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The Theory of Quaternion Orthogonal Designs

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Abstract

Over the past several years, there has been a renewed interest in complex orthogonal designs for their application in space-time block coding. Motivated by the success of this application, this paper generalizes the definition of complex orthogonal designs by introducing orthogonal designs over the quaternion domain. This paper builds a theory of these novel quaternion orthogonal designs, offers examples, and provides several construction techniques. These theoretical results, along with the results of preliminary simulations, lay the foundation for developing applications of these designs as orthogonal space-time-polarization block codes.

EDICS: SPC-STCD

I. Introduction: Motivation and Preliminaries

Space-time block codes built from generalized complex orthogonal designs can be viewed as a generalization of Alamouti's scheme [1] and were introduced by Tarokh *et al.* [18]. These space-time block codes are particularly attractive because they can provide full transmit diversity while requiring a very simple decoupled maximum-likelihood (ML) decoding method [13], [18]. The combination of space and time diversity moved the capacity of wireless communication systems towards the theoretical limits, and this technique has been adopted in the 3G standard in the form of an Alamouti code and in the newly proposed standard for wireless LANs IEEE 802.11n. The success of the combined diversity achieved by complex orthogonal space-time block codes motivated us to consider alternative definitions of orthogonal designs that would permit the combination of space and time diversity with additional forms of diversity. We expect that the combination of additional forms of diversity, for example frequency and/or polarization diversity, will further improve capacity.

Polarization diversity has been studied in the past [4], and Collins has provided a detailed assessment of the diversity gain under Rayleigh fading [5]. Recently, the demand for high rates in mobile communications has raised interest in applying polarization diversity, often together with other forms of diversity. It has been shown that polarization diversity can significantly add to the performance improvements offered by other diversity techniques and be nearly as effective as spatial diversity for base station antennas without a noticeable increase in their dimensions [14],

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[6]. In addition, the use of dual-polarized antennas for both transmitter and receiver combined with channel coding has been previously studied [22]. Polarization diversity is usually considered separately from the other forms of diversity and there is no well known mechanism of utilizing it jointly with the other forms rather than through a simple concatenation. However, it is intuitively clear, and shown through space-time coding, that that it is possible to achieve better performance by using combined rather than concatenated forms of diversity.

The work of Isaeva and Sarytchev [12] implies that a signal having components in two orthogonal polarizations can be modeled by means of quaternions, whereby two complex constellations orthogonal to each other on a polarization plane form a quaternion, or hyper-complex, number [2]. Using this notation, any rotation on the polarization plane can be represented in a similar way as a rotation on a complex plane. Similarly, any change to the polarization bases, *e.g.*, due to differences between transmit and receive antenna alignments, can be represented in the quaternion notation as long as the polarization bases are orthogonal [12].

Hence, we propose to develop orthogonal designs with quaternion elements to be used as a basis for orthogonal space-time-polarization block codes. The generalization from real orthogonal designs to complex orthogonal designs was straight-forward, while this further generalization is significantly complicated by the non-commuting nature of the quaternions. The proposed application of these quaternion designs would allow polarization diversity to be jointly combined with space and time diversities. Our preliminary simulations indicate that the proposed quaternion orthogonal space-time-polarization block codes perform similarly to complex orthogonal space-time block codes that use twice as many transmit antennas and twice as many receive antennas. In addition to further studying this savings on dimension, other potential performance gains must be further addressed in future work.

The original definition of orthogonal design proposed by Geramita, Geramita and Seberry Wallis [10] concerned only square matrices defined on real variables. An orthogonal design (OD) of order n and type $(s_1, s_2, \ldots, s_u)$ on commuting real variables $x_1, x_2, \ldots, x_u$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, is an $n \times n$ matrix A with entries in the set $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ satisfying $AA^T = \left(\sum_{h=1}^{u} s_h x_h^2\right) I_n$, where T denotes the transpose of a matrix and I_n is the identity matrix of order n. Qualitatively, we say that the rows of A are formally orthogonal and each row has precisely s_h entries of the type $\pm x_h$. Geramita, Geramita and Seberry Wallis also showed that $A^T A = \left(\sum_{h=1}^{u} s_h x_h^2\right) I_n$ and so our qualitative description of A applies equally well to the columns of A [10]. Geramita and Geramita [11] first studied orthogonal designs in the complex domain as $n \times n$ matrices A of type $(s_1, s_2, \ldots, s_u)$ with entries in the set $\{0, \pm x_1, \pm \mathbf{i} x_1, \pm x_2, \pm \mathbf{i} x_2, \ldots, \pm x_u, \pm \mathbf{i} x_u\}$ satisfying $A^H A = AA^H = \left(\sum_{h=1}^{u} s_h x_h^2\right) I_n$, where H denotes the Hermitian transpose. This definition can be generalized to include $r \times n$ rectangular designs. We say that these designs are complex orthogonal designs (CODs) on real variables $x_1, x_2, \ldots, x_u$. A variation more widely used in the signals processing literature is the generalized complex orthogonal design A of size $r \times n$ and type $(s_1, s_2, \ldots, s_u)$ with entries in the set of complex variables $\{0, \pm z_1, \pm z_1^*, \pm \mathbf{i} z_1, \pm \mathbf{i} z_1^*, \pm z_2, \pm z_2^*, \pm \mathbf{i} z_2, \pm \mathbf{i} z_2^*, \ldots, \pm z_u, \pm z_u^*, \pm \mathbf{i} z_u, \pm \mathbf{i} z_u^*\}$ satisfying $A^H A = \left(\sum_{h=1}^{u} s_h |z_h|^2\right) I_n$ [13], [18]. We say that these designs are complex orthogonal designs (CODs) on complex variables $z_1, z_2, \ldots, z_u$. A number of variations on these definitions has been considered. For example, if an entry of a design is a linear combination of the permitted variables, then the design is said to be with linear processing.

We recall here the properties of the quaternions that are critical to our discussion and refer the reader to Altmann's book [2] for more information. The non-commutative quaternions $\mathcal{Q} = \{\pm 1, \ \pm \mathbf{i}, \ \pm \mathbf{j}, \ \pm \mathbf{k}\}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. A *quaternion variable* $\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$, where a_1, a_2, a_3, a_4 are real variables has a *quaternion conjugate* defined by $\mathbf{a}^Q = a_1 - a_2\mathbf{i} - a_3\mathbf{j} - a_4\mathbf{k}$. It follows that $\mathbf{a}^Q\mathbf{a} = \mathbf{a}\mathbf{a}^Q = |\mathbf{a}|^2$ is real. Given a matrix $A = (\mathbf{a}_{\ell,m})$,

where $\mathbf{a}_{\ell,m}$ are quaternion variables, its *quaternion transpose* is $A^Q = (\mathbf{a}^Q_{m,\ell})$. It follows from this definition of the quaternion transpose that the product $A^Q B$ consists of taking the inner product of the columns of A and the columns of B, where the inner product of quaternion variables $\mathbf{a}$ and $\mathbf{b}$, is said to be $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^Q \mathbf{b}$. This notion will now be developed as we define quaternion matrices whose columns are mutually orthogonal under this inner product.

