of diagonals in Fell algebras. In particular, we show that $\delta(C_0(X) \rtimes G)$ is trivial.

2. Preliminaries

For a C^*-algebra A, let $\widetilde{A}$ denote the C^*-algebra $A + \mathbb{C}1$ obtained by adjoining a unit. If B is a C^*-subalgebra of A, we regard $\widetilde{B}$ as a unital C^*-subalgebra of $\widetilde{A}$ (so, $1_{\widetilde{B}} = 1_{\widetilde{A}}$).

Given a Hilbert space H, denote by $\mathcal{K}(H)$ the C^*-algebra of compact operators on H. For $\xi, \eta \in H$, let $\Theta_{\xi,\eta} \in \mathcal{K}(H)$ be the rank-one operator defined by $\Theta_{\xi,\eta}(\zeta) = (\zeta|\eta)\xi$.

A C^*-algebra A is *liminary* if $\pi(A) = \mathcal{K}(H_\pi)$ for every irreducible representation π. If B is an abelian C^*-algebra we freely identify B and $C_0(\widehat{B})$.

Let G be a Hausdorff topological groupoid with unit space $G^{(0)}$. We denote the range and source maps by $r, s : G \to G^{(0)}$ and the set of composable pairs of G by $G^{(2)}$. Let U be a subset of the unit space. We write UG, GU and UGU for $r^{-1}(U)$, $s^{-1}(U)$ and $r^{-1}(U) \cap s^{-1}(U)$; U is called *full* if $s(UG) = G^{(0)}$. A subset T of G is a *G-set* if the restrictions of s and r to T are one-to-one. We implicitly identify units of G with the associated identity morphisms throughout.

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A groupoid is *principal* if the map $\gamma \mapsto (r(\gamma), s(\gamma))$ is one-to-one. A groupoid is *étale* if the range map (equivalently the source map) is a local homeomorphism. If G is an étale groupoid then the unit space $G^{(0)}$ is open in G and for each $u \in G^{(0)}$ the fibre $r^{-1}(u)$ is discrete.

A topological space X is *locally compact* if every point of X has a compact neighbourhood in X; and X is *locally Hausdorff* if every point of X has a Hausdorff neighbourhood.

3. Fell and Type I_0 algebras

In this section we show that the classes of Fell and Type I_0 C^*-algebras coincide.

Let A be a C^*-algebra. A positive element a of A is *abelian* if the hereditary C^*-subalgebra $\overline{aAa}$ generated by a is commutative. If A is generated as a C^*-algebra by its abelian elements then A is said to be of *Type* I_0 [34, §6.1]. An irreducible representation π_0 of A satisfies *Fell's condition* if there exist $b \in A^+$ and an open neighbourhood U of $[\pi_0]$ in $\widehat{A}$ such that $\pi(b)$ is a rank-one projection whenever $[\pi] \in U$; this property goes back as far as [18]. If every irreducible representation of A satisfies Fell's condition then A is said to be a *Fell algebra* [6, §3]. That the Fell algebras coincide with the Type I_0 C^*-algebras is a consequence of the following lemma which is stated in [6, §3].

Lemma 3.1. *Let A be a C^*-algebra and π_0 an irreducible representation of A. Then there exists an abelian element a of A such that $\pi_0(a) \neq 0$ if and only if π_0 satisfies Fell's condition.*

Proof. Suppose $a \in A^+$ is an abelian element such that $\pi_0(a) \neq 0$. By rescaling we may assume that $\|\pi_0(a)\| = 1$. By [34, Lemma 6.1.3], $\operatorname{rank}(\pi(a)) \leqslant 1$ for all irreducible representations π of A. Since $[\pi] \mapsto \|\pi(a)\|$ is lower semicontinuous there exists a neighbourhood U of $[\pi_0]$ in $\widehat{A}$ such that $\|\pi(a)\| > 1/2$ when $[\pi] \in U$. In particular, the spectrum $\sigma(\pi(a))$ of $\pi(a)$ is $\{0, \lambda_\pi\}$ for some $\lambda_\pi > 1/2$. Fix $f \in C([0, \|a\|])$ such that f is identically zero on $[0, 1/8]$ and is identically one on $[1/4, \|a\|]$. Set $b = f(a)$. If $[\pi] \in U$ then

$$\sigma\big(\pi(b)\big) = \sigma\big(\pi\big(f(a)\big)\big) = f\big(\sigma\big(\pi(a)\big)\big) = f\big(\{0, \lambda_\pi\}\big) = \{0, 1\}.$$

Since $\operatorname{rank}(\pi(a)) = 1$, $\pi(b)$ is a rank-one projection. Thus π_0 satisfies Fell's condition.

Now suppose that π_0 satisfies Fell's condition. Then there exist $a \in A^+$ and an open neighbourhood U of $[\pi_0]$ in $\widehat{A}$ such that $\pi(a)$ is a rank-one projection when $[\pi] \in U$. Let J be the closed ideal of A such that $\widehat{J} = U$. There exists $x \in J^+$ such that $\pi_0(axa) \neq 0$ (choose an approximate identity $\{e_\lambda\}$ for J and note that $\pi_0(e_\lambda) \to 1$ in $B(H)$). Now $\pi(axa) = 0$ whenever $[\pi] \notin \widehat{J}$, and $\operatorname{rank}(\pi(axa)) \leqslant \operatorname{rank}(\pi(a)) \leqslant 1$ when $[\pi] \in \widehat{J}$. Thus $\operatorname{rank}(\pi(axa)) \leqslant 1$ for all irreducible representations of A, and hence axa is an abelian element by [34, Lemma 6.1.3]. □

It is well known that if $p \in M(A)$ is a projection then Ap is an $\overline{ApA}$–pAp-imprimitivity bimodule (see, for example, [37, Example 3.6]). More generally we have:

Lemma 3.2. *Let A be a C^*-algebra. If $b \in A$ is self-adjoint then $\overline{Ab}$ is an $\overline{AbA}$–$\overline{bAb}$-imprimitivity bimodule with actions given by multiplication in A and*

$${}_{\overline{AbA}}\langle ab, cb\rangle = ab^2c^* \quad \textit{and} \quad \langle ab, cb\rangle_{\overline{bAb}} = ba^*cb.$$

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Proof. The actions and inner products are restrictions of those on the standard A–A-bimodule A, so we need only check that both inner products are full. The right inner product is full because products are dense in A and the left inner product is full because $b \in C^*(\{b^2\}) \subset A$, so $\overline{AbA} = \overline{Ab^2A}$. □

By [6, Corollary 3.4], the spectrum of a Fell algebra is locally Hausdorff. So Fell algebras may be regarded as locally continuous-trace C^*-algebras; since they are generated by their abelian elements, they may also be regarded as locally Morita equivalent to a commutative C^*-algebra. We make this precise in the following theorem.

Theorem 3.3. *Let A be a C^*-algebra. The following are equivalent*:

1. *A is of Type I_0*;
2. *there exists a collection $\{I_a\colon a \in S\}$ of ideals of A such that A is generated by these ideals and each I_a is Morita equivalent to a commutative C^*-algebra*;
3. *A is a Fell algebra.*

Proof. ((1) ⇒ (2)) Suppose A is Type I_0. Let S be the set of abelian elements of A. For each $a \in S$, the sub-C^*-algebra $\overline{aAa}$ is commutative, and by Lemma 3.2, $\overline{aAa}$ is Morita equivalent to the ideal $I_a := \overline{AaA}$ generated by a. Since A is generated by S it is also generated, as a C^*-algebra, by the collection of ideals $\{I_a\colon a \in S\}$.

((2) ⇒ (3)) Assume (2). Fix an irreducible representation π_0 of A. Since A is generated by the ideals I_a there exists a_0 such that π_0 does not vanish on I_{a_0}. Morita equivalence preserves the property of being a continuous-trace C^*-algebra, so I_{a_0} is itself a continuous-trace C^*-algebra. The restriction of π_0 to I_{a_0} is an irreducible representation which satisfies Fell's condition in I_{a_0} (since I_{a_0} is a continuous-trace C^*-algebra). So there exist $b \in I_{a_0}^+$ and a neighbourhood U of $\widehat{I_{a_0}}$ such that $\pi(b)$ is a rank-one projection whenever $[\pi] \in U$. Now $b \in A^+$ and U can be viewed as an open subset of $\widehat{A}$, so π_0 satisfies Fell's condition in A.

((3) ⇒ (1)) Suppose that A is a Fell algebra. By Lemma 3.1, for each irreducible representation π of A there exists an abelian element $a_\pi \in A$ such that $\pi(a_\pi) \neq 0$. Let B be the C^*-algebra generated by the set S of all abelian elements of A, so that $B = \overline{\operatorname{span}}\{a_1 \cdots a_n\colon n \in \mathbb{N},\ a_i \in S\}$. Since $\pi|_B \neq 0$ for all $\pi \in \widehat{A}$, B is not contained in any proper ideal of A. But B is an ideal of A by [34, 6.1.7] (the largest Type I_0 ideal in fact), so $B = A$ and A is Type I_0. □

4. Green's theorem for free Cartan transformation groups.

Throughout this section let G be a second-countable, locally compact, Hausdorff group acting continuously on a second-countable, locally compact, Hausdorff space X. We will prove a generalisation of Green's theorem for free group actions which are not proper but only locally proper. Green's theorem says that if a group G acts freely and properly on a space X, then the crossed product $C_0(X) \rtimes G$ is Morita equivalent to $C_0(X/G)$; it follows that the Dixmier–Douady invariant of the continuous-trace C^*-algebra $C_0(X) \rtimes G$ is trivial. In Section 7, we will establish the analogous result for locally proper actions and our generalisation of the Dixmier–Douady classification.

Recall from [33, Definition 1.1.2] that X is a *Cartan G-space* if each point of X has a wandering neighbourhood U; that is, a neighbourhood U such that $\{s \in G\colon s \cdot U \cap U \neq \emptyset\}$ is relatively

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compact in G. If X is a Cartan G-space with a free action of G then we will just say that (G, X) is a free Cartan transformation group.

The action of G on X is *proper* if every compact subset of X is wandering. Equivalently, the action is proper if the map $\phi : G \times X \to X \times X$ given by $\phi(g, x) = (g \cdot x, x)$ is proper in the sense that the inverse images of compact sets are compact. If U is a wandering neighbourhood in X, then the action of G on the saturation $G \cdot U$ of U is proper by [33, Proposition 1.2.4].

If G acts freely on X and $x, y \in X$ with $G \cdot x = G \cdot y$, then there is a unique $\tau(x, y) \in G$ such that

$$y = \tau(x, y) \cdot x; \tag{4.1}$$

this defines a function τ from

$$X \times_{G\backslash X} X := \{(x, y) \in X \times X\colon\ G \cdot x = G \cdot y\}$$

to G. If X is a free Cartan G-space, then τ is continuous by [33, Theorem 1.1.3].

The next lemma follows from [9, I.10.1 Proposition 2].

Lemma 4.1. *Suppose that the group G acts freely on X. Then the action of G on X is proper if and only if $\phi : G \times X \to X \times X$, $(g, x) \mapsto (g \cdot x, x)$ is a homeomorphism onto a closed subset of $X \times X$.*

Lemma 4.2. *Suppose that (G, X) is a free Cartan transformation group.*

1. *There exists a covering $\{U_i\colon\ i \in I\}$ of X by G-invariant open sets such that (G, U_i) is proper for each i.*
2. *Let $\{U_i\colon\ i \in I\}$ be a cover as in* (1)*, and let $W := \bigsqcup_i U_i$ be the topological disjoint union of the U_i. Then the map $\phi : G \times W \to W \times W$, $(g, x) \mapsto (g \cdot x, x)$ is a homeomorphism onto a closed subset of $W \times W$.*

Proof. (1) If U is a wandering neighbourhood in X, then its saturation $G \cdot U$ is a proper G-space by [33, Proposition 1.2.4]. So choose a cover $\{V_i\colon\ i \in I\}$ of open wandering neighbourhoods in X and then take $U_i = G \cdot V_i$ for all I.

(2) The action of G on W is $g \cdot x^i = (g \cdot x)^i$, where, for $x \in U_i \subset X$, we write x^i for the corresponding element in the copy of U_i in W. The action of G on W is free because the action of G on X is free.

Since the action on W is continuous, so is ϕ. Since the action on W is free, ϕ is one-to-one. The inverse $\phi^{-1} : \operatorname{range}\phi \to G \times W$ is given by $\phi^{-1}(y, x) = (\tau(x, y), x)$. The map $\tau : X \times_{G\backslash X} X \to G$ of (4.1) is continuous because (G, X) is Cartan, so ϕ^{-1} is continuous. To see that the range of ϕ is closed, suppose that

$$\left(g_n \cdot x_n^{i_n}, x_n^{i_n}\right)$$

is a sequence in $\operatorname{range}\phi$ converging to (y, x^j). Then $x_n^{i_n} \to x^j$, so $i_n = j$ eventually. Since U_j is G-invariant, $g_n \cdot x_n^{i_n} \in U_j$ eventually as well. Since $g_n \cdot x_n^{i_n} \to y$ it follows that $y \in U_j$. In

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particular, $G \cdot x_n^j$ converges to both $G \cdot x^j$ and $G \cdot y$. But the action of G on U_j is proper, so $G\backslash X$ is Hausdorff and hence $y \in G \cdot x$ as required. □

The following definitions are from [31, §2]. Let Γ be a locally compact, Hausdorff groupoid and Z a locally compact space. We say Γ *acts on the left of* Z if there is a continuous open map $\rho : Z \to \Gamma^{(0)}$ and a continuous map $(\gamma, x) \mapsto \gamma \cdot x$ from $\Gamma * Z = \Gamma\ {}_s{*}_\rho\ Z := \{(\gamma, x) \in \Gamma \times Z\colon s(\gamma) = \rho(x)\}$ to Z such that

1. $\rho(\gamma \cdot x) = r(\gamma)$ for $(\gamma, x) \in \Gamma\ {}_s{*}_\rho\ Z$;
2. if $(\gamma_1, x) \in \Gamma\ {}_s{*}_\rho\ Z$ and $(\gamma_2, \gamma_1) \in \Gamma^{(2)}$ then $(\gamma_2\gamma_1) \cdot x = \gamma_2 \cdot (\gamma_1 \cdot x)$;
3. $\rho(x) \cdot x = x$ for $x \in Z$.

Right actions of Γ on Z are defined similarly, except that we use $\sigma : Z \to \Gamma^{(0)}$ and $Z\ {}_\sigma{*}_r\ \Gamma := \{(x, \gamma) \in Z \times \Gamma\colon \sigma(x) = r(\gamma)\}$. An action of Γ on the left of Z is said to be *free* if $\gamma \cdot x = x$ implies that $\gamma = \rho(x)$, and is said to be *proper* if the map $\Gamma\ {}_s{*}_\rho\ Z \to Z \times Z : (\gamma, x) \mapsto (\gamma \cdot x, x)$ is proper.

Definition 4.3. If Γ_1 and Γ_2 are groupoids then an *equivalence from* Γ_1 *to* Γ_2 is a triple (Z, ρ, σ) where

1. Z carries a free and proper left-action of Γ_1 with fibre map $\rho : Z \to \Gamma_1^{(0)}$, and a free and proper right-action of Γ_2 with fibre map $\sigma : Z \to \Gamma_2^{(0)}$;
2. the actions of Γ_1 and Γ_2 on Z commute; and
3. ρ and σ induce bijections of Z/Γ_2 onto $\Gamma_1^{(0)}$ and of $\Gamma_1\backslash Z$ onto $\Gamma_2^{(0)}$, respectively.

Since ρ and σ are continuous open maps Definition 4.3(3) implies that ρ and σ induce homeomorphisms $Z/\Gamma_2 \cong \Gamma_1^{(0)}$ and $\Gamma_1\backslash Z \cong \Gamma_2^{(0)}$. We will often just say that Z is a Γ_1–Γ_2-equivalence, leaving the fibre maps σ, ρ implicit. The main theorem of [31] says that if Γ_1 and Γ_2 are groupoids with Haar systems and Z is a Γ_1–Γ_2-equivalence, then $C_c(Z)$ can be completed to a $C^*(\Gamma_1)$–$C^*(\Gamma_2)$-imprimitivity bimodule [31, Theorem 2.8], so the full groupoid C^*-algebras of Γ_1 and Γ_2 are Morita equivalent.

If (G, X) is a transformation group we view $G \times X$ as a transformation-group groupoid with composable elements

$$(G \times X)^{(2)} = \big\{\big((g, x), (h, y)\big) \in (G \times X) \times (G \times X)\colon x = h \cdot y\big\}$$

and $(g, h \cdot y)(h, y) = (gh, y)$; the inverse is given by $(g, x)^{-1} = (g^{-1}, g \cdot x)$. We identify the unit space $(G \times X)^{(0)} = \{e\} \times X$ with X, so $s(g, x) = x$ and $r(g, x) = g \cdot x$ for all $(g, x) \in G \times X$.

Suppose that (G, X) is a free Cartan transformation group. Let $\{U_i\colon i \in I\}$ be a covering of X by G-invariant open sets such that (G, U_i) is proper for each i; then each $V_i := G\backslash U_i$ is locally compact and Hausdorff. Let $q : X \to G\backslash X$ be the quotient map. For each i, denote by $q_i : U_i \to V_i$ the restricted quotient map, and let $\psi_i : V_i \to q(U_i) \subseteq G\backslash X$ be the inclusion homeomorphism. Let $Y := \bigsqcup_i V_i$ be the topological disjoint union of the V_i, and define $\psi : Y \to G\backslash X$ by $\psi|_{V_i} = \psi_i$. Then ψ is a local homeomorphism and Y is locally compact and Hausdorff.

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Lemma 4.4. *Suppose that X is a free Cartan G-space, and adopt the notation of the preceding paragraph.*

1. *The formula $(g,x)\cdot(x,y)=(g\cdot x,y)$ defines a free left action of the groupoid $G\times X$ on $X*Y=\{(x,y)\in X\times Y\colon\ q(x)=\psi(y)\}$.*
2. *The formula $(g,x)\cdot x=g\cdot x$ defines a free and proper left action of the groupoid $G\times X$ on $W:=\bigsqcup_i U_i$.*
3. *There is a homeomorphism $\alpha: W\to X*Y$ such that $(g,x)\cdot\alpha(z)=\alpha((g,x)\cdot z)$ for all g,x,z.*

Proof. (1) Define $\rho: X*Y\to(G\times X)^{(0)}$ by $\rho(x,y)=x$. Then

$$(G\times X)*(X*Y)=\big\{\big((g,x),\big(x',y\big)\big)\colon\ x=s(g,x)=\rho\big(x',y\big)=x'\big\}.$$

It is straightforward to check that the formula $(g,x)\cdot(x,y):=(g\cdot x,y)$ defines a free action of $G\times X$ on the left of $X*Y$.

