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A new class of fully nonlinear curvature flows

James A. McCoy
University of Wollongong, jamesm@uow.edu.au

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Abstract

This article is about contraction by fully nonlinear curvature flows of convex hypersurfaces. As with previously considered flows, including the quasilinear mean curvature flow and a broad class of fully nonlinear curvature flows, solutions exist for a finite time and contract to a point. As with some other flows, including the mean curvature flow, a smaller class of fully nonlinear flows and the Gauss curvature flow for surfaces, under a suitable rescaling the solutions converge exponentially to spheres. We provide an introduction to the study of curvature flow of hypersurfaces with extensive references to most of the main areas of study and applications to other areas of mathematics and beyond. The main points of interest in this particular work are the allowance of nonsmooth initial data and its consequences and that the only second derivative requirement on the speed is weaker than a requirement of convexity or concavity. We outline new results in both cases of smooth and nonsmooth initial data. These results are joint work with Ben Andrews and Zheng Yu.

Keywords

class, curvature, flows, fully, nonlinear

Disciplines

Physical Sciences and Mathematics

A NEW CLASS OF FULLY NONLINEAR CURVATURE FLOWS

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1. Background on curvature flows and their applications

We are interested in the changing shape of smooth hypersurfaces without boundary, where points $X \in \mathbb{R}^{n+1}$ on the hypersurface move in their normal directions with speeds dependent on the curvature of the hypersurface at point X. By *hypersurfaces*, we mean n-dimensional surfaces in $\mathbb{R}^{n+1}$; these are also referred to as simply *surfaces* when $n = 2$. It is also possible to consider n-dimensional surfaces evolving in other ambient spaces ([1], [8], [10], [35], [54], [77], [91], [94, 95], [105], [151], [157]), or with higher codimension ([6], [150], [164], [165], [172]); such considerations introduce additional complications into

the analysis with fewer techniques and results available. We will not pursue such situations in this article.

In the case of curves in the plane, that is $n = 1$, the analysis of curvature flows also has a somewhat different character. Techniques are available which are unique to this situation, and curves have only one curvature so a speed dependent only on curvature is a function of one variable. Evolving curves have been extensively studied (see, for example, [2], [3], [14, 15, 19, 20], [24], [42], [68, 69], [70], [82], [116], [122], [130], [137], [140, 141], [163], [167, 168]). One remarkable result for plane curves [82] is that if a condition of convexity on the initial closed curve is relaxed to a requirement only of embeddedness (ie no self-intersections), then the resulting curve 'unwraps' under the 'curve shortening flow' (where the flow speed is equal to the curvature), becomes convex, then earlier results gives that it shrinks to a point [70], becoming increasingly round as it does so [69]. The unwrapping result does not generalise for $n \geq 2$, but the result for convex initial data does, and in different ways (e.g. [9, 16, 17, 21, 22], [43, 44], [92]. The roundness result does not always hold for $n \geq 2$, it depends on the particular flow or class of flows; it has been shown that under a certain flow the extinction point is asymptotically an ellipse [18] rather than a sphere.

With $n \geq 2$, flows of hypersurfaces by their curvature find wide ranging modelling applications outside mathematics: tumbling stones on the beach ([16], [67], [108], [156]), image analysis ([5], [46], [90], [120], [121], [130], [139], [140, 141]), crystal growth ([41], [78]), general relativity ([99, 100, 101]), annealing of metals [131], flame propagation [149] and motion of phase boundaries ([27], [62], [80], [83]). Furthermore, such flows have fruitful applications in other areas of mathematics, including convex geometry ([13], [93], [126, 128]), affine geometry ([12, 15]), and topology ([93, 98], [102, 103], [155], [169, 170, 171]).

In this article we will be concerned with second order parabolic curvature flows. A smaller collection of literature exists concerning higher order flows, particularly fourth order parabolic-type flows. The most common of these are the surface diffusion flow ([58], [59], [60, 61], [72, 73], [110], [123], [131], [124, 125]), with applications in physics and the Willmore flow [113, 114, 115], with applications in topology. Others are associated with affine geometry ([15], [51], [141]). An important tool in the analysis of second order flows, the maximum principle, is generally not available in higher order settings so the analysis relies more heavily on integral estimates. We shall not discuss higher order flows further in this article. Recently a hyperbolic mean curvature flow has been introduced ([89]), replacing the first time derivative of the position vector by the second time derivative. We will not discuss such flows either.

In this article we are only concerned with compact, convex initial hypersurfaces without boundary; when the convexity condition is dropped, curvature flows may develop singularities before shrinking to a point. Classifying

singularities and extending flows beyond singularities is presently a very active area of research ([65], [71], [97, 98], [102, 103, 104], [166], [170, 171]) and recently a process of 'surgery' for mean curvature flow has been established ([104]) in one of several parallels between mean curvature flow and the Ricci flow of metrics. Surgery for Ricci flow received much fame recently with Perelman's proof of the Poincaré and Geometrization Conjectures [135, 136]; surgery for mean curvature flow is more closely modelled on the process initiated 20 years earlier by Hamilton [84] in his pioneering work on the Ricci flow.

