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# On the multiplication theorems of Hadamard matrices of generalized quaternion type using $M$ -structures<sup>1</sup>

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**Abstract.** We show that  $M$ -structures can be extended to Hadamard matrices of *generalized quaternion type* and obtain multiplication type theorems which preserve the structure.

## 1. Introduction

The concept of  $M$ -structures generalizes a number of concepts in Hadamard matrices, including Williamson matrices, Goethals-Seidel matrices, Wallis-Whiteman matrices and generalized quaternion matrices. We found many symmetric Williamson matrices and many Hadamard matrices using the concept of  $M$ -structures [4], [5], [6]. Furthermore, the concept of  $M$ -structures leads to the new concept of strong Kronecker products introduced by Jennifer Seberry and Xian-mo Zhang [8]. This was used by Craigen, Seberry and Zhang [1] to prove that if there exist Hadamard matrices of orders  $4p$ ,  $4q$ ,  $4r$ , and  $4s$ , then we have an Hadamard matrix of order  $16pqrs$ .

An orthogonal matrix of order  $4t$  can be divided into sixteen  $t \times t$  blocks  $M_{ij}$ . This partitioned matrix is said to be an  $M$ -structure. If the orthogonal matrix can be partitioned into sixty-four blocks  $M_{ij}$ , it will be called a 64 block  $M$ -structure.

First we give some definitions.

**Definition 1:** The matrices  $X$  and  $Y$  are said to be *amicable matrices* if

$$XY^t = YX^t,$$

where  $X^t$  and  $Y^t$  are the transpose matrices of  $X$  and  $Y$  respectively.

**Definition 2:** *Williamson matrices* of order  $w$  are four circulant symmetric matrices  $A, B, C, D$  which have entries 1 or  $-1$  and which satisfy

$$AA^t + BB^t + CC^t + DD^t = 4wI_w$$

where  $I_w$  is a unit matrix of order  $w$ .

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Definition 3: *Williamson-type matrices* of order  $w$  are four pairwise amicable matrices  $A, B, C, D$  which have entries 1 or  $-1$  and which satisfy

$$AA^t + BB^t + CC^t + DD^t = 4wI_w.$$

A generalized quaternion group  $Q_s$ , of order  $2^{s+2}$  is a group generated by the two elements  $\rho, j$  such that

$$\rho^{2^{s+1}} = 1, j^2 = \rho^{2^s}, j\rho j^{-1} = \rho^{-1}.$$

Let  $G$  be a semi-direct product of a cyclic group of an odd order  $n$  by the generalized quaternion group  $Q_s$  of order  $2^{s+2}$ . That is,  $G$  is generated by  $\rho, \xi$  and  $j$  with the relations

$$\rho^{2^s} = -1, j^2 = -1, j\rho j^{-1} = \rho^{-1}, \rho\xi\rho^{-1} = \xi, j\xi j^{-1} = \xi^{-1}, \xi^n = 1.$$

We consider the ring  $\mathcal{R}$  obtained from the group ring  $\mathbb{Z}G$  by identifying the elements  $\pm 1$  in the center of  $Q_s$  with  $\pm 1$  of the rational integer ring  $\mathbb{Z}$ . Put  $\mathcal{H} = \{\rho^k \zeta^l : 0 \leq k \leq 2^s - 1, 0 \leq l \leq n - 1\}$  and choose the basis  $\mathcal{L} = \mathcal{H} \cup \mathcal{H}j$  of  $\mathcal{R}$ . An element  $\xi$  in  $\mathcal{R}$  takes the following form.

$$\xi = \sum_{k=0}^{2N-1} \sum_{l=0}^{n-1} \alpha_{k,l} \zeta^l \rho^k + \sum_{k=0}^{2N-1} \sum_{l=0}^{n-1} b_{k,l} \zeta^l \rho^k j = \alpha + \beta j, \quad N = 2^{s-1}, \quad (1)$$

where

$$\alpha = \sum_{k=0}^{2N-1} \sum_{l=0}^{n-1} \alpha_{k,l} \zeta^l \rho^k \quad \text{and} \quad \beta = \sum_{k=0}^{2N-1} \sum_{l=0}^{n-1} b_{k,l} \zeta^l \rho^k.$$