Definition 1: A *quaternion orthogonal design (QOD) on commuting real variables* x_1, x_2, $\ldots$, x_u of type $(s_1, s_2, \ldots, s_u)$ is an $r \times n$ matrix A with entries from $\{0, \pm\mathbf{q}_1 x_1, \pm\mathbf{q}_2 x_2, \ldots, \pm\mathbf{q}_u x_u\}$, $\mathbf{q}_h \in \mathcal{Q}$, that satisfies $A^Q A = \left(\sum_{h=1}^{u} s_h x_h^2\right) I_n$. This design is denoted by $QOD(r, n;\ s_1, s_2, \ldots, s_u)$. When $r = n$, we have $A^Q A = A A^Q = \left(\sum_{h=1}^{u} s_h x_h^2\right) I_n$. Similarly, we define a *QOD A on commuting complex variables* $z_1, z_2, \ldots, z_u$ to have entries from the set $\{0, \pm z_1, \pm z_1^*, \pm z_2, \pm z_2^*, \ldots, \pm z_u, \pm z_u^*\}$ including possible multiplications on the left and/or right by quaternion elements $\mathbf{q} \in \mathcal{Q}$, and to satisfy $A^Q A = (\sum_{h=1}^{u} s_h |z_h|^2) I_n$. Finally, we define a *QOD A on non-commuting quaternion variables* $\mathbf{a}_1$, $\mathbf{a}_2, \ldots, \mathbf{a}_u$ to have entries from $\{0, \pm\mathbf{a}_1, \pm\mathbf{a}_1^Q, \pm\mathbf{a}_2, \pm\mathbf{a}_2^Q, \ldots, \pm\mathbf{a}_u, \pm\mathbf{a}_u^Q\}$ including possible multiplications on the left and/or right by quaternion elements $\mathbf{q} \in \mathcal{Q}$ and to satisfy $A^Q A = (\sum_{h=1}^{u} s_h |\mathbf{a}_h|^2) I_n$. We can generalize these definitions to allow the design entries to be real linear combinations of the permitted variables and their quaternion multipliers, in which case we say the design is with linear processing.

Example 2: The matrix $X = \begin{bmatrix} -x_1 & x_2\mathbf{i} \\ -x_2\mathbf{j} & x_1\mathbf{k} \end{bmatrix}$ is a QOD on real variables x_1, x_2. The matrix $Z = \begin{bmatrix} \mathbf{i}z_1 & \mathbf{i}z_2 \\ -\mathbf{j}z_2^* & \mathbf{j}z_1^* \end{bmatrix}$ is a QOD on complex variables z_1, z_2. Finally, $A = \begin{bmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{a} \end{bmatrix}$ is the most obvious example of a QOD on the quaternion variable $\mathbf{a}$. QODs on quaternion variables are the most difficult to construct.

In Section II, we provide two construction techniques that utilize real and complex orthogonal designs to build quaternion orthogonal designs. In Section III, we define *quaternion-commutivity* and *quaternion amicable designs* in an effort to develop construction techniques that do not rely on existing real and complex orthogonal designs. In Section IV, we describe simulation results of a quaternion orthogonal design applied as a space-time-polarization block code. The paper is concluded and some open problems are described in Section V.

II. Construction Techniques Using Real and Complex Designs

In this section, we provide two construction techniques that use existing real and complex orthogonal designs to build quaternion orthogonal designs. The constructions demonstrate that QODs exist, and the resulting examples help build intuition concerning the behavior of these novel designs. The non-commutativity of the quaternions prevents simple generalizations of construction techniques for ODs or CODs. However, the new techniques that we propose for constructing QODs can be readily modified to construct ODs or CODs.

In Subsection II-A, we define quaternion permutation matrices and use these matrices, along with existing ODs and CODs, to construct QODs. In Subsection II-B, we define symmetric-paired complex orthogonal designs and use these designs to build QODs.

A. Quaternion Permutation Matrices

We present here our most simple, and potentially most useful, method for constructing quaternion orthogonal designs. The construction requires only readily available real or complex orthogonal designs and what we define below as quaternion permutation matrices.

Definition 3: An $n \times n$ matrix M is a *quaternion permutation matrix* if it has one nonzero element $\mathbf{q} \in \mathcal{Q}$ per row and per column and if $M^Q M = M M^Q = I_n$.

Theorem 4: Let D be an $r \times n$ orthogonal design or complex orthogonal design of type $(s_1, s_2, \ldots, s_u)$. Let M be an $r \times r$ and N be an $n \times n$ quaternion permutation matrix. Then MDN is an $r \times n$ quaternion orthogonal design of type $(s_1, s_2, \ldots, s_u)$.

Proof: If D is real, write $D^Q D = D^T D = \sum_{h=1}^{u} s_h |x_h|^2 I_n = \gamma I_n$. If D is complex, write $D^Q D = D^H D = \sum_{h=1}^{u} s_h |z_h|^2 I_n = \gamma I_n$. In either case, the coefficient γ is real. Now, consider

$$\begin{aligned}(MDN)^Q(MDN) &= (N^Q D^Q M^Q)(MDN) \\ &= N^Q D^Q M^Q MDN \\ &= N^Q \gamma I_n N = \gamma N^Q N \\ &= \gamma I_n\end{aligned}$$

which clearly satisfies the required orthogonality constraint so that MDN is a quaternion orthogonal design. ■

We note that Theorem 4 holds for any quaternion matrices M and N such that $M^Q M = I_r$ and $N^Q N = I_n$. Our restriction to matrices that have exactly one nonzero quaternion element per column and per row is meant to show the simplicity of this idea and to facilitate the construction of such quaternion permutation matrices. This construction shows that orthogonal designs over the quaternion domain exist and can be constructed in a straight-forward way.

Example 5: For simplicity, let N be the identity matrix. Let $M = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{j} \end{bmatrix}$. Let $D = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}$ be an OD on real variables x_1, x_2. Then, $MD = \begin{bmatrix} x_1\mathbf{i} & x_2\mathbf{i} \\ -x_2\mathbf{j} & x_1\mathbf{j} \end{bmatrix}$ is QOD on real variables x_1, x_2. Now, let $D' = \begin{bmatrix} x_1\mathbf{i} & x_2 \\ x_2 & x_1\mathbf{i} \end{bmatrix}$ be a COD on real variables x_1, x_2. Then $MD' = \begin{bmatrix} -x_1 & x_2\mathbf{i} \\ -x_2\mathbf{j} & x_1\mathbf{k} \end{bmatrix}$ is a QOD on real variables x_1, x_2. Finally, let $D'' = \begin{bmatrix} z_1 & z_2 \\ z_2^* & -z_1^* \end{bmatrix}$ be a COD on complex variables z_1, z_2. Then, $MD'' = \begin{bmatrix} \mathbf{i}z_1 & \mathbf{i}z_2 \\ -\mathbf{j}z_2^* & \mathbf{j}z_1^* \end{bmatrix}$ is a QOD on complex variables z_1, z_2.