(2) As earlier, for $x\in U_i\subset X$, we write x^i for the corresponding element of $U_i\subset W$. Define $\rho': W\to(G\times X)^{(0)}$ by $\rho'(x^i)=x$. Then

$$(G\times X)*W=\bigsqcup\nolimits_i\big\{\big((g,x),x^i\big)\colon\ (g,x)\in G\times U_i\big\},$$

and the formula $(g,x)\cdot x^i:=(g\cdot x)^i$ defines a free action of $(G\times X)$ on W.

By Lemma 4.1, to see that the action is proper it suffices to verify that

$$\phi:(G\times X)*W\to W\times W,\quad \big((g,x),x^i\big)\mapsto\big((g\cdot x)^i,x^i\big)$$

is a homeomorphism of $(G\times X)*W$ onto a closed subset of $W\times W$.

Let $\tau: X*_{G\backslash X}X\to G$ be as in (4.1). Then τ is continuous since X is a Cartan G-space. So $\phi:(G\times X)*W\to\operatorname{range}\psi$ is invertible with continuous inverse

$$(y,x)\mapsto\big(\big(\tau(x,y),x\big),x\big).$$

That the range of ϕ is closed is precisely Lemma 4.2(2).

(3) Define $\alpha: W\to X*Y$ by $\alpha(x^i)=(x,q_i(x^i))$. Clearly α is continuous and one-to-one with continuous inverse $(x,q_i(x^i))\mapsto x$. To see that α is onto, notice that $(x,y)\in X*Y$ for $y\in V_i$ if and only if $y=q_i(x^i)$. That α is equivariant is a simple calculation:

$$(g,x)\cdot\alpha\big(x^i\big)=(g,x)\cdot\big(x^i,q_i\big(x^i\big)\big)=\big(g\cdot x^i,q_i\big(x^i\big)\big)=\alpha\big(g\cdot x^i\big)=\alpha\big((g,x)\cdot x^i\big).\qquad\square$$

Recall that under the relative topology

$$R(\psi)=\big\{(y_1,y_2)\in Y\times Y\colon\ \psi(y_1)=\psi(y_2)\big\}$$

is a principal groupoid with range and source maps $s(y_1,y_2)=y_2$, $r(y_1,y_2)=y_1$, composition $(y_1,y_2)(y_2,y_3)=(y_1,y_3)$ and inverses $(y_1,y_2)^{-1}=(y_2,y_1)$; $R(\psi)$ is étale because ψ is a local homeomorphism. We identify $R(\psi)^{(0)}$ with Y via $(y,y)\mapsto y$.

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Theorem 4.5. *Let (G, X) be a free Cartan G-space. Then the transformation-group groupoid $G \times X$ is equivalent to the groupoid $R(\psi)$ described in the preceding paragraph. More specifically, resume the notation of Lemma* 4.4 *and define fibre maps $\rho : X * Y \to (G \times X)^{(0)}$ and $\sigma : X * Y \to R(\psi)^{(0)}$ by*

$$\rho(x, y) = x \quad \textit{and} \quad \sigma(x, y) = y.$$

Then the space $X * Y$ is a $(G \times X)$–$R(\psi)$ equivalence under the actions

$$(g, x) \cdot (x, y) = (g \cdot x, y) \quad \textit{and} \quad (x, y) \cdot (y, z) = (x, z).$$

Proof. We need to verify (1)–(3) of Definition 4.3. By Lemma 4.4, the left action of $G \times X$ on $X * Y$ is free and proper. It is easy to check that the right action of $R(\psi)$ on $X * Y$ is free and proper, verifying (1).

To verify (2), we calculate:

$$\begin{aligned}\big((g, x) \cdot (x, y)\big) \cdot (y, z) &= (g \cdot x, y) \cdot (y, z) = (g \cdot x, z) = (g, x) \cdot (x, z) \\ &= (g, x) \cdot \big((x, y) \cdot (y, z)\big).\end{aligned}$$

It remains to verify (3). Since both ρ and σ are surjective, we need only show that both induce injections. Suppose that $\rho(x, y) = \rho(x', y')$. Then certainly $x = x'$. Since $(x, y), (x, y') \in X * Y$ we have $\psi(y) = q(x) = \psi(y')$, so $(x, y) = (x, y') \cdot (y', y)$ with $(y, y') \in R(\psi)$. Hence ρ induces an injection. Similarly, suppose $\sigma(x, y) = \sigma(x', y')$. Then $y = y'$. Also, $q(x) = \psi(y) = q(x')$, so there exists $g \in G$ such that $g \cdot x = x'$. Thus $(g, x) \cdot (x, y) = (g \cdot x, y) = (x', y)$ and $(g, x) \in G \times X$. Hence, σ induces an injection. □

We now obtain an analogue of Green's beautiful theorem for free transformation groups: if G acts freely and properly on X then $C_0(X) \rtimes G$ and $C_0(G\backslash X)$ are Morita equivalent [19, Theorem 14]. If the action is only locally proper then $G\backslash X$ may not be Hausdorff, so that $C_0(G\backslash X)$ is not a C^*-algebra – the groupoid C^*-algebra $C^*(R(\psi))$ serves as its replacement in this case.

Corollary 4.6. *Suppose that (G, X) is a free Cartan transformation group. Then the transformation-group C^*-algebra $C_0(X) \rtimes G$ is Morita equivalent to the groupoid C^*-algebra $C^*(R(\psi))$.*

Proof. Since $R(\psi)$ is étale, $R(\psi)$ has a Haar systems given by counting measures. A natural Haar system for $G \times X$ is $\{\mu \times \delta_x\colon x \in X\}$, where μ is a left Haar measure on G and δ_x is point-mass measure. So the $(G \times X)$–$R(\psi)$ equivalence of Theorem 4.5 induces a Morita equivalence of full groupoid C^*-algebras by [31, Theorem 2.8]. Since $C^*(G \times X)$ and $C_0(X) \rtimes G$ are isomorphic [38, remarks on p. 59] the result follows. □

Let (G, X) be a free Cartan transformation group. Then $C_0(X) \rtimes G$ is a Fell algebra by [22]. Since the property of being a Fell algebra is preserved under Morita equivalence by [24], $C^*(R(\psi))$ is also a Fell algebra. Alternatively, by [14, Theorem 7.9] a principal-groupoid C^*-algebra is a Fell algebra if and only if the groupoid is Cartan in the sense that every unit has a wandering neighbourhood (see Definition 7.3 of [14]); it is straightforward to verify the existence of wandering neighbourhoods in $R(\psi)$.

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5. Fell algebras, the extension property and C^*-diagonals

In this section we show how to construct from a separable Fell algebra A a Morita equivalent C^*-algebra C containing a diagonal subalgebra in the sense of [27]. The bulk of the work is to show that diagonal subalgebras of separable Fell algebras can be characterised as those abelian subalgebras which possess the extension property. We start by verifying that the different notions of diagonals in nonunital C^*-algebras which appear in the literature coincide.

5.1. Diagonals in nonunital C^*-algebras

Let A be a C^*-algebra and B a C^*-subalgebra of A. Recall that $P : A \to B$ is a *conditional expectation* if P is a linear, norm-decreasing, positive map such that $P|_B = \mathrm{id}_B$ and $P(ab) = P(a)b$, $P(ba) = bP(a)$ for all $a \in A$ and $b \in B$. We say P is *faithful* if $P(a^*a) = 0$ implies $a = 0$.

Remark 5.1. There are two other equivalent characterisations of a conditional expectation:

1. $P : A \to B$ is a linear idempotent of norm 1;
2. $P : A \to B$ is a linear, norm-decreasing, completely positive map such that $P|_B = \mathrm{id}_B$ and $P(ab) = P(a)b$, $P(ba) = bP(a)$ for all $a \in A$ and $b \in B$.

Our definition above implies (1); that (1) implies (2) is in [42] (see, for example [7, Theorem II.6.10.2]), and (2) implies our definition since completely positive maps are positive.

Definition 5.2. Let A be a separable C^*-algebra and let B be an abelian C^*-subalgebra of A. A *normaliser* n of B in A is an element $n \in A$ such that $n^*Bn, nBn^* \subset B$; the collection of normalisers of B is denoted by $N(B)$. A normaliser n is *free* if $n^2 = 0$; the collection of free normalisers of B is denoted by $N_f(B)$. We say B is *diagonal* or that (A, B) is a *diagonal pair* if

(D1) B contains an approximate identity for A;
(D2) there is a faithful conditional expectation $P : A \to B$; and
(D3) $\ker(P) = \overline{\operatorname{span}}\, N_f(B)$.

In [27, Definition 1.1], a pair (A, B) of unital C^*-algebras is said to be a diagonal pair if $1_B = 1_A$ and (D2) and (D3) are satisfied, and a nonunital pair (A, B) is said to be a diagonal pair if the minimal unitisations form a diagonal pair $(\widetilde{A}, \widetilde{B})$ (recall we identify $1_{\widetilde{B}}$ with $1_{\widetilde{A}}$). If A is unital then (D1) implies that B contains the unit of A, so [27, Definition 1.1] and Definition 5.2 agree for unital A. We will use the next two lemmas to show in Corollary 5.6 that [27, Definition 1.1] and Definition 5.2 (which is the definition implicitly used in [26]) also coincide if A is nonunital.

Lemma 5.3. *Let A be a C^*-algebra with C^*-subalgebra B and let $P : A \to B$ be a conditional expectation. Then $\widetilde{P} : \widetilde{A} \to \widetilde{B}$ defined by $\widetilde{P}((a, \lambda)) = (P(a), \lambda)$ is also a conditional expectation. Moreover, P is faithful if and only if $\widetilde{P}$ is.*

Proof. Since $P : A \to B$ is a conditional expectation it is completely positive by Remark 5.1. By [13, Lemma 3.9], $\widetilde{P}$ is also completely positive, and the proof of [13, Lemma 3.9] shows that $\widetilde{P}$

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is norm-decreasing. Since $\widetilde{P}(1_{\widetilde{A}}) = 1_{\widetilde{A}}$ and since $\widetilde{P}$ is idempotent, $\widetilde{P}$ is an idempotent of norm 1 and hence is a conditional expectation by Remark 5.1.

Now suppose that P is faithful and that $(a, \lambda) \in \widetilde{A}^+$ with $\widetilde{P}(a, \lambda) = 0$. Since $\widetilde{P}(a, \lambda) = (P(a), \lambda)$, we have $\lambda = 0$ and $P(a) = 0$. Since $a \in A^+$ and P is faithful, $a = 0$ also. Hence, $\widetilde{P}$ is faithful. Conversely, if $\widetilde{P}$ is faithful then so is its restriction P. □

Lemma 5.4. *Let A be a C^*-algebra and B an abelian C^*-subalgebra of A. Suppose that B contains an approximate identity for A. Then $n^*n \in B$ for all $n \in N(B)$. If, in addition, $P : A \to B$ is a conditional expectation, then $P(n) = 0$ for all $n \in N_f(B)$.*

Proof. Fix $n \in N(B)$ and let $(b_i)_{i \in I}$ be an approximate identity for A contained in B. Then we have $n^*n = \lim_{i \in I} n^*b_i n \in B$.

Now fix $n \in N_f(B)$. Set $a_k = (n^*n)^{1/k}$. A standard spatial argument using the polar decomposition of n shows that $na_k \to n$. To see that $P(n) = 0$, it suffices by continuity to show that $P(na_k) = 0$ for all k. Fix k. Then

$$P(na_k) = P(n)a_k = a_k P(n) = P(a_k n)$$

since $a_k \in B$ and P is a conditional expectation. Since $n \in N_f(B)$, we have $(n^*n)n = n^*n^2 = 0$ and it follows that $a_k n = (n^*n)^{1/k} n = 0$. Hence, $P(na_k) = 0$ as required. □

Lemma 5.5. *Suppose that (A, B) is a diagonal pair with expectation $P : A \to B$. Let $\widetilde{P} : \widetilde{A} \to \widetilde{B}$ be the conditional expectation of Lemma* 5.3. *Then*

$$N_f(\widetilde{B}) = \big\{(n, 0)\colon\ n \in N_f(B)\big\} \quad \textit{and} \quad \ker \widetilde{P} = \overline{\operatorname{span}}\, N_f(\widetilde{B}).$$

Proof. Fix $n \in N_f(B)$ and $(b, \mu) \in \widetilde{B}$. Then $(n, 0)^2 = 0$ and

$$\big(n^*, 0\big)(b, \mu)(n, 0) = \big(n^*bn + \mu n^*n, 0\big) \in \widetilde{B}$$

by Lemma 5.4. Similarly, $(n, 0)(b, \mu)(n^*, 0) \in \widetilde{B}$. Hence, $(n, 0) \in N_f(\widetilde{B})$, giving $\{(n, 0)\colon\ n \in N_f(B)\} \subset N_f(\widetilde{B})$. Now fix $c = (n, \lambda) \in N_f(\widetilde{B})$. Since $(n^2 + 2\lambda n, \lambda^2) = c^2 = 0$, we have $\lambda = 0$ and $n^2 = 0$. We now verify that n normalises B. Fix $b \in B$. Then since $(n, 0) \in N_f(\widetilde{B})$ and $(b, 0) \in \widetilde{B}$ we have

$$\big(n^*bn, 0\big) = (n, 0)^*(b, 0)(n, 0) \in \widetilde{B}.$$

Hence $n^*bn \in B$. Similarly, $nbn^* \in B$. This proves $N_f(\widetilde{B}) = \{(n, 0)\colon\ n \in N_f(B)\}$.

Since (A, B) is a diagonal pair, we have $\ker P = \overline{\operatorname{span}}\, N_f(B)$. Hence,

$$\ker \widetilde{P} = \big\{(a, 0)\colon\ a \in \ker P\big\} = \overline{\operatorname{span}}\big\{(n, 0)\colon\ n \in N_f(B)\big\} = \overline{\operatorname{span}}\, N_f(\widetilde{B}). \qquad \square$$

Corollary 5.6. *Let A be a nonunital C^*-algebra and let B be an abelian C^*-subalgebra of A. Then (A, B) is a diagonal pair in the sense of Definition* 5.2 *if and only if $(\widetilde{A}, \widetilde{B})$ is a diagonal pair in the sense of* [27, *Definition* 1.1].

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Proof. First suppose that (A, B) is diagonal with conditional expectation $P : A \to B$. We have $1_{\widetilde{A}} \in \widetilde{B}$ by definition of the inclusion of $\widetilde{B}$ in $\widetilde{A}$. Lemma 5.3 implies that $\widetilde{P} : \widetilde{A} \to \widetilde{B}$ is faithful. Moreover, by Lemma 5.5 we have $\ker \widetilde{P} = \overline{\operatorname{span}}\, N_f(\widetilde{B})$. Thus $(\widetilde{A}, \widetilde{B})$ is a diagonal pair in the sense of [27, Definition 1.1].

Conversely, suppose $(\widetilde{A}, \widetilde{B})$ is a diagonal pair, in the sense of [27, Definition 1.1], with conditional expectation $Q : \widetilde{A} \to \widetilde{B}$. Since Q is faithful, $P := Q|_A$ is also a faithful conditional expectation, and $Q = \widetilde{P}$.

As in the proof of Lemma 5.5 if $(n, \lambda) \in N_f(\widetilde{B})$, then $\lambda = 0$ and $n \in N_f(B)$. So

$$N'_f(B) := \{n \in A\colon\ (n, 0) \in N_f(\widetilde{B})\} \subset N_f(B).$$

By assumption $\ker \widetilde{P} = \overline{\operatorname{span}}\, N_f(\widetilde{B})$. By definition of $\widetilde{P}$, we have $\ker \widetilde{P} = \{(a, 0)\colon\ a \in \ker P\}$. Hence $\ker P = \overline{\operatorname{span}}\, N'_f(B) \subset \overline{\operatorname{span}}\, N_f(B)$.

Fix an approximate identity $(b_i)_{i\in I}$ for B; we claim it is also an approximate identity for A. Since $A = B + \ker P$ and $\ker P = \overline{\operatorname{span}}\, N'_f(B)$, it suffices to show that $nb_i \to n$ for each $n \in N'_f(B)$. Fix $n \in N'_f(B)$. Since $(n, 0) \in N_f(\widetilde{B})$, we have $(n^*, 0)(0, 1)(n, 0) = (n^*n, 0) \in \widetilde{B}$, so $n^*n \in B$. Since $n^*nb_i \to n^*n$, it follows that $nb_i \to n$ also, so $(b_i)_{i\in I}$ is an approximate identity for A.

Lemma 5.4 now gives $\overline{\operatorname{span}}\, N_f(B) \subset \ker P$, and hence $\ker P = \overline{\operatorname{span}}\, N_f(B)$. $\square$

Corollary 5.6 above ensures, in particular, that we may apply the results of [27] to our diagonal pairs, and we shall do so without further comment.

5.2. Diagonals in Fell algebras and the extension property

Building on the seminal work of Kadison and Singer [25], Anderson defined the extension property for a pair of unital C^*-algebras [3, Definition 3.3] as follows. Let A be a unital C^*-algebra and B a C^*-subalgebra with $1_A \in B$. Then B is said to have the extension property relative to A if each pure state of B has a unique extension to a pure state of A (equivalently, each pure state of B has a unique extension to a state of A – this extension is then necessarily pure). If B is abelian and has the extension property relative to A then B must be maximal abelian by the Stone–Weierstass Theorem [3, p. 311]. The converse is false: for example, Cuntz has shown that the canonical maximal abelian subalgebra of $\mathcal{O}_n$ does not have the extension property [15, Proposition 3.1]; the next example shows this can happen even in a Fell algebra.

Example 5.7. Let $B = C([-1, 1])$ and let $G = \{0, 1\}$ act on $[-1, 1]$ by $g \cdot x = (-1)^g x$. Then the crossed product $A = B \rtimes G$ is generated by B and a self-adjoint unitary U which does not commute with B. That A is a Fell algebra follows from, for example, [23, Lemma 5.10]. Moreover, B is a maximal abelian subalgebra of A. By [46, Theorem 5.3] the spectrum of A is homeomorphic to $\{\pi_{-1}, \pi_1\} \cup (0, 1]$ where $t_n \to \pi_1, \pi_{-1}$ for $t_n \in (0, 1]$ if and only if $t_n \to 0$ in $\mathbb{R}$. In particular, π_{-1} and π_1 cannot be separated by disjoint open sets. The π_i are one-dimensional representations determined by $\pi_j(f) = f(0)$ for $f \in B$ and $\pi_j(U) = j$. Hence π_1, π_{-1} are distinct pure states of A which restrict to evaluation at 0 on B. Thus B is a maximal abelian subalgebra but does not have the extension property.