We define the conjugate  $\xi = \bar{\alpha} - \beta j$  of  $\xi = \alpha + \beta j$  based on the automorphism  $\tau : \rho \rightarrow \rho^{-1}, \zeta \rightarrow \zeta^{-1}$  of  $G$ . Furthermore, we define the norm  $\mathcal{N}(\xi) = \xi \bar{\xi}$  so that:

$$\begin{aligned} \mathcal{N}(\xi) &= \alpha \bar{\alpha} + \beta \bar{\beta} \\ \mathcal{N}(\xi\eta) &= \mathcal{N}(\xi)\mathcal{N}(\eta) \quad \text{for } \xi, \eta \in \mathcal{R}. \end{aligned}$$

For an arbitrary element  $\xi \in \mathcal{R}$  we construct the right regular representation matrix  $R(\xi)$ , defined by

$$(\rho^k \zeta^l \xi) = R(\xi)(\rho^k \zeta^l).$$

More precisely, for an element  $\xi$  of  $\mathcal{R}$  with the form (1) the right regular representation matrix  $R(\xi)$  is given by

$$R(\xi) = \begin{pmatrix} A & B \\ -B^t & A^t \end{pmatrix}$$

$$A = \begin{pmatrix} A_0 & A_1 & \dots & A_{2N-1} \\ -A_{2N-1} & A_0 & \dots & A_{2N-2} \\ \vdots & \vdots & & \vdots \\ -A_1 & -A_2 & \dots & A_0 \end{pmatrix}$$

$$B = \begin{pmatrix} B_0 & B_1 & \dots & B_{2N-1} \\ -B_{2N-1} & B_0 & \dots & B_{2N-2} \\ -B_{2N-2} & B_{2N-1} & \dots & B_{2N-3} \\ \vdots & \vdots & & \vdots \\ -B_1 & -B_2 & \dots & B_0 \end{pmatrix}$$

where  $A_k = \sum_{l=0}^{n-1} a_{k,l} T^l$  and  $B_k = \sum_{l=0}^{n-1} a_{k,l} T^l$  are the circulant matrices of order  $n$  where  $T$  denotes the basic circulant matrix of order  $n$

$$T = \begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & 1 & \\ & & & & 1 \\ & & & & \ddots \\ & & & & & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since  $R(\xi) = R(\xi)^t$ , we have

$$R(\xi) R(\xi) = R(\xi) R(\bar{\xi}) = R(\xi \bar{\xi}) = \begin{pmatrix} AA^t + BB^t & 0 \\ 0 & AA^t + BB^t \end{pmatrix}$$

Definition 4: If an element in  $\mathcal{R}$  which is given by the equation (1) above satisfies

- (i) all the coefficients  $a_{k,l}, b_{k,l}$  are from  $\{1, -1\}$  and
- (ii)  $\mathcal{N}(\xi) = 2^{s+1}n = 4nN$ ,

then the right regular representation matrix  $R(\xi)$  becomes an Hadamard matrix of order  $2^{s+1}n = 4nN$ , which is called an *Hadamard matrix of generalized quaternion type*.

Similarly, if the following conditions are satisfied:

- (iii)  $a_{k,k} = 0$  and all other coefficients  $a_{k,l}, b_{k,l}$  are from  $\{1, -1\}$  and
- (iv)  $\mathcal{N}(\xi) = 2^{s+1}n - 1 = 4nN - 1$ ,

then  $R(\xi)$  is a *C-matrix* of order  $2^{s+1}n = 4nN$ , which we call a *C-matrix of generalized quaternion type*.

We abbreviate *generalized quaternion type* as *GQ type* for convenience sake.