Other settings in which curvature flows have been investigated include axially symmetric surfaces ([28], [29], [50], [109]), entire graphs ([53], [55, 56]) and radial graphs ([132], [157], [159]). Hypersurfaces with boundary have also been considered ([4], [49, 50], [54], [81], [96], [133, 134], [144], [145]). In each setting one may also explore the possibility of 'self-similar' or homothetic solutions such as those which evolve purely by translation or by scaling ([4], [7], [26], [41], [42], [48], [87], [109], [111], [120], [122], [142], [144], [150], [152], [161, 162, 160, 163], [173]). Such solutions are often closely related to singular behaviour under suitable rescaling. A particularly interesting self-similar solution of mean curvature flow is the 'shrinking doughnut' [26]. We will not pursue self-similar solutions here, except to remark where self-similar subsolutions are used in constructing appropriate barriers for use in the comparison principle for second order parabolic equations.

In all settings described above, much work has additionally been done on numerically modelling the shape of evolving hypersurfaces.

2.   Introduction and notation

Let M_0 be a compact, convex, n-dimensional hypersurface without boundary, $n \geq 2$, embedded in $\mathbb{R}^{n+1}$. We represent M_0 by the embedding $X_0 : \mathbb{S}^n \to \mathbb{R}^{n+1}$.

Writing now $X : \mathbb{S}^n \times [0, T)$, we consider curvature contraction flows of the form

$$(1) \qquad \frac{\partial X}{\partial t}(p, t) = -F\left(\mathcal{W}(p, t)\right) \nu(p, t), \, p \in \mathbb{S}^n, \, t \in (0, T)$$

with initial data

$$X(p, 0) = X_0(p) \text{ for } p \in \mathbb{S}^n.$$

In this setting $X(p, t)$ is the position vector of a point on the evolving hypersurface $M_t = X(\mathbb{S}^n, t)$ and $\nu(p, t)$ is the outward unit normal to the hypersurface at this point. When the speed of evolution F is positive, points $X(p, t)$ move under the flow (1) inwards with speed F and the evolving

hypersurface contracts. Curvature expansion flows have also been considered ([74], [158, 159]) but we will not discuss such flows in this article. Other settings such as level sets and formulations in terms of weak or viscosity solutions have also been considered ([6], [32], [33], [38, 39], [40], [57], [63, 64, 65, 66], [79], [81], [99, 100], [106, 107], [108], [121], [148], [129], [154]), but we will not adopt either of these approaches here. Other speed modifications which have also been considered include the addition of a global factor $h(t)$ to the speed ([18], [35], [36], [93], [117], [118], [126, 127, 128]) and the multiplication of the speed by a factor independent of curvature but dependent on the normal vector ([18], [30], [31], [32], [46], [47], [57], [78], [79], [130], [154], [173]), so-called 'anisotropic' curvature flows.

Under quite general conditions in our situation, the evolving hypersurface M_t remains convex, so we may parametrise over $\mathbb{S}^n$. There are various ways to do this, for example via the Gauss map ([11], [25], [88], [156], [158]) or as a radial graph ([127, 128], [157], [159]).

The speed F of the evolution depends upon the curvature of the hypersurface M_t. The *principal curvatures* κ_i, $1 \le i \le n$ of M_t at point $X(p,t)$ are the eigenvalues of the *Weingarten map* $\mathcal{W}$ of M_t at the point $X(p,t)$. The matrix of the Weingarten map may be written down in terms of first and second spatial derivatives of X, while the normal ν may be found in terms of first spatial derivatives only. Hence (1) is system of $(n+1)$ second order partial differential equations.

Specifically, tangent vectors to M_t are given by $\overline{\nabla}_i X$ for each $i = 1, \dots, n$ and the metric on M_t by

$$g_{ij}(p,t) = \left\langle \overline{\nabla}_i X(p,t), \overline{\nabla}_j X(p,t) \right\rangle.$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product of vectors in $\mathbb{R}^{n+1}$ and $\overline{\nabla}$ is the derivative on $\mathbb{S}^n$. Components of the matrix of the *second fundamental form* of M_t are

$$h_{ij}(p,t) = -\left\langle \nu(p,t), \overline{\nabla}_i \overline{\nabla}_j X(p,t) \right\rangle.$$

The components of the *inverse metric*, denoted g^{ij}, are obtained as the entries of the matrix $(g_{ij})^{-1}$. The matrix of the Weingarten map is then given by the matrix product $(g^{ij})(h_{ij})$. For details of hypersurface geometry, we refer the reader to [138] or [153].

Some examples of possible speeds F are the *mean curvature*, given by

$$H = \text{trace}(\mathcal{W}) = \kappa_1 + \kappa_2 + \dots + \kappa_n$$

and the *Gauss curvature*, given by

$$K = \det(\mathcal{W}) = \kappa_1 \kappa_2 \dots \kappa_n.$$

A smooth hypersurface with each κ_i everywhere positive is called *strictly convex*, while if instead it is only true that each $\kappa_i \ge 0$ everywhere, the

hypersurface is called *weakly convex*. Later we will also use the notation $\kappa_{\min}$ and $\kappa_{\max}$ to denote respectively the smallest and the largest principal curvature.

