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Abstract

Some permanents of circulant (0,1) matrices are computed. Three methods are used. First, the permanent of a Kronecker product is computed by directly counting diagonals. Secondly, Lagrange expansion is used to calculate a recurrence for a family of sparse circulants. Finally, a "complement expansion" method is used to calculate a recurrence for a permanent of a circulant with few zero entries. Also, a bound on the number of different permanents of circulant matrices with a given row sum is obtained.

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SOME REMARKS ON THE PERMANENTS OF
CIRCULANT (0,1) MATRICES

Peter Eades, Cheryl E. Praeger and Jennifer R. Seberry

ABSTRACT. Some permanents of circulant (0,1) matrices are computed. Three methods are used. First, the permanent of a Kronecker product is computed by directly counting diagonals. Secondly, Lagrange expansion is used to calculate a recurrence for a family of sparse circulants. Finally, a "complement expansion" method is used to calculate a recurrence for a permanent of a circulant with few zero entries.

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1. *Introduction.*

The permanent of an $n \times n$ matrix $A = (a_{ij})$, $0 \le i, j \le n-1$, is

$$per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)} \tag{1.1}$$

where S_n denotes the group of permutations on $\{0,1,\ldots,n-1\}$.

The problem of computing the permanent of a matrix has a long history (see Minc [1]). In particular, the permanent of a (0,1) matrix has received much attention because it counts various combinatorial structures - for instance, perfect matchings of bipartite graphs, cycle covers of graphs, and systems of distinct representatives. Recently Valiant [8] has shown that the problem of computing the permanent of a (0,1) matrix belongs to class of intractable counting problems.

Metropolis, Stein, and Stein [2] present a method of calculating permanents of circulant matrices with first row

$$(\underbrace{1,1,\ldots,1}_{k\text{ ones}},\underbrace{0,0,\ldots,0}_{(n-k)\text{ zeros}}). \tag{1.2}$$

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In this paper we present some methods for computing permanents of many (0,1) circulant matrices which are not of the form (1.2). Also, an equivalence relation on circulant (0,1) matrices is used to bound the number of distinct values that can be attained by circulant (0,1) matrices of given order and row sum.

2. *Equivalence and the Distribution of Permanents.*

Two notational conventions will be adopted. First, circulant $n \times n$ (0,1) matrices will be identified with subsets of the integers modulo n, that is, of $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$, so that the matrix $A = (a_{ij})$, $0 \le i, j \le n-1$, is identified with $S = \{i \mid a_{0i} = 1\}$. For example, the permanent of

$$\begin{bmatrix} 1&1&0&1&0&1&0 \\ 0&1&1&0&1&0&1 \\ 1&0&1&1&0&1&0 \\ 0&1&0&1&1&0&1 \\ 1&0&1&0&1&1&0 \\ 0&1&0&1&0&1&1 \\ 1&0&1&0&1&0&1 \end{bmatrix}$$

may be written as $per\ (0,1,3,5)$. (The order n is understood from the context.) We say that A is the *incidence matrix* of S and that S *realizes* A.

Secondly, arithmetic on indices of circulant matrices and on elements of $\{0,1,\ldots,n-1\}$ is taken modulo n.

Two $n \times n$ circulant matrices $A = (a_{ij})$ and $B = (b_{ij})$ are *equivalent* if there are integers $x, y \in \mathbb{Z}_n$ such that x is prime to n and, for each $j \in \mathbb{Z}_n$,

$$a_{0,xj+y} = b_{0,j}. \tag{2.1}$$

Note that two (0,1) circulants cannot be equivalent unless their row sums are the same.

Note that the definition of permanent (1.1) immediately implies that interchanging the rows and columns of a matrix does not alter its permanent. Now if (2.1) holds and C_x and T denote the permutation matrices which represent

$$i \to xj$$

and

$$j \to j+1,$$

respectively, then $C_x A C_x^t\, T^y = B$.