Let us express the conditions (i), (ii) in terms of the component matrices  $A_k$ , and  $B_k$ :

$$\sum_{k=0}^{2N-1} A_k A_k^t + \sum_{k=0}^{2N-1} B_k B_k^t = 4nNI,$$

$$\sum_{k=0}^{t-1} (A_k A_{2N-t+k}^t + B_{2N-t+k}^t) - \sum_{k=0}^{2N-t-1} (A_k^t A_{k+t} + B_k^t B_{k+t}) = 0 \text{ for } 1 \leq t \leq 2N-1.$$

In particular, in the case  $N = 1$  the conditions will become

$$\begin{aligned} A_0 A_0^t + A_1 A_1^t + B_0 B_0^t + B_1 B_1^t &= 4nI, \\ A_0 A_1^t - A_1 A_0^t + B_0 B_1^t + B_1 B_0^t &= 0. \end{aligned}$$

Moreover, suppose that  $A_0, A_1, B_0$  and  $B_1$  are symmetric, that is, Williamson matrices, or suppose they are pairwise amicable, that is, Williamson-type matrices, then the second condition is trivial.

## 2. $M$ -structure Hadamard matrices

We consider Hadamard matrices of GQ type as an  $M$ -structure. Namely, an Hadamard matrix  $H$  of GQ type is partitioned into sixteen blocks,

$$H = \begin{pmatrix} A & B \\ -B^t & A^t \end{pmatrix} = \begin{pmatrix} C_0 & C_1 & D_0 & D_1 \\ -C_1 & C_0 & -D_1 & D_0 \\ -D_0^t & D_1^t & C_0^t & -C_1^t \\ -D_1^t & -D_0^t & C_1^t & C_0^t \end{pmatrix} \quad (3)$$

where

$$\begin{aligned} C_0 &= \begin{pmatrix} A_0 & A_1 & \dots & A_{N-1} \\ -A_{2N-1} & A_0 & \dots & A_{N-2} \\ -A_{N+1} & -A_{N+2} & \dots & A_0 \end{pmatrix}, & C_1 &= \begin{pmatrix} A_N & A_{N+1} & \dots & A_{2N-1} \\ -A_{N-1} & A_N & \dots & A_{2N-2} \\ -A_1 & -A_2 & \dots & A_N \end{pmatrix}, \\ D_0 &= \begin{pmatrix} B_0 & B_1 & \dots & B_{N-1} \\ -B_{2N-1} & B_0 & \dots & B_{N-2} \\ -B_{N+1} & -B_{N+2} & \dots & B_0 \end{pmatrix}, & D_1 &= \begin{pmatrix} B_N & B_{N+1} & \dots & B_{2N-1} \\ -B_{N-1} & B_N & \dots & B_{2N-2} \\ -B_1 & -B_2 & \dots & B_N \end{pmatrix}, \end{aligned}$$

Since  $H$  is an Hadamard matrix, the component matrices  $C_0, C_1, D_0, D_1$  satisfy the following equations,

$$\begin{cases} C_0 C_0^t + C_1 C_1^t + D_0 D_0^t + D_1 D_1^t = C_0^t C_0 + C_1^t C_1 + D_0^t D_0 + 0 + D_1^t D_1 = 4nNI \\ C_0 C_1^t - C_1 C_0^t + D_1 D_1^t - D_1 D_0^t = C_0^t C_1 - C_1^t C_0 + D_0^t D_1 - D_1^t D_0 = 0 \\ C_0 D_0 - C_1 D_1 - D_0 C_0 + D_1 C_1 = 0 \\ C_0 D_1 + C_1 D_0 - D_0 C_1 - D_1 C_0 = 0 \end{cases} \quad (4)$$

An Hadamard matrix having the form (3) will be called an  $M$ -structure Hadamard matrix of GQ type.

## 3. Paley type 1 matrix

The Paley type 1 matrix can be changed into the form of a  $C$ -matrix of GQ type and is defined as follows (see [3]).