B. Symmetric-Paired Designs

We now present a construction technique that uses specially chosen pairs of complex orthogonal designs of size $r \times n$ to build $r \times n$ quaternion orthogonal designs.

Definition 6: Two complex orthogonal designs A and B are said to be *symmetric-paired designs* if AB^H is symmetric and/or if $A^H B$ is symmetric.

We recall that complex amicable designs A and B satisfy $AB^H = BA^H$ and/or $A^H B = B^H A$ [21]. Hence, the proposed definition of symmetric-paired designs has a resemblance to amicability. The concept of amicability is revisited in Section III-B.

Lemma 7: Let A and B be matrices whose entries include 0 and arbitrary complex variables. (For example, A and B could be complex orthogonal designs.) If AB^H is a symmetric matrix, then $AB^H\mathbf{q} = \mathbf{q}BA^H$, where $\mathbf{q} \in \{\pm\mathbf{j}, \pm\mathbf{k}\}$. Similarly, if $A^H B$ is a symmetric matrix, then $A^H B\mathbf{q} = \mathbf{q}B^H A$, where $\mathbf{q} \in \{\pm\mathbf{j}, \pm\mathbf{k}\}$.

Proof: Suppose that AB^H is a symmetric matrix and denote $AB^H = [\alpha_{st}]$, where $\alpha_{st} = a_{st} + b_{st}\mathbf{i}$ is a complex variable and so a_{st}, b_{st} are real variables. Then $BA^H = (AB^H)^H =$

$([\alpha_{st}])^H = [\alpha_{ts}^*]$. Thus

$$\begin{aligned}
\mathbf{q}BA^H - AB^H\mathbf{q} &= \mathbf{q}[\alpha_{ts}^*] - [\alpha_{st}]\mathbf{q} \\
&= [\mathbf{q}(a_{ts} - b_{ts}\mathbf{i})] - [(a_{st} + b_{st}\mathbf{i})\mathbf{q}] \\
&= [\mathbf{q}(a_{ts} - a_{st}) - \mathbf{q}\mathbf{i}(b_{ts}) - \mathbf{i}\mathbf{q}(b_{st})] \\
&= 0,
\end{aligned}$$

which follows since the symmetry of AB^H implies that $a_{ts} = a_{st}$ and $b_{ts} = b_{st}$, and since $\mathbf{i}\mathbf{q} = -\mathbf{q}\mathbf{i}$ for $\mathbf{q} \in \{\pm\mathbf{j}, \pm\mathbf{k}\}$. It follows that $\mathbf{q}BA^H = AB^H\mathbf{q}$, where $\mathbf{q} \in \{\pm\mathbf{j}, \pm\mathbf{k}\}$.

It follows similarly that if A^HB is symmetric, then $\mathbf{q}B^HA = A^HB\mathbf{q}$, where $\mathbf{q} \in \{\pm\mathbf{j}, \pm\mathbf{k}\}$. ■

Example 8: Consider the following CODs on real variables x_1, x_2, x_3, x_4 : $A = \begin{bmatrix} x_1 & x_2\mathbf{i} \\ x_2\mathbf{i} & x_1 \end{bmatrix}$ and $B = \begin{bmatrix} x_3 & x_4\mathbf{i} \\ x_4\mathbf{i} & x_3 \end{bmatrix}$. One can confirm that $AB^H = \begin{bmatrix} x_1x_3 + x_2x_4 & (-x_1x_4 + x_2x_3)\mathbf{i} \\ (x_2x_3 - x_1x_4)\mathbf{i} & x_1x_3 + x_2x_4 \end{bmatrix}$ is symmetric. Lemma 7 then implies that $AB^H\mathbf{j} = \mathbf{j}BA^H$, which we now confirm directly:

$$\begin{aligned}
AB^H\mathbf{j} &= \begin{bmatrix} x_1x_3 + x_2x_4 & (-x_1x_4 + x_2x_3)\mathbf{i} \\ (x_2x_3 - x_1x_4)\mathbf{i} & x_1x_3 + x_2x_4 \end{bmatrix}\mathbf{j} \\
&= \begin{bmatrix} (x_1x_3 + x_2x_4)\mathbf{j} & (-x_1x_4 + x_2x_3)\mathbf{k} \\ (x_2x_3 - x_1x_4)\mathbf{k} & (x_1x_3 + x_2x_4)\mathbf{j} \end{bmatrix} \\
&= \mathbf{j}\begin{bmatrix} x_1x_3 + x_2x_4 & (x_1x_4 - x_2x_3)\mathbf{i} \\ (x_1x_4 - x_2x_3)\mathbf{i} & x_1x_3 + x_2x_4 \end{bmatrix} \\
&= \mathbf{j}BA^H
\end{aligned}$$

It is similar to show that A^HB is also symmetric, which implies that $\mathbf{j}B^HA = A^HB\mathbf{j}$.

Theorem 9: Let A and B be complex orthogonal designs $COD(n, n; s_1, s_2, \ldots, s_u)$ and $COD(n, n;\ t_1,\ t_2,\ \ldots,\ t_u)$ respectively, on commuting complex variables $z_1, \ldots, z_u$. If A^HB is symmetric, then $A+B\mathbf{j}$ is a quaternion orthogonal design $QOD(n, n; s_1+t_1, s_2+t_2, \ldots, s_u+t_u)$ on the complex variables $z_1, \ldots, z_u$. Specifically, $(A + B\mathbf{j})^Q(A + B\mathbf{j}) = \sum_{h=1}^{u}(s_h + t_h)|z_h|^2 I_n$.

Proof: Since A and B are complex, we have $A^Q = A^H$ and $B^Q = B^H$. Then,

$$\begin{aligned}
(A + B\mathbf{j})^Q(A + B\mathbf{j}) &= (A^Q - \mathbf{j}B^Q)(A + B\mathbf{j}) \\
&= A^QA - \mathbf{j}B^QA + A^QB\mathbf{j} - \mathbf{j}B^QB\mathbf{j} \\
&= A^QA + B^QB - \mathbf{j}B^QA + A^QB\mathbf{j} \\
&= A^HA + B^HB - \mathbf{j}B^HA + A^HB\mathbf{j} \\
&= A^HA + B^HB \text{ (by Lemma 7)} \\
&= \sum_{h=1}^{u} s_hz_hz_h^*I + \sum_{h=1}^{u} t_hz_hz_h^*I \\
&= \sum_{h=1}^{u}(s_h + t_h)|z_h|^2 I.
\end{aligned}$$

Hence, if A and B are CODs on complex variables $z_1, z_2, \ldots, z_u$ and A^HB is symmetric, then $A + B\mathbf{j}$ is a quaternion orthogonal design on complex variables $z_1, z_2, \ldots, z_u$. ■

Corollary 10: Let A and B be complex orthogonal designs $COD(n, n;\ s_1, s_2,\ \ldots, s_u)$ and $COD(n, n;\ t_1, t_2, \ldots, t_u)$ respectively, on commuting real variables $x_1, \ldots, x_u$. If A^HB is symmetric, then $A + B\mathbf{j}$ is a quaternion orthogonal design $QOD(n, n; s_1 + t_1, s_2 + t_2, \ldots, s_u + t_u)$ on real variables $x_1, \ldots, x_u$. Specifically, $(A + B\mathbf{j})^Q(A + B\mathbf{j}) = \sum_{h=1}^{u}(s_h + t_h)x_h^2 I_n$.

