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Hadamard matrices of order $8 \pmod{16}$ with maximal excess

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\[ o(4m(m - 1)) \leq 4(m - 1)^2(2m + 1). \]

Kharaghani in 'An infinite class of Hadamard matrices of maximal excess' (to appear) showed this maximal excess can be attained if \( m \) is the order of a skew-Hadamard matrix. We give another proof of Kharaghani's result, by generalizing an example of Farmakis and Kounias, 'The excess of Hadamard matrices and optimal designs', Discrete Math. 67 (1987) 165-176, and further show that the maximal excess of the bound is attained if \( m \equiv 2 \) (mod 4) is the order of a conference matrix.

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Hadamard matrices of order $\equiv 8 \pmod{16}$ with maximal excess

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Dedicated to Professor R.G. Stanton on the occasion of his 68th birthday.

Abstract


Kounias and Farmakis, in ‘On the excess of Hadamard matrices’, Discrete Math. 68 (1988) 59-69, showed that the maximal excess (or sum of the elements) of an Hadamard matrix of order $h$, $a(h)$ for $h = 4m(m - 1)$ is given by

$$a(4m(m - 1)) = 4(m - 1)^2(2m + 1).$$

Kharaghani in ‘An infinite class of Hadamard matrices of maximal excess’ (to appear) showed this maximal excess can be attained if $m$ is the order of a skew-Hadamard matrix.

We give another proof of Kharaghani’s result, by generalizing an example of Farmakis and Kounias, ‘The excess of Hadamard matrices and optimal designs’, Discrete Math. 67 (1987) 165-176, and further show that the maximal excess of the bound is attained if $m = 2 \pmod{4}$ is the order of a conference matrix.

1. Introduction

An Hadamard matrix, $H$, of order $h$ has elements $\pm 1$ and satisfies $HH^T = hI_h$. If $H = I + S$ where $S^T = -S$ then $H$ is called skew-Hadamard. The most recent results on skew-Hadamard matrices are given by Seberry [12].

A conference matrix, $C$, of order $n = 2 \pmod{4}$ has zero diagonal, other elements $\pm 1$, and satisfies $CC^T = (n - 1)I_n$. Conference matrices are known for

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the orders:
(i) \( q'^2 + 1, q = 1 \) (mod 4) is a prime power;
(ii) \((s-1)^2t + 1, s \) is the order of a skew-Hadamard matrix and \( t \) a positive integer;
(iii) \( q^2(q + 2) + 1, q = 4t - 1 \) a prime power, \( q + 3 \) the order of a conference matrix, Mathon [9];
(iv) \( 5.9^{2r+1} + 1, r \geq 0 \) a nonnegative integer, Seberry and Whitman [14];
(v) \((c-1)^t + 1, c \) the order of a conference matrix and \( t \) a nonnegative integer.
Conference matrices cannot exist unless \( n - 1 \) is the sum of two squares: thus they cannot exist for orders 22, 34, 58, 70, 78, 94. The first few undecided orders are 66, 86, 118, 146, 154 and 186.
Let \( e \) denote the \( 1 \times q \) matrix of ones, \( J \) the \( q \times q \) matrix of ones and \( I \) the identity matrix. The skew-Hadamard and conference matrices of order \( q + 1 \) can be denoted
\[
\begin{bmatrix}
1 \\
-e^T \\
I + Q 
\end{bmatrix}
\] and
\[
\begin{bmatrix}
e \\
e^T \\
X 
\end{bmatrix}
\]
where \( Q^T = -Q, X^T = X, QQ^T = qI - J, XX^T = qI - J, eQ = 0, \) and \( eX = 0. \) \( Q \) and \( X \) are called the core of the matrices from which they have been derived.
The excess of a Hadamard matrix, \( H, \) denoted \( \sigma(H) \) is the sum of all its elements. \( \sigma(n) \) is used to denote the maximal excess of all Hadamard matrices of order \( n. \) This concept has been studied by a number of authors (see [1–8, 10–11, 13, 15–16]) and some theoretical bounds of \( \sigma(n) \) obtained and Hadamard matrices meeting that bound constructed. In this paper we give a new family of Hadamard matrices whose excess is maximal.

2. The constructions
Let \( q = 1 \) (mod 4) and \( X^T = X, eX = 0, XX^T = qI - J, \) where \( X \) is the core of a conference matrix. Then consider the following matrix:

\[
\begin{array}{cccccccc}
-A \times 1 & -A \times 1 & A \times 1 & -A \times 1 & B \times e & B \times e & C \times e & C \times e \\
A \times 1 & -A \times 1 & A \times 1 & A \times 1 & C \times e & C \times e & -B \times e & -B \times e \\
-A \times 1 & A \times 1 & A \times 1 & A \times 1 & B \times e & -B \times e & C \times e & -C \times e \\
A \times 1 & A \times 1 & A \times 1 & A \times 1 & C \times e & -C \times e & -B \times e & B \times e \\
\end{array}
\]
Hadamard matrices of order $= 8 \pmod{16}$

(this is in fact an orthogonal design $OD(4(q+1); 4, 2q, 2q)$). We replace $A$ by $J_q$, $B$ by $I+X$ and $C$ by $I-X$ to get the required Hadamard matrix of order $4q(q+1)$ and excess $4q^2(2q+3)$.