Let A be a unital C^*-algebra, and let be B be a maximal abelian subalgebra of A. Then B has the extension property relative to A if and only if there exists a conditional expectation

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$P : A \to B$ such that for each pure state h of B, the state $h \circ P$ is its unique pure state extension to A [3, Theorem 3.4]. By [5, Theorem 2.4] B, whether or not it is maximal abelian, has the extension property relative to A if and only if $A = B + \overline{\operatorname{span}}[B, A]$ where $[B, A] = \{ba - ab: a \in A,\ b \in B\}$. The techniques used in the proof imply that the extension property is equivalent to the requirement that $B + \operatorname{span}[B, A]$ be dense in A (if f is a state on A which restricts to a pure state on B, then $f(ab) = f(a)f(b) = f(ba)$ for all $a \in A$, $b \in B$ and hence f vanishes on $\operatorname{span}[B, A]$).

We use the following definition of the extension property for nonunital C^*-algebras.

Definition 5.8. Let B be a C^*-subalgebra of a C^*-algebra A. As in [26, §2], we say that B *has the extension property relative to* A if

1. B contains an approximate identity for A; and
2. every pure state of B extends uniquely to a pure state of A.

By [27, Proposition 1.4], if (A, B) is a diagonal pair, then B has the extension property relative to A.

Remark 5.9.

1. The extension property as presented in [5, Definition 2.5] seems slightly different to Definition 5.8: in the former B is said to have the extension property relative to A if pure states of B extend uniquely to pure states of A and no pure state of A annihilates B. As noted in [45, §2] these two definitions are equivalent: it follows from [1, Lemma 2.32] that B contains an approximate identity for A if and only if no pure state of A annihilates B.
2. Let B be an abelian C^*-subalgebra of a nonunital C^*-algebra A. By [5, Remark 2.6(iii)] B has the extension property relative to A if and only if $\widetilde{B}$ has the extension property relative to $\widetilde{A}$ (and B is maximal abelian in A if and only if $\widetilde{B}$ is maximal abelian in $\widetilde{A}$). Moreover, as in the unital case, B has the extension property relative to A if and only if $B + \operatorname{span}[B, A]$ is dense in A.

Notation 5.10. Let B be an abelian C^*-subalgebra of a C^*-algebra A, and suppose that B has the extension property relative to A. By the discussion above, B is maximal abelian and there exists a unique conditional expectation $P : A \to B$. Moreover, for each pure state h of B, the state $h \circ P$ is its unique pure state extension to A. For this reason, we say that the extension property is *implemented by* P. The map $x \mapsto x \circ P$ is a weak*-continuous map from the set of pure states of B (which may be identified with $\widehat{B}$) to the pure states of A.

Of course $x \circ P$ determines a GNS triple (π_x, H_x, ξ_x). That is, π_x is an irreducible representation of A on the Hilbert space H_x, the unit vector ξ_x is cyclic vector for π_x, and $x \circ P(a) = (\pi_x(a)\xi_x|\xi_x)$ for all $a \in A$. Let $\psi = \psi_P : \widehat{B} \to \widehat{A}$ be the map which takes $x \in \widehat{B}$ to the unitary equivalence class $[\pi_x] \in \widehat{A}$. We call ψ the *spectral map* associated to the inclusion $B \subset A$.

Since diagonal pairs have the extension property, it follows from the above that if (A, B) is a diagonal pair, then the conditional expectation from B to A is unique. We use this frequently: given a diagonal pair (A, B), we will without comment refer to *the* expectation $P : B \to A$ and use that the extension property is implemented by $x \mapsto x \circ P$.

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There is some overlap between Lemma 5.11 and [4, Proposition 2.10].

Lemma 5.11. *Suppose that A is a separable C^*-algebra, let B be an abelian C^*-subalgebra with the extension property relative to A implemented by $P : A \to B$, and let $\psi : \widehat{B} \to \widehat{A}$ be the spectral map. Suppose that π is an irreducible representation of A such that $\pi(A) = \mathcal{K}(H_\pi)$. Then $\psi^{-1}(\{[\pi]\})$ is a discrete countable subset of $\widehat{B}$, and there exist a listing $\{x_\lambda\colon \lambda \in \Lambda\}$ of $\psi^{-1}(\{[\pi]\})$ and a basis $\{\xi_\lambda\colon \lambda \in \Lambda\}$ of H_π such that $x_\lambda \circ P = (\pi(\cdot)\xi_\lambda|\xi_\lambda)$ for all $\lambda \in \Lambda$ and*

$$\pi(b) = \sum_{\lambda\in\Lambda} x_\lambda(b)\Theta_{\xi_\lambda,\xi_\lambda} \quad \textit{for all } b \in B. \tag{5.1}$$

Furthermore, if A is liminary, then ψ is surjective and $P : A \to B$ is faithful.

Proof. We begin by identifying a basis $\{\xi_\lambda\colon \lambda \in \Lambda\}$ for H_π and points $\{x_\lambda\colon \lambda \in \Lambda\}$ in $\widehat{B}$ satisfying (5.1). We then show that the x_λ form a discrete set which coincides with $\psi^{-1}(\{[\pi]\})$.

We have $\pi(B)$ maximal abelian in $\pi(A)$ by [5, Corollary 3.2]. Since $\pi(A) = \mathcal{K}(H_\pi)$, we have $B/\ker\pi \cong \pi(B) = \overline{\operatorname{span}}\{\Theta_{\xi_\lambda,\xi_\lambda}\colon \lambda \in \Lambda\}$ for some orthonormal basis $\{\xi_\lambda\colon \lambda \in \Lambda\}$ of H_π[1]; and Λ is countable because A is separable. Since each one-dimensional subspace $\operatorname{span}\{\xi_\lambda\}$ is invariant under $\pi(B)$, it determines an irreducible representation of B given by point evaluation at $x_\lambda \in \widehat{B}$. The set $\{x_\lambda\colon \lambda \in \Lambda\}$ is discrete because for each λ there exists b_λ such that $\pi(b_\lambda) = \Theta_{\xi_\lambda,\xi_\lambda}$ which forces $x_\mu(b_\lambda) = 0$ for $\lambda \neq \mu$. The formula (5.1) follows from the definition of the x_λ.

Fix $\lambda \in \Lambda$. Then for $b \in B$,

$$\big(\pi(b)\xi_\lambda|\xi_\lambda\big) = \Big(\sum_{\mu\in\Lambda} x_\mu(b)\Theta_{\xi_\mu,\xi_\mu}(\xi_\lambda)|\xi_\lambda\Big) = x_\lambda(b) = x_\lambda \circ P(b).$$

Hence $x_\lambda \circ P = (\pi(\cdot)\xi_\lambda|\xi_\lambda)$ for all $\lambda \in \Lambda$ by the extension property. Thus $\psi(x_\lambda) = [\pi]$, and it follows that $\{x_\lambda\colon \lambda \in \Lambda\} \subset \psi^{-1}(\{[\pi]\})$.

For the other inclusion, let $x \in \psi^{-1}(\{[\pi]\})$. Since the GNS representation associated to $x \circ P$ is equivalent to π, we may assume that $x \circ P(\cdot) = (\pi(\cdot)\xi|\xi)$ for some unit vector $\xi \in H_\pi$. Using (5.1) we get

$$x(b) = \sum_{\lambda\in\Lambda} x_\lambda(b)\big|(\xi|\xi_\lambda)\big|^2 \quad \text{for all } b \in B.$$

Suppose that there exist λ_i such that $(\xi|\xi_{\lambda_i}) \neq 0$ for $i = 1, 2$. Since $\{x_\lambda\colon \lambda \in \Lambda\}$ is discrete we can find $b_i \in B$ such that $x_\lambda(b_i) = 0$ unless $\lambda = \lambda_i$. Now $x(b_1b_2) = 0$ but

$$x(b_1)x(b_2) = x_{\lambda_1}(b_1)\big|(\xi|\xi_{\lambda_1})\big|^2 x_{\lambda_2}(b_2)\big|(\xi|\xi_{\lambda_2})\big|^2 \neq 0$$

[1] This standard fact follows from the Spectral Theorem. Specifically, the Spectral Theorem implies firstly that the C^*-algebra generated by each self-adjoint $T \in \mathcal{K}(H)$ is equal to the C^*-algebra generated by its spectral projections. So any abelian C^*-subalgebra D of $\mathcal{K}(H)$ is spanned by commuting finite-dimensional projections. A minimal subprojection of any of these spanning projections then commutes with D. So if D is maximal abelian, then it is spanned by a maximal family of mutually orthogonal minimal projections on H.

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which is impossible. It follows that there is precisely one λ such that $(\xi|\xi_\lambda) \neq 0$, and hence that $x = x_\lambda$. Thus $\{x_\lambda\colon \lambda \in \Lambda\} = \psi^{-1}(\{[\pi]\})$.

Now suppose that A is liminary and let π be an irreducible representation of A. Then $\pi(A) = \mathcal{K}(H_\pi)$, so the above argument shows that $\psi^{-1}(\{[\pi]\})$ is nonempty. Therefore, ψ is surjective. It remains to prove that P is faithful. Fix $a \in A^+ \setminus \{0\}$. There is an irreducible representation π on a Hilbert space H_π with $\pi(a) \neq 0$. Then with $\{\xi_\lambda\colon \lambda \in \Lambda\}$ and $\psi^{-1}(\{[\pi]\}) = \{x_\lambda\colon \lambda \in \Lambda\}$ as in the statement of the lemma we have

$$\pi\big(P(a)\big) = \sum_{\lambda\in\Lambda} x_\lambda\big(P(a)\big)\Theta_{\xi_\lambda,\xi_\lambda} = \sum_{\lambda\in\Lambda}\big(\pi(a)\xi_\lambda|\xi_\lambda\big)\Theta_{\xi_\lambda,\xi_\lambda} \neq 0.$$

Hence $P(a) \neq 0$ and P is faithful. $\square$

Lemma 5.12. *Let A be a separable liminary C^*-algebra and B an abelian C^*-subalgebra with the extension property relative to A, and let ψ be the spectral map. Let U be an open subset of $\widehat{B}$ and let $J = \{b \in B\colon y(b) = 0$ for all $y \in \widehat{B} \setminus U\} \lhd B$. Let $I = \overline{AJA}$ be the ideal of A generated by J. Then*

$$\widehat{I} = \big\{[\pi] \in \widehat{A}\colon \pi|_J \neq 0\big\} = \psi(U). \tag{5.2}$$

Proof. Since I is generated by J, we have

$$I = \bigcap\big\{\ker\pi\colon [\pi] \in \widehat{A},\ I \subseteq \ker\pi\big\} = \bigcap\big\{\ker\pi\colon [\pi] \in \widehat{A},\ J \subseteq \ker\pi\big\},$$

which gives $\widehat{I} = \{[\pi] \in \widehat{A}\colon \pi|_J \neq 0\}$.

To prove that $\psi(U) \subset \{[\pi] \in \widehat{A}\colon \pi|_J \neq 0\}$, let P be the unique conditional expectation from A to B. Fix $x \in U$, and let $\pi \in \psi(x)$. Since $x \in U = \widehat{J}$, there is an element $b \in J$ such that $x(b) \neq 0$. Since $x \circ P$ is a pure state associated with π there is a unit vector $\xi \in H_\pi$ such that $x \circ P(a) = (\pi(a)\xi|\xi)$ for all $a \in A$. But $x \circ P(b) = x(b) \neq 0$, so $\pi(b) \neq 0$. Hence, $\psi(U) \subset \{[\pi] \in \widehat{A}\colon \pi|_J \neq 0\}$.

To see that $\{[\pi] \in \widehat{A}\colon \pi|_J \neq 0\} \subset \psi(U)$, fix an irreducible representation π of A with $\pi(f) \neq 0$ for some $f \in J$. Since A is liminary, $\pi(A) = \mathcal{K}(H_\pi)$ so Lemma 5.11 implies that $\psi^{-1}([\pi])$ is a countable discrete set $\{x_\lambda\colon \lambda \in \Lambda\} \subset \widehat{B}$, and there is a basis $\{\xi_\lambda\colon \lambda \in \Lambda\}$ for H_π such that $\pi(b) = \sum_{\lambda\in\Lambda} x_\lambda(b)\Theta_{\xi_\lambda,\xi_\lambda}$ for all $b \in B$. Since $\pi(f) \neq 0$ and $\widehat{J} = U$, there exists $\lambda \in \Lambda$ such that $x_\lambda \in U$ and $f(x_\lambda) \neq 0$. Thus $[\pi] = \psi(x_\lambda) \in \psi(U)$. Hence $\{[\pi] \in \widehat{A}\colon \pi|_J \neq 0\} = \psi(U)$. $\square$

The following lemma is used implicitly in the proof of [27, Theorem 3.1].

Lemma 5.13. *Let A be a separable liminary C^*-algebra and B an abelian C^*-subalgebra with the extension property relative to A. Let ψ be the spectral map.*

1. *Suppose that $f, g \in B^+$ have the property that the restriction of ψ to* $\operatorname{supp} f \cup \operatorname{supp} g$ *is injective. Then $gAf \subset B$.*
2. *If $f, g \in B^+$ have the property that the restrictions of ψ to* $\operatorname{supp} f$ *and* $\operatorname{supp} g$ *are injective, then $gAf \subset N(B)$.*

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Proof. Let $P : A \to B$ be the unique conditional expectation.

(1) Fix $a \in A$ and an irreducible representation $\pi : A \to \mathcal{B}(H_\pi)$. It suffices to show that

$$\pi\big(P(gaf)\big) = \pi(gaf). \tag{5.3}$$

Since P is an expectation with $f, g \in P(A)$, we have $P(gaf) = gP(a)f$, so (5.3) is trivial if $\pi(f) = 0$ or $\pi(g) = 0$. So we suppose that $\pi(f), \pi(g) \neq 0$ and we verify that $\pi(gaf) = \pi(gP(a)f)$.

Since A is liminary, we may use Lemma 5.11 to obtain a listing $\psi^{-1}(\{[\pi]\}) = \{x_\lambda\colon \lambda \in \Lambda\}$ and a basis $\{\xi_\lambda\colon \lambda \in \Lambda\}$ of H_π such that $x_\lambda \circ P = (\pi(\cdot)\xi_\lambda|\xi_\lambda)$ for all $\lambda \in \Lambda$ and $\pi(b) = \sum_{\lambda\in\Lambda} x_\lambda(b)\Theta_{\xi_\lambda,\xi_\lambda}$ for all $b \in B$. Since $\psi(x_\lambda) = [\pi]$ for all λ, and since ψ restricts to an injection on $\operatorname{supp} f \cup \operatorname{supp} g$ there exists a unique $\lambda \in \Lambda$ such that $x_\lambda \in \operatorname{supp} f \cup \operatorname{supp} g$. Thus $\pi(f) = x_\lambda(f)\Theta_{\xi_\lambda,\xi_\lambda}$ and $\pi(g) = x_\lambda(g)\Theta_{\xi_\lambda,\xi_\lambda}$. Hence

$$\begin{aligned}\pi(gaf) &= \big(x_\lambda(g)\Theta_{\xi_\lambda,\xi_\lambda}\big)\pi(a)\big(x_\lambda(f)\Theta_{\xi_\lambda,\xi_\lambda}\big) = x_\lambda(g)\big(\pi(a)\xi_\lambda|\xi_\lambda\big)x_\lambda(f)\Theta_{\xi_\lambda,\xi_\lambda}\\ &= x_\lambda(g)x_\lambda\big(P(a)\big)x_\lambda(f)\Theta_{\xi_\lambda,\xi_\lambda} = \pi\big(gP(a)f\big).\end{aligned}$$

So $gaf = P(gaf)$ and hence $gAf \subset B$.

(2) Fix $a \in A$ and set $n := gaf$. Then for every $b \in B$ we have $n^*bn = f(a^*gbga)f \in B$ by (1). Thus, $n^*Bn \subset B$ and symmetrically $nBn^* \subset B$. Hence, $n = gaf \in N(B)$. □

Our next result, Theorem 5.14, extends [26, Theorem 2.2] from continuous-trace C^*-algebras to Fell algebras; indeed our proof follows similar lines. There is also some overlap with [4, Proposition 3.3] and [11, Proposition 4.1]. Example 5.15 below shows that Theorem 5.14 cannot be extended to bounded-trace C^*-algebras.

Theorem 5.14. *Let A be a separable Fell algebra and let B be an abelian C^*-subalgebra with the extension property relative to A. Then*

1. *the spectral map ψ is a local homeomorphism, and*
2. *(A, B) is a diagonal pair.*

Proof. (1) We must prove that ψ is continuous, open, surjective and locally injective. Continuity follows from the observation that $\phi \mapsto \phi \circ P$ is a weak*-continuous map from the state space of B to that of A. That ψ is an open map follows from Lemma 5.12 and the surjectivity of ψ follows from Lemma 5.11.

To show that ψ is locally injective we argue as in [28, Theorem 2]. Suppose that ψ fails to be locally injective at $x \in \widehat{B}$. Then there exist sequences $(y_n)_{n=1}^\infty$, $(z_n)_{n=1}^\infty$ in $\widehat{B}$ such that $y_n, z_n \to x$ and, for all n, $y_n \neq z_n$ and $\psi(y_n) = \psi(z_n)$. Let $\pi \in \psi(x)$. Since A is a Fell algebra there exists a Hausdorff neighbourhood V of $[\pi]$ in $\widehat{A}$ [6, Corollary 3.4]. Let I be the ideal of A such that $\widehat{I} = V$. Since V is Hausdorff, I is a continuous trace C^*-algebra. Let $U = \psi^{-1}(V)$ and let J be the ideal of B such that $\widehat{J} = U$. We have $J \subset I$ by Lemma 5.12.

By Lemma 5.11, $\psi^{-1}(\{[\pi]\}) = \{x_\lambda\colon \lambda \in \Lambda\}$ is discrete and countable, and there exists a basis $\{\xi_\lambda\colon \lambda \in \Lambda\}$ of H_π such that $\pi(b) = \sum_{\lambda\in\Lambda} x_\lambda(b)\Theta_{\xi_\lambda,\xi_\lambda}$ for all $b \in B$. Choose $b \in C_c(\widehat{B})^+$ such that $x(b) > 0$ and $\operatorname{supp} b \subset U$. Since $\pi \in \psi(x)$, we have $x = x_\mu$ for some $\mu \in \Lambda$, and since $\psi^{-1}(\{[\pi]\})$ is discrete, we may choose $g \in C_c(\widehat{B})^+$ such that $x_\lambda(g) = 0$ unless $\lambda = \mu$. So

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$\pi(bg)$ is a positive multiple of Θ_{ξ_μ,ξ_μ}, so $f := \frac{1}{x_\mu(bg)} bg \in C_c(\widehat{B})^+$ satisfies $\pi(f) = \Theta_{\xi_\mu,\xi_\mu}$ and $x(f) = 1$.