Example 11: The designs $A = \begin{bmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{bmatrix}$ and $B = \begin{bmatrix} z_2 & z_1 \\ z_1^* & -z_2^* \end{bmatrix}$, where z_1 and z_2 are commuting complex variables, are $COD(2,2;1,1)$ designs that are equivalent to Alamouti's code [1]. It is straight-forward to verify that $A^H B$ is symmetric. Then, $A + B\mathbf{j}$ gives $\begin{bmatrix} z_1 + z_2\mathbf{j} & z_2 + z_1\mathbf{j} \\ -z_2^* + z_1^*\mathbf{j} & z_1^* - z_2^*\mathbf{j} \end{bmatrix}$, which is a $QOD(2,2;2,2)$ with linear processing on complex variables z_1, z_2. Defining a quaternion variable $\mathbf{a} = z_1 + z_2\mathbf{j}$, we see: $A + B\mathbf{j} = \begin{bmatrix} \mathbf{a} & -\mathbf{iak} \\ \mathbf{ja} & -\mathbf{kak} \end{bmatrix}$. Hence, $A + B\mathbf{j}$ can also be considered a QOD on the one quaternion variable $\mathbf{a}$.

Example 12: Since our examples have been restricted to 2×2 designs thus far, we provide an example using maximum rate, minimum decoding delay CODs of size 4×4. To show variety, we allow a scalar multiple of one of our complex variables.

$$A = \begin{bmatrix} z_3 & 0 & -z_1 & -2z_2 \\ 0 & z_3 & 2z_2^* & -z_1^* \\ z_1^* & -2z_2 & z_3^* & 0 \\ 2z_2^* & z_1 & 0 & z_3^* \end{bmatrix}$$

is a COD satisfying $A^H A(|z_1|^2 + 4|z_2|^2 + |z_3|^2)I$. Also,

$$B = \begin{bmatrix} -z_1 & -2z_2 & z_3 & 0 \\ 2z_2^* & -z_1 & 0 & z_3 \\ z_3^* & 0 & z_1^* & -2z_2 \\ 0 & z_3^* & 2z_2^* & z_1^* \end{bmatrix}$$

is a COD satisfying $B^H B = (|z_1|^2 + 4|z_2|^2 + |z_3|^2)I$.

It follows that

$$\begin{aligned} A^H B &= \left(|z_1|^2 + 4|z_2|^2 + |z_3|^2\right) \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \left(A^H B\right)^T, \end{aligned}$$

so $A^H B$ is symmetric. Then Lemma 7 implies that $\mathbf{j}B^H A = AB^H\mathbf{j}$. So, according to Theorem 9,

$$A + B\mathbf{j} = \begin{bmatrix} z_3^* - z_1^*\mathbf{j} & 2z_2\mathbf{j} & z_1 + z_3\mathbf{j} & 2z_2 \\ -2z_2^*\mathbf{j} & z_3^* - z_1\mathbf{j} & -2z_2^* & z_1^* + z_3\mathbf{j} \\ -z_1^* + z_3^*\mathbf{j} & 2z_2 & z_3 + z_1\mathbf{j} & 2z_2\mathbf{j} \\ -2z_2^* & -z_1 + z_3^*\mathbf{j} & -2z_2^*\mathbf{j} & z_3 + z_1^*\mathbf{j} \end{bmatrix}$$

satisfies the orthogonality constraint $(A + \mathbf{j}B)^Q(A + \mathbf{j}B) = 2(|z_1|^2 + 4|z_2|^2 + |z_3|^2)I$. Hence, $A + B\mathbf{j}$ is a $QOD(4,4;2,8,2)$ with linear processing on the complex variables z_1, z_2, z_3. It is possible to rewrite this example using quaternion variables $\mathbf{a}$, $\mathbf{b}$, where (for example) $\mathbf{a} = z_1 + z_3\mathbf{j}$ and $2z_2 = \mathbf{b} - \mathbf{ibi}$. We omit the details for brevity.

We note that in applying Theorem 9 (similarly Corollary 10), A and B need not contain the same variables, since we can simply take the union of their variables as the set of possible variables and set certain coefficients s_h and t_h to be zero as necessary. We illustrate with the following examples:

Example 13: We apply Theorem 9 to the following complex orthogonal designs on complex variables z_1, z_2: $A = \begin{bmatrix} z_1 & 0 \\ 0 & -z_1^* \end{bmatrix}$, and $B = \begin{bmatrix} z_2 & 0 \\ 0 & -z_2^* \end{bmatrix}$. We can say that A is a

$COD(2,2;1,0)$ and B is a $COD(2,2;0,1)$, with both designs defined on the complex variables $\{z_1, z_2\}$. It is straight-forward to verify that $A^H B$ is symmetric. Then, $A + B\mathbf{j}$ gives $\begin{bmatrix} z_1 + z_2\mathbf{j} & 0 \\ 0 & -z_1^* - z_2^*\mathbf{j} \end{bmatrix}$, which is a $QOD(2,2;1,1)$ with linear processing on complex variables z_1, z_2. Writing $\mathbf{a} = z_1 + z_2\mathbf{j}$ we have $A + B\mathbf{j} = \begin{bmatrix} \mathbf{a} & 0 \\ 0 & \mathbf{jaj} \end{bmatrix}$, which shows that $A + B\mathbf{j}$ can also be considered a QOD on the quaternion variable $\mathbf{a}$.

Example 14: We apply Corollary 10 to the CODs A and B from Example 8. We can denote A as a $COD(2,2;1,1,0,0)$ and B as a $COD(2,2;0,0,1,1)$ on the real variables $\{x_1, x_2, x_3, x_4\}$. Then, $A + B\mathbf{j} = \begin{bmatrix} x_1 + x_3\mathbf{j} & x_2\mathbf{i} + x_4\mathbf{k} \\ x_2\mathbf{i} + x_4\mathbf{k} & x_1 + x_3\mathbf{j} \end{bmatrix}$. It can be shown that $(A + B\mathbf{j})^Q(A + B\mathbf{j}) = \sum_{i=1}^4 x_i^2 I$, so that $A + B\mathbf{j}$ is a $QOD(2;1,1,1,1)$ with linear processing on real variables x_1, x_2, x_3, x_4. Alternatively, writing $\mathbf{a} = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, we have $A + B\mathbf{j} = \frac{1}{2}\begin{bmatrix} \mathbf{a} - \mathbf{jaj} & \mathbf{a} + \mathbf{jaj} \\ \mathbf{a} + \mathbf{jaj} & \mathbf{a} - \mathbf{jaj} \end{bmatrix}$, which shows that our matrix can also be considered a QOD with linear processing on the quaternion variable $\mathbf{a}$.

