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Abstract
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Matrices of this kind which have zero diagonal and other elements \( \pm 1 \) give rise to skew-Hadamard and n-type matrices; we show that the existence of a skew-Hadamard (n-type) matrix of order h implies the existence of skew-Hadamard (n-type) matrices of orders \((h - 1)^5 + 1\) and \((h - 1)^7 + 1\). Finally we show that, although there are matrices B with elements other than \( \pm 1 \) and 0, the equations force considerable restrictions on the elements of B.

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On Integer Matrices Obeying Certain Matrix Equations

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We discuss integer matrices $B$ of odd order $v$ which satisfy

$$B^T = \pm B, BB^T = vI - J, BJ = 0.$$ 

Matrices of this kind which have zero diagonal and other elements $\pm 1$ give rise to skew-Hadamard and $n$-type matrices; we show that the existence of a skew-Hadamard ($n$-type) matrix of order $h$ implies the existence of skew-Hadamard ($n$-type) matrices of orders $(h - 1)^2 + 1$ and $(h - 1)^2 + 1$.

Finally we show that, although there are matrices $B$ with elements other than $\pm 1$ and 0, the equations force considerable restrictions on the elements of $B$.

1. INTRODUCTION

Some interesting theorems have been discovered by H. J. Ryser [6–8], K. Majindar [5] and more recently by W. G. Bridges and H. J. Ryser [1] on integer matrices. Other specialized matrices have been studied and some of these results may be found in Marshall Hall Jr.’s book [4], G. Szekeres [9], J. M. Goethals and J. J. Seidel [2] and Jennifer Wallis [10,11].

In this paper I propose to look at some integer matrices satisfying very restrictive matrix equations.

2. PRELIMINARIES

An Hadamard matrix $H$, of order $h = 2$ or $\phi = 0 \pmod{4}$, has every element $\pm 1$ or $-1$ and satisfies $HH^T = hI_h$, where $I$ denotes the identity matrix. An Hadamard matrix $H = I + R$ of order $h$ will be called skew-Hadamard if $R$ has zero diagonal, $R^T = -R$ and $RR^T = (h - 1)I_h$. Skew-Hadamard matrices are discussed in [2], [3], [4], and [10].

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A matrix $N = I + P$ of order $n = 2 \pmod{4}$ is called $n$-type if it has every element $+1$ or $-1$, $P$ has zero diagonal, $P^T = P$ and $PP^T = (n - 1)I_n$. These matrices are discussed in [2] and [11].

K. Goldberg has proved in [3] that, if there is a skew-Hadamard matrix of order $h$ (Goldberg refers to "type 1" matrices—we use the more recent nomenclature "skew-Hadamard"), there is a skew-Hadamard matrix of order $(h-1)^3 + 1$. We examine the fifth and seventh powers.

$M \times N$ is the Kronecker product of $M$ and $N$, and we use $J$ to denote the matrix with every element $+1$.

The core of a skew-Hadamard of $n$-type matrix is found by altering the rows and columns of the matrix until it can be written in the form

$$\begin{bmatrix} 0 & e \\ \pm e^T & W \end{bmatrix} + I,$$

where $W$, the core, has zero diagonal and $+1$ or $-1$ elsewhere, and $e = (1, 1, \ldots, 1)$. If $W$ is of order $h$ then $WJ = 0$, $WW^T = hI - J$, $W^T = -W$ if $h \equiv 3 \pmod{4}$ and $W^T = W$ if $h \equiv 1 \pmod{4}$.

We will mean by the notation $\sum' A \times B \times C \times \cdots D$, where $\times$ is the Kronecker product, the sum obtained by circulating the letters formally; thus

$$\sum' A \times B \times C = A \times B \times C + B \times C \times A + C \times A \times B.$$

The sum $\sum'$ over $x$ letters has $x$ terms in the sum.

For convenience we restate some results from Jennifer Wallis [11] and K. Goldberg [3]:

**Lemma 2.1** [11, class II]. If $h$ is the order of a skew-Hadamard matrix there is a $n$-type matrix of order $(h-1)^3 + 1$.

**Lemma 2.2** [11, class II]. If $h$ is the order of an $n$-type matrix there is an $n$-type matrix of order $(h-1)^3 + 1$.

**Lemma 2.3** [11, class III]. If $h$ is the order of an $n$-type matrix there is an $n$-type matrix of order $(h-1)^3 + 1$.