Let $q = 3 \pmod{4}$, $Q^T = -Q$, $eQ = 0$, $QQ^T = qI - J$, where $Q$ is the core of a skew-Hadamard matrix. We generalize an example of Farmakis and Kounias [3] and consider the matrix:

$$
\begin{array}{cccc}
A \times 1 & -A \times 1 & -A \times 1 & -A \times 1 \\
A \times 1 & -A \times 1 & A \times 1 & A \times 1 \\
A \times 1 & A \times 1 & -A \times 1 & A \times 1 \\
A \times 1 & A \times 1 & A \times 1 & -A \times 1 \\
\end{array}
\begin{array}{cccc}
B \times e & B \times e & B \times e & B \times e \\
B \times e & B \times e & -B \times e & -B \times e \\
B \times e & -B \times e & B \times e & -B \times e \\
B \times e & -B \times e & -B \times e & B \times e \\
\end{array}
\begin{array}{cccc}
-A \times I + B \times Q & A \times I + B \times Q & A \times I - B \times Q & A \times I + B \times Q \\
-A \times I + B \times Q & A \times I - B \times Q & A \times I + B \times Q & A \times I - B \times Q \\
A \times I + B \times Q & -A \times I + B \times Q & A \times I + B \times Q & A \times I - B \times Q \\
A \times I - B \times Q & A \times I + B \times Q & -A \times I - B \times Q & A \times I + B \times Q \\
\end{array}
\begin{array}{cccc}
B \times e^T & B \times e^T & B \times e^T & B \times e^T \\
-B \times e^T & -B \times e^T & B \times e^T & B \times e^T \\
-B \times e^T & B \times e^T & B \times e^T & -B \times e^T \\
-B \times e^T & -B \times e^T & -B \times e^T & B \times e^T \\
\end{array}
\begin{array}{cccc}
-A \times I + B \times Q & A \times I + B \times Q & A \times I - B \times Q & A \times I + B \times Q \\
-A \times I + B \times Q & A \times I - B \times Q & A \times I + B \times Q & A \times I - B \times Q \\
A \times I + B \times Q & -A \times I + B \times Q & A \times I + B \times Q & A \times I - B \times Q \\
A \times I - B \times Q & A \times I + B \times Q & -A \times I - B \times Q & A \times I + B \times Q \\
\end{array}
\begin{array}{cccc}
A \times I + B \times Q & A \times I - B \times Q & A \times I + B \times Q & A \times I - B \times Q \\
A \times I - B \times Q & A \times I + B \times Q & -A \times I - B \times Q & A \times I + B \times Q \\
\end{array}
$$

(3. Numerical results)

It can be seen by inspection that the matrices constructed above have the row-sum vector

$$
(2qe_q(3q+4), 2(q+2)e_q),
$$

where $q$ is odd.

The first few values obtained are given in the table, the values for $h = 8$ and $h = 48$ were known ([15, 10]), the values for $h = 0 \pmod{16}$ arise from [6] but the remaining results are new.

Thus we have shown the following theorem.

**Theorem 1.** If there is a skew-Hadamard matrix of order $m = 0 \pmod{4}$ or a conference matrix of order $m = 2 \pmod{4}$ then there is a Hadamard matrix of order $4m(m-1)$ ($= 0 \pmod{16}$ and $= 8 \pmod{16}$ respectively) whose excess meets the Kounias—Farmakis bound, i.e.

$$
\sigma(4(m(m-1))) = 4(m-1)^2(2m+1).
$$
C. Koukouvinos, J. Seberry

\[
q \quad h = 4q(q + 1) \quad o(h) = 4q^2(2q + 3) \quad \text{Comment}
\]

\begin{array}{ccc}
1 & 8 & 20 & [15] \\
3 & 48 & 324 & [10] \\
5 & 120 & 1300 & \text{New} \\
7 & 224 & 3332 & [6] \\
9 & 360 & 6804 & \text{New} \\
11 & 528 & 12100 & [6] \\
13 & 728 & 19604 & \text{New} \\
17 & 1224 & 42772 & \text{New} \\
21 & & & \text{No conference matrix exists for order 22} \\
23 & 2208 & 103684 & [6]
\end{array}

References

[13] J. Seberry, SBIBD \((4k^2, 2k^2 + k, k^2 + k)\) and Hadamard matrices of order \( 4k^2 \) with maximal excess are equivalent, Graphs Combin. 5 (1989) 373–383.