We have $\operatorname{supp}(f) \subset \operatorname{supp}(b) \subset U$, so $f \in J \subset I$. Since f has compact support it belongs to the Pedersen ideal of I and hence is a continuous trace element in I. For each n, fix $\pi_n \in \psi(y_n)$. Since $\psi(y_n) \to \psi(x)$ we have

$$\lim_{n\to\infty} \operatorname{Tr}\bigl(\pi_n(f)\bigr) = \operatorname{Tr}\bigl(\pi(f)\bigr) = \operatorname{Tr}\Theta_{\xi_\mu,\xi_\mu} = 1.$$

But $\{y_n, z_n\} \in \psi^{-1}(\{\psi(y_n)\})$, so by Lemma 5.11 and the positivity of f we also have

$$\operatorname{Tr}\bigl(\pi_n(f)\bigr) \geqslant y_n(f) + z_n(f) \to 2x(f) = 2,$$

which results in a contradiction. Thus ψ is a local homeomorphism.

(2) Since $\psi : \widehat{B} \to \widehat{A}$ is a local homeomorphism by (1), the collection

$$\mathcal{U}(\psi) := \{U \subset \widehat{B} \text{ open: } \psi|_U \text{ is injective}\}$$

forms an open cover of $\widehat{B}$. Since B is separable, $\widehat{B}$ is second-countable and hence paracompact. It follows by [26, Lemma 2.1] (see also the Shrinking Lemma [37, Lemma 4.32]) that there is a countable, locally finite refinement $\mathcal{V} := \{V_n\colon n \geqslant 0\}$ of $\mathcal{U}$ such that $V_i \cap V_j \neq \emptyset$ implies $V_i \cup V_j \in \mathcal{U}(\psi)$.

By definition of the extension property, B contains an approximate identity for A. Since A is liminary, we may apply Lemma 5.11 to conclude that P is faithful. By Lemma 5.4 we have $N_f(B) \subset \ker P$, so it remains to show that every element in $\ker(P)$ may be approximated by sums of elements in $N_f(B)$.

By [37, Lemma 4.34] there exists a partition of unity subordinate to $\mathcal{V}$; that is, there exists a sequence $(f_n)_{n=0}^\infty$ in B such that $\operatorname{supp} f_n \subset V_n$, $0 \leqslant f_n \leqslant 1$ and $\sum_{n=0}^\infty x(f_n) = 1$ for all $x \in \widehat{B}$. For each $n \geqslant 0$, let $g_n = \sum_{j=0}^n f_j$. For a compact subset K of $\widehat{D}$, the local finiteness of $\mathcal{V}$ implies that $K \cap V_j = \emptyset$ for all but finitely many j. Hence there exists $n \geqslant 0$ such that $g_n(x) = 1$ for all $x \in K$. Therefore, $(g_n)_{n=0}^\infty$ is an approximate identity for B and hence for A. Fix $a \in A$. Then $g_n a g_n \to a$ and $P(g_n a g_n) = g_n P(a) g_n$ for all n. Fix $x \in \widehat{B}$. Since $x(f_i)x(f_j) = 0$ whenever $x \notin V_i \cap V_j$, we obtain

$$x \circ P(g_n a g_n) = x\bigl(g_n P(a) g_n\bigr) = \sum_{\{0\leqslant i,j\leqslant n\colon x\in V_i\cap V_j\}} x(f_i)\bigl(x \circ P(a)\bigr)x(f_j).$$

Hence

$$P(g_n a g_n) = \sum_{\{0\leqslant i,j\leqslant n\colon V_i\cap V_j\neq\emptyset\}} f_i P(a) f_j.$$

Suppose that $V_i \cap V_j \neq \emptyset$. Then $V_i \cup V_j \in \mathcal{U}(\psi)$, so $\psi|_{V_i\cup V_j}$ is injective, and Lemma 5.13 gives $f_i a f_j \in B$. Hence $f_i P(a) f_j = P(f_i a f_j) = f_i a f_j$ and we have

$$P(g_n a g_n) = \sum_{\{0\leqslant i,j\leqslant n\colon V_i\cap V_j\neq\emptyset\}} f_i a f_j,$$

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and

$$(I-P)(g_n a g_n) = \sum_{\{0\leqslant i,j\leqslant n:\ V_i\cap V_j=\emptyset\}} f_i a f_j. \tag{5.4}$$

Suppose now that $a \in \ker P$. Then $g_n a g_n \in \ker P$. Since $g_n a g_n \to a$, it suffices to show that $g_n a g_n$ may be expressed as a sum of free normalisers. Using (5.4) and $P(g_n a g_n) = 0$ we have

$$g_n a g_n = \sum_{\{0\leqslant i,j\leqslant n:\ V_i\cap V_j=\emptyset\}} f_i a f_j.$$

Since $\psi|_{V_k}$ is injective and $\operatorname{supp} f_k \subset V_k$ for all k, Lemma 5.13 gives $f_i a f_j \in N(B)$. If $V_i \cap V_j = \emptyset$, then $f_i f_j = 0$ so that $(f_i a f_j)^2 = 0$. Thus $f_i a f_j \in N_f(B)$ as required, and (A, B) is a diagonal pair. □

We now give an example of a bounded-trace C^*-algebra A with a maximal abelian subalgebra B such that (A, B) has the extension property but is not a diagonal pair. Thus Theorem 5.14 cannot be extended from Fell algebras to bounded-trace C^*-algebras.

Example 5.15. Let $C := \{f \in C([0,1], M_2)\colon\ f(0) \in \mathbb{C}I_2\}$, and let D be the subalgebra of C consisting of functions f such that each $f(t)$ is a diagonal matrix. Then C is a bounded-trace algebra, but is not a Fell algebra, and D is an abelian C^*-subalgebra. Each pure state of D has the form $d \mapsto d(t)_{i,i}$ for some $t \in [0,1]$ and $i \in \{1,2\}$, and then $c \mapsto c(t)_{i,i}$ is the unique extension to a pure state of C, so D has the extension property relative to C.

For $t > 0$, let $u_t := \left(\begin{smallmatrix} \cos(1/t) & \sin(1/t) \\ -\sin(1/t) & \cos(1/t) \end{smallmatrix}\right) \in M_2(\mathbb{C})$. Define $\alpha \in \operatorname{Aut}(C)$ by

$$\alpha(f)(t) = \begin{cases} u_t f(t) u_t^* & \text{if } t > 0, \\ f(0) & \text{if } t = 0. \end{cases}$$

Let

$$A := M_2(C) \quad \text{and} \quad B := \left\{ \begin{pmatrix} d_1 & 0 \\ 0 & \alpha(d_2) \end{pmatrix} \colon\ d_1, d_2 \in D \right\}.$$

Then A is not a Fell algebra but has bounded trace; B is abelian, and B has the extension property relative to A because each of D and $\alpha(D)$ has the extension property relative to C. The unique faithful conditional expectation $P : A \to B$ is given by

$$P := \begin{pmatrix} \phi & 0 \\ 0 & \alpha\phi\alpha^{-1} \end{pmatrix},$$

where ϕ is the canonical expectation from C onto D: $\phi(c)(t) = \left(\begin{smallmatrix} c_{1,1}(t) & 0 \\ 0 & c_{2,2}(t) \end{smallmatrix}\right)$.

We claim that B is not diagonal in A. First observe that if n is a normaliser of D in C, then there exists $\lambda(n) \in \mathbb{C}$ such that $n(0) = \lambda(n) I_2$ by definition of C. Hence the off-diagonal entries of $n(t)$ go to zero as t goes to zero. Since n is a normaliser, for $t > 0$ the matrix $n(t)$ is either diagonal, or a linear combination of the off-diagonal matrix units. In particular, if $n(0) \neq 0$ then

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by continuity, $n(t)$ is diagonal for t in some neighbourhood of 0. If n is a *free* normaliser, then each $n(t)^2 = 0$, and it follows from the above that $n(0) = 0$.

Now suppose that

$$n = \begin{pmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{pmatrix} \in A$$

is a normaliser of B. We claim that $n_{1,2}(0) = 0$. Note that for $t > 0$, each of $n_{1,1}(t)$, $u_t^* n_{2,2}(t) u_t$ is a normaliser of $D(t)$ and each of $n_{1,2}(t)u_t$, and $u_t^* n_{2,1}(t)$ is a normaliser of $D(t)$ in $C(t)$. Suppose for contradiction that $n_{1,2}(0) \neq 0$. Since $n_{1,2}(0)$ is diagonal, there exists ε such that for $t < \varepsilon$, both $\|n_{1,2}(t)\| > \|n_{1,2}(0)\|/\sqrt{2}$ and $|(n_{1,2}(t))_{i,j}| < \|n_{1,2}(0)\|/2$ for $i \neq j$. Choose $t_0 < \varepsilon$ such that $u_{t_0} = \left(\begin{smallmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{smallmatrix}\right)$. Since $n_{1,2}(t_0)u_{t_0}$ is a normaliser of $D(t_0)$, it is either diagonal, or a linear combination of off-diagonal matrix units, and since $t_0 < \varepsilon$, there is an entry of $n_{1,2}(t_0)u_{t_0}$ with modulus at least $\|n_{1,2}(0)\|/\sqrt{2}$. It follows by choice of u_{t_0} that $n_{1,2}(t_0) = (n_{1,2}(0)u_{t_0})u_{t_0}^*$ has at least one off-diagonal entry of modulus greater than $(\|n_{1,2}(0)\|/\sqrt{2})(1/\sqrt{2}) = \|n_{1,2}(0)\|/2$, contradicting the choice of ε. Hence $n_{1,2}(0) = 0$ as claimed.

The function $f : [0, 1] \to M_4$ given by

$$f(t) = \begin{pmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{pmatrix}$$

belongs to $\ker(P) \subset A$. But $f(0)_{1,2} \neq 0$, so f is not in the closed span of the normalisers of B in A, and hence is not in the closed span of the free normalisers. In particular B is not diagonal in A.

We will show that every separable Fell algebra is Morita equivalent to one with a diagonal subalgebra; to do this we need:

Lemma 5.16. *Let A be a separable Fell algebra. Then there exists a countable set of abelian elements of A which generate A as an ideal.*

Proof. By Lemma 3.1, for every irreducible representation π of A there exists an abelian element a_π of A such that $\pi(a_\pi) \neq 0$. For each π, the set $U_\pi := \{[\sigma] \in \widehat{A}\colon \sigma(a_\pi) \neq 0\}$ is an open neighbourhood of $[\pi]$ in $\widehat{A}$. Since A is separable, the topology for $\widehat{A}$ has a countable base [16, Proposition 3.3.4]. So there exists a countable subset $S := \{a_{\pi_i}\colon i \in \mathbb{N}\}$ such that $\{U_{\pi_i}\colon i \in \mathbb{N}\}$ is an open cover of $\widehat{A}$. Let I be the ideal generated by S. Since $\sigma(a_{\pi_i}) \neq 0$ when $[\sigma] \in U_{\pi_i}$, it follows that $\pi|_I \neq 0$ for every irreducible representation π of A. Hence $I = A$. □

Theorem 5.17. *Let A be a separable Fell algebra, and let $\{a_i\colon i \in \mathbb{N}\} \subset A$ be a set of abelian norm-one elements which generate A as an ideal. Let $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$, and denote the canonical matrix units in $\mathcal{K}$ by $\{\Theta_{ij}\colon i, j \in \mathbb{N}\}$. Set*

$$a := \sum_{i=1}^{\infty} \frac{1}{i} a_i \otimes \Theta_{ii} \in A \otimes \mathcal{K}.$$

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Then

1. *the hereditary subalgebra* $C := \overline{a(A \otimes \mathcal{K})a}$ *generated by* a *is Morita equivalent to* A;
2. $D := \bigoplus_{i \in \mathbb{N}} \overline{a_i A a_i} \otimes \Theta_{ii}$ *is a* C^**-diagonal in* C; *and*
3. *the conditional expectation* $P : C \to D$ *is given by* $P(c) = \bigoplus_{i \in \mathbb{N}} (1 \otimes \Theta_{ii}) c (1 \otimes \Theta_{ii})$.

Proof. To prove (1), it suffices to show that C is full or, equivalently, that $\overline{(A \otimes \mathcal{K})a(A \otimes \mathcal{K})} = A \otimes \mathcal{K}$. Since A is generated by the a_i, it suffices to show that for all $i, j, k \in \mathbb{N}$

$$a_i \otimes \Theta_{jk} \in \overline{(A \otimes \mathcal{K})a(A \otimes \mathcal{K})}.$$

Fix $i, j, k \in \mathbb{N}$ and let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for A. Then

$$a_i \otimes \Theta_{jk} = i \lim_{\lambda \in \Lambda} (e_\lambda \otimes \Theta_{ji}) a (e_\lambda \otimes \Theta_{ik}) \in \overline{(A \otimes \mathcal{K})a(A \otimes \mathcal{K})}$$

as required.

For (2), first observe that D is commutative because each a_i is an abelian element. Since A is a Fell algebra so is C. So by Theorem 5.14, to see that D is diagonal in C, it suffices to prove that D has the extension property relative to C. By Remark 5.9(2), it is enough to show that $D + \operatorname{span}[D, C]$ is dense in C.

Sums of the form

$$\sum_{j,k=1}^{n} a_j b_{jk} a_k \otimes \Theta_{jk},$$

with $b_{jk} \in A$, are dense in C. It therefore suffices to show that elements of the form $a_j b_{jk} a_k \otimes \Theta_{jk}$ with $j \neq k$ may be approximated by elements in $[D, C]$. Fix $c := a_j b_{jk} a_k \otimes \Theta_{jk}$ with $j \neq k$. For $n \in \mathbb{N}$, let $d_n := a_j^{1/n} \otimes \Theta_{jj} \in D$. Since $a_j^{1/n} a_j \to a_j$ as $n \to \infty$,

$$[d_n, c] = d_n c - c d_n = a_j^{1/n} a_j b_{jk} a_k \otimes \Theta_{jk} \xrightarrow{n \to \infty} a_j b_{jk} a_k \otimes \Theta_{jk} = c.$$

Hence c may be approximated by the commutators $[d_n, c]$, and so $D + \operatorname{span}[D, C]$ is dense in C.

For (3), observe that the formula given for P determines a norm-decreasing projection of C onto D. This is then a conditional expectation by Remark 5.1, and is the unique expectation from C to D as discussed in Notation 5.10. □

6. Fell algebras and twisted groupoid C^*-algebras

In [27, Theorem 3.1] Kumjian showed that if (A, B) is a diagonal pair, then A is isomorphic to a twisted groupoid C^*-algebra. Here we combine this with the results of Section 5 to show that up to Morita equivalence every Fell algebra arises as a twisted groupoid C^*-algebra, and conversely determine for which twists the associated twisted groupoid C^*-algebra is Fell. We start with some background from [27, §2].

A $\mathbb{T}$-*groupoid* Γ is a locally compact, Hausdorff groupoid Γ equipped with a free range- and source-preserving action of the circle group $\mathbb{T}$ such that $(t_1 \cdot \gamma_1)(t_2 \cdot \gamma_2) = (t_1 t_2) \cdot (\gamma_1 \gamma_2)$

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whenever (γ_1, γ_2) is a composable pair in Γ. The quotient groupoid $\Gamma/\mathbb{T}$ is Hausdorff because $\mathbb{T}$ is compact.

Recall that a sequence $K^{(0)} \hookrightarrow K \xrightarrow{i} G \xrightarrow{q} H$ of groupoids is *exact* if q is a surjective groupoid homomorphism which restricts to an isomorphism of unit spaces, and i is an isomorphism of K onto $\ker(q) = \{g \in G\colon q(g) \in H^{(0)}\}$. A *topological twist* or just *twist* is a $\mathbb{T}$-groupoid Γ such that there is an exact sequence

$$\Gamma^{(0)} \to \Gamma^{(0)} \times \mathbb{T} \to \Gamma \xrightarrow{q} R$$

of groupoids in which R is a principal, étale groupoid (a *relation* in the terminology of [27]). Note that $\Gamma^{(0)} = R^{(0)}$. We often abbreviate the exact sequence to $\Gamma \to R$. Twists $\Gamma_1 \xrightarrow{q_1} R$ and $\Gamma_2 \xrightarrow{q_2} R$ over the same relation R are *isomorphic* if there is a $\mathbb{T}$-equivariant isomorphism $\pi : \Gamma_1 \to \Gamma_2$ such that $q_2 \circ \pi = q_1$; we call π a *twist isomorphism*. A twist $\Gamma \xrightarrow{q} R$ is said to be *trivial* if q has a continuous section which is a groupoid homomorphism. A trivial twist over R is isomorphic to the cartesian-product groupoid $R \times \mathbb{T}$ [27, Remark 4.2].

We outline in Appendix A the construction of the twisted groupoid C^*-algebra $C^*_{\mathrm{r}}(\Gamma; R)$ associated to a twist, and also prove there that the C^*-algebra of a trivial twist is isomorphic to the reduced groupoid C^*-algebra $C^*_{\mathrm{r}}(R)$ of R. In brief, $C^*_{\mathrm{r}}(\Gamma; R)$ is a C^*-completion of the collection of $C_c(\Gamma; R)$ of compactly supported $\mathbb{T}$-equivariant functions on Γ; the closure of the algebra of sections in $C_c(\Gamma; R)$ which are supported on $\mathbb{T} \cdot \Gamma^0$ can be identified with $C_0(\Gamma^{(0)})$, and restriction of functions extends to a conditional expectation $P : C^*_{\mathrm{r}}(\Gamma; R) \to C_0(\Gamma^{(0)})$.

For our classification theorem, a key tool will be the following theorem, proved in [27].

Theorem 6.1. (See [27, Theorem 3.1].) *Let A be a separable C^*-algebra with diagonal B, and let $Y := \widehat{B}$. Then there exist a twist $\Gamma \to R$, a homeomorphism $\phi : Y \to \Gamma^{(0)}$, and an isomorphism $\pi : A \to C^*_{\mathrm{r}}(\Gamma; R)$ such that the following diagram commutes*

$$\begin{array}{ccc} B & \xrightarrow{\phi_*} & C_0(\Gamma^{(0)}) \\ \subseteq\big\downarrow & & \subseteq\big\downarrow \\ A & \xrightarrow{\pi} & C^*_{\mathrm{r}}(\Gamma; R). \end{array} \tag{6.1}$$

Since we need the details below, we now sketch the construction of the twist Γ from a unital diagonal pair (A, B) given in [27, Theorem 3.1]; Remark 6.2 below explains how the construction works for nonunital diagonal pairs. Let $Y = \widehat{B}$ and set

$$\Gamma_0 = \big\{(a, y) \in N(B) \times Y\colon\ y\big(a^*a\big) > 0\big\}.$$

For $y \in Y$ we continue to write y for the unique state extension to A, and then for each $(a, y) \in \Gamma_0$, we define $[a, y] : A \to \mathbb{C}$ by $[a, y](c) = y(a^*c)y(a^*a)^{-1/2}$. Then each $[a, y]$ belongs to the dual space A^* of A, and the following are equivalent: (1) $[a, y] = [c, y]$; (2) $y(a^*c) > 0$; (3) there exist $b_1, b_2 \in B$ with $y(b_1), y(b_2) > 0$ such that $ab_1 = cb_2$. Set

$$\Gamma := \big\{[a, y]\colon\ (a, y) \in \Gamma_0\big\} \subset A^*. \tag{6.2}$$

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Define a $\mathbb{T}$-action on Γ by scalar multiplication: $t \cdot [a, y] = [\bar{t}a, y]$; this agrees with scalar multiplication on A^* but not with the convention used in [39, §5].