**Lemma 2.4** [3]. If $h$ is the order of a skew-Hadamard there is a skew-Hadamard matrix of order $(h-1)^3 + 1$.

My thanks go to W. D. Wallis who suggested the line of proof of the next lemma and theorem:
DEFINITION. Let $A = [a_{ij}]$ and $C = [c_{ij}]$ be two matrices of order $n$. The Hadamard product $A \ast C$ if $A$ and $C$ is given by

$$A \ast C = [a_{ij}c_{ij}].$$

**Lemma 2.5.** When $A$, $B$, $C$, $D$ are matrices of the same order

$$(A \times B) \ast (C \times D) = (A \ast C) \times (B \ast D).$$

**Theorem 2.6.** Suppose $I$, $J$, and $W$ are of order $h$, where $W$ is a matrix with zero diagonal and $+1$ or $-1$ elsewhere. Suppose $A$ is a matrix of order $q = h^p$ which is of the form

$$A = B_1 + B_2 + \cdots + B_k$$

where each $B_i$ is a Kronecker product of $p$ terms $I$, $J$ or $W$ in some order, such that

(a) each $B_i$ contains at least one term $W$,
(b) for any two summands $B_i$ and $B_j$ there is a position $r$ such that one of the summands has $I$ in the $r$-th position and the other has $W$ in that position,

and suppose $A$ satisfies

$$AA^T = qI_q - J_q.$$

Then $A$ has zero diagonal and $+1$ or $-1$ in every other place.

**Proof.** Clearly

$$W \ast I = I \ast W = 0;$$

so, by part (b) of the hypothesis and Lemma 2.5,

$$B_i \ast B_j = B_j \ast B_i = 0$$

whenever $i \neq j$.

Hence no two $B_i$ have non-zero elements in the same position, so each non-zero element of $A$ comes from exactly one of the $B_i$; since each $B_i$ is a $(0, 1, -1)$ matrix it follows that $A$ is a $(0, 1, -1)$ matrix. But

$$AA^T = qI_q - J_q.$$

So, if $A = [a_{ij}]$,

$$\sum_j a_{ij}^2 = q - 1$$

for any $i$; therefore at most one element in each row of $A$ is zero. $W$ appears in each $B_i$, and $W$ has zero diagonal, so each $B_i$—and consequently $A$—has zero diagonal. So we have the result.
3. n-TYPE AND SKEW-HADAMARD MATRICES OF ORDER \((h - 1)^n + 1\)

WHERE \(n = 5\) AND 7

THEOREM 3.1. If \(h\) is the order of a skew-Hadamard (n-type) matrix then there is a skew-Hadamard (n-type) matrix of order \((h - 1)^n + 1\).

Proof. Let \(W\) be the core of the skew-Hadamard (n-type) matrix. Then \(WJ = 0\), \(WWT = (h - 1)I_{h-1} - J_{h-1}\) and \(WT = -W\) for \(h \equiv 0\) (mod 4) and \(WT = W\) for \(h \equiv 2\) (mod 4).

Now if \(I, W,\) and \(J\) are all of order \(h - 1\) write

\[
B = W \times W \times W \times W \times W + \sum' I \times J \times I \times J \times W
\]

We have

\[
B = W \times W \times W \times W \times W + \sum' I \times J \times W \times W \times W.
\]

Then, since \(WJ = 0\), \(B(J \times J \times J \times J \times J) = 0\), and if \(WT = -W\) then \(B^T = -B\) but if \(WT = W\) then \(B^T = B\).

It is easy to see that, if we multiply one summand of \(B\) by the transpose of a different summand, then either \(WJ\) or \(JW\) appears as one term of the Kronecker product expansion, so the product is zero; thus

\[
BB^T = WWT \times WWT \times WWT \times WWT \times WWT
\]

substituting for \(WWT\), and using the distributive law for + over ×, we find

\[
BB^T = (h - 1)^n I_{(h-1)^n} - J_{(h-1)^n}.
\]

B satisfies Theorem 2.6, so it has zero diagonal and 1 and -1 elsewhere. So B is the core of the required matrix of order \((h - 1)^n + 1\).

THEOREM 3.2. If \(h\) is the order of a skew-Hadamard (n-type) matrix then there is a skew-Hadamard (n-type) matrix of order \((h - 1)^2 + 1\).