The QODs produced in Examples 11, 12, and 14 clearly contain no zero entries. As space-time block codes with no (or few) zero entries have practical advantages when used in signal processing [17], we expect that QODs with no zero entries will have similar advantages. However, the quaternion case is complicated because although each entry in the above examples is nonzero, it is not the case that each entry is a full quaternion variable $\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$, with all nonzero real components a_h. It would likely be useful if all entries were nonzero in at least one of the polarization planes. Further work must be completed to determine any practical implementation advantages of these designs with no zero entries.

III. Quaternion Construction Techniques

The goal of this section is to develop theory that can be used to construct QODs without a reliance on ODs or CODs. Although the OD- and COD-dependent construction techniques of Section II provide insight into quaternion orthogonality and contribute to a library of examples of QODs, the resulting QODs are necessarily limited by the structure of the OD and COD building blocks. In practical applications, the goal will be to maximize the rate and minimize the decoding delay of the QODs. By relying on CODs in constructions, we may be limiting the rate of the resulting QODs due to limitations of the CODs. Therefore, it is of practical importance to develop QOD construction techniques that do not rely on existing ODs or CODs.

This section also differs from Section II in that the focus is on QODs that are defined naturally on quaternion variables, as opposed to QODs that are defined naturally on real or complex variables and then manipulated to be written using quaternion variables. QODs on more than one quaternion variable appear to be the most difficult type of QODs to construct.

In Subsection III-A, we define *quaternion-commutivity* as a way to overcome certain of the difficulties presented by the non-commutivity of the quaternions. In Subsection III-B, we define *quaternion amicable designs*, which we expect to be building blocks in future QOD constructions. In Subsection III-C, we investigate certain conditions required to build 2×2 QODs on two quaternion variables.

A. Quaternion-Commuting Variables

The non-commutivity of the quaternions is clearly a hurdle in developing construction techniques for QODs, as it prevents simple generalizations of known construction techniques for ODs and CODs. In this subsection, we investigate a possible condition that would allow us to keep

the benefits of including the $\mathbf{j}$ and $\mathbf{k}$ components of the quaternion variables while restricting the quaternion variables so that they behave more like commuting variables.

Definition 15: Let $\mathbf{a}, \mathbf{b}$ be two quaternion variables. Then $\mathbf{a}$ and $\mathbf{b}$ are said to *quaternion-commute* when $\mathbf{ab}^Q = \mathbf{ba}^Q$ and/or $\mathbf{a}^Q\mathbf{b} = \mathbf{b}^Q\mathbf{a}$.

Our next example shows the utility of this definition as applied to two quaternion variables.

Example 16: Given the success of the Alamouti code represented by the COD $Z = \begin{bmatrix} z_1 & z_2 \\ z_2^* & -z_1^* \end{bmatrix}$, it is natural to start with $A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b}^Q & -\mathbf{a}^Q \end{bmatrix}$ and look for modification that would convert A into a QOD on quaternion variables $\mathbf{a}, \mathbf{b}$. Of course, A only satisfies the orthogonality constraint if $\mathbf{a}$ and $\mathbf{b}$ commute. However, the modification $A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & -\mathbf{a} \end{bmatrix}$ satisfies the orthogonality constraint if $\mathbf{a}$ and $\mathbf{b}$ are chosen to be quaternion-commuting variables. In this case,

$$\begin{aligned} AA^Q &= \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & -\mathbf{a} \end{bmatrix} \begin{bmatrix} \mathbf{a}^Q & \mathbf{b}^Q \\ \mathbf{b}^Q & -\mathbf{a}^Q \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{aa}^Q + \mathbf{bb}^Q & \mathbf{ab}^Q - \mathbf{ba}^Q \\ \mathbf{ba}^Q - \mathbf{ab}^Q & \mathbf{bb}^Q + \mathbf{aa}^Q \end{bmatrix} \\ &= (\mathbf{aa}^Q + \mathbf{bb}^Q)I \end{aligned}$$

and similarly for A^QA. Thus A is a QOD on the quaternion-commuting variables $\mathbf{a}$ and $\mathbf{b}$.

We now address the natural question of which quaternion variables quaternion-commute. To do this, we must introduce some notation:

Definition 17: Let $\mathbf{a}, \mathbf{b}$ be quaternion variables with $\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$ and $\mathbf{b} = b_1 + b_2\mathbf{i} + b_3\mathbf{j} + b_4\mathbf{k}$, where a_h, b_h are real variables $1 \leq h \leq 4$. Then, we denote the product $\mathbf{ab}^Q = \sigma(\mathbf{a}, \mathbf{b}) = \sigma_1 + \sigma_2\mathbf{i} + \sigma_3\mathbf{j} + \sigma_4\mathbf{k}$, where

$$\begin{aligned} \sigma_1 = \sigma_1(\mathbf{a}, \mathbf{b}) &= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4, \\ \sigma_2 = \sigma_2(\mathbf{a}, \mathbf{b}) &= -a_1b_2 + a_2b_1 - a_3b_4 + a_4b_3, \\ \sigma_3 = \sigma_3(\mathbf{a}, \mathbf{b}) &= -a_1b_3 + a_2b_4 + a_3b_1 - a_4b_2, \\ \sigma_4 = \sigma_4(\mathbf{a}, \mathbf{b}) &= -a_1b_4 - a_2b_3 + a_3b_2 + a_4b_1. \end{aligned}$$

Similarly, $\mathbf{a}^Q\mathbf{b} = \tau(\mathbf{a}, \mathbf{b}) = \tau_1 + \tau_2\mathbf{i} + \tau_3\mathbf{j} + \tau_4\mathbf{k}$, where

$$\begin{aligned} \tau_1 = \tau_1(\mathbf{a}, \mathbf{b}) &= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4, \\ \tau_2 = \tau_2(\mathbf{a}, \mathbf{b}) &= a_1b_2 - a_2b_1 - a_3b_4 + a_4b_3 \\ \tau_3 = \tau_3(\mathbf{a}, \mathbf{b}) &= a_1b_3 + a_2b_4 - a_3b_1 - a_4b_2 \\ \tau_4 = \tau_4(\mathbf{a}, \mathbf{b}) &= a_1b_4 - a_2b_3 + a_3b_2 - a_4b_1. \end{aligned}$$

Lemma 18: Two quaternion variables $\mathbf{a}$ and $\mathbf{b}$ quaternion-commute with $\mathbf{ab}^Q = \mathbf{ba}^Q$ (and/or $\mathbf{a}^Q\mathbf{b} = \mathbf{b}^Q\mathbf{a}$) if and only if $\mathbf{ab}^Q$ is real (and/or $\mathbf{a}^Q\mathbf{b}$ is real).