LEMMA 2.2. *If A and B are equivalent circulant matrices, then per* (A) = *per* (B). □

This lemma gives an upper bound on the number of distinct permanents. Suppose that Λ_n^k denotes the set of k-subsets *of* $\mathbb{Z}_n$. Note that the equivalence relation on circulant matrices defined by (2.1) induces an equivalence relation on Λ_n^k: two k-subsets S and T are equivalent if there are integers x and y such that $(x,n) = 1$ and $x\,S + y = T$. Let $f(n,k)$ denote the number of equivalence classes of Λ_n^k, and let $e(n,k)$ denote the number of distinct permanents of elements of Λ_n^k.

LEMMA 2.3. *Suppose that* ϕ *denotes the Euler* ϕ *function. Then*

$$e(n,k) \le f(n,k) \le \frac{1}{n} \sum_{d|(n,k)} \phi\,(d)\binom{n/d}{k/d}$$

Proof. Razen, Seberry, and Wehrhahn [5] consider the equivalence relation on Λ_n^k defined by shifts, that is, $S \simeq T$ if there is $y \in \mathbb{Z}_n$ such that $S + y = T$. They show that the number of equivalence classes under $\simeq$ is

$$\frac{1}{n} \sum_{d|(n,k)} \phi(d) \binom{n/d}{k/d}$$

Since $S \simeq T$ implies equivalence in the sense of (2.1), the inequality follows. □

The converse of Lemma 2.2 is false (for instance, *per* $(0,1,2,3)$ = *per* $(0,1,3,4)$ in order 7). Hence the inequality $e(n,k) \le f(n,k)$ is strict in some cases.

Lemma (2.2) does show, however, that in computing permanents of circulant matrices we usually need only consider matrices whose $(0,0)th$ and $(0,1)th$ entries are 1. For as long as S contains two elements whose difference is prime to the order, there is a set containing the elements 0 and 1 in the equivalence class containing S.

Remark. Burnside's Lemma may be used to compute $f(n,k)$ as follows. For each pair $a,b \in \mathbb{Z}_n$ with a prime to n let $\chi_{a,b}$ be the map $\Lambda_n^k \to \Lambda_n^k$ given by $\chi_{a,b}\,(S) = a\,S + b$; and let G be the set of all these $\chi_{a,b}$. Since G is a subgroup of the group of all permutations of Λ_n^k, Burnside's Lemma gives $f(n,k)$ in terms of the numbers of sets in Λ_n^k fixed setwise by each of the $\chi_{a,b}$. In the case where n is prime these numbers are easily computed and for $1<k<n$ we obtain:

$$f(n,k) = \frac{1}{n(n-1)}\left[\binom{n}{k} + n\sum_{\substack{d|(k,n-1)\\ d>1}} \phi(d)\binom{(n-1)/d}{k/d} + n\sum_{\substack{d|(k-1,n-1)\\ d>1}} \phi(d)\binom{(n-1)/d}{(k-1)/d}\right].$$

3. *Some Permanents.*

Valiant [8] has shown that, in complexity terms, the problem of computing the permanent of a (0,1) matrix is at least as difficult as counting the number of accepting computations of any nondeterministic polynomial-time Turing machine. Further, no polynomial time algorithm for computing permanents of circulant (0,1) matrices is known.

However, the circulant problem seems to be much easier than the general problem. In this section three methods are displayed by example. First, permanents of Kronecker products are investigated. Secondly, a direct Lagrange expansion is used to generate a recurrence for $per\ (0,1,4)$ in order n. Finally, inclusion-exclusion is used to calculate $per\ (1,3,4,\ldots,\ n-1)$.

Two Kronecker Products

LEMMA 3.1. *For each* (0,1) *matrix* A, $per\ (A \times I_m) = (per\ (A))^m$

Proof. By interchanging rows and columns we can transform $A\times I_m$ into $I_m\times A$, which is a direct sum of m copies of A. The result follows from the fact that the permanent of a direct sum is the product of the permanents of the summands [3]. □

Lemma 3.1 may be used to calculate $per\ (a,b)$, that is, the permanent of a circulant matrix with 2 nonzero entities in each row. For if

$$gcd\ (n,a-b) = g,$$

then $\{a,b\}$ is equivalent to $\{0,g\}$. Let T denote the $(n/g) \times (n/g)$ permutation matrix which represents the permutation $i \to i+1$; then $\{0,g\}$ realizes

$$(I_{(n/g)} + T)\ \ I_g .$$

But $per\ (I_{(n/g)} + T) = 2$; so $per\ (a,b) = 2^g$.