Proof. Proceed as in the previous theorem but use instead

\[
B = W \times W \times W \times W \times W \times W
\]

\[
\sum' I \times J \times [W \times W + I \times J] \times [W \times W + I \times J] \times W.
\]

COROLLARY 3.3. If \(h\) is the order of a skew-Hadamard matrix then there is a skew-Hadamard matrix of order \((h - 1)^3 + 1\) where \(c = 3^35^k7^e\), \(j, k, e\) are non-negative integers.

This follows from Theorems 2.4, 3.1, and 3.2.
Corollary 3.4. If $h$ is the order of a skew-Hadamard matrix then there is an $n$-type matrix of order $(h - 1)^n + 1$ where $c = 2^{i}3^{j}5^{k}7^{e}$, $i$ a positive integer, $j$, $k$, $e$ non-negative integers.

This follows from Theorems 2.1, 2.2, 2.3, 3.1, and 3.2.

So we see, writing $W^n$ to mean $W \times W \times \cdots \times W$ the Kronecker product of $n$ $W$'s, that the formulae for the third, fifth, and seventh powers may be written as

$$W^n + \sum IJ[W^z + IJ]^{(n-3)/2} W,$$

where juxtaposition denotes Kronecker product. This formula breaks down for 9 and 11.

4. The Nature of Matrices Satisfying the Equation for the Core of Skew-Hadamard Matrices

In the preceding sections we have seen that solutions to the equations

$$AA^T = vI - J, \quad AJ = 0 = JA, \quad A^T = -A \quad (1)$$

exist for an infinitude of matrices $A$ of order $v$. In these cases it is clear that $A$ is a $(0, 1, -1)$ matrix with zero diagonal and 1 or $-1$ elsewhere. We now show that it is possible to have a matrix $A$ satisfying (1) but which is not a $(0, 1, -1)$ matrix. Consider

$$A = \begin{bmatrix} 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 \end{bmatrix};$$

then $AA^T = 7I - J$, $AJ = 0 = JA$ and $A^T = -A$.

However there are considerable restrictions on the elements of $A$ as Theorem 2.6 and the following theorem show:

Theorem 4.1. Let $A$ be an integer matrix of order $v$, where $v \equiv 3 \pmod{4}$, satisfying

$$AA^T = vI - J, \quad AJ = 0 = JA, \quad A^T = -A.$$
ON INTEGER MATRICES OBEYING CERTAIN MATRIX EQUATIONS

Suppose $-a$ is the least element of $A$; define

$$B = A + a(J - I);$$

write $e$ for the g.c.d. of non-zero elements of $B$. Then either $e = 1$ or $A$ has zero diagonal and $+1$ or $-1$ elsewhere.

Proof. $A^T = -A$, so $A$ has zero diagonal, and hence $B$ has zero diagonal. $B$ satisfies

$$BB^T = (v + a^2)I + [a^2(v - 2) - 1]J,$$
$$BJ = a(v - 1)J.$$

Consider the matrix $C = (1/e)B$. $C$ is an integer matrix and

$$CC^T = \frac{v + a^2}{e^2}I + \frac{a^2(v - 2) - 1}{e^2}J,$$
$$CJ = \frac{a(v - 1)}{e}J.$$

The element $-a$ occurs in $A$; and since $A^T = -A$ the element $a$ must occur in $A$ and must be the greatest element of $A$. Then $2a$ is the greatest element of $B$ and $2a/e$, which is therefore an integer, the greatest element of $C$. $e | 2a$, so there are two possibilities:

(i) $e$ is odd, $e | a$;
(ii) $e$ is even.

If (i) is true, since

$$\frac{a^2(v - 2) - 1}{e^2}$$

is an integer we have $e^2 | 1$ and so $e = 1$.

Now consider (ii). $v \equiv 3 \pmod{4}$, so since $e^2 | v + a^2$ and $e$ is even we must have $a$ odd. Since $e$ is even, every element of $B$ is even; all the non-diagonal elements of $A$ differ from elements of $B$ by $a$, an odd number, so they are odd.

Now $AA^T = vI - J$, so (evaluating the $i$-th diagonal element)

$$\sum_{i=1}^n a_{ii} = v - 1.$$
Each $a_{ij}$ is an odd integer, unless $j = i$; so $a_{ij}^2 \geq 1$, and

$$\sum_{j=1}^{v} a_{ij}^2 \geq v - 1,$$

with equality only when each $a_{ij}$ (except $a_{ii}$) is $\pm 1$. This holds for every $i$, so we have the result.

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