Proof: Using the σ notation developed in Definition 17, we see that just as $\mathbf{ab}^Q = \sigma_1 + \sigma_2\mathbf{i} + \sigma_3\mathbf{j} + \sigma_4\mathbf{k}$, we have $\mathbf{ba}^Q = (\mathbf{ab}^Q)^Q = \sigma_1 - \sigma_2\mathbf{i} - \sigma_3\mathbf{j} - \sigma_4\mathbf{k}$. Thus:

$$\begin{aligned} \mathbf{ab}^Q = \mathbf{ba}^Q &\iff \mathbf{ab}^Q - \mathbf{ba}^Q = 0 \\ &\iff (\sigma_1 + \sigma_2\mathbf{i} + \sigma_3\mathbf{j} + \sigma_4\mathbf{k}) - (\sigma_1 - \sigma_2\mathbf{i} - \sigma_3\mathbf{j} - \sigma_4\mathbf{k}) = 0 \\ &\iff 2(\sigma_2\mathbf{i} + \sigma_3\mathbf{j} + \sigma_4\mathbf{k}) = 0 \\ &\iff \sigma_2 = \sigma_3 = \sigma_4 = 0. \end{aligned}$$

Hence $\mathbf{ab}^Q$ is real. The case for $\mathbf{a}^Q\mathbf{b} = \mathbf{b}^Q\mathbf{a}$ follows similarly with $\mathbf{a}^Q\mathbf{b} = \mathbf{b}^Q\mathbf{a} \iff \tau_2 = \tau_3 = \tau_4 = 0$. (We observe that for both $\mathbf{ab}^Q = \mathbf{ba}^Q$ *and* $\mathbf{a}^Q\mathbf{b} = \mathbf{b}^Q\mathbf{a}$ to hold, we require $a_\ell b_m = b_\ell a_m$, for all $\ell \neq m$, $\ell, m = 1, 2, 3, 4$.) ■

Example 19: Suppose $\mathbf{a}$ and $\mathbf{b}$ are quaternion-commuting variables such that $\mathbf{ab}^Q = \mathbf{ba}^Q$ is real. Then, $D = \begin{bmatrix} \mathbf{a} & \mathbf{jb} \\ \mathbf{ib} & -\mathbf{ka} \end{bmatrix}$ is a QOD on these specific quaternion-commuting variables $\mathbf{a}$ and $\mathbf{b}$. This follows as

$$\begin{aligned} DD^Q &= \begin{bmatrix} \mathbf{a} & \mathbf{jb} \\ \mathbf{ib} & -\mathbf{ka} \end{bmatrix} \begin{bmatrix} \mathbf{a}^Q & -\mathbf{b}^Q\mathbf{i} \\ -\mathbf{b}^Q\mathbf{j} & \mathbf{a}^Q\mathbf{k} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{aa}^Q + \mathbf{bb}^Q & -\mathbf{ab}^Q\mathbf{i} + \mathbf{jba}^Q\mathbf{k} \\ \mathbf{iba}^Q + \mathbf{kab}^Q\mathbf{j} & \mathbf{aa}^Q + \mathbf{bb}^Q \end{bmatrix} \\ &= (|\mathbf{a}|^2 + |\mathbf{b}|^2)I_2. \end{aligned}$$

It also holds that $D^QD = (\mathbf{a}^Q\mathbf{a} + \mathbf{b}^Q\mathbf{b})I_2 = (|\mathbf{a}|^2 + |\mathbf{b}|^2)I_2$. We will later show in condition 2) of Example 24 that the proposed matrix D is also a QOD if the quaternion variables $\mathbf{a}$, $\mathbf{b}$ are instead restricted so that only $\sigma_3 = 0$, where $\sigma_3 = \sigma_3(\mathbf{a}, \mathbf{b})$ is as defined in Definition 17.

We have demonstrated that quaternion-commuting variables can be used to build QODs, however Lemma 18 shows that quaternion-commuting variables are subject to very strong restrictions. Therefore, this construction method is likely to be difficult for sizes greater than 2, as these would require more than two mutually quaternion-commuting variables. However, this notion of quaternion-commuting variables might be useful in conjunction with other conditions on certain of the quaternion variables.

B. Amicable Designs

Real amicable designs were defined over a quarter of a century ago [15], [16], [19], [20], and yet many questions concerning these designs, as well as complex amicable designs, remain unsolved.

Definition 20: Two square real orthogonal designs A and B are said to be *amicable* if $A^TB = B^TA$ and $AB^T = BA^T$. Two generalized complex orthogonal designs A and B are said to be *complex amicable* if $AB^H = BA^H$ and/or $A^HB = B^HA$.

Example 21: One can confirm that $A = \begin{bmatrix} x_1 & x_2 \\ -\mathbf{i}x_2 & \mathbf{i}x_1 \end{bmatrix}$ and $B = \begin{bmatrix} x_3 & x_4 \\ \mathbf{i}x_4 & -\mathbf{i}x_3 \end{bmatrix}$ are CODs on real variables x_1, x_2, x_3, x_4. Furthermore, A and B are complex amicable designs. This can be seen by noting that

$$A^HB = \begin{bmatrix} x_1x_3 - x_2x_4 & x_1x_4 + x_2x_3 \\ x_2x_3 + x_1x_4 & -x_1x_3 + x_2x_4 \end{bmatrix}$$

is real and symmetric. This implies that $A^HB = (A^HB)^T = (A^HB)^H = B^HA$. So, $A^HB = B^HA$. One can also verify that $AB^H = BA^H$.

Real amicable designs have been used to build complex orthogonal designs, and there has been a renewed interest in amicable designs due to the discovery that they can be utilized in space-time coding [8], [9], [21], [23]. Motivated by the utility of the real and complex cases, we propose a definition for quaternion amicable designs and investigate their existence in the 2×2 case.

Definition 22: Let A and B be quaternion orthogonal designs. If $AB^Q = BA^Q$, then say A and B are *quaternion amicable designs*.

Lemma 23: Suppose $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}$ are quaternion variables such that

$$\begin{aligned}
\mathbf{a}\mathbf{x}^Q &= \mathbf{x}\mathbf{a}^Q, \\
\mathbf{b}\mathbf{y}^Q &= \mathbf{y}\mathbf{b}^Q, \\
\tau_2(\mathbf{a}, \mathbf{b}) &= a_1b_2 - a_2b_1 - a_3b_4 + a_4b_3 = 0, \\
\tau_2(\mathbf{x}, \mathbf{y}) &= x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3 = 0, \\
\tau_2(\mathbf{a}, \mathbf{y}) &= a_1y_2 - a_2y_1 - a_3y_4 + a_4y_3 = 0, \\
\tau_2(\mathbf{b}, \mathbf{x}) &= b_1x_2 - b_2x_1 - b_3x_4 + b_4x_3 = 0.
\end{aligned}$$

Then the matrices

$$A = \begin{bmatrix} \mathbf{a} & \mathbf{b}\mathbf{j} \\ \mathbf{b}\mathbf{i} & -\mathbf{a}\mathbf{k} \end{bmatrix} \text{ and } B = \begin{bmatrix} \mathbf{x} & -\mathbf{y}\mathbf{j} \\ \mathbf{y}\mathbf{i} & \mathbf{x}\mathbf{k} \end{bmatrix}$$

are QODs on the specific quaternion variables $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y}$. (Given $\tau_2(\mathbf{a}, \mathbf{b}) = 0$, A is orthogonal; given $\tau_2(\mathbf{x}, \mathbf{y}) = 0$, B is orthogonal.) Also, $AB^Q = BA^Q$ so A and B are quaternion amicable designs.