Some known results for curvature flows of the form (1) with smooth strictly convex initial hypersurfaces M_0 are:

- Hypersurfaces shrink to round points in a finite time T under flows with

  - $F = H$ [92],
  - $F = K^{\frac{1}{n}}$, $F = R^{\frac{1}{2}}$ [43, 44],
  - a class of fully nonlinear F [9], [21], [22],
  - $F = K$, $n = 2$ (Firey's conjecture) [16],
  - Various other fully nonlinear speeds when $n = 2$ [143].

- A larger class of flows shrink hypersurfaces to points, but roundness is unknown, untrue or requires extra conditions on the initial hypersurface M_0:

  - $F = K$ [16], [156],
  - $F = K^\alpha$, $\alpha > 0$ [43],
  - $F = H^k$, $k > 0$ [146, 147],
  - a large class of fully nonlinear flows [88].

By a 'round point' we mean that if the evolving hypersurfaces M_t are rescaled to keep some geometric property such as the surface area of M_t or the enclosed volume fixed under the flow, that is, set

(2) $$\tilde{X}(p, t) = \psi(t) X(p, t)$$

and adopt a rescaled time parameter $\tau = -\frac{1}{2} \log\left(1 - \frac{t}{T}\right)$, it is possible to show smooth exponential convergence of the rescaled solution hypersurface $\tilde{M}_\tau$ to the sphere as $\tau \to \infty$. This requires additional estimates which generally hold for a smaller class of curvature flow.

Another interesting class of flows of convex hypersurfaces arises by adding a positive 'global term' $h(t)$ into (1) such that some geometric property of the evolving hypersurface, such as its volume or surface area, is preserved under the flow. The position vector then evolves according to

$$\frac{\partial X}{\partial t}(p, t) = \{h(t) - F(\mathcal{W}(p, t))\} \nu(p, t), \, p \in \mathbb{S}^n, \, t \in (0, T).$$

Corresponding hypersurfaces converge to spheres in infinite time under the volume preserving mean curvature flow [93], the surface area preserving mean curvature flow [126], the mixed volume preserving mean curvature flow [127],

a class of constrained fully nonlinear flows [128] and a different class of fully nonlinear volume preserving curvature flows [36]. In these examples, $h(t)$ is some kind of average of F over the evolving hypersurface. One consequence of this is that some points move inwards whilst others move outwards under the flow. The first two flows above provide an approach to the isoperimetric problem (recently an alternative approach for smooth compact hypersurfaces has been described in [148]). The third example above puts the first two into a class of mean curvature flows with the same behaviour, while the fully nonlinear speeds used in the fourth example facilitated a new proof of the Minkowski inequalities of convex geometry [34]. Some constrained curvature flows have also been considered in other settings ([18], [28], [29], [90], [137]). Recently more general functions $h(t)$ have been investigated ([118], [117]). There it was shown that if h is small enough, hypersurfaces shrink to a point, while if h is large surfaces will expand indefinitely. The examples above fall into the intermediate case where solutions converge to a sphere. More general h should find applications in modelling in the future. More general terms have also been incorporated into the flow speed in different settings ([1], [24], [38, 39], [79], [80], [90]).

Many speeds $F(\mathcal{W})$ are defined in terms of the elementary symmetric functions of the principal curvatures.

Definition 2.1. *For any $k = 1, \ldots, n$, the k-th elementary symmetric function of the principal curvatures is given by*

$$E_k = \sum_{1 \le i_1 < \ldots < i_k \le n} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_k}.$$

Some examples of F written in terms of the elementary symmetric functions are the mean curvature $E_1 = H = \text{trace}(\mathcal{W})$, the scalar curvature $E_2 = R$ and the Gauss curvature $E_n = K = \det(\mathcal{W})$. Some other possible flow speeds F are

- $F = E_k^{1/k}$ for any $k = 1, \ldots n$,

- The *power means* $F = H_r = \left(\sum_{i=1}^n \kappa_i^r\right)^{\frac{1}{r}}$ for any $r \ne 0$,

- $F = \frac{E_{k+1}}{E_k}$, for any $k = 0, 1, \ldots, n-1$.

- $F = \left(\frac{E_k}{E_l}\right)^{\frac{1}{k-l}}$, $n \ge k > l \ge 0$.

- 'Some' positive linear combinations and weighted means of the above. Which combinations are allowable depends upon any second spatial derivative requirements on F needed for existence of a solution to the flow equation. Roughly speaking there are fewer such requirements when $n = 2$ or when the initial hypersurface is already 'close enough' to a sphere.

The flow speeds F in the above list are all homogeneous of degree one in the principal curvatures. This is an important property of the most well studied class of F. The homogeneity of degree one is particularly useful in obtaining various estimates to use with the maximum principle applied to the evolution equations of various geometric quantities associated to M_t evolving by (1).