Similarly, *per* $(0,m,2m,\ldots,(m-1)\ell)$ in order ℓm is $(\ell!)^m$. This matrix achieves the upper bound in Minc's conjecture (see [3]).

The following theorem gives a formula for a less trivial Kronecker product.

THEOREM 3.2. *Suppose that* J *is the* $m\times m$ *matrix of ones and that* T *is the* $\ell\times\ell$ *permutation matrix which represents* $j \to j+1$. *Then*

$$\text{per}\ (\ (I+T)\times J) = (m!)^{\ell}\sum_{f=0}^{m}\binom{m}{f}^{\ell}.$$

Proof. A *nonzero diagonal* of an $n\times n$ matrix $A = (a_{ij})$ is a permutation matrix $P = (P_{ij})$ such that $P_{ij} \neq 0$ implies $a_{ij} \neq 0$. The permanent of a (0,1) matrix is its number of nonzero diagonals. To prove Theorem (3.2) we count the nonzero diagonals of $(I+T)\times J$.

Denote $(I+T)\times J$ by X. If blank denotes zero, then X has the form

$$\begin{bmatrix} J & J & & & \\ & J & J & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ J & & & & J \end{bmatrix}$$

so that each nonzero diagonal of X has the form

$$\begin{bmatrix} D_0 & E_0 & & & \\ & D_1 & E_1 & & \\ & & D_2 & E_2 & \\ & & & \ddots & \ddots \\ E_{\ell-1} & & & & D_{\ell-1} \end{bmatrix}$$

where D_i and E_i are $m\times m$ (0,1) matrices with at most one nonzero entry per row and column.

If D_0 has f nonzero entries, then since D is a permutation matrix, E_0 has $(m-f)$ nonzero entries; this forces D_1 to have f nonzero entries, and so on, so that each D_i has f nonzero entries and each E_i

has $(m-f)$ nonzero entries. Let $\mathcal{D}_f$ denote the set of diagonals of X for which D_0 has f nonzero entries. Then

$$per\ (X) = \sum_{f=0}^{m} |\mathcal{D}_f| .$$

We establish a one-one correspondence between $\mathcal{D}_f$ and the set A_f of ℓ-tuples

$$((A_0,S_0),\ (A_1,S_1),\dots,(A_{\ell-1}\ ,\ S_{\ell-1})) ,$$

where each A_i is an $m\times m$ permutation matrix and each S_i is an f-subset of $\{0,1,\dots,m-1\}$, that is, $S_i \in \Lambda_f^m$. Since $|A_f| = ((m!)\ \binom{m}{f})^{\ell}$, once this correspondence has been established, the theorem will follow.

If $((A_0,S_0),\ (A_1,S_1),\dots,\ (A_{\ell-1}\ ,\ S_{\ell-1})) \in A_f$, then the correspond-corresponding diagonal

$$\begin{bmatrix} D_0 & E_0 & & \\ & D_1 & E_1 & \\ & & \ddots & \ddots \\ E_{\ell-1} & & & D_{\ell-1} \end{bmatrix}$$

is constructed as follows. If $0 \le i \le \ell-1$, then

$$(E_i)_{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha \notin S_i \text{ and } (A_i)_{\alpha\beta} = 1, \\ 0 \text{ otherwise.} \end{cases}$$

Note that in the form of the permutation matrix D, each D_i is vertically adjacent to E_{i-1} and horizontally adjacent to E_{i-1}. Thus the nonzero rows of D_i correspond to the zero rows of E_i, and the non-zero columns of D_i correspond to the zero columns of E_{i-1}. To define D_i precisely, we first define U_i and V_i by

$$U_i = \{u : \text{the } u \text{ th column of } E_{i-1} \text{ is zero}\}$$

and

$$V_i = \{v : \text{for some } \alpha \in S_i\ ,\ (A_i)_{\alpha v} = 1\}$$

We order the elements of U as $u_1,u_2,\ \dots\ ,\ u_{f-1}$, so that $u_1 < u_2 < \dots < u_{f-1}$, and similarly for V.