Proof. Let $\mathbf{a}\mathbf{i}\mathbf{y}^Q = \alpha_1 + \alpha_2\mathbf{i} + \alpha_3\mathbf{j} + \alpha_4\mathbf{k}$. Then $(\mathbf{a}\mathbf{i}\mathbf{y}^Q)^Q = \alpha_1 - \alpha_2\mathbf{i} - \alpha_3\mathbf{j} - \alpha_4\mathbf{k}$. Now, $\mathbf{y}\mathbf{i}\mathbf{a}^Q = -(\mathbf{a}\mathbf{i}\mathbf{y}^Q)^Q$. Hence $\mathbf{y}\mathbf{i}\mathbf{a}^Q = -\alpha_1 + \alpha_2\mathbf{i} + \alpha_3\mathbf{j} + \alpha_4\mathbf{k}$. But $\alpha_1 = a_1y_2 - a_2y_1 - a_3y_4 + a_4y_3 = \tau_2(a, y) = 0$. So $\mathbf{a}\mathbf{i}\mathbf{y}^Q = \mathbf{y}\mathbf{i}\mathbf{a}^Q$. Likewise, it can be shown that $\mathbf{b}\mathbf{i}\mathbf{x}^Q = \mathbf{x}\mathbf{i}\mathbf{b}^Q$, given $\tau_2(b, x) = 0$. Then,

$$\begin{aligned}
AB^Q &= \begin{bmatrix} \mathbf{a} & \mathbf{b}\mathbf{j} \\ \mathbf{b}\mathbf{i} & -\mathbf{a}\mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{x}^Q & -\mathbf{i}\mathbf{y}^Q \\ \mathbf{j}\mathbf{y}^Q & -\mathbf{k}\mathbf{x}^Q \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{a}\mathbf{x}^Q - \mathbf{b}\mathbf{y}^Q & -\mathbf{a}\mathbf{i}\mathbf{y}^Q - \mathbf{b}\mathbf{i}\mathbf{x}^Q \\ \mathbf{b}\mathbf{i}\mathbf{x}^Q + \mathbf{a}\mathbf{i}\mathbf{y}^Q & \mathbf{b}\mathbf{y}^Q - \mathbf{a}\mathbf{x}^Q \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{x}\mathbf{a}^Q - \mathbf{y}\mathbf{b}^Q & -\mathbf{y}\mathbf{i}\mathbf{a}^Q - \mathbf{x}\mathbf{i}\mathbf{b}^Q \\ \mathbf{x}\mathbf{i}\mathbf{b}^Q + \mathbf{y}\mathbf{i}\mathbf{a}^Q & \mathbf{y}\mathbf{b}^Q - \mathbf{x}\mathbf{a}^Q \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{x} & -\mathbf{y}\mathbf{j} \\ \mathbf{y}\mathbf{i} & \mathbf{x}\mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{a}^Q & -\mathbf{i}\mathbf{b}^Q \\ -\mathbf{j}\mathbf{b}^Q & \mathbf{k}\mathbf{a}^Q \end{bmatrix} \\
&= BA^Q.
\end{aligned}$$

Thus, we have provided some conditions (albeit restrictive) under which 2×2 quaternion amicable designs exist. Many open questions remain concerning the construction and theory of quaternion amicable designs (as well as for real and complex amicable designs). A question of particular interest is to determine if small examples of quaternion amicable designs can be used to build larger QODs.

C. Size Two Quaternion Orthogonal Designs

Thus far, it has proven difficult to construct QODs on more than one quaternion variable. In order to build such designs without the use of ODs or CODs, we must develop an understanding of the forced constraints on the quaternion variables appearing in these designs. In this subsection, we study constraints under which we can design 2×2 QODs on two quaternion variables.

Example 24: Consider the matrix $D = \begin{bmatrix} \mathbf{a} & t\mathbf{b} \\ s\mathbf{b} & -u\mathbf{a} \end{bmatrix}$, with $s, t, u \in \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ and quaternion variables $\mathbf{a}, \mathbf{b}$. Then

$$\begin{aligned}
DD^Q &= \begin{bmatrix} \mathbf{a} & t\mathbf{b} \\ s\mathbf{b} & -u\mathbf{a} \end{bmatrix} \begin{bmatrix} \mathbf{a}^Q & -\mathbf{b}^Q s \\ -\mathbf{b}^Q t & \mathbf{a}^Q u \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{a}\mathbf{a}^Q + \mathbf{b}\mathbf{b}^Q & -\mathbf{a}\mathbf{b}^Q s + t\mathbf{b}\mathbf{a}^Q u \\ s\mathbf{b}\mathbf{a}^Q + u\mathbf{a}\mathbf{b}^Q t & \mathbf{a}\mathbf{a}^Q + \mathbf{b}\mathbf{b}^Q \end{bmatrix}
\end{aligned}$$

For this to be a $QOD(2;1,1)$, we require the off-diagonal elements to equal 0. We observe

$$\begin{aligned}(-\mathbf{ab}^Q s + t\mathbf{ba}^Q u)^Q &= (-\mathbf{ab}^Q s)^Q + (t\mathbf{ba}^Q u)^Q \\ &= s\mathbf{ba}^Q + u\mathbf{ab}^Q t.\end{aligned}$$

Thus if one of the diagonal elements equals 0, it follows that the other element is 0. An analysis shows that for $s\mathbf{ba}^Q + u\mathbf{ab}^Q t = 0$ to hold for *arbitrary* quaternion variables $\mathbf{a}, \mathbf{b}$, it would be required that the following four equations are simultaneously satisfied:

$$\begin{aligned}s + ut &= 0, \\ s\mathbf{i} &= u\mathbf{i}t, \\ s\mathbf{j} &= u\mathbf{j}t, \\ s\mathbf{k} &= u\mathbf{k}t.\end{aligned}$$