Definition 5: Let  $q$  be a prime power,  $q \equiv 3 \pmod{4}$ ,  $F = GF(q)$  the finite field of  $q$  elements,  $K = GF(q^2)$  a quadratic extension over  $F$ , and  $K^\times$  and  $F^\times$  the multiplicative groups of  $K$  and  $F$  respectively. Furthermore, let  $\eta$  be a

generator of  $K^\times$ ,  $\gamma = \eta^{(q+1)/2}$  and let  $N_{K/F}$  and  $S_{K/F}$  denote the relative norm and relative trace from  $K$  to  $F$  respectively. Denote by  $\psi$  the quadratic character of  $F$ . Then the matrix

$$P = (\psi(N_{K/F}\alpha)\psi(S_{K/F}\gamma^{-1}\beta\alpha^{-1}))_{\alpha,\beta \in K^\times/F^\times}$$

is called the *Paley type 1 matrix*.

We recall here the definition of Seidel-equivalence of matrices.

**Definition 6:** If a square matrix  $A$  can be obtained from a square matrix  $B$  by a sequence of two kinds of operations:

- (i) multiplying the row and the corresponding column by  $-1$  simultaneously,
  - (ii) interchanging two rows and the corresponding two columns simultaneously,
- then  $A$  will be said to be *Seidel-equivalent* to  $B$ .

**Theorem 1.** *The Paley type 1 matrix is Seidel equivalent to a C-matrix of GQ type with some additional properties:*

- (i)  $A$  is skew symmetric;
- (ii)  $B_{2N-m-1} = -B_m^t$  for  $m = 0, \dots, N-1$  where  $q+1 = 2^{s+1}n$ ,  $s \geq 1$ ,  $n$  odd,  $N = 2^{s+1}$ .

**Proof:** See [11]. ■

#### 4. Infinite series of Hadamard matrices of generalized quaternion type

Yamada constructed some infinite series of Hadamard matrices of GQ type [11]. In this section we show these constructions of finite series.

Let  $q$  be a power of a prime  $p$ ,  $F = GF(q)$  denote a finite field of  $q$  elements,  $K = GF(q^t)$  an extension of  $F$  of degree  $t$ ,  $t \geq 2$ . Let  $\eta$  be a generator of  $K^\times$  and let  $S_K$  and  $S_F$  denote the absolute trace in  $K$  and  $F$ . Furthermore, let  $S_{K/F}$  and  $N_{K/F}$  be the relative trace and relative norm from  $K$  to  $F$  respectively.

**Definition 7:** Let  $\chi$  be a character of  $F$  and  $\zeta_p = e^{2\pi i/p}$ , then the Gauss sum  $\tau_F(\chi)$  is defined by

$$\tau_F(\chi) = \sum_{\alpha \in F} \chi(\alpha) \zeta_p^{S_F \alpha}.$$

If  $\chi$  is a nonprincipal character of  $K$ , then the ratio

$$\theta_\chi = \frac{\tau_K(\chi)}{\tau_F(\chi)}$$

of two Gauss sums is called the *relative Gauss sum associated with  $\chi$* .

The following theorem on the relative Gauss sum is very useful.

**Theorem 2.** Suppose that  $\chi$  is a character of  $K$  inducing in  $F$  a nonprincipal character. Then the relative Gauss sum associated with  $\chi$  can be written in the following form

$$\theta_\chi = \sum_{\alpha \in K^*/F^*} \chi(\alpha) \bar{\chi}(S_{K/F}\alpha),$$

and we have the norm relation

$$\theta_\chi \bar{\theta}_\chi = q^{t-1}.$$

Proof: See [11]. ■

Using Theorem 2 for the case  $t = 2$ , we give infinite series of Hadamard matrices of GQ type.

**Theorem 3.** Let  $q + 1 = 2^s n$ ,  $s \geq 2$ ,  $n$  odd,  $\rho$  a primitive  $2^{s+1}$ th root of unity and  $w$  an arbitrary  $n$ th root of unity. Put  $\chi = \chi_{2^{s+1}} \chi_n$  where  $\chi_{2^{s+1}}(\eta) = \rho$ ,  $\chi_n(\eta) = w$ , so that  $\chi$  induces a quadratic character  $\psi$  in  $F$ .

Then for the relative Gauss sum  $\theta_\chi$  we have

$$\theta_\chi = \alpha + \beta \rho^n \quad \alpha, \beta \in \mathbf{Z}[\rho^2, w],$$

and the right regular representation matrix of

$$\gamma = \alpha \pm i + \beta j$$

gives an Hadamard matrix of GQ type of order  $2^s n$  where  $i$  is a primitive fourth root of unity.