By [27, Proposition 1.6], for each $a \in N(B)$, there is a homeomorphism

$$\sigma_a : \{y \in Y\colon\ y(a^*a) > 0\} \to \{y \in Y\colon\ y(aa^*) > 0\}$$

such that $y(a^*ba) = \sigma_a(y)(baa^*)$ for all $b \in B$ and all y in the domain of σ_a. The set Γ, with source and range maps defined by $s([d, y]) = y$ and $r([d, y]) = \sigma_d(y)$, and partial multiplication defined by $[a, \sigma_c(y)][c, y] = [ac, y]$, is a $\mathbb{T}$-groupoid. The quotient groupoid $R = \Gamma/\mathbb{T}$ is a principal étale groupoid, and $\Gamma \to R$ is a twist satisfying the requirements of Theorem 6.1. The class of this twist is the negative of the one constructed in [39, §5].

Remark 6.2. The construction outlined above is for unital diagonal pairs (A, B). However, as mentioned in the proof of [27, Theorem 3.1], the construction may be applied to nonunital pairs as follows. When (A, B) is a nonunital diagonal pair, one applies the above construction to the diagonal pair $(\widetilde{A}, \widetilde{B})$ to obtain a twist $\widetilde{\Gamma} \to \widetilde{R}$ with unit space

$$\widetilde{\Gamma}^{(0)} = \widetilde{R}^{(0)} = \widehat{B} \cup \{\infty\}.$$

It is straightforward to see that $\widehat{B} \subset \widetilde{\Gamma}^{(0)}$ is an open invariant subset, so we may restrict both $\widetilde{\Gamma}$ and $\widetilde{R}$ to $\widehat{B}$ to obtain a twist $\Gamma \to R$. It is routine to check that $C^*_{\mathrm{r}}(\Gamma; R)$ may be identified with an ideal $I \lhd C^*_{\mathrm{r}}(\widetilde{\Gamma}; \widetilde{R}) \cong A$ for which the quotient is isomorphic to $C^*_{\mathrm{r}}(\mathbb{T}; \{1\}) = \mathbb{C}$. Hence I coincides with $A \lhd \widetilde{A}$, and it is clear from the construction that this identification takes $I \cap C_0(\widetilde{\Gamma}^{(0)}) = C_0(\Gamma^{(0)})$ to B. In particular, there is an isomorphism $\pi : A \to C^*_{\mathrm{r}}(\Gamma; R)$ which makes the diagram (6.1) commute.

We claim that Γ is still described by (6.2). This is not obvious right off the bat: by definition the elements of Γ are of the form $[n, x]$ where n is a normaliser of $\widetilde{B}$ in $\widetilde{A}$, and x belongs to $\widehat{B} \subset \widehat{\widetilde{B}}$. So we must show that if $n = (n', \lambda) \in \widetilde{A}$ normalises $\widetilde{B}$ and $x \in s(n) \setminus \{\infty\}$, then $[n, x] = [(m, 0), x]$ for some normaliser m of B in A.

Fix $u \in \Gamma^{(0)}$. Then u has the form $[b_0, x]$ where $b_0 \in \widetilde{B}^+$ and $x \in \widehat{\widetilde{B}}$. Moreover, if $x \neq \infty$, then there exists $b \in B^+ \subset \widetilde{B}^+$ such that $b(x) > 0$, and then $[b, x] = [b_0, x]$. Now for any $n \in N(\widetilde{B}) \subset \widetilde{A}$ and any $x \in \{y \in \widehat{\widetilde{B}}\colon\ y(n^*n) > 0\}$, we can express $s([n, x]) = [b, x]$ where $b \in B \subset \widetilde{B}$, and then $[n, x] = [n, x][b, x] = [nb, x]$. We have $nb \in A$ because A is an ideal in $\widetilde{A}$, and nb also normalises B: for $c \in B$,

$$(nb)^*c(nb) = b^*(n^*cn)b \quad \text{and} \quad (nb)c(nb)^* = n(bcb^*)n^*, \tag{6.3}$$

and both belong to B because n normalises $\widetilde{B}$ and A is an ideal in $\widetilde{A}$.

Proposition 6.3. *Let (A, B) be a diagonal pair such that A is a separable Fell algebra, and let $\Gamma \to R$ be the twist constructed from (A, B) as above. Let $\psi : \widehat{B} \to \widehat{A}$ be the spectral map. Then for $x, y \in Y$, there exists $\alpha \in R$ such that $r(\alpha) = x$ and $s(\alpha) = y$ if and only if $\psi(x) = \psi(y)$. Furthermore, the map $\alpha \mapsto (r(\alpha), s(\alpha))$ is a topological groupoid isomorphism from R onto $R(\psi)$.*

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Proof. Let P be the conditional expectation from A to B. Recall that for $x \in \widehat{B}$, we have $\psi(x) = \psi_P(x) = [\pi_x]$ where (π_x, H_x, ξ_x) is the GNS triple associated with the pure state $x \circ P$; so we have $x \circ P(a) = (\pi_x(a)\xi_x|\xi_x)$ for all $a \in A$ (see Notation 5.10).

First fix $\alpha \in R$, and let $x = r(\alpha)$ and $y = s(\alpha)$. By definition of $R = \Gamma/\mathbb{T}$, there exists $n \in N(B)$ such that $y(n^*n) > 0$ and $x = \sigma_n(y)$. By scaling n we may assume that $y(n^*n) = 1$. Since $n^*n \in B$ it follows that for $b \in B$, we have $y(b) = y(b)y(n^*n) = y(bn^*n)$. So by definition of σ_n, we have $x(b) = y(n^*bn)$ for all $b \in B$. Since $y(n^*n) = 1$, the vector $\eta_y := \pi_y(n)\xi_y$ has norm 1. Now $x(b) = y(n^*bn) = (\pi_y(b)\eta_y|\eta_y)$ for all $b \in B$. Since B has the extension property relative to A, $x \circ P$ and $a \mapsto (\pi_y(a)\eta_y|\eta_y)$ coincide on A. Hence, π_y and π_x are unitarily equivalent, whence $\psi(x) = \psi(y)$.

Conversely, suppose $\psi(x) = \psi(y)$. Then the GNS representations π_x and π_y are unitarily equivalent, so there is an irreducible representation $\pi : A \to B(H)$ and unit vectors $\xi, \eta \in H$ such that $y(P(\cdot)) = (\pi(\cdot)\xi|\xi)$ and $x(P(\cdot)) = (\pi(\cdot)\eta|\eta)$. Since $\pi(A) = \mathcal{K}(H)$, there exists $a \in A$ such that $\pi(a) = \Theta_{\eta,\xi}$. Since A is a Fell algebra, Theorem 5.14(1) implies that ψ is a local homeomorphism, so there exist open neighbourhoods U of y and V of x such that $\psi|_U$ and $\psi|_V$ are injective. Fix norm-one positive functions f, g with compact support such that $\operatorname{supp}(f) \subset U$, $\operatorname{supp}(g) \subset V$, and $f(y) = g(x) = 1$. Then $\pi(f)\xi = \xi$ and $\pi(g)\eta = \eta$. Let $n := gaf$. Then $y(n^*n) = (\pi(gaf)\xi|\pi(gaf)\xi) = (\eta|\eta) = 1$ and $y(n^*bn) = (\pi(b)\pi(gaf)\xi|\pi(gaf)\xi) = x(b)$ for all $b \in B$. Lemma 5.13 implies that $n \in N(B)$, so $\alpha := q([n, y]) \in R$ with $r(\alpha) = x$ and $s(\alpha) = y$.

It remains to prove that the map $\Upsilon : \alpha \mapsto (r(\alpha), s(\alpha))$ is a homeomorphism. It follows from the above that Υ is surjective, and it is injective since R is principal. It is continuous because the range and source maps are continuous from R to Y. We must now show that Υ is open. For this, fix $\alpha = q([a_0, x]) \in R$. Fix neighbourhoods U_0 of $\sigma_{a_0}(x)$ and V_0 of x in $\widehat{B}$ such that $\psi|_{U_0}$ and $\psi|_{V_0}$ are homeomorphisms, and fix $b, c \in B^+$ with $\operatorname{supp} b \subset U_0$ and $\operatorname{supp} c \subset V_0$ such that $b(\sigma_{a_0}(x)) = c(x) = 1$. As in (6.3), the element $a := ba_0c$ is a normaliser of B, and

$$U := \{y \in \widehat{B}\colon aa^*(y) > 0\} \subset U_0 \quad \text{and} \quad V := \{y \in \widehat{B}\colon a^*a(y) > 0\} \subset V_0.$$

Hence $[a, x] = [a_0, x]$, so $W := \{q([a, y])\colon y \in V\}$ is an open neighbourhood of $[a, x]$ (see [27, p. 982]). Since $\psi(\sigma_a(y)) = \psi(y)$ for all y, and since ψ is injective on U and V, we have $\Upsilon(W) = U *_\psi V = (U \times V) \cap R(\psi)$, and hence $\Upsilon(W)$ is open in the relative topology. So each $\alpha \in R$ has a neighbourhood W such that $\Upsilon(W)$ is open, and it follows that Υ is open. □

The following is a rewording of [27, Definition 5.5].

Definition 6.4. Twists $\Gamma_i \to R_i$ $(i = 1, 2)$ are *equivalent* if there exist a twist $\Gamma \to R$ and maps $\iota_i : R_i \to R$ such that

1. each $U_i := \iota_i(R_i^{(0)})$ is a full (see p. 2) and open subset of $R^{(0)}$;
2. $R^{(0)} = U_1 \sqcup U_2$; and
3. each ι_i is an isomorphism onto $U_i R U_i$ and the pullback $\iota_i^*(\Gamma)$ is isomorphic to Γ_i.

We call $\Gamma \to R$ a *linking twist*.

The following lemma will be used in the proof of Theorem 6.6 below.

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Lemma 6.5. *Let (C_1, D_1) and (C_2, D_2) be diagonal pairs and suppose that C_1 and C_2 are separable Fell algebras. Then C_1 and C_2 are Morita equivalent if and only if the associated twists obtained from Theorem* 6.1 *are equivalent.*

Proof. For the "only if" implication, let X be a C_1–C_2-imprimitivity bimodule, and let L be the associated linking algebra. Let $q_1, q_2 \in M(L)$ be the multiplier projections such that $q_i L q_i \cong C_i$ and $q_1 L q_2 \cong X$, and identify the C_i and X with subsets of L under these isomorphisms. By [27, Proposition 5.4], it suffices to show that $D := D_1 \oplus D_2$ is a diagonal in L. Since L is a Fell algebra, by Theorem 5.14 it suffices to show that D has the extension property relative to L. Let x be a pure state of D. Since $\widehat{D} = \widehat{D}_1 \sqcup \widehat{D}_2$, x is a pure state of D_i for either $i = 1$ or $i = 2$ (but not both) and thus extends uniquely to a pure state of $C_i = q_i L q_i$ because (C_i, D_i) is a diagonal pair. Since all states extend uniquely from hereditary subalgebras [34, Proposition 3.1.6], x has a unique extension to L, so D has the extension property relative to L as required.

The "if" implication is [27, Proposition 5.4]. □

Theorem 6.6.

1. *Suppose that A is a separable Fell algebra. Then there exists a locally compact, Hausdorff space Y, a local homeomorphism $\psi : Y \to \widehat{A}$ and a $\mathbb{T}$-groupoid Γ such that*

$$Y \to Y \times \mathbb{T} \to \Gamma \to R(\psi)$$

   is a twist, and the twisted groupoid C^*-algebra $C^*_{\mathrm{r}}(\Gamma; R(\psi))$ is Morita equivalent to A. Moreover, any two such twists are equivalent.
2. *Let Y be a locally compact, Hausdorff space, X a locally compact, locally Hausdorff space, and $\psi : Y \to X$ a local homeomorphism. Let $Y \to Y \times \mathbb{T} \to \Gamma \to R(\psi)$ be a twist such that Γ is second-countable. Then $A := C^*_{\mathrm{r}}(\Gamma; R(\psi))$ is a Fell algebra. Let $B := C_0(Y)$, and identify B with a subalgebra of A with conditional expectation $P : A \to B$ as on p.* 1563. *Then there is a homeomorphism $h : X \to \widehat{A}$ such that $h \circ \psi = \psi_P$.*

Proof. (1) Let A be a Fell algebra. By Theorem 5.17 there exists a diagonal pair (C, D) such that C is Morita equivalent to A. Let $Y = \widehat{D}$. By Theorem 6.1 there is a twist $Y \to Y \times \mathbb{T} \to \Gamma \to R$ such that C is isomorphic to $C^*_{\mathrm{r}}(\Gamma; R)$ via an isomorphism which carries D to $C_0(\Gamma^{(0)})$. By Proposition 6.3, $R \cong R(\psi)$, where $\psi : Y \to \widehat{C}$ is the spectral map. Hence A is Morita equivalent to $C^*_{\mathrm{r}}(\Gamma; R(\psi))$.

Now suppose that $Y' \to Y' \times \mathbb{T} \to \Gamma' \to R(\psi')$ is another twist such that A is Morita equivalent to $C^*_{\mathrm{r}}(\Gamma'; R(\psi'))$. Let $(C_1, D_1) = (C^*_{\mathrm{r}}(\Gamma; R(\psi)), C_0(Y))$ and $(C_2, D_2) = (C^*_{\mathrm{r}}(\Gamma'; R(\psi')), C_0(Y'))$. Then each C_i is Morita equivalent to A. So Lemma 6.5 implies that the twists are equivalent.

(2) The pair (A, B) satisfies (D2) and (D3) by Theorem 2.9 of [27] and that it satisfies (D1) is shown in Appendix A, so (A, B) is a diagonal pair. We will show that for each $y \in Y$ there exists $f_y \in C_0(Y)$ such that f_y is abelian in A and $y(f_y) > 0$, and then use Theorem 3.3 to see that A is a Fell algebra.

Fix $y \in Y$. There exists a neighbourhood U of y in Y such that $\psi|_U$ is injective. Let $f_y \in C_c(Y)$ be a positive element of A with support contained in U such that $f_y(y) \neq 0$. To see that $f_y g f_y \in C_c(Y)$ for any g in the dense subalgebra $C_c(\Gamma; R)$ of A, we identify f_y, g with

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sections of the complex line bundle L over R as outlined in Appendix A. Note that f_y has support in $R^{(0)}$. A straightforward calculation yields

$$f_y g f_y(\rho) = f_y\big(r(\rho)\big) g(\rho) f_y\big(s(\rho)\big)$$

for $\rho \in R(\psi)$. Now let $\rho = (y_1, y_2) \in \operatorname{supp} f_y g f_y$. Then $\psi(y_1) = \psi(y_2)$ and $y_1, y_2 \in \operatorname{supp} f_y \subset U$ gives $y_1 = y_2$, so ρ is a unit. Thus $f_y g f_y \in C_c(Y)$. In particular, the hereditary subalgebra $\overline{f_y A f_y}$ is contained in $C_0(Y)$, hence is abelian. Thus f_y is abelian in A and $y(f_y) > 0$ as claimed.

For each $y \in Y$, set $I_y := \overline{A f_y A}$. Then each I_y is Morita equivalent to the abelian algebra $\overline{f_y A f_y}$ by Lemma 3.2. Let J be the ideal of A generated by the I_y (this ideal is also the C^*-subalgebra of A generated by the I_y), and let $I = J \cap C_0(Y)$, so I is an ideal of $C_0(Y)$. Then I is the set of functions vanishing on some closed subset K_I of Y. But for each $y \in Y$, we have $f_y \in I$ and $y(f_y) \neq 0$. Hence $K_I = \emptyset$, that is $I = C_0(Y)$. In particular, $C_0(Y) \subset J$. Since $C_0(Y)$ is diagonal in A, it contains an approximate identity for A, so A is generated as a C^*-algebra by the I_y. Theorem 3.3 now implies that A is a Fell algebra.

It remains to prove that there is a homeomorphism $h : X \to \widehat{A}$ such that $h \circ \psi = \psi_P$. We have $R(\psi) \cong R(\psi_P)$ by Proposition 6.3. Given $y, y' \in Y$, we have

$$y, y' \in \psi^{-1}\big(\{x\}\big) \quad \Leftrightarrow \quad \big(y, y'\big) \in R(\psi) = R(\psi_P) \quad \Leftrightarrow \quad \psi_P(y) = \psi_P\big(y'\big).$$

It follows that the assignment

$$x \mapsto \psi_P(y) \quad \text{where } y \in \psi^{-1}\big(\{x\}\big)$$

gives a well-defined injective function $h : X \to \widehat{A}$ such that $h \circ \psi = \psi_P$. Since A is liminal ψ_P is surjective by Lemma 5.11, so $h \circ \psi = \psi_P$ implies h is surjective. Moreover, that $h \circ \psi = \psi_P$ and that ψ, ψ_P are local homeomorphisms implies that h is continuous and open. Thus h is a homeomorphism. □

7. A Dixmier–Douady theorem for Fell algebras

Recall that a *sheaf* of abelian groups over a topological space X is a pair (B, π) where B is a topological space and $\pi : B \to X$ is a local homeomorphism such that for each $x \in X$ the fibre $B_x := \pi^{-1}(\{x\})$ is an abelian group. Of particular importance are the constant sheaf $\underline{\mathbb{Z}}_X$ over X whose every fibre is $\mathbb{Z}$, and the sheaf $\mathcal{S}_X$ of germs of continuous $\mathbb{T}$-valued functions on X (see Notation 7.3 for details). When the base space X is clear from context, we will often suppress the subscript, and denote these $\underline{\mathbb{Z}}$ and $\mathcal{S}$ respectively.

Our strategy for defining an analogue of the Dixmier–Douady invariant for a Fell algebra A is as follows. We first choose a twist $\Gamma \to R$ whose C^*-algebra is Morita equivalent to A. The results of [28] show that Γ determines an element of a *twist group* associated to R and that this in turn determines an element of the second equivariant-sheaf cohomology group $H^2(R, \mathcal{S})$. We show that $H^2(R, \mathcal{S}) \cong H^2(\widehat{A}, \mathcal{S})$ to obtain an element $\delta(A)$ of $H^2(\widehat{A}, \mathcal{S})$ which we regard as an analogue of the Dixmier–Douady invariant for A. The bulk of the work in the section goes towards proving that this assignment does not depend on our choice of twist $\Gamma \to R$.