Here, as part of joint work with Ben Andrews and Zheng Yu [25] we mainly consider speeds of the particular form

$$(3) \qquad F(\mathcal{W}) = n^{\beta - \delta} \left(\frac{E_n}{E_{n-1}} \right)^{\beta} G(\mathcal{W})^{\gamma} \left(\frac{E_1}{E_0} \right)^{\delta},$$

where $\beta, \delta > 0$, $\gamma \geq 0$ are constants with $\beta + \gamma + \delta = \alpha$ and the function $G(\mathcal{W})$ satisfies some natural conditions. The class with $F \equiv G$ is closed under taking weighted geometric means, hence F as in (3) is allowable. The $\beta, \delta > 0$ are used for curvature pinching estimates and roundness in case of strictly convex M_0 and for a lower displacement bound and curvature bounds in the case of nonsmooth weakly convex M_0.

The precise conditions on G we require are

(i) $G(\mathcal{W}) = g(\kappa(\mathcal{W}))$, where g is a smooth, symmetric function of the eigenvalues $\kappa(\mathcal{W})$ of $\mathcal{W}$.

(ii) g is defined on the positive cone

$$\Gamma = \{\kappa = (\kappa_1, \ldots, \kappa_n) : \kappa_i > 0 \text{ for all } i\}.$$

(iii) g is strictly increasing in each argument: $\frac{\partial g}{\partial \kappa_i} > 0$ everywhere in Γ.

(iv) g is positive and normalised such that $g(1, \ldots, 1) = 1$.

(v) g is homogeneous of degree one: $g(k\kappa) = k f(\kappa)$ for any $k > 0$ and all $\kappa \in \Gamma$.

(vi) g is *inverse-concave*, that is, the function $\tilde{g}$ is concave, where

$$\tilde{g}(x_1, \ldots, x_n) = -g\left(x_1^{-1}, \ldots, x_n^{-1}\right).$$

Since g is symmetric in its arguments, it suffices to denote the n principal curvatures simply by $\kappa(\mathcal{W})$. Condition (vi) is a weaker condition than convexity of g, Indeed, one can show that if g is convex, then g is also inverse-concave [21], however, other g may be inverse-concave and concave, convex or neither. Condition (iii) above, and more generally a condition that first partial derivatives of f are strictly positive, ensures that (1) is a parabolic equation. Short time existence of a solution to (1) is then well known (see, for example, [76] or [80]). One fixes a particular tangential diffeomorphism

by adding tangential terms to equation (1), thereby removing the degeneracy in the equation.

In view of (3), F is a symmetric function of the principal curvatures and is also written as $f(\kappa_1, \ldots, \kappa_n)$. That f is homogeneous of degree α means

$$f(k\kappa_1, \ldots, k\kappa_n) = k^\alpha f(\kappa_1, \ldots, \kappa_n)$$

for any $k > 0$ and the famous *Euler homogeneous function theorem* gives that

$$\sum_{i=1}^n \frac{\partial f}{\partial \kappa_i} \kappa_i = \alpha f.$$

In the special case $\alpha = 1$, it is easy to check that the first partial derivatives of f are homogeneous of degree zero, a property which implies that the elliptic operator associated to f,

$$\mathcal{L} = \dot{F}^{kl}(\mathcal{W}) \overline{\nabla}_k \overline{\nabla}_l,$$

which appears in all evolution equations derived from (1), is in fact 'uniformly elliptic', provided we have a 'curvature pinching' estimate (see, for example, [9]). Such estimates for our class of flows will be discussed below. In the special case of mean curvature flow, uniform ellipticity is immediate since the matrix $\left(\dot{F}^{kl}(\mathcal{W}) \right)$ is precisely the identity matrix, independent of the curvature, and the operator $\mathcal{L}$ is just the Laplace-Beltrami operator on M_t. The divergence structure of the equations in this case further assists in the analysis in the setting of convex hypersurfaces ([92], [127], [128]). An induction argument can be used to obtain curvature derivative estimates, while in more general cases, regularity results for fully nonlinear parabolic PDE are required. Also significantly in the case of the mean curvature flow, Huisken established a beautiful monotonicity formula using the unique structure of the mean curvature [97]. This formula assists in the analysis of singularities in more general settings than that of convex hypersurfaces. We will not pursue classification of singularities in this article, except to describe some cases where our solutions are asymptotically spherical. As mentioned in the introduction, the analysis of curvature flow singularities is a very active area of research.

We denote by $\left(\dot{F}^{kl} \right)$ the matrix of first partial derivatives of F with respect to the components of its argument:

$$\left. \frac{\partial}{\partial s} F(A + sB) \right|_{s=0} = \dot{F}^{kl}(A) B_{kl}.$$

Similarly for the second partial derivatives of F we write

$$\left. \frac{\partial^2}{\partial s^2} F(A + sB) \right|_{s=0} = \ddot{F}^{kl,rs}(A) B_{kl} B_{rs}.$$

Unless otherwise indicated, in this paper we always sum over repeated indices. Also, throughout this paper we will always evaluate partial derivatives of F at $\mathcal{W}$ and partial derivatives of f at $\kappa\left(\mathcal{W}\right)$. At a particular point of M_t, the derivatives of F are related to the derivatives of f. In particular, if we choose coordinates at the point such that the Weingarten map is a diagonal matrix, then the matrix $\left(\dot{F}^{kl}\left(\mathcal{W}\right)\right)$ is diagonal and equal to $\left(\frac{\partial f}{\partial \kappa_k}\left(\kappa\left(\mathcal{W}\right)\right)\delta_{kl}\right)$, while the second derivatives are also related. We state this relationship more generally:

Theorem 2.1. *Let f be a C^2 symmetric function defined on a symmetric region $\Omega \subset \mathbb{R}^n$. Let $\overline{\Omega} = \{A \in Sym\left(n\right) : \kappa\left(A\right) \in \Omega\}$ and define $F : \overline{\Omega} \to \mathbb{R}$ by $F\left(A\right) = f\left(\lambda\left(A\right)\right)$. Then at any diagonal $A \in \overline{\Omega}$ with distinct eigenvalues, the second derivative of F in direction $B \in Sym\left(n\right)$ satisfies*

$$\ddot{F}^{kl,rs} B_{kl} B_{rs} = \frac{\partial^2 f}{\partial \kappa_k \partial \kappa_l} B_{kk} B_{ll} + 2 \sum_{k<l} \frac{\frac{\partial g}{\partial \kappa_k} - \frac{\partial g}{\partial \kappa_l}}{\kappa_k - \kappa_l} B_{kl}^2.$$

In the above, $Sym\left(n\right)$ denotes the set of $n \times n$ symmetric matrices, with eigenvalues $\kappa\left(A\right)$. For proofs of this theorem we refer the reader to [21] or [75]. In practice, one may assume the eigenvalues κ_i are distinct; otherwise approximations can be used.

Several function spaces and associated norms on $\mathbb{S}^n$ and on $\mathbb{S}^n \times [0, T)$ are needed in the setting of convex hypersurfaces. In the analysis of regularity of curvature flows, these function spaces are as used, for example, in [128] and [158, 159]. For $k \in \mathbb{N}$, $C^k\left(\mathbb{S}^n\right)$ is the Banach space of real valued functions on $\mathbb{S}^n$ which are k-times continuously differentiable, equipped with the norm

$$\|u\|_{C^k(\mathbb{S}^n)} = \sum_{|\beta| \le k} \sup_{\mathbb{S}^n} \left|\overline{\nabla}^\beta u\right|.$$

Here β is a standard multi-index for partial derivatives. We further define, for $\alpha \in (0, 1]$, $C^{k,\alpha}$ to be the space of functions $u \in C^k\left(\mathbb{S}^n\right)$ such that the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{S}^n)} = \|u\|_{C^k(\mathbb{S}^n)} + \sup_{|\beta|=k} \sup_{\substack{x, y \in \mathbb{S}^n \\ x \ne y}} \frac{\left|\overline{\nabla}^\beta u\left(x\right) - \overline{\nabla}^\beta u\left(y\right)\right|}{|x-y|^\alpha}$$

is finite. Here $|x - y|$ is the distance between x and y in $\mathbb{S}^n$.

On the space-time $\mathbb{S}^n \times I$, $I = [a, b] \subset \mathbb{R}$, we denote by $C^k\left(\mathbb{S}^n \times I\right)$ the space of real valued functions u which are k-times continuously differentiable with respect to x and $\left[\frac{k}{2}\right]$-times continuously differentiable with respect to t

such that the norm

$$\|u\|_{C^k(\mathbb{S}^n \times I)} = \sum_{|\beta|+2r \leq k} \sup_{\mathbb{S}^n \times I} \left| \overline{\nabla}^\beta D_t^r u \right|$$

is finite. Here $\left[\frac{k}{2}\right]$ is the largest integer not greater than $\frac{k}{2}$. We also denote by $C^{k,\alpha}(\mathbb{S}^n \times I)$ the space of functions in $C^k(\mathbb{S}^n \times I)$ such that the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{S}^n \times I)} = \|u\|_{C^k(\mathbb{S}^n \times I)}$$

$$+ \sup_{|\beta|+2r=k} \sup_{\substack{(x,s),(y,t) \in \mathbb{S}^n \times I \\ (x,s) \neq (y,t)}} \frac{\left| \overline{\nabla}^\beta D_t^r u(x,s) - \overline{\nabla}^\beta D_t^r u(y,s) \right|}{\left(|x-y|^2 + |s-t| \right)^{\frac{\alpha}{2}}}$$

is finite.

3.   Smooth, strictly convex initial data

We have the following new result for strictly convex initial data:

Theorem 3.1 ([25], '08). *Let F be given by (3), with $\alpha \in (0,1]$. The solution of (1) exist for a finite time $T > 0$ and the M_t's converge to a point as $t \to T$. If $\alpha = 1$, then under a suitable rescaling, the hypersurfaces converge exponentially in the C^∞ topology to the unit sphere with respect to the rescaled time parameter $\tau = -\frac{1}{2}\ln\left(1 - \frac{t}{T}\right)$.*

Note that if $\alpha = 1$ and G is additionally concave, or if $n = 2$, the result is known [21], [22].

Elements of the proof. Here we focus on the case $\alpha = 1$, details for other α appear in [25]. Many computations are done rewriting M_t as an evolving graph over the sphere.