Proof: See [11]. ■

**Corollary 1.** Let  $\alpha, \beta$  be as in Theorem 3. Then the right regular representation matrix of

$$\gamma = (\alpha - i + \beta \rho^n j)(1 - j) = (1 - j)(\theta_\chi + ij)$$

is an Hadamard matrix of GQ type of order  $2^{s+1} n$ . In particular, if  $s = 1$  then we get an Hadamard matrix of Turyn's type [9], [10].

Proof: See [11]. ■

**Theorem 4.** Let  $q + 1 = 2n$ ,  $n$  odd and  $\rho$  a primitive octic root of unity. Let  $\eta$  and  $w$ , be as in Theorem 3. Put  $\chi = \chi_8 \chi_n$ ,  $\chi_8(\eta) = \rho$ . So that  $\chi$  induces a biquadratic character in  $F$ .

The right regular representation matrix of

$$\tau = (\theta_\chi + \rho^t j)(1 + i)(1 + j), \quad t = 1, 3, 5, 7,$$

gives an Hadamard matrix of GQ type of order  $8n$ . We may change the order of factors  $\theta_\chi + \rho^t j$ ,  $1 + i$  and  $1 + j$  arbitrarily.

Proof: See [11]. ■

On the other hand, if there exists an Hadamard matrix of GQ type of order  $2^s n$ , we can double its order.

**Theorem 5.** Assume that the right regular representation matrix of  $\xi = \alpha + \beta j$  in  $\mathcal{R}$  is an Hadamard matrix of GQ type of order  $2^s n$ . Let  $\rho$  be a primitive  $2^{s+1}$ th root of unity. Then

$$\gamma = (\alpha + \beta j)(1 + \rho^t j) \text{ for } t = 1, 3, 5, \dots, 2^s - 1,$$

generates an Hadamard matrix of GQ type of order  $2^{s+1} n$ . We can exchange the order of two factors  $\alpha + \beta j$  and  $1 + \rho^t j$ .

Proof: See [11]. ■

## 5. Main theorems

**Theorem 6.** Let  $H$  be an  $M$ -structure Hadamard matrix of GQ type of order  $4n$ ,

$$H = \begin{pmatrix} C_0 & C_1 & D_0 & D_1 \\ -C_1 & C_0 & -D_1 & D_0 \\ -D_0^t & D_1^t & C_0^t & -C_1^t \\ -D_1^t & -D_0^t & C_1^t & C_0^t \end{pmatrix}.$$

Furthermore, let  $T_0, T_1, T_2$  and  $T_3$  be matrices of order  $m$  which have entries 0, 1 or  $-1$  which satisfy

- (i)  $T_i \wedge T_j, i \neq j$  ( $\wedge$  the Hadamard product);
- (ii)  $\sum_{i=0}^3 T_i$  is a matrix whose entries are  $\pm 1$  or  $-1$ ;
- (iii)  $\sum_{i=0}^3 T_i T_i^t = \sum_{i=0}^3 T_i^t T_i = m I_m$ ;
- (iv)  $T_0 T_1^t - T_1 T_0^t + T_2 T_3^t - T_3 T_2^t = T_0^t T_1 - T_1^t T_0 + T_2^t T_3 - T_3^t T_2 = 0,$   
 $T_0 T_2 - T_2 T_0 - T_1 T_3 + T_3 T_1 = T_0 T_3 - T_3 T_0 + T_1 T_2 - T_2 T_1 = 0.$

Then we have an  $M$ -structure Hadamard matrices of GQ type of  $4nm$ .