We recall [28, Definition 0.6]. Let G be an étale groupoid and B a sheaf over $G^{(0)}$. An *action* of G on B is a continuous map $\alpha : G * B := \{(\gamma, b)\colon\ \gamma \in G,\ b \in B_{s(\gamma)}\} \to B$, $(\gamma, b) \mapsto \alpha_\gamma(b)$

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such that each $\alpha_\gamma : B(s(\gamma)) \to B(r(\gamma))$ is an isomorphism of abelian groups and $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1} \circ \alpha_{\gamma_2}$ when $(\gamma_1, \gamma_2) \in G^{(2)}$. It is common practice to suppress the α and write γb for $\alpha_\gamma(b)$, and we shall do so henceforth. A sheaf B over $G^{(0)}$ with such an action is called a *G-sheaf*. A *G-sheaf morphism* $f : B_1 \to B_2$ is a sheaf morphism such that $f(\gamma b) = \gamma f(b)$ for $\gamma \in G$ and $b \in B$. We will frequently regard the sheaves $\underline{\mathbb{Z}}_{G^{(0)}}$ and $\mathcal{S}_{G^{(0)}}$ as G-sheaves with trivial action [20].

Fix a topological groupoid G, a locally compact, Hausdorff space Y, and a continuous open surjection $\psi : Y \to G^{(0)}$. As in [28, §0.5], we may construct a groupoid G^ψ with unit space $(G^\psi)^{(0)} = Y$ as follows:

$$G^\psi := \big\{(x, g, y)\colon\ x, y \in Y,\ g \in G,\ \psi(x) = r(g) \text{ and } \psi(y) = s(g)\big\},$$

with structure maps

$$r(x, g, y) = x, \qquad s(x, g, y) = y, \qquad (x, g, y)^{-1} = \big(y, g^{-1}, x\big),$$

and

$$(x, g, y)(y, h, z) = (x, gh, z),$$

and with the relative topology inherited from the product topology on $Y \times G \times Y$. We identify Y with the unit space $(G^\psi)^{(0)}$ via the map $x \mapsto (x, \psi(x), x)$. There is then a groupoid homomorphism $\pi_\psi : G^\psi \to G$ given by

$$\pi_\psi(x, g, y) = g. \tag{7.1}$$

For the next result, recall the definition of a groupoid equivalence from Definition 4.3.

Lemma 7.1. *Let G_1 and G_2 be second-countable, locally compact, Hausdorff groupoids, and let (Z, ρ, σ) be an equivalence from G_1 to G_2. Then for each $(x, g, y) \in G_1^\rho$, there exists a unique element $\omega(x, g, y) \in G_2$ such that*

$$x \cdot \omega(x, g, y) = g \cdot y.$$

Moreover, the map ω is a homomorphism from G_1^ρ to G_2, and $\Omega_{\rho,\sigma} : (x, g, y) \mapsto (x, \omega(x, g, y), y)$ is an isomorphism from G_1^ρ to G_2^σ.

Proof. Fix $(x, g, y) \in G_1^\rho$. Then $\rho(x) = r(g) = \rho(g \cdot y)$. Since ρ induces to a bijection from Z/G_2 to $G_1^{(0)}$, it follows that x and $g \cdot y$ belong to the same G_2-coset. Since Z is a principal G_2-space, there exists a unique element $\omega(x, g, y) \in G_2$ such that $\sigma(x) = r(\omega(x, g, y))$ and $x \cdot \omega(x, g, y) = g \cdot y$.

Since σ is G_1-invariant, we have $\sigma(y) = \sigma(g \cdot y) = \sigma(x \cdot \omega(x, g, y))$. In particular, $\sigma(y) = s(\omega(x, g, y))$, and hence $(x, \omega(x, g, y), y) \in G_2^\sigma$. An argument symmetric to that of the preceding paragraph shows that g is uniquely determined by $\omega(x, g, y)$ and the formula $x \cdot \omega(x, g, y) = g \cdot y$. Hence $\Omega_{\rho,\sigma}$ is a bijection.

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To see that ω is a homomorphism, we first check that it maps units to units and that it intertwines the range and source maps. This will imply that ω maps composable pairs to composable pairs. Let $(x, \rho(x), x) \in (G_1^\rho)^{(0)}$. Since $x \cdot \sigma(x) = \rho(x) \cdot x$ we have $\omega(x, \rho(x), x) = \sigma(x)$; so ω preserves units. For $(x, g, y) \in G_1^\rho$, we see as above that $r(\omega(x, g, y)) = \sigma(x)$ and $s(\omega(x, g, y)) = \sigma(y)$. Thus, ω maps composable pairs to composable pairs. Now let $(x, g, y), (y, h, z) \in G_1^\rho$ be a composable pair; then

$$x \cdot \omega(x, g, y)\omega(y, h, z) = g \cdot y \cdot \omega(y, h, z) = g \cdot (h \cdot z) = (gh) \cdot z,$$

so the uniqueness assertion of the first paragraph implies that

$$\omega(x, g, y)\omega(y, h, z) = \omega(x, gh, z) = \omega\big((x, g, y)(y, h, z)\big).$$

Hence, ω is a homomorphism.

It is immediate that $\Omega_{\rho,\sigma}$ preserves composable pairs. So to see that $\Omega_{\rho,\sigma}$ is also a homomorphism, we calculate

$$\begin{aligned}\Omega_{\rho,\sigma}(x, g, y)\Omega_{\rho,\sigma}(y, h, z) &= \big(x, \omega(x, g, y), y\big)\big(y, \omega(y, h, z), z\big)\\ &= \big(x, \omega(x, g, y)\omega(y, h, z), z\big) = \Omega_{\rho,\sigma}(x, gh, z).\end{aligned}$$

The map $\Omega_{\rho,\sigma}$ is continuous because the structure maps on the groupoid equivalence Z are continuous. Reversing the rôles of G_1 and G_2 in the above yields a continuous inverse $\Omega_{\rho,\sigma}^{-1} = \Omega_{\sigma,\rho}$, so $\Omega_{\rho,\sigma}$ is a homeomorphism. $\square$

Let $\psi : Y \to G^{(0)}$ be a local homeomorphism as before, and let π_ψ^* be the pullback functor from the category $\mathsf{Sh}(G)$ of G-sheaves to the category $\mathsf{Sh}(G^\psi)$ of G^ψ-sheaves. So

$$\pi_\psi^*(B) = \big\{(y, b)\colon\ y \in Y,\ b \in B_{\psi(y)}\big\},$$

and for a morphism $f : B_1 \to B_2$ of G-sheaves, $\pi_\psi^*(f)(y, b) = (y, f(b))$. Let $R(\psi)$ be the equivalence relation on Y induced by ψ. We may regard $R(\psi)$ as a subgroupoid of G^ψ by identifying it with $\{(x, \psi(x), y)\colon\ (x, y) \in R(\psi)\}$. Hence, for a G^ψ-sheaf B the action of G^ψ on B restricts to an action of $R(\psi)$ on B.

By [28, Theorem 0.9], π_ψ^* is a category equivalence between $\mathsf{Sh}(G)$ and $\mathsf{Sh}(G^\psi)$. Indeed, the proof of [28, Theorem 0.9] shows that the "inverse" functor F^ψ is defined as follows. For a G^ψ-sheaf B, $F^\psi(B)$ is the quotient sheaf $B/R(\psi) \in \mathsf{Sh}(G)$. Since morphisms between G-sheaves are equivariant maps, each morphism f of G^ψ-sheaves descends to a morphism $F^\psi(f)$ of G-sheaves. Specifically, $[(y, b)] \mapsto b$ is a natural isomorphism from $F^\psi \circ \pi_\psi^*$ to $\mathrm{id}_{\mathsf{Sh}(G)}$, and $(y, [c]) \mapsto c$ is a natural isomorphism from $\pi_\psi^* \circ F^\psi$ to $\mathrm{id}_{\mathsf{Sh}(G^\psi)}$. Moreover $\pi_\psi^*(\underline{\mathbb{Z}}_{G^{(0)}})$ is isomorphic to $\underline{\mathbb{Z}}_Y$.

Lemma 7.2. *Let G be a groupoid, and let U be a full open subset of $G^{(0)}$, and let $\iota_U : UGU \to G$ be the inclusion map. The functor $\iota_U^* : \mathsf{Sh}(G) \to \mathsf{Sh}(UGU)$ is an equivalence of categories such that the UGU-sheaves $\iota_U^*(\underline{\mathbb{Z}}_{G^{(0)}})$ and $\underline{\mathbb{Z}}_U$ are isomorphic.*

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Proof. It is straightforward to verify that GU is a G–UGU equivalence under the structure maps $\rho := r|_{GU}$ and $\sigma := s|_{GU}$ inherited from G. Hence Lemma 7.1 provides an isomorphism $\Omega_{\sigma,\rho}$ from $(UGU)^\sigma$ to G^ρ, and hence an equivalence of categories $\Omega^*_{\sigma,\rho}$ from $\mathsf{Sh}(G^\rho)$ to $\mathsf{Sh}((UGU)^\sigma)$. Composing with the category equivalences π^*_ρ and F^σ discussed above, we obtain an equivalence of categories $F^\sigma \Omega^*_{\sigma,\rho}\pi^*_\rho : \mathsf{Sh}(G) \to \mathsf{Sh}(UGU)$. We show that ι^*_U is naturally isomorphic to $F^\sigma \Omega^*_{\sigma,\rho}\pi^*_\rho$. It then follows that ι^*_U is also an equivalence of categories.

Fix $B \in \mathsf{Sh}(G)$. Then $\iota^*_U(B) = \{(u,b)\colon u \in U,\ b \in B_u\}$ and

$$F^\sigma \Omega^*_{\sigma,\rho}\pi^*_\rho(B) = F^\sigma\big(\big\{(x,b)\colon x \in GU,\ b \in B_{\rho(x)}\big\}\big) = \big\{[x,b]\colon x \in GU,\ b \in B_{\rho(x)}\big\}.$$

The map $[g] \mapsto \sigma(g)$ from $GU/R(\sigma) \to U$ is a bijection. It follows that

$$F^\sigma \Omega^*_{\sigma,\rho}\pi^*_\rho(B) = \big\{[u,b]\colon u \in U,\ b \in B_u\big\},$$

and that $t_B : [u,b] \mapsto (u,b)$ is an isomorphism from $F^\sigma \Omega^*_{\sigma,\rho}\pi^*_\rho(B)$ to $\iota^*_U(B)$. It is routine to see that for a morphism f of G-sheaves, $F^\sigma \Omega^*_{\sigma,\rho}\pi^*_\rho(f)[u,b] = [u, f(b)]$, and $\iota^*_U(f)(u,b) = (u, f(b))$, so the family of maps t_B constitute a natural isomorphism from $F^\sigma \Omega^*_{\sigma,\rho}\pi^*_\rho$ to ι^*_U.

It remains to check that $\iota^*_U(\underline{\mathbb{Z}}_{G^{(0)}}) \cong \underline{\mathbb{Z}}_U$. We have

$$\begin{aligned}\iota^*_U(\underline{\mathbb{Z}}_{G^{(0)}}) &= \big\{(x,n,y)\colon x \in U,\ (n,y) \in \mathbb{Z} \times G^{(0)},\ \iota_U(x) = y\big\}\\ &= \big\{(x,n,x)\colon x \in U,\ n \in \mathbb{Z}\big\},\end{aligned}$$

and the latter is isomorphic to $\underline{\mathbb{Z}}_U$ via $(x,n,x) \mapsto (n,x)$. $\square$

For the next lemma, we need some notation.

Notation 7.3. Given a topological space X, continuous $\mathbb{T}$-valued functions f, g defined on open subsets of X, and a point $x \in X$, we write $f \sim_x g$ if there exists an open neighbourhood W of x with $W \subset \mathrm{dom}(f) \cap \mathrm{dom}(g)$ such that $f|_W = g|_W$. We denote by $[f]^X_x$ the equivalence class of f under $\sim_x$; this is called the *germ* of f at x. The sheaf $\mathcal{S}_X$ has fibres

$$\mathcal{S}_x := \big\{[f]^X_x\colon\ f \in C(U,\mathbb{T}) \text{ for some open neighbourhood } U \text{ of } x\big\},$$

with group operation $[f]^X_x + [g]^X_x := [(f|_{\mathrm{dom}(f)\cap\mathrm{dom}(g)})(g|_{\mathrm{dom}(f)\cap\mathrm{dom}(g)})]^X_x$. For each open set $U \subset X$ and function $f \in C(U,\mathbb{T})$, let $O^X_{f,U} := \{[f]^X_x\colon\ x \in U\}$. The topology on $\mathcal{S}_X$ has basis $\{O^X_{f,U}\colon\ U \subset X \text{ is open},\ f \in C(U,\mathbb{T})\}$. Fix an open subset U of X. The pullback sheaf $\iota^*_U(\mathcal{S}_X)$ is equal to $\{(u,[f]^X_u)\colon u \in U,\ [f]^X_u \in \mathcal{S}_X\}$ with the relative topology inherited from $X \times \mathcal{S}_X$; we regard $\iota^*_U(\mathcal{S})$ as the restriction of $\mathcal{S}$ to U.

Lemma 7.4. *Let X and Y be second-countable, locally compact spaces such that X is locally Hausdorff and Y is Hausdorff. Let ψ be a local homeomorphism from Y onto an open subset of X. There is an isomorphism $\phi : \psi^*(\mathcal{S}_X) \to \mathcal{S}_Y$ determined by $\phi(y,[f]^X_{\psi(y)}) = [f \circ \psi]^Y_y$.*

In particular, if U is an open subset of a second-countable, locally compact, Hausdorff space X with inclusion map $\iota_U : U \to X$, then there is an isomorphism $\phi : \iota^*_U(\mathcal{S}_X) \to \mathcal{S}_U$ determined by $\phi(u,[f]^X_u) = [f]^U_u$.

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Proof. To see that the formula for ϕ is well defined, suppose that $(y, [f]^X_{\psi(y)}) = (z, [g]^X_{\psi(z)})$. Then $y = z$, and there exists an open neighbourhood V of $\psi(y)$ in X such that $f|_V = g|_V$. Let $U := \psi^{-1}(V)$. Then U is an open neighbourhood of y, and $(f \circ \psi)|_U = (g \circ \psi)|_U$ because f and g agree on $\psi(U)$. Hence $[f \circ \psi]^Y_y = [g \circ \psi]^Y_z$. It is routine to check that ϕ is a sheaf morphism.

For surjectivity, fix an open subset $U \subset Y$, a function $f \in C(U, \mathbb{T})$ and a point $y \in U$. We must show that $[f]^Y_y$ belongs to the image of ϕ. Choose a subneighbourhood $V \subset U$ of y such that $\psi|_V$ is a homeomorphism, and define $g \in C(\psi(V), \mathbb{T})$ by $g := f \circ (\psi|_V)^{-1}$. By definition,

$$\phi\big(y, [g]^X_{\psi(y)}\big) = [g \circ \psi]^Y_y = \big[f \circ (\psi|_V)^{-1} \circ \psi\big]^Y_y = [f|_V]^Y_y = [f]^Y_y.$$

For injectivity, suppose that $\phi(y, [f]^X_{\psi(y)}) = \phi(z, [g]^X_{\psi(z)})$. Then $y = z$, and there is an open $U \subset X$ such that $\psi(y) \in U$ and $(f \circ \psi)|_U = (g \circ \psi)|_U$. Since ψ is a local homeomorphism, $\psi(U)$ is an open neighbourhood of $\psi(y)$ in X. Moreover, for $x \in \psi(U)$, say $x = \psi(z)$, we have $f(x) = f \circ \psi(z) = g \circ \psi(z) = g(x)$, so f and g agree on $\psi(U)$. Hence $[f]^X_{\psi(y)} = [g]^X_{\psi(y)}$, so $(y, [f]^X_{\psi(y)}) = (z, [g]^X_{\psi(z)})$, and hence ϕ is injective. To see that ϕ is a homeomorphism, recall that the basic open sets in $\mathcal{S}_X$ are those of the form

$$O^X_{f,U} := \big\{[f]^X_u \colon u \in U\big\}$$

where U ranges over open subsets of X and f ranges over continuous $\mathbb{T}$-valued functions on U. Since ψ is a local homeomorphism, the family of open sets $\{O^Y_{f,V} \colon \psi|_V \text{ is a homeomorphism}\}$ is a basis for the topology on $\mathcal{S}_Y$. The basic open neighbourhoods in $\pi^*_\psi(\mathcal{S}_X)$ are by definition of the form

$$W * O^X_{f,U} = \big(W \times O^X_{f,U}\big) \cap \pi^*_\psi(\mathcal{S}_X) = \bigcup_{w \in W} \big\{\big(w, [f]^X_{\psi(w)}\big) \colon \psi(w) \in W\big\},$$

where $W \subset Y$ is open and $O^X_{f,U}$ is a basic open set in $\mathcal{S}_X$. We calculate:

$$\begin{aligned}\phi\big(W * O^X_{f,U}\big) &= \bigcup_{w \in W} \big\{[f \circ \psi]^Y_w \colon \psi(w) \in U\big\} \\ &= O^Y_{f \circ \psi, \psi^{-1}(U) \cap W}.\end{aligned}$$

If $V \subset Y$ and $\psi|_V$ is a homeomorphism, then for $f \in C(V)$,

$$\begin{aligned}\phi^{-1}\big(O^Y_{f,V}\big) &= \phi^{-1}\big(O^Y_{f \circ (\psi|_V)^{-1} \circ \psi, V}\big) = \phi^{-1}\big(\big\{\big[f \circ (\psi|_V)^{-1} \circ \psi\big]^Y_y \colon y \in V\big\}\big) \\ &= \big\{\big[f \circ (\psi|_V)^{-1}\big]^X_{\psi(y)} \colon y \in V\big\} = O^X_{f \circ (\psi|_V)^{-1}, \psi(V)},\end{aligned}$$

which is a basic open set because ψ is open. Hence both ϕ and ϕ^{-1} carry basic open sets to basic open sets, and ϕ is a homeomorphism.

For the second statement, apply the first to $\iota_U : U \to X$. $\square$

Recall from [28, p. 215] that given a groupoid G and a G-sheaf B, for each $n \in \mathbb{N}$, the nth equivariant-cohomology group $H^n(G, B)$ is defined by $H^n(G, B) := \operatorname{Ext}^n_G(\underline{\mathbb{Z}}, B)$ (see [21] for an alternative definition of sheaf cohomology of étale groupoids).