Then

$$(D_i)_{\alpha\beta} = \begin{cases} 1 \text{ if } \alpha \in S_i \text{ and } \beta = u_j \text{ where } (A_i)\alpha v_j = 1 \\ 0 \text{ otherwise.} \end{cases}$$

We illustrate this correspondence in the following example. Suppose that $\ell = 3$ and $m = 4$, and consider the 3 - tuple

$$((A_0, S_0), (A_1, S_1), (A_2, S_2)),$$

where

$$A_0 = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \end{bmatrix},$$

and

$$S_0 = \{0,1\}, \quad S_1 = \{2,3\}, \quad S_2 = \{0,3\}.$$

First we compute E_0. The nonzero rows of E_0 correspond to the complement of $S_0 = \{0,1\}$ in $\{0,1,2,3\}$ that is, only rows 2 and 3 are nonzero. The positions of the nonzero entries in these rows correspond to the positions of the nonzero entries in the corresponding rows of A_0.

Hence

$$E_0 = \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \end{bmatrix}.$$

Similarly,

$$E_1 = \begin{bmatrix} 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}.$$

Next we compute D_0. The nonzero rows of D_0 correspond to $S_0 = \{0,1\}$; that is, only the first two rows are nonzero. The nonzero columns of D_0 correspond to the zero columns of E_2; that is, only the first two columns of D_0 are nonzero. Thus only the top left 2×2 submatrix of D_0 contains nonzero entries. This submatrix is, in fact, the same as the submatrix of A_0 found by deleting the rows of A_0 which correspond to the complement of S_0 and the columns which correspond to the nonzero entries in the deleted rows. Thus

$$D_0 = \begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}.$$

Similarly,

$$D_1 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&1&0&0 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix}$$

Thus the nonzero diagonal corresponding to

$((A_0, S_0), (A_1, S_1), (A_2, S_2))$ is

$$\left[\begin{array}{cccc|cccc|cccc}
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0
\end{array}\right]$$

It is tedious but not difficult to show that this correspondence is one-one and onto. □

As an example using Theorem 3.2. consider the $2\ell x 2\ell$ incidence matrix of $\{0,1,\ell,\ell+1\}$. By re-arranging rows and columns this matrix can be transformed to

$$\begin{bmatrix} J & J & & & \\ & J & J & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ J & & & & J \end{bmatrix}$$

where J is the 2×2 matrix of ones. Hence, using Theorem 3.2., $per\ (0,1,\ell,\ell+1) = 2^{\ell+1} + 2^{2\ell}$. Similarly in order 3ℓ

$$per\ (0,1,\ell,\ell+1,2\ell,2\ell+1)+2^{\ell+1}3^{\ell}(1+3^{\ell}).$$

Remark. It seems likely that a similar argument could be used to prove that

$$per\ ((I+T)\ x\ A) = (per\ A)^{\ell} \sum_{f=0}^{m} \binom{m}{f}^{\ell}$$

for each (0,1) matrix A.

A Recurrence from Lagrange Expansion.

The next theorem illustrates the direct use of a Lagrange expansion to obtain a recurrence relation.

THEOREM 3.3. *Let $Q_n = per\ (0,1,4)$ for order $n \geq 6$. Then for $n \geq 24$,*

$$\begin{aligned} Q_n = {} & Q_{n-1} + Q_{n-3} + 2Q_{n-4} - 4Q_{n-5} - 4Q_{n-6} \\ & - 2Q_{n-7} - 2Q_{n-8} + 6Q_{n-10} + 2Q_{n-11} + 2Q_{n-12} \\ & - 4Q_{n-13} - 4Q_{n-14} - Q_{n-15} - Q_{n-16} + Q_{n-17} + Q_{n-18}. \end{aligned}$$

For $6 \leq n \leq 24$, the value of Q_n is given in the following table.