Proof: We define the matrices  $\alpha, \beta, \gamma$  and  $\delta$  as follow.

$$\begin{aligned} \alpha &= T_0 \times C_0 - T_1 \times C_1 - T_2 \times D_0^t - T_3 \times D_1^t, \\ \beta &= T_0 \times C_1 + T_1 \times C_0 + T_2 \times D_1^t - T_3 \times D_0^t, \\ \gamma &= T_0 \times D_0 - T_1 \times D_1 + T_2 \times C_0^t + T_3 \times C_1^t, \\ \delta &= T_0 \times D_1 + T_1 \times D_0 - T_2 \times C_1^t + T_3 \times C_0^t, \end{aligned}$$

It is easily verified that  $\alpha, \beta, \gamma$  and  $\delta$  satisfy the equation (4). Hence

$$\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & -\delta & \gamma \\ -\gamma^t & \delta^t & \alpha^t & -\beta^t \\ \delta^t & -\gamma^t & \beta^t & \alpha^t \end{pmatrix}$$

is an  $M$ -structure Hadamard matrix of GQ type of order  $4nm$ . ■

**Corollary 2.** Let  $q$  be a prime power and  $q + 1 = 2^s n$ ,  $n$  odd. Let  $m_1, \dots, m_r$ , be the orders of Williamson matrices or type 1 Williamson type matrices. Then

- (i) when  $s \geq 2$ , there exists an  $M$ -structure Hadamard matrix of GQ type of order  $2^r m_1 \dots m_r (q + 1) = 2^{r+s} r_1 \dots m_r n$ ;
- (ii) when  $s = 1$ , there exists an  $M$ -structure Hadamard matrix of GQ type of order  $2^{r+1} m_1 \dots m_r (q + 1) = 2^{r+2} m_1 \dots m_r n$ .

**Proof:** From Theorems 1 and 3 there exists an  $M$ -structure Hadamard matrix of GQ type of order  $2^s n$ . From Corollary 1 an  $M$ -structure of Hadamard matrix of GQ type of order  $4n$  exists. Let  $W_1, W_2, W_3$  and  $W_4$  be Williamson matrices of order  $m$  or type 1 Williamson-type matrices of order  $m$ . Put

$$X_1 = 2(W_1 + W_2), X_2 = \frac{1}{2}(W_1 - W_2), Y_1 = \frac{1}{2}(W_3 + W_4), Y_2 = \frac{1}{2}(W_3 - W_4).$$

Further, put

$$T_0 = \begin{pmatrix} X_1 & \\ & X_1 \end{pmatrix}, T_1 = \begin{pmatrix} X_2 & \\ & X_2 \end{pmatrix}, T_2 = \begin{pmatrix} & Y_1 \\ Y_1 & \end{pmatrix}, T_3 = \begin{pmatrix} & Y_2 \\ Y_2 & \end{pmatrix},$$

then  $T_0, T_1, T_2, T_3$  satisfy the conditions of Theorem 6. ■

**Theorem 7.** Let  $H$  be an  $M$ -structure Hadamard matrix of GQ type of order  $4n$ ,

$$H = \begin{pmatrix} C_0 & C_1 & D_0 & D_1 \\ -C_1 & C_0 & -D_1 & D_0 \\ -D_0^t & D_1^t & C_0^t & -C_1^t \\ -D_1^t & -D_0^t & C_1^t & C_0^t \end{pmatrix}.$$

Furthermore, let  $T_0$  and  $T_1$  be the matrices of order  $m$  which have entries 0, 1 or  $-1$  and satisfy

- (i)  $T_0 \wedge T_1 = 0$ , ( $\wedge$  the Hadamard product);
- (ii)  $T_0 + T_1$  is a matrix which has entries 1 or  $-1$ ;
- (iii)  $T_0 T_0^t + T_1 T_1^t = T_0^t T_0 + T_1^t T_1 = mI$ ;
- (iv)  $T_0 T_1^t - T_1 T_0^t = T_0^t T_1 - T_1^t T_0 = 0$ .

Then we have an  $M$ -structure Hadamard matrix of GQ type of order  $4nm$ .