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Proposition 7.5. *Let G be a second-countable, locally compact, Hausdorff, étale groupoid, B a G-sheaf, and U a full open subset of $G^{(0)}$. Then the inclusion $\iota_U : UGU \to G$ induces an isomorphism $\iota_U^* : H^*(G, B) \to H^*(UGU, \iota_U^*(B))$, so in particular an isomorphism $\iota_U^* : H^2(G, \mathcal{S}_{G^{(0)}}) \to H^2(UGU, \mathcal{S}_U)$.*

Proof. Note that $\iota_U^*(\underline{\mathbb{Z}}_{G^{(0)}}) = \underline{\mathbb{Z}}_U$ by Lemma 7.2. So the first isomorphism follows from applying [28, Proposition 1.8] to the groupoid homomorphism $\iota_U : UGU \to G$. In particular, there is an isomorphism $\iota_U^* : H^2(G, \mathcal{S}_{G^{(0)}}) \to H^2(UGU, \iota_U^*(\mathcal{S}_{G^{(0)}}))$. Now Lemma 7.4 and the naturality of the homology functor H^* imply that $H^2(UGU, \iota_U^*(\mathcal{S}_{G^{(0)}})) \cong H^2(UGU, \mathcal{S}_U)$. □

Corollary 7.6. *Let X be a second-countable, locally compact, locally Hausdorff space. For $i = 1, 2$ fix a second-countable, locally compact, Hausdorff space Y_i and a local homeomorphism $\psi_i : Y_i \to X$. Let $Y = Y_1 \sqcup Y_2$, and define $\psi : Y \to X$ by $\psi|_{Y_i} = \psi_i$. Then for each i, the inclusion map $\iota_{Y_i} : R(\psi_i) \to R(\psi)$ induces an isomorphism $\iota_{Y_i}^* : H^2(R(\psi), \mathcal{S}_Y) \to H^2(R(\psi_i), \mathcal{S}_{Y_i})$. In particular $\iota_{1,2} := \iota_{Y_2}^* \circ (\iota_{Y_1}^*)^{-1}$ is an isomorphism from $H^2(R(\psi_1), \mathcal{S}_{Y_1})$ to $H^2(R(\psi_2), \mathcal{S}_{Y_2})$.*

Proof. The Y_i are full in $R(\psi)^{(0)}$, and $Y_i R(\psi) Y_i = R(\psi_i)$. The result now follows from Proposition 7.5. □

Let $\Gamma \to R$ be a twist, R' be a principal étale groupoid, and $\varphi : R' \to R$ be a continuous groupoid homomorphism. Then the pullback twist $\varphi^*(\Gamma)$ is the fibred product $R' *_\varphi \Gamma$ with structure maps $r(\alpha, \gamma) = r(\alpha)$ and $s(\alpha, \gamma) = s(\alpha)$, and with coordinatewise operations; it is regarded as a twist over R' under the surjection $(\alpha, \gamma) \mapsto \alpha$.

Recall from [28, Remark 2.9] that given a twist $\Gamma \xrightarrow{q} R$ there is an extension

$$\mathcal{S}_{R^{(0)}} \to \underline{\Gamma} \to R$$

such that $\underline{\Gamma}$ is the groupoid consisting of germs of continuous local sections of the surjection $\Gamma \to R$. Such extensions are called *sheaf twists*, and the group of isomorphism classes of sheaf twists over R is denoted $T_R(\mathcal{S})$ (see [28, Definition 2.5]). Pullbacks of sheaf twists are defined in a manner analogous to that of the preceding paragraph. By the discussion in [28, Section 2.9], the assignment $\Gamma \mapsto \underline{\Gamma}$ determines an isomorphism $\theta_R : [\Gamma] \mapsto [\underline{\Gamma}]$ from the group $\mathrm{Tw}(R)$ of isomorphism classes of twists over R to $T_R(\mathcal{S})$. Moreover, suppose that R is a principal étale groupoid, Γ is a twist over R, and U is a full open subset of $X = R^{(0)}$. Then an argument nearly identical to that of Lemma 7.4 shows that $[\phi]_u^U \mapsto (u, [\phi]_u^X)$ determines an isomorphism $\underline{\iota_U^*(\Gamma)} \cong \iota_U^*(\underline{\Gamma})$. Hence, using Lemma 7.4 to identify $\iota_U^*(\mathcal{S}_X)$ with $\mathcal{S}_U$, we see that the diagram

$$\begin{array}{ccc} \mathrm{Tw}(R) & \xrightarrow[\iota_U^*]{} & \mathrm{Tw}(URU) \\ \Big\downarrow \theta_R & & \Big\downarrow \theta_{URU} \\ T_R(\mathcal{S}_X) & \xrightarrow[\iota_U^*]{} & T_{URU}(\mathcal{S}_U) \end{array} \tag{7.2}$$

commutes.

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The long exact sequence of [28, Theorem 3.7] yields a boundary map ∂^1 from the first derived functor Z^1_R of the cocycle functor to $H^2(R,\mathcal{S})$. By [28, Corollary 3.4], the twist group $T_R(\mathcal{S})$ is naturally isomorphic to Z^1_R, so each twist Γ over R determines an element $\partial^1([\underline{\Gamma}]) \in H^2(R,\mathcal{S})$.

Theorem 7.7. *Fix a separable Fell algebra A. For each of $i = 1,2$ suppose that (C_i, D_i) is a diagonal pair, and that H_i is an A–C_i-imprimitivity bimodule with Rieffel homeomorphism $h_i : \widehat{C_i} \to \widehat{A}$. For each i, let $\psi_i : \widehat{D_i} \to \widehat{C_i}$ be the spectral map, and let Γ_i be a twist associated to (C_i, D_i) as in Theorem 6.1. For each i, let $\tilde{\psi}_i := h_i \circ \psi_i : \widehat{D_i} \to \widehat{A}$. Then the isomorphism $\iota_{1,2} : H^2(R(\tilde{\psi}_1), \mathcal{S}_{\widehat{D}_1}) \to H^2(R(\tilde{\psi}_2), \mathcal{S}_{\widehat{D}_2})$ of Corollary 7.6 carries $\partial^1([\underline{\Gamma}_1])$ to $\partial^1([\underline{\Gamma}_2])$.*

Proof. Since each C_i is Morita equivalent to A, each C_i is a separable Fell algebra, and Lemma 6.5 implies that Γ_1 and Γ_2 are equivalent twists. Let $\Gamma \to R$ be a linking twist (see Definition 6.4). Then in particular, each $\Gamma_i \cong \widehat{D_i}\Gamma\widehat{D_i} \cong i^*_{\widehat{D}_i}(\Gamma)$. Let $Y := \widehat{D}_1 \sqcup \widehat{D}_2$ and define $\psi : Y \to \widehat{A}$ by $\psi|_{\widehat{D}_i} = \tilde{\psi}_i$. Since $\iota_{1,2} = \iota^*_{\widehat{D}_2} \circ (\iota^*_{\widehat{D}_1})^{-1}$ by definition, it suffices to show that, for each of $i = 1,2$, the isomorphism

$$\iota^*_{\widehat{D}_i} : H^2\big(R(\psi), \mathcal{S}_Y\big) \to H^2\big(R(\tilde{\psi}_i), \mathcal{S}_{\widehat{D}_i}\big)$$

obtained from the first statement of Corollary 7.6 carries $\partial^1([\underline{\Gamma}])$ to $\partial^1([\underline{\Gamma}_i])$.

The naturality of the long exact sequence of [28, Theorem 3.7] together with [28, Corollary 3.4] implies that the right-hand square of the diagram

$$\begin{array}{ccccc}
\mathrm{Tw}(R(\tilde{\psi}_i), \mathbb{T}) & \longrightarrow & T_{R(\tilde{\psi}_i)}(\mathcal{S}_{\widehat{D}_i}) & \xrightarrow[\partial^1]{} & H^2(R(\tilde{\psi}_i), \mathcal{S}_{\widehat{D}_i}) \\
\uparrow{\scriptstyle \iota^*_{\widehat{D}_i}} & & \uparrow{\scriptstyle \iota^*_{\widehat{D}_i}} & & \uparrow{\scriptstyle \iota^*_{\widehat{D}_i}} \\
\mathrm{Tw}(R(\psi), \mathbb{T}) & \longrightarrow & T_{R(\psi)}(\mathcal{S}_Y) & \xrightarrow[\partial^1]{} & H^2(R(\psi), \mathcal{S}_Y)
\end{array}$$

commutes; the left-hand square is an instance of (7.2). Since Γ is a linking twist for the Γ_i, the maps $\iota^*_{\widehat{D}_i}$ on the left of the diagram carry $[\underline{\Gamma}]$ to $[\underline{\Gamma}_i]$. Since the diagram commutes, it follows that the maps $\iota^*_{\widehat{D}_i}$ on the right of the diagram carry $\partial^1([\underline{\Gamma}])$ to $\partial^1([\underline{\Gamma}_i])$. $\square$

If X and Y are topological spaces, and $\psi : Y \to X$ is a local homeomorphism, then we may regard X as a groupoid whose only elements are units, and there is then an induced groupoid homomorphism $\pi_\psi : R(\psi) \to X$ given by $\pi_\psi(y,z) = \psi(y)$.

Proposition 7.8. *Let X be a second-countable, locally compact, locally Hausdorff space, let Y be a second-countable, locally compact, Hausdorff space, and let $\psi : Y \to X$ be a local homeomorphism. Then $\pi^*_\psi : \mathsf{Sh}(X) \to \mathsf{Sh}(R(\psi))$ is an equivalence of categories such that $\pi^*_\psi(\underline{\mathbb{Z}}_X) = \underline{\mathbb{Z}}_Y$ and $\pi^*_\psi(\mathcal{S}_X) \cong \mathcal{S}_Y$. Moreover, π^*_ψ determines an isomorphism $\pi^*_\psi : H^*(X, \mathcal{S}_X) \to H^*(R(\psi), \mathcal{S}_Y)$. Finally, under the hypotheses of Theorem 7.7, $\iota_{1,2} \circ \pi^*_{\tilde{\psi}_1} = \pi^*_{\tilde{\psi}_2}$.*

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Proof. Regard X as a groupoid with unit space X whose only morphisms are units. Then

$$X^{\psi} = \{(y, x, z)\colon\ \psi(y) = x = \psi(z)\} \cong R(\psi),$$

and under this identification the map $\pi_{\psi} : (y, x, z) \mapsto x$ of (7.1) agrees with the map $\pi_{\psi} : R(\psi) \to X$ described above.

By [28, Proposition 0.8 and Theorem 0.9], π_{ψ}^{*} is an equivalence of categories which takes $\underline{\mathbb{Z}}$ to $\underline{\mathbb{Z}}$. Moreover, Lemma 7.4 implies that π_{ψ}^{*} takes $\mathcal{S}$ to $\mathcal{S}$ also. That π_{ψ}^{*} determines an isomorphism of cohomologies follows from [28, Proposition 1.8].

It remains to show that $\iota_{1,2} \circ \pi_{\tilde{\psi}_2}^{*} = \pi_{\tilde{\psi}_1}^{*}$. For this, let $Y_i = \widehat{D}_i$ for $i = 1, 2$, let $Y := Y_1 \sqcup Y_2$ and define $\tilde{\psi} : Y \to \widehat{A}$ by $\tilde{\psi}|_{Y_i} = \tilde{\psi}_i$ as in Corollary 7.6. Consider the diagrams below

$$\begin{tikzcd}
 & Y \arrow[dd, "\tilde{\psi}"] & \\
Y_1 \arrow[ur, "\iota_{Y_1}"] \arrow[dr, "\tilde{\psi}_1"'] & & Y_2 \arrow[ul, "\iota_{Y_2}"'] \arrow[dl, "\tilde{\psi}_2"] \\
 & \widehat{A} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R(\tilde{\psi}) \arrow[dd, "\pi_{\tilde{\psi}}"] & \\
R(\tilde{\psi}_1) \arrow[ur, "\iota_{Y_1}"] \arrow[dr, "\pi_{\tilde{\psi}_1}"'] & & R(\tilde{\psi}_2). \arrow[ul, "\iota_{Y_2}"'] \arrow[dl, "\pi_{\tilde{\psi}_2}"] \\
 & \widehat{A} &
\end{tikzcd}$$

The diagram on the left commutes by definition, and it follows that the diagram on the right commutes also. Recall that $\iota_{1,2} = (\iota_{Y_2}^{*})^{-1} \circ \iota_{Y_1}^{*}$ by definition. Thus functoriality and naturality of the cohomology exact sequence, and that $\pi_{\tilde{\psi}}^{*}$ takes $\mathcal{S}$ to $\mathcal{S}$ ensure that $\iota_{1,2} \circ \pi_{\tilde{\psi}_1}^{*} = \pi_{\tilde{\psi}_2}^{*}$ as required. $\square$

Theorem 7.7 and Proposition 7.8 ensure that we may specify a well-defined invariant as follows.

Definition 7.9. Let A be a separable Fell algebra. Let (C, D) be a diagonal pair such that C is Morita equivalent to A, fix an A–C-imprimitivity bimodule, and let $h : \widehat{C} \to \widehat{A}$ be its Rieffel homeomorphism. Let $\psi : \widehat{D} \to \widehat{C}$ be the spectral map, and $\tilde{\psi} := h \circ \psi : \widehat{D} \to \widehat{A}$. Let Γ be the twist associated to (C, D) as in Theorem 6.1. Then we define

$$\delta(A) := \left(\pi_{\tilde{\psi}}^{*}\right)^{-1}\left(\partial^{1}\left([\underline{\Gamma}]\right)\right) \in H^{2}(\widehat{A}, \mathcal{S}).$$

Remark 7.10. It seems difficult to establish that our invariant $\delta(A)$ coincides with the original Dixmier–Douady invariant of A when A is a continuous-trace C^{*}-algebra. The issue is that the boundary map ∂^{1} which takes the class of a twist over $R(\tilde{\psi})$ to an element of $H^{2}(R(\tilde{\psi}), \mathcal{S})$ is defined by abstract nonsense. Nevertheless our invariant does classify Fell algebras up to spectrum-preserving Morita equivalence (see Theorem 7.13), and this generalises the original Dixmier–Douady theorem of [17].

Proposition 7.11. *Let (G, X) be a free Cartan transformation group. Then $\delta(C_0(X) \rtimes G) = 0$ as an element of $H^{2}(X/G, \mathcal{S})$.*

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Proof. By Corollary 4.6, $C_0(X) \rtimes G$ is Morita equivalent to a groupoid C^*-algebra $C^*(R)$, where R is a principal, étale groupoid. By the remarks following Corollary 4.6, $C^*(R)$ is a Fell algebra. Thus the reduced C^*-algebra $C^*_{\mathrm{r}}(R)$ is also a Fell algebra and hence is nuclear. By [2, Corollary 6.2.14], since R is principal and $C^*_{\mathrm{r}}(R)$ is nuclear, R is measurewise amenable, and thus $C^*(R) = C^*_{\mathrm{r}}(R)$ by [2, Proposition 6.1.8].

By Lemma A.1, $C^*_{\mathrm{r}}(R)$ is isomorphic to the C^*-algebra $C^*_{\mathrm{r}}(R \times \mathbb{T}; R)$ of the trivial twist $\Gamma := R \times \mathbb{T} \to R$. The associated sheaf twist $\underline{\Gamma}$ is therefore also trivial and hence $\partial^1([\underline{\Gamma}]) = 0$. It follows that $\delta(C_0(X) \rtimes G) = 0$ also. □

To prove our classification theorem, we need another lemma.

Lemma 7.12. *Let X be a second-countable, locally compact, locally Hausdorff space. For $i = 1, 2$, let Y_i be a second-countable, locally compact, Hausdorff space, and let $\psi_i : Y_i \to X$ be a local homeomorphism. For $i = 1, 2$, let $\Gamma_i \to R(\psi_i)$ be a twist, and suppose that the isomorphism $\iota_{1,2}$ of Corollary 7.6 carries $\partial^1([\underline{\Gamma}_1])$ to $\partial^1([\underline{\Gamma}_2])$. Then there exist a locally compact, Hausdorff space Z and local homeomorphisms $\rho_i : Z \to Y_i$ such that $\psi_1 \circ \rho_1 = \psi_2 \circ \rho_2$ and $\pi^*_{\rho_1}(\Gamma_1) \cong \pi^*_{\rho_2}(\Gamma_2)$ as twists over $R(\psi_1 \circ \rho_1)$. In particular, Γ_1 and Γ_2 are equivalent twists.*

Proof. Let $Y := Y_1 * Y_2 = \{(y_1, y_2) \in Y_1 \times Y_2 \colon \psi_1(y_1) = \psi_2(y_2)\}$. For each i, let $\phi_i : Y \to Y_i$ be the projection map; then $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ is a local homeomorphism from Y to X.

We claim that each Γ_i is twist-equivalent to $\pi^*_{\phi_i}(\Gamma_i)$. To see this, we first observe that for $i = 1, 2$, the assignment

$$\big(x, \big(\phi_i(x), \phi_i(y)\big), y\big) \mapsto (x, y)$$

is an isomorphism from $R(\psi_i)^{\phi_i}$ to $R(\psi_i \circ \phi_i)$, and the assignment $((x, y), g) \mapsto (x, g, y)$ is an isomorphism from $\pi^*_{\phi_i}(\Gamma_i)$ to $\Gamma_i^{\phi_i}$. By [27, Proposition 5.7], each Γ_i is equivalent to $\Gamma_i^{\phi_i}$, so the isomorphisms above complete the proof of the claim.

Since

$$\iota_{1,2}\big(\partial^1\big([\pi^*_{\phi_1}(\underline{\Gamma}_1)]\big)\big) = \partial^1\big([\pi^*_{\phi_2}(\underline{\Gamma}_2)]\big),$$

[28, Proposition 3.9] implies that there exist a locally compact, Hausdorff space Z and a local homeomorphism $\tau : Z \to Y$ such that $\pi^*_\tau(\pi^*_{\phi_1}(\underline{\Gamma}_1))$ and $\pi^*_\tau(\pi^*_{\phi_2}(\underline{\Gamma}_2))$ are isomorphic sheaf twists. Since each $\pi^*_{\phi_i \circ \tau}(\underline{\Gamma}_i) = \pi^*_\tau(\pi^*_{\phi_i}(\underline{\Gamma}_i))$, it follows that with $\rho_i := \phi_i \circ \tau : Z \to Y_i$, we have $\pi^*_{\rho_1}(\underline{\Gamma}_1) \cong \pi^*_{\rho_2}(\underline{\Gamma}_2)$, and hence by naturality

$$\pi^*_{\rho_1}(\Gamma_1) \cong \pi^*_{\rho_2}(\Gamma_2). \tag{7.3}$$

For the final assertion, we apply the claim above with ϕ_i replaced with ρ_i to see that each Γ_i is twist-equivalent to $\pi^*_{\rho_i}(\Gamma_i)$, and then invoke (7.3). □

Theorem 7.13. *Let A_1 and A_2 be separable Fell algebras. Then A_1 and A_2 are Morita equivalent if and only if there is a homeomorphism $h : \widehat{A}_1 \to \widehat{A}_2$ such that the induced isomorphism $h^* : H^2(\widehat{A}_2, \mathcal{S}) \to H^2(\widehat{A}_1, \mathcal{S})$ carries $\delta(A_2)$ to $\delta(A_1)$.*

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Proof. First suppose that H is an A_2–A_1-imprimitivity bimodule and let $h : \widehat{A}_1 \to \widehat{A}_2$ be the associated Rieffel homeomorphism. Let (C, D) be a diagonal pair together with an A_2–C-imprimitivity bimodule K, and let $k : \widehat{C} \to \widehat{A}_2$ be the Rieffel homeomorphism associated to K. Let $\psi : \widehat{D} \to \widehat{C}$ be the spectral map.