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
Q_n	17	31	33	45	65	91	113	159	229	304	417	581	809	1105	1537	2137	2953	4097	5697

Proof. Let C_n, H_n, G_n, and K_n denote the following permanents. For $n > 4$

$$C_n = per\begin{bmatrix} 1&0&0&1 \\ 1&1&0&0&1 \\ &1&1&0&0&1 \\ &&&\ddots \\ &&&1&1&0&0&1 \\ &&&&1&1&0&0 \\ &&&&&1&1&0 \\ &&&&&&1&1 \end{bmatrix},$$

for $n > 7$

$$G_n = per\begin{bmatrix} 1&1 \\ 1&0&1 \\ &1&0&1 \\ &1&1&0&1 \\ &&1&0&0&1 \\ &&&1&0&0&1 \\ &&&1&1&0&0&1 \\ &&&&1&1&0&0&1 \\ &&&&&\ddots \\ &&&&&1&1&0&0&1 \\ &&&&&&&&1&1 \end{bmatrix},$$

and for $n > 6$

$$H_n = per\begin{bmatrix} 1&0&1 \\ 1&1&0&1 \\ &1&0&0&1 \\ &&1&0&0&1 \\ &&1&1&0&0&1 \\ &&&1&1&0&0&1 \\ &&&&\ddots \\ &&&&1&1&0&0&1 \\ &&&&&&1&1&0 \\ &&&&&&&1&1 \end{bmatrix}$$

and

$$K_n = per \begin{bmatrix} 1\;0\;1 & & & & \\ 1\;0\;0\;1 & & & & \\ \;\;1\;0\;0\;1 & & & & \\ \;\;1\;1\;0\;0\;1 & & & & \\ & \;\;1\;1\;0\;0\;1 & & & \\ & & \ddots & & \\ & & & 1\;1\;0\;0\;1 & \\ & & & \;\;1\;1\;0\;0 & \\ & & & \;\;\;\;1\;1\;0 & \end{bmatrix}$$

Note that except for the first and last few rows, the above matrices resemble the incidence matrix of {0,1,4}.

For small values of n the numbers C_n, G_n, H_n, and K_n are defined explicitly as follows.

$$C_1 = C_2 = C_3 = 1,\; C_4 = 2,$$

$$G_1 = G_3 = G_4 = 1,\; G_2 = G_7 = 2,\; G_5 = G_6 = 3,$$

$$H_1 = 0,\; H_2 = H_3 = H_4 = 1,\; H_5 = 2,\; H_6 = 3,$$

$$K_1 = K_3 = K_4 = 1,\; K_2 = 0,\; K_5 = K_6 = 2\,.$$

Repeated Lagrange expansion of these matrices gives for $n > 4$,

$$C_n = C_{n-1} + C_{n-4},$$

$$G_n = G_{n-3} + G_{n-4},$$

$$H_n = K_n + H_{n-4},$$

and for $n > 6$,

$$K_n = K_{n-2} + K_{n-3} + K_{n-4} - K_{n-6}\,.$$

Lagrange expansion of $per\ (0,1,4)$ gives

$$\begin{aligned} Q_n = {} & C_n + 4\,C_{n-4} + 3G_{n-5} + G_{n-9} \\ & + H_{n-2} + 2H_{n-5} + 3H_{n-6} \\ & + 2K_{n-1} + 2K_{n-4} - 2K_{n-5} + 2\,. \end{aligned}$$

Solving these equations gives the desired result. □

Remark. Theorem 3.3. illustrates a method that can be applied to calculate *per (S)* for sets $S \in \Lambda_n^k$ as long as the largest element d of S is small. The incidence matrix of S has the form

$$F_{A,B} = \begin{matrix} & \cdots\cdots \\ a_{d-1} & \cdots\cdots \\ a_{d-2}a_{d-1},\ldots. & \\ \vdots & \\ a_1\ a_2\ \ldots\ a_{d-1} & \\ a_0\ a_1\ \ldots\ a_{d-1} & \end{matrix}$$

where A and B are $d \times d$ (0,1) matrices.