Definition 3.1. *The* support function *of M_t, $h : \mathbb{S}^n \times [0,T) \to \mathbb{R}$ is given by*

$$h(p,t) = \langle X(p,t), \nu(p,t) \rangle .$$

Geometrically, the support function gives the perpendicular distance from the origin to the plane tangent to M_t at point $X(p,t)$. The inverse of the Weingarten map, $\mathcal{W}^{-1} = (r_{ij})$, is well defined for convex hypersurfaces and is related to the support function by

$$(4) \qquad\qquad r_{ij} = \overline{\nabla}_i \overline{\nabla}_j h + \overline{g}_{ij} h.$$

Setting

$$\Phi\left(\mathcal{W}^{-1}\right) = -F\left(\mathcal{W}\right) \Leftrightarrow \varphi\left(r_1, \ldots, r_n\right) = -f\left(\kappa_1, \ldots, \kappa_n\right),$$

it is easy to check that φ has positive first derivatives and a straightforward computation [21] shows that the property that F is inverse concave implies precisely that Φ is concave. The evolution (1) is 'equivalent' to

$$(5) \qquad \frac{\partial h}{\partial t} = \Phi\left(\mathcal{W}^{-1}\right) = \Phi\left(\overline{\nabla}_i \overline{\nabla}_j h + \overline{g}_{ij} h\right).$$

with initial data M_0 having corresponding support function

$$h\left(x, 0\right) := h_0\left(x\right).$$

Above $\overline{\nabla}$ and $\overline{g}$ are the standard derivative and metric on $\mathbb{S}^n$. By equation (5) being 'equivalent' we mean that one can add a tangential diffeomorphism to the flow (1) such that equation (5) holds and representation of M_t by

$$X\left(p, t\right) = h\left(p, t\right) \nu\left(p, t\right) + \nabla^{\mathbb{S}^n} h\left(p, t\right)$$

is preserved. For a calculation which verifies that X may be represented in terms of the support function this way, we refer the reader to [11] or [158].

It is very useful to work with the form (5) of the flow equation as the regularity theory is well developed for nonlinear concave Φ [112]. Short-time existence of a solution to (5) on $\mathbb{S}^n \times (0, T)$ is well known. To proceed further, we need several flow-independent estimates coming from the structure of F. These results also appear in [25].

Lemma 3.1. *Let F have form (3) with $\alpha = 1$. If $\mathrm{trace}\left(\mathcal{W}^{-2}\right) \cdot \Phi^2 \leq C^2$ for some $C > 0$ then*

$$\frac{\kappa_{\max}}{\kappa_{\min}} \leq C.$$

Proof. From the assumption,

$$\Phi^2 \leq \frac{C^2}{r_1^2 + \ldots + r_n^2} \leq C^2 \kappa_{\min}^2 \implies -\Phi = F \leq C\kappa_{\min}.$$

In view of (3), this implies

$$\kappa_{\min}^{\beta+\gamma} \kappa_{\max}^{\delta} \leq C\kappa_{\min},$$

so, since $\beta + \gamma + \delta = 1$, the result follows.   □

Although we concentrate on the $\alpha = 1$ case here, we include also the estimate below for later use, coming from the precise form of F.

Lemma 3.2. *Let $F \leq \overline{F}$ have form (3), with $0 < \alpha < 1$. Then*

$$\kappa_{\max} \leq c\left(n\right) \kappa_{\min}^{1-\frac{\alpha}{\delta}} \overline{F}^{\frac{1}{\delta}}.$$

A key new ingredient in the $\alpha = 1$ case is the evolution of the function $V\left(\mathcal{W}^{-1}\right) = \text{trace}\left(\mathcal{W}^{-2}\right) \Phi^2$. It provides a 'pinching estimate' on the ratio of the principal curvatures of the evolving hypersurface M_t and also facilitates showing that under an appropriate rescaling the M_t approach a sphere.

Under equation (5) and indeed more generally under (1), we may compute the corresponding evolutions of various geometric quantities associated with M_t (see, e.g. [9], [11]).

Lemma 3.3. *Under the flow given by (5), with $\alpha = 1$, we have, as long as the solution continues to exist, the corresponding evolution equations:*

(i) $\frac{\partial}{\partial t}\Phi = \dot{\Phi}^{kl}\overline{\nabla}_k\overline{\nabla}_l\Phi - \dot{\Phi}^{kl}\overline{g}_{kl}\Phi,$

(ii) $\frac{\partial}{\partial t}r_{ij} = \dot{\Phi}^{kl}\overline{\nabla}_k\overline{\nabla}_l r_{ij} + \ddot{\Phi}^{kl,pq}\overline{\nabla}_i r_{pq}\overline{\nabla}_j r_{kl} - \dot{\Phi}^{kl}\overline{g}_{kl}r_{ij},$ *and*

(iii) *for V defined as above*

$$(6) \qquad \frac{\partial}{\partial t}V = \dot{\Phi}^{kl}\overline{\nabla}_k\overline{\nabla}_l V + 2\Phi^2\ddot{\Phi}^{kl,pq}r^{ij}\overline{\nabla}_i r_{kl}\overline{\nabla}_j r_{pq}$$
$$- 2\dot{\Phi}^{ij}\left(r_{pq}\overline{\nabla}_i\Phi + \Phi\overline{\nabla}_i r_{pq}\right)\left(r^{pq}\overline{\nabla}_j\Phi + \Phi\overline{\nabla}_j r^{pq}\right).$$