**Proof:** We define the matrices  $\alpha, \beta, \gamma$  and  $\delta$  as follows.

$$\begin{aligned} \alpha &= T_0 \times C_0 - T_1 \times C_1, \\ \beta &= T_0 \times C_1 + T_1 \times C_0 \\ \gamma &= T_0 \times D_0 - T_1 \times D_1 \\ \delta &= T_0 \times D_1 + T_1 \times D_0 \end{aligned}$$

Then,  $\alpha, \beta, \gamma$  and  $\delta$  satisfy the equation (4). ■

**Corollary 3.** Let  $q$  be a prime power and  $q + 1 = 2^s n$ ,  $n$  odd. Let  $p_i$  be a prime power and  $p_i \equiv 1 \pmod{4}$  for  $1 \leq i \leq r$ . Then

- (i) when  $s \geq 2$  there exists an  $M$ -structure Hadamard matrix of GQ type of order  $(p_1 + 1)(p_2 + 1) \dots (p_r + 1)(q + 1)$ ;
- (ii) when  $s = 1$  there exists an  $M$ -structure Hadamard matrix of GQ type of order  $2(p_1 + 1)(p_2 + 1) \dots (q + 1)$ .

**Proof:** An Hadamard matrix of order  $2(p_i + 1)$  obtained from the Paley type 2 matrix has a form

$$\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}.$$

Then  $T_0 = \frac{1}{2}(X + Y)$ ,  $T_1 = \frac{1}{2}(X - Y)$  satisfy the conditions of Theorem 7. ■

**Corollary 4.** Let  $q$  be a prime power and  $q + 1 = 2^s n$ ,  $n$  odd. Let  $m_1, \dots, m_r$  be the orders of Williamson type (not necessarily circulant or type 1) matrices. Then

- (i) when  $s \geq 2$ , there exists an  $M$ -structure Hadamard matrix of GQ type of order  $2^r m_1 \dots m_r (q + 1) = 2^{r+s} r_1 \dots m_r n$ ;
- (ii) when  $s = 1$ , there exists an  $M$ -structure Hadamard matrix of GQ type of order  $2^{r+1} m_1 \dots m_r (q + 1) = 2^{r+2} m_1 \dots m_r n$ .

**Proof:** Let  $W_1, W_2, W_3$  and  $W_4$  be Williamson type matrices of order  $m$ . Then

$$\begin{pmatrix} W_1 & W_2 & W_3 & W_4 \\ -W_2 & W_1 & -W_4 & -W_3 \\ -W_3 & W_4 & W_1 & W_2 \\ W_4 & W_3 & -W_2 & W_1 \end{pmatrix}$$

is an Hadamard matrix which has an  $M$ -structure

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

where

$$X = \begin{pmatrix} W_1 & W_2 \\ -W_2 & W_1 \end{pmatrix}, \quad Y = \begin{pmatrix} W_3 & W_4 \\ -W_4 & W_3 \end{pmatrix}.$$

Then  $T_0 = \frac{1}{2}(X + Y)$ ,  $T_1 = \frac{1}{2}(X - Y)$  satisfy the conditions of Theorem 7. ■

**Corollary 5.** Let  $q$  be a prime power and  $q + 1 = 2^s n$ ,  $n$  odd. Suppose there exists a symmetric  $C$ -matrix of order  $p_i + 1$  and there exists a symmetric Hadamard matrix of order  $p_i - 1$  for  $1 \leq i \leq r$ . Then

- (i) when  $s \geq 2$  we have an  $M$ -structure Hadamard matrix of GQ type of order  $2^r p_1 \dots p_r (q + 1) = 2^{r+s} p_1 \dots p_r n$ ;
- (ii) when  $s = 1$  we have an  $M$ -structure Hadamard matrix of GQ type of order  $2^{r+1} p_1 \dots p_r (q + 1) = 2^{r+2} p_1 \dots p_r n$ .

**Proof:** If there exists a symmetric  $C$ -matrix of order  $p_i + 1$  and there exists a symmetric Hadamard matrix of  $p_i - 1$ , then there exists a symmetric Hadamard matrix of order  $4p_i$  having a form

$$\begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}.$$

for  $1 \leq i \leq l$ .  $T_0 = \frac{1}{2}(X + Y)$  and  $T_1 = \frac{1}{2}(X - Y)$  satisfy the conditions of Theorem 7. ■

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