For each $n > 2d$ let F_n denote the set of $n \times n$ matrices $F_{A,B}$ where A and B range over all $d \times d$ (0,1) matrices and $a_0, a_1, \ldots, a_{d-1}$ are fixed. Lagrange expansion about the first row of each element of F_n gives the permanents of the elements of F_n in terms of the permanents of elements of F_{n-1}. With appropriate initial conditions this recurrence can be solved to find *per (S)*.

It seems that the problem with this approach is that $|F_n| = 2^{2d^2}$ is large; however, in practice the number of elements of F_n needed to compute *per (S)* is much smaller than 2^{2d^2} and so the approach is feasible.

Further Remark. *Per* $(0,1,x)$ has been calculated for various values of x (see [2] and [4]). It would be a fruitful exercise to study the complexity of this subclass of permanents.

Recurrence from Complement Expansion.

The simplest form of the inclusion-exclusion principle can be used to obtain recurrence relations for circulant matrices with few nonzero entries. The principle for permanents can be stated as follows.

Let A denote an $n \times n$ (0,1) matrix with $(i,j)^{th}$ entry 1. Suppose that B is obtained from A by changing the $(i,j)^{th}$ entry to zero, and C is obtained by deleting the i^{th} row and the j^{th} column. Then we can "expand A about (i,j)" by putting $per\ (A) = per\ (B) + per\ (C)$.

This may be used to show that if D_n is the permanent of the $n \times n$ matrix $J-I$, then $D_n = (n-1)\ (D_{n-1} + D_{n-2})$. This is a well known recurrence for the derangement numbers [6]. Further, the Menage numbers [6] may be obtained as the permanents of the incidence matrices of $\{2,3,\ldots,n-1\}$ in this way. The following theorem extends the latter result.

THEOREM 3.4. *Suppose that n is even and R_n denotes per $(1,3,4,\ldots,n-1)$, and M_n denotes the n^{th} Menage number. Then $R_n = M_n + 2$.*

Proof. Let Z_n denote the $n \times n$ (0,1) matrix with zeros on and just above the main diagonal. Let ℓ denote $[\frac{1}{2}n]-1$. (Recall that matrices are indexed $0,1,\ldots,n-1$). Let X_n, Y_n, W_n, denote Z_n with the $(\ell,\ell)^{th}$, $(\ell,\ell+1)^{th}$, $(n-1,1)^{th}$ entries changed to one, one, zero respectively, and let U_n denote Y_n with both the $(\ell,1)^{th}$ and $(n-1,\ell+1)^{th}$ entries changed to zero. For example, for $n = 4$,

$$Z_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \qquad X_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

$$Y_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad W_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad U_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Note that $per\ (W_n)$ is the n^{th} Menage number and that the incidence matrix of $\{1,3,4,\ldots,n-1\}$ can be rearrranged to form U_n.

Expanding X_n, Y_n, and Z_n, about their $(\ell,\ell)^{th}$, $(\ell,\ell+1)^{th}$, and $(n-1,1)^{th}$ entries, respectively, we find that

$$per\ (X_n) = per\ (Z_n) + per\ (Y_{n-1}),$$
$$per\ (Y_n) = per\ (Z_n) + per\ (X_{n-1}),$$
$$per\ (Z_n) = per\ (W_n) + per\ (Z_{n-1}).$$

Expanding Y_n about the $(\ell,1)^{th}$ and $(n-1,\ell)^{th}$ entries gives

$$per\ (Y_n) = per\ (U_n) + 2\ per\ (Y_{n-1}) - per\ (Y_{n-2}).$$

Solving these equations gives

$$per\ (U_n) = per\ (W_n) + 2\ ,$$

which is the desired result. □

COROLLARY 3.5. *If* $gcd\ (n,x) = 2$ *then* $per\ (0,1,2,\ldots,x-1,x+1,\ldots,n-1) = M_n + 2.$ □

Remark. The value of M_n can be calculated easily. For instance, the following formula is given by Riordan [6].

$$M_n = \sum_{j=0} (-1)^j \frac{2n}{2n-j} \binom{2n-j}{j} (n-j)!$$

Further Remark. It seems plausible from numerical evidence that $per\ (0,1,2,\ldots,x-1,x+1,\ldots,n-1)$ could be evaluated for various values of $gcd(n,x)$ by similar methods.

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