Proof. Under equation (5), the Weingarten map of M_t evolves according to

$$\frac{\partial}{\partial t}r_{ij} = \overline{\nabla}_i\overline{\nabla}_j\Phi + \overline{g}_{ij}\Phi,$$

where we have differentiated equation (4) with respect to t. Contracting this equation with $\dot{\Phi}^{ij}$ yields (i). Interchanging second covariant derivatives on the first term and using symmetry to obtain (ii). For details of Codazzi- and Simons'-type identities applying in this setting we refer the reader to [17]. Equation (iii) now follows by a straightforward computation using (i) and (ii). $\qquad\square$

Corollary 3.1. *The maximum of V over M_t, as a function of time t, is non-increasing under the flow.*

Proof. The second line of equation (6) is non-positive since the matrix $\left(\dot{\Phi}^{ij}\right)$ is positive definite. The $\ddot{\Phi}$ term is non-positive since Φ is concave. At a maximum of V the $\overline{\nabla}^2$ term is non-positive. A result of Hamilton [85] then gives that for almost every t such that the solution to the flow equation exists,

$$\frac{d}{dt}\max_{M_t} V \leq 0,$$

so $\max_{M_t} V$ is non-increasing. The above may also be written more formally in terms of the Dini derivative (see, for example, [45]).   □

Corollary 3.2. *While the solution to* (5) *exists, there exists a constant* C, *depending only on the initial hypersurface* M_0, *such that the curvatures of* M_t *satisfy*

$$\frac{\kappa_i}{\kappa_j} \geq C \text{ for all } 1 \leq i, j \leq n.$$

Proof. From Corollary 3.1,

$$\text{trace} \left(\mathcal{W}^{-2} \right) \cdot \Phi^2 \big|_{(x,t)} \leq C\left(M_0 \right) = \max_{M_0} \left[\text{trace} \left(\mathcal{W}^{-2} \right) \cdot \Phi^2 \right].$$

The result now follows from Lemma 3.1.   □

The above 'curvature pinching' result has many useful consequences. In particular, it implies that (5) is a uniformly parabolic partial differential equation; arguments as in [9], using the regularity theory of Krylov for such fully nonlinear equations with concave Φ [112], or [23], imply that the solution hypersurfaces converge to a point. In particular, one obtains $C^{2,\alpha}$ smoothness on $\mathbb{S}^n \times [0, T)$ up to the extinction time using [112] or [23]; higher regularity then follows by Schauder estimates (e.g. [119]).

One may obtain estimates on the time T at which the solution hypersurface has contracted to a point using the comparison principle for parabolic evolution equations. Solutions which are initially disjoint remain so, as long as they both exist, so comparing the evolving M_t with interior and exterior spheres shrinking under the same equation, whose radius $r(t)$ may be written down explicitly (see (7)), provides upper and lower estimates on the extinction time T.

We now rescale the evolving hypersurface M_t to examine its shape asymptotically towards the final time T. Again this argument is for $\alpha = 1$, and we use part (iii) of Lemma 3.3. We rescale the position vector as in (2), where $\tau = -\frac{1}{2} \log \left(1 - \frac{t}{T} \right)$ is the rescaled time parameter, to keep, for example, the surface area of $\tilde{M}_\tau = \tilde{X}\left(\mathbb{S}^n, \tau \right)$ constant. Since the function V is homogeneous degree zero in the principal curvatures, it is unchanged under rescaling and satisfies exactly the same form of equation as Lemma 3.3, (iii). We apply the strong maximum principle to this equation: if a maximum of V were obtained at some (p_0, τ_0), $\tau_0 > 0$, then V would necessarily be identically constant. Then (6) would become

$$0 \equiv 2\Phi^2 \ddot{\Phi}^{kl,pq} r^{ij} \overline{\nabla}_i r_{kl} \overline{\nabla}_j r_{pq} - 2\dot{\Phi}^{ij} \left(r_{pq} \overline{\nabla}_i \Phi + \Phi \overline{\nabla}_i r_{pq} \right) \left(r^{pq} \overline{\nabla}_j \Phi + \Phi \overline{\nabla}_j r^{pq} \right).$$

Each of these terms is non-positive, so each must be identically zero. Diagonalising $\mathcal{W}$, the second term implies

$$\overline{\nabla}_i r_{pq} = -\frac{r_p}{\Phi} \delta_{pq} \overline{\nabla}_i \Phi,$$

where δ_{pq} is equal to 1 if $p = q$ and 0 otherwise. Substituting this into the first term gives

$$2\Phi^2 \dddot{\Phi}^{kl,pq} r^{ij} \overline{\nabla}_i r_{kl} \overline{\nabla}_j r_{pq} = 2r_i \frac{\partial^2 \phi}{\partial r_k \partial r_l} r_k \overline{\nabla}_i \Phi r_l \overline{\nabla}_i \Phi = 4r_i \left(\overline{\nabla}_i \Phi \right)^2 \Phi,$$

since φ is homogeneous of degree -1. For details of why, in the diagonalised setting, the first equality above holds, we refer the reader to [21]. Now the right hand-side can only be equal to zero if Φ is identically constant. But this means that $\overline{\nabla}_i r_{pq} \equiv 0$ which implies that $\tilde{M}_{\tau_0}$ is a sphere. A stability argument similar to those in [18], [127] and [128] gives that the $\tilde{M}_\tau$ converge exponentially to $\mathbb{S}^n$ as $\tau \to \infty$.