Let $\tilde{\psi}_2 := k \circ \psi : \widehat{D} \to \widehat{A}_2$, and let $\Gamma_2 \to R$ be the twist obtained from (C, D) as in Theorem 6.1. Note that $R = R(\tilde{\psi}_2)$ by Proposition 6.3. By definition, $\delta(A_2) = (\pi^*_{\tilde{\psi}_2})^{-1}(\partial^1([\underline{\Gamma}_2])) \in H^2(\widehat{A}_2, \mathcal{S})$. Let $\widetilde{H}$ be the dual bimodule of H, and observe that $K \otimes_{A_2} \widetilde{H}$ is a C–A_1-imprimitivity bimodule with Rieffel homeomorphism $h^{-1} \circ k$. Let $\tilde{\psi}_1 := h^{-1} \circ k \circ \psi : \widehat{D} \to \widehat{A}_1$, and let Γ_1 be the twist over $R(\tilde{\psi}_1)$ obtained from (C, D) as in Theorem 6.1. Again by definition, $\delta(A_1) = (\pi^*_{\tilde{\psi}_1})^{-1}(\partial^1([\underline{\Gamma}_1])) \in H^2(R(\tilde{\psi}_1), \mathcal{S})$. Since $\tilde{\psi}_2 = h \circ \tilde{\psi}_1$, Theorem 7.7 and Proposition 7.8 imply that the induced isomorphism $h^* : H^2(\widehat{A}_2, \mathcal{S}) \to H^2(\widehat{A}_1, \mathcal{S})$ carries $\delta(A_2)$ to $\delta(A_1)$.

Now suppose that there is a homeomorphism $h : \widehat{A}_1 \to \widehat{A}_2$ such that the induced isomorphism $h^* : H^2(\widehat{A}_2, \mathcal{S}) \to H^2(\widehat{A}_1, \mathcal{S})$ carries $\delta(A_2)$ to $\delta(A_1)$. Let (C_i, D_i) be diagonal pairs with C_i Morita equivalent to A_i, let $\psi_i : \widehat{D}_i \to \widehat{C}_i$ be the spectral maps, and let $\Gamma_i \to R(\psi_i)$ be the associated twists. Proposition 7.8 and the hypothesis that h^* carries $\delta(A_2)$ to $\delta(A_1)$ ensures that the induced map (also denoted h^*) from $H^2(R(\psi_2), \mathcal{S})$ to $H^2(R(\psi_1), \mathcal{S})$ satisfies

$$h^*\big(\partial^1\big([\underline{\Gamma}_2]\big)\big) = \partial^1\big([\underline{\Gamma}_1]\big).$$

Hence we may regard Γ_1 as a twist over $R(h \circ \psi_1)$ with the same image under ∂^1 as Γ_2. Lemma 7.12 therefore implies that Γ_1 and Γ_2 are equivalent twists, and then Lemma 6.5 implies that C_1 and C_2 are Morita equivalent, whence A_1 and A_2 are also Morita equivalent. □

Recall that if X is the spectrum of a C^*-algebra then X is locally compact, and every open subset of X is itself locally compact (because it is the spectrum of an ideal); such spaces are called *locally quasi-compact* in [16, §3.3].

Remark 7.14. Let X be a second-countable, locally Hausdorff space such that every open subset of X is locally compact. We will show that every element of $H^2(X, \mathcal{S})$ arises as the class of a Fell algebra with spectrum X. To do this, it is convenient to work with Čech cohomology, rather than sheaf cohomology, of a locally compact, Hausdorff "desingularisation" Y of X.

It is observed in [37, Hooptedoodle 4.16], with reference to [44, §5.23], that all reasonable sheaf-cohomology theories coincide over Hausdorff paracompact spaces. Specifically, by Theorem 5.32 of [44] and the subsequent corollary, any two sheaf cohomologies over Hausdorff paracompact spaces satisfying [44, Axioms 5.18] are canonically isomorphic. Warner demonstrates in [44, §5.33] that Čech cohomology satisfies these axioms. All but one of these axioms are automatically satisfied by the sheaf cohomology used here because it is defined in terms of derived functors; the remaining axiom (property (b) of [44, Axioms 5.18]) requires that $H^q(Y, B) = 0$ for $q > 0$ if B is a fine sheaf, and this follows from [43, Proposition 4.36].

For an introduction to Čech cohomology, see [37, Chapter 4]. Given a covering $\{U_i : i \in I\}$ of a space Y, and given $i, j, k \in I$, we write U_{ijk} for the intersection $U_i \cap U_j \cap U_k$.

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Lemma 7.15. *Let Y be a second-countable, locally compact, Hausdorff space. For each $a \in H^2(Y, \mathcal{S})$, there exist a locally compact, Hausdorff space Z and a local homeomorphism $\phi : Z \to Y$ such that $\phi^*(a) = 0 \in H^2(Z, \mathcal{S})$.*

Proof. By Remark 7.14, we may regard a as an element of $\check{H}^2(Y, \mathcal{S})$. So there exists a covering $\mathcal{U} = \{U_i\colon i \in I\}$ of Y by open sets and a 2-cocycle $c = \{c_{ijk} : U_{ijk} \to \mathbb{T} \mid i, j, k \in I\}$ such that a is equal to the class of c in $\check{H}^2(Y, \mathcal{S})$.

Let $Z := \bigcup_{i \in I}(\{i\} \times U_i) \subset I \times Y$, and let $\phi : Z \to Y$ be the projection onto the second coordinate. Let $V_i := \{i\} \times U_i \subset Z$ for each i. Then $\mathcal{V} = \{V_i\colon i \in I\}$ is a refinement of the pullback cover $\{\phi^{-1}(U_i)\colon i \in I\}$ to a cover by mutually disjoint sets; in particular the only nonempty triple overlaps are those of the form V_{iii}. Since $\check{H}^2(Z, \mathcal{S})$ is the direct limit over covers of Z of the cocycle group, the class of $\phi^*(c)$ is equal to the class of its image $i_{\mathcal{U},\mathcal{V}}(\phi^*(c))$ in the cocycle group for $\mathcal{V}$. Since the V_i are pairwise disjoint, $i_{\mathcal{U},\mathcal{V}}(\phi^*(c))$ amounts to a continuous circle-valued function on each V_{iii}, and so is a coboundary (specifically, the coboundary of itself regarded as a 1-cochain). □

Next we require notation for the forgetful functor which takes an equivariant Γ-sheaf B to an ordinary sheaf B^0 over Γ^0 by forgetting the Γ-action. Note that $B^0 = \jmath^*(B)$ where $\jmath : \Gamma^{(0)} \to \Gamma$ is the inclusion map. The pullback functor induces the homomorphism $\jmath_n^* : H^n(\Gamma, B) \to H^n(\Gamma^{(0)}, B^0)$ which appears in the long exact sequence of [28, Theorem 3.7].

Proposition 7.16. *Let X be a second-countable, locally Hausdorff space such that every open subset of X is locally compact. Then for each $a \in H^2(X, \mathcal{S})$ there exist a locally compact, Hausdorff space Z, a local homeomorphism $\psi : Z \to X$ and a twist Γ over $R(\psi)$ such that $a = (\pi_\psi^*)^{-1}(\partial^1([\underline{\Gamma}]))$. In particular, for each $a \in H^2(X, \mathcal{S})$, there exists a separable Fell algebra A such that $\widehat{A} = X$ and $a = \delta(A)$.*

Proof. Choose a countable open cover $\{U_i\}$ of X consisting of Hausdorff subsets of X and let $Y := \bigsqcup_i U_i$. Since every open subset of X is locally compact, each U_i is locally compact, and hence Y is locally compact and Hausdorff. The inclusion map $\theta : Y \to X$ is a local homeomorphism. Let $b := \pi_\theta^*(a) \in H^2(R(\theta), \mathcal{S})$. By Lemma 7.15, there exist a second-countable, locally compact, Hausdorff space Z and a local homeomorphism $\phi : Z \to Y$ such that $\phi^*(\jmath_2^*(b)) = 0$. Let $\psi := \theta \circ \phi : Z \to X$. Then by naturality of the long exact sequence, $\jmath_2^*(\pi_\psi^*(a)) = 0 \in H^2(Z, \mathcal{S})$. By exactness, it follows that there is a twist Γ over $R(\psi)$ such that $\pi_\psi^*(a) = \partial^1([\underline{\Gamma}])$. Let $A := C_{\mathrm{r}}^*(\Gamma; R(\psi))$. By Theorem 6.6(2), A is a Fell algebra and its spectrum is homeomorphic to X. After identifying $\widehat{A}$ with X, $\delta(A) = a$ by Definition 7.9. □

Remark 7.17. There is a notion of a Brauer group $\mathrm{Br}(G)$ for a locally compact, Hausdorff groupoid G [29]. Moreover, [29, Proposition 11.3] implies that if G is étale, then $\mathrm{Br}(G) \cong H^2(G, \mathcal{S})$. If Z is a groupoid equivalence of locally compact, Hausdorff groupoids G and H, then Z determines an isomorphism between $H^2(G, \mathcal{S})$ and $H^2(H, \mathcal{S})$ [29, Theorem 4.1]. Thus $H^2(X, \mathcal{S})$ is canonically isomorphic to $\mathrm{Br}(R(\psi))$ for any local homeomorphism ψ from a locally compact, Hausdorff space onto X (the isomorphism of Proposition 7.5 is a special case of [29, Theorem 4.1]).

Though it would have been natural to identify $\mathrm{Br}(X)$ with $H^2(X, \mathcal{S})$ for a locally compact, locally Hausdorff space X, we have chosen not to use the notation $\mathrm{Br}(X)$ nor the term Brauer

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group as the notion has not yet been extended to non-Hausdorff spaces (to say nothing of non-Hausdorff groupoids). To justify the use of the term it would first be necessary to formulate a notion of balanced tensor product for Fell algebras with spectra identified with X. We leave the details for future work.

Appendix A

Let Γ be a $\mathbb{T}$-groupoid and $\Gamma \xrightarrow{q} R$ a twist (see p. 1562 for the definition of a twist). The details of the construction of the twisted groupoid C^*-algebra $C^*_{\mathrm{r}}(\Gamma; R)$ may be found in §2 of [27]. The idea is that a dense subalgebra is identified with continuous compactly supported sections of an associated line bundle, and then convolution and involution are defined by virtue of the Fell bundle structure of the line bundle (as in [39, §5] but our conventions differ slightly). We briefly review the construction for the convenience of the reader. Since R is an étale groupoid, we may use the standard Haar system consisting of counting measures.

Define a line bundle $L = L(\Gamma)$ over R by taking the quotient of $\mathbb{C} \times \Gamma$ by the diagonal action of $\mathbb{T}$ — that is, L consists of equivalence classes of the equivalence relation $(z, \gamma) \sim (tz, t \cdot \gamma)$ for $t \in \mathbb{T}$. Then L is a complex line-bundle over R with bundle map $[(z, \gamma)] \mapsto q(\gamma)$. As usual, we denote by L_ρ the fibre over $\rho \in R$.

The following Fell bundle structure on L is implicit in [27]. Given a composable pair (ρ_1, ρ_2) of elements in R and elements $[(z_i, \gamma_i)] \in L_{\rho_i}$, we define the product in $L_{\rho_1\rho_2}$ by

$$\big[(z_1, \gamma_1)\big]\big[(z_2, \gamma_2)\big] = \big[(z_1 z_2, \gamma_1\gamma_2)\big];$$

and involution is defined by $[(z, \gamma)] \in L_{q(\gamma)} \mapsto [(\bar{z}, \gamma^{-1})] \in L_{q(\gamma^{-1})}$. It is straightforward to check that these operations are well defined.

Now define

$$C_c(\Gamma; R) := \big\{ f \in C_c(\Gamma)\colon\ f(t \cdot \gamma) = t f(\gamma) \text{ for } t \in \mathbb{T} \text{ and } \gamma \in \Gamma \big\}.$$

Each $f \in C_c(\Gamma; R)$ determines a section $\tilde{f}$ of L by the formula $\tilde{f}(q(\gamma)) := [(f(\gamma), \gamma)]$ (it is straightforward to check that this map is well defined). Moreover, given $\gamma \in \Gamma$, and $z \in \mathbb{T}$ the element z is uniquely determined by γ and $[z, \gamma]$, so $f \mapsto \tilde{f}$ is a bijection. Hence we may endow $C_c(\Gamma; R)$ with the structure of a $*$-algebra by the following formulae for $f, g \in C_c(\Gamma; R)$

$$(f * g)\tilde{}\,(\rho) = \sum_{\alpha\beta=\rho} \tilde{f}(\alpha)\tilde{g}(\beta) \quad \text{and} \quad \tilde{f}^*(\rho) = \tilde{f}\big(\rho^{-1}\big)^*.$$

These operations match up with the convolution and involution on $C_c(\Gamma; R)$ used in, for example, [32,39]. To keep our notation simple we identify each element of $C_c(\Gamma; R)$ with the corresponding compactly supported continuous section of the Fell bundle L (as in [27,39]).

Note that the map, $(x, z) \mapsto [(x, z)]$ gives a trivialisation $R^{(0)} \times \mathbb{C} \cong L|_{R^{(0)}}$ and hence L is trivial over $R^{(0)}$; thus we may identify $C_c(R^{(0)})$ with the abelian subalgebra $\{f \in C_c(\Gamma; R)\colon \operatorname{supp} f \subset R^{(0)}\}$, so $C_c(\Gamma; R)$ may be regarded as a right $C_c(R^{(0)})$-module under right-multiplication. Moreover, the restriction map $P : C_c(\Gamma; R) \to C_c(R^{(0)})$ is a $C_c(R^{(0)})$-module morphism. For $f, g \in C_c(\Gamma; R)$, the formula $\langle f, g\rangle = P(f^* g)$ defines an inner product on $C_c(\Gamma; R)$, and the completion $H(\Gamma; R)$ of $C_c(\Gamma; R)$ in the norm $\|f\| = \|\langle f, f\rangle\|_\infty^{1/2}$

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is a right-Hilbert $C_0(R^{(0)})$-module. Finally, left multiplication by $f \in C_c(\Gamma; R)$ extends to an adjointable operator $\phi(f)$ on $H(\Gamma; R)$; this defines a $*$-homomorphism $\phi : C_c(\Gamma; R) \to \mathcal{L}(H(\Gamma; R))$. The twisted groupoid C^*-algebra $C^*_{\mathrm{r}}(\Gamma; R)$ is defined to be the completion of $C_c(\Gamma; R)$ in the operator norm, and $C_0(R^{(0)})$ is identified with the closure of $C_c(R^{(0)})$ in $C^*_{\mathrm{r}}(\Gamma; R)$.

We show that $(C^*_{\mathrm{r}}(\Gamma; R), C_0(R^{(0)}))$ is a diagonal pair in the sense of Definition 5.2. This follows from [27, Proposition 2.9] and Corollary 5.6 once we establish that $C_0(R^{(0)})$ contains an approximate identity for $C^*_{\mathrm{r}}(\Gamma; R)$. For this, let $(K_n)_{n=1}^{\infty}$ be an increasing sequence of compact subsets of $R^{(0)}$ such that $R^{(0)} = \bigcup_n K_n$, and for each $n \in \mathbb{N}$, fix $g_n \in C_c(R^{(0)})$ such that $g_n|_{K_n} = 1$. Since $(fg_n)(\rho) = f(\rho)g_n(s(\rho))$ for each compactly supported section f, the g_n form an approximate identity for $C^*_{\mathrm{r}}(\Gamma; R)$.

For the next result, recall from [28, Remark 4.2] that a trivial twist over R is isomorphic to $R \times \mathbb{T} \to R$. The reduced norm on $C_c(R)$ is variously defined in the literature; see, for example, [38, p. 82] and [2, p. 146], and also [41, §3] for a discussion of the equivalence of these two definitions. There are also two definitions of the reduced norm on $C_c(\Gamma; R)$: one using the operator norm outlined above and the other based on induced representations from point evaluations on $C_0(R^{(0)})$ used in [39, p. 40]. The equivalence of the two follows from the observation that

$$\big(\pi_x(f)\xi|_{s^{-1}(x)}|\eta|_{s^{-1}(x)}\big)_{H_x} = \overline{\langle \phi(f)\xi, \eta\rangle_{C_0(R^{(0)})}(x)}$$

for compactly supported sections $f \in C_c(\Gamma; R)$ and $\xi, \eta \in C_c(\Gamma; R) \subset H(\Gamma; R)$ (see also the discussion in [39, p. 40]).

Lemma A.1. *If $\Gamma \to R$ is a trivial twist, then $C^*_{\mathrm{r}}(\Gamma; R) \cong C^*_{\mathrm{r}}(R)$.*

Proof. Suppose that Γ is a trivial twist. Then we may identify L and $\mathbb{C} \times R$ and therefore $C_c(R, L)$ and $C_c(R)$. It is routine to check that this identification preserves the $*$-algebra structure defined above. So we just need to check that for $f \in C_c(R)$ we have $\|f\|_r = \|\phi(f)\|$. Let $f \in C_c(R)$. By the definition given in [38, p. 82], we have

$$\|f\|_r = \sup_{\mu} \big\|\mathrm{Ind}_\mu(f)\big\|$$

where μ ranges over all Radon measures on $R^{(0)}$. Denote by $\pi_\mu : C_0(R^{(0)}) \to \mathcal{B}(L^2(R^{(0)}, \mu))$ the usual representation by multiplication operators. The discussion on p. 81 of [38] shows that the induced representation Ind_μ is given on

$$H(\Gamma; R) \otimes_{\pi_\mu} L^2\big(R^{(0)}, \mu\big)$$

by the formula

$$\mathrm{Ind}_\mu(f)(\xi \otimes g) = \phi(f)\xi \otimes g.$$

Hence, $\|\mathrm{Ind}_\mu(f)\| \leqslant \|\phi(f)\|$ and so $\|f\|_r \leqslant \|\phi(f)\|$. Now, let μ be a measure with full support; then π_μ is faithful and hence the corresponding representation of $\mathcal{K}(H(\Gamma; R))$ is also faithful. Since $\mathcal{L}(H(\Gamma; R)) = M(\mathcal{K}(H(\Gamma; R)))$, this shows that Ind_μ is faithful on $C^*_{\mathrm{r}}(\Gamma; R)$. Hence, $\|\phi(f)\| = \|\mathrm{Ind}_\mu(f)\| \leqslant \|f\|_r$. $\square$

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