4. Non-smooth convex initial data

Here the initial hypersurface M_0 is the boundary of a convex region $\Omega_0 \subset \mathbb{R}^{n+1}$. It is known that solutions become immediately strictly convex under the mean curvature flow (for smooth weakly convex initial data in [44] and for general convex M_0 by Andrews), flows by powers of Gauss curvature, $F = K^\alpha$, $0 < \alpha \le \frac{1}{n}$ [17], flows with speed $F = \left(\frac{K}{E_k} \right)^\alpha$ [13] and flows with speed $F = \frac{E_{k+1}}{E_k}$, in the case of smooth weakly convex initial data [52]. We extend this list to flows with our speed (3) and $\alpha \subset (0, 1]$. In the case $\alpha = 1$, F must satisfy an additional property in order that solutions immediately become strictly convex. We have the following short-time existence result:

Theorem 4.1 ([25], '08). *Let F have the form (3), $\alpha \in (0, 1]$. If $\alpha = 1$ suppose also*

$$\lim_{x \to \infty} f(1, \dots, 1, x) = \infty.$$

For any open, bounded convex region $\Omega_0 \subset \mathbb{R}^{n+1}$, there is a solution to (1) on a finite time interval $(0, T)$, unique up to composition with an arbitrary time-independent diffeomorphism, such that M_t is smooth and strictly convex for $t > 0$ and M_t converges in Hausdorff distance to $\partial\Omega_0$ as $t \to 0$.

Remark 4.1. *Given M_t above is strictly convex for a small $t_0 > 0$, Theorem 3.1 describes the long time behaviour.*

Idea of proof. We approximate $M_0 = \partial\Omega_0$ by a sequence $\{M_0^\varepsilon\}$ of smooth, strictly convex, with uniformly bounded inradius r_- and circumradius R_+, such that $M_0^\varepsilon \to M_0$ in Hausdorff distance as $\varepsilon \to 0$. By comparison with the flow of B_{r_-} contained inside all these hypersurfaces, we get a short existence time t_0 independent of ε. For balls, under (1), $r(t)$ satisfies

$$(7) \qquad \qquad \frac{d}{dt} r = -r^{-\alpha},$$

which can be solved explicitly to give a suitable t_0. We now flow each M_0^ε by (1), and estimate displacement, speed and curvature, independent of ε. Specifically, an upper displacement bound is obtained by comparison with a suitable interior ball of radius $r(t)$. A lower displacement bound follow by comparison with an exterior ball, if $0 < \alpha < 1$, or a graphical barrier, if $\alpha = 1$ and $\lim_{x\to\infty} f(1,\ldots,1,x) = \infty$. An upper speed bound is obtained while the lower displacement bound holds, using the function $Z = \frac{F}{u - r_0}$, first introduced in [156]. A lower speed bound then follows from the Harnack inequality [11]; this requires F to be inverse concave and the lower displacement bound. The lower curvature bound $\kappa_{\min} \geq \min\left(C_1 t^{n-1}, C_2\right)$ follows using (3) and the above speed bounds. The upper curvature bound the follows from pinching (Lemma 3.1 or Lemma 3.2) and the lower curvature bound. Now $C^{2,\alpha}$ regularity follows by theory for uniformly parabolic PDE [112] and higher regularity is clear by standard Schauder theory (see, eg. [119]). The Arzela-Ascoli theorem gives a subsequence $M_t^{\varepsilon_k}$ converging to a solution M_t on $\mathbb{S}^n \times (0, t_0)$.

More details of the above proof appear in [25]. The key for obtaining the lower curvature estimate is clearly the lower displacement bound. In an alternative situation, flat sides may persist under the curvature flow. This is known to occur in the Gauss curvature flow [86], flows by powers $p > \frac{1}{n}$ of Gauss curvature and the harmonic mean curvature flow of surfaces [37]. In [25] we provide the following characterisation of when flat sides persist under a curvature flow (1):

Theorem 4.2. *Let M_0 be the boundary of an open convex region $\Omega_0 \subset \mathbb{R}^{n+1}$ and speed F monotone and homogeneous of degree α. If $\alpha > 1$, or $\alpha = 1$ and $\lim_{x\to\infty} f(1,\ldots,1,x) < \infty$, then flat sides persist for some time under the flow. If $0 < \alpha < 1$, or $\alpha = 1$ and $\lim_{x\to\infty} f(1,\ldots,1,x) = \infty$, then hypersurfaces become immediately strictly convex.*

Idea of proof. In each case we either construct a self-similar subsolution with a flat side, or obtain a lower displacement bound, which implies strict convexity of the hypersurface M_t.

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James McCoy

Centre for Pure Mathematics
School of Mathematics and Applied Statistics
University of Wollongong
Wollongong, NSW 2522, AUSTRALIA

E-mail: jamesm@uow.edu.au