However, a lengthy analysis shows that it is not possible to simultaneously satisfy the above four equations, and therefore we cannot find solutions to $s\mathbf{ba}^Q + u\mathbf{ab}^Q t = 0$ that hold for arbitrary quaternion variables $\mathbf{a}, \mathbf{b}$. Nevertheless, by imposing certain restrictions on the quaternion variables $\mathbf{a}, \mathbf{b}$, we can still obtain conditions under which $s\mathbf{ba}^Q + u\mathbf{ab}^Q t = 0$ so that the proposed D is a QOD. The following are conditions under which $s\mathbf{ba}^Q + u\mathbf{ab}^Q t = 0$, where the $\sigma_h = \sigma_h(\mathbf{a}, \mathbf{b})$ are the components of $\mathbf{ab}^Q$ as given in Definition 17:

1) $s = \mathbf{i}, t = \pm\mathbf{k}, u = \mp\mathbf{j}, \sigma_4 = 0.$
2) $s = \mathbf{i}, t = \pm\mathbf{j}, u = \pm\mathbf{k}, \sigma_3 = 0.$
3) $s = \mathbf{j}, t = \pm\mathbf{k}, u = \pm\mathbf{i}, \sigma_4 = 0.$
4) $s = \mathbf{j}, t = \pm\mathbf{i}, u = \mp\mathbf{k}, \sigma_2 = 0.$
5) $s = \mathbf{k}, t = \pm\mathbf{j}, u = \mp\mathbf{i}, \sigma_3 = 0.$
6) $s = \mathbf{k}, t = \pm\mathbf{i}, u = \pm\mathbf{j}, \sigma_2 = 0.$

Hence, given quaternion variables and coefficients that satisfy one of the above six sets of constraints, we can form a QOD of the form $D = \begin{bmatrix} \mathbf{a} & t\mathbf{b} \\ s\mathbf{b} & -u\mathbf{a} \end{bmatrix}$. We note that solution 2) was previously presented by Finlayson, Seberry, Wysocki and Xia [7]. We have deliberately chosen to consider only solutions where s has positive sign, due to our Lemma 25 below.

Lemma 25: Let D be the matrix $\begin{bmatrix} \mathbf{a} & t\mathbf{b} \\ s\mathbf{b} & -u\mathbf{a} \end{bmatrix}$, and let $\mathbf{ab}^Q = \sigma_1 + \sigma_2\mathbf{i} + \sigma_3\mathbf{j} + \sigma_4\mathbf{k}$. Given the condition that $\sigma_p = 0$, for some p = 1, 2, 3, 4, if the use of s, t, u, where $s, t, u \in \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ implies that D is a QOD on quaternion variables $\mathbf{a}$ and $\mathbf{b}$, then the use of $-s, t, -u$ and $-s, -t, u$ also implies that D is a QOD.

Lemma 25 is motivated by the preceding Example 24, and its proof is straight-forward. Lemma 25 and Example 24 conspicuously do not consider scaling the (1,1) position with an element from $\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. We now consider this situation:

Example 26: Consider the matrix $D' = \begin{bmatrix} r\mathbf{a} & t\mathbf{b} \\ s\mathbf{b} & -u\mathbf{a} \end{bmatrix}$, with $r, s, t, u \in \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. Then

$$\begin{aligned}D'D'^Q &= \begin{bmatrix} r\mathbf{a} & t\mathbf{b} \\ s\mathbf{b} & -u\mathbf{a} \end{bmatrix} \begin{bmatrix} -\mathbf{a}^Q r & -\mathbf{b}^Q s \\ -\mathbf{b}^Q t & \mathbf{a}^Q u \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{aa}^Q + \mathbf{bb}^Q & -r\mathbf{ab}^Q s + t\mathbf{ba}^Q u \\ -s\mathbf{ba}^Q r + u\mathbf{ab}^Q t & \mathbf{aa}^Q + \mathbf{bb}^Q \end{bmatrix}\end{aligned}$$

As before, for this to be a $QOD(2;1,1)$, the off-diagonal elements must equal zero. We can achieve this by placing conditions on $\mathbf{a}$ and $\mathbf{b}$ through restricting the terms $\sigma_h(\mathbf{a}, \mathbf{b})$ as follows:

1) $r = s, \sigma_1 = 0.$

2) $r = \mathbf{i}, s = \mathbf{j}, t = -\mathbf{j}, u = \mathbf{i}, \sigma_4 = 0$.
3) $r = \mathbf{i}, s = \mathbf{k}, t = -\mathbf{k}, u = \mathbf{i}, \sigma_3 = 0$.
4) $r = \mathbf{j}, s = \mathbf{k}, t = -\mathbf{k}, u = \mathbf{j}, \sigma_2 = 0$.

In this example, we have limited our list to cases with positive r and s due to the following Lemma 27; the proof of this lemma is straight-forward.

Lemma 27: Let D be the matrix $\begin{bmatrix} r\mathbf{a} & -s\mathbf{b} \\ s\mathbf{b} & -r\mathbf{a} \end{bmatrix}$, and let $\mathbf{ab}^Q = \sigma_1 + \sigma_2\mathbf{i} + \sigma_3\mathbf{j} + \sigma_4\mathbf{k}$. Given the condition $\sigma_h = 0$ for some $1 \leq h \leq 4$, if the use of r, s imply that D is a QOD on quaternion variables $\mathbf{a}$ and $\mathbf{b}$, then the use of the $r' = -r, s' = s$; $r' = r, s' = -s$; $r' = -r, s' = -s$ and $r' = s, s' = r$ also imply that D is a QOD.

To see the distinction between QODs of the types introduced in Examples 24 and 26, let D be the $QOD(2;1,1)$ $\begin{bmatrix} r\mathbf{a} & t\mathbf{b} \\ s\mathbf{b} & -u\mathbf{a} \end{bmatrix}$ for some $r, s, t, u \in \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. We can write $D = r\begin{bmatrix} \mathbf{a} & t'\mathbf{b} \\ s'\mathbf{b} & -u'\mathbf{a} \end{bmatrix} = rD'$, where $t', s', u' \in \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. However, the conditions required to make D a QOD are not consistent with the conditions required for D' to be a QOD. Therefore, the designs introduced in Examples 24 and 26 are not equivalent.

A very similar analysis can be done using quaternion pre-multipliers on the right, rather than the left, of the entries of a $QOD(2,2;1,1)$. Despite the differences due to the non-commutivity of the quaternions, the constraints are similar. We omit the details for brevity.

IV. Simulation Results

In this section, we provide some numerical results concerning the implementation of a quaternion orthogonal design as an "orthogonal space-time-polarization block code (OSTPBC)." As with traditional complex orthogonal space-time block codes [18], OSTPBCs can be used in scenarios involving transmission of complex signals from several transmit antennas to one or more receive antennas, as shown in Fig. 1. The only difference is that with OSTPBCs the antennas are dual polarized antennas. Hence, in case of a single receive antenna and N transmit antennas, each channel is described by the channel gain matrix $H^{(m)} = \begin{bmatrix} h_{11}^{(m)} & h_{12}^{(m)} \\ h_{21}^{(m)} & h_{22}^{(m)} \end{bmatrix}$, for $m = 1, 2, \ldots, N$, where h_{11} and h_{22} are complex channel gains for signals received with the same polarization as they were transmitted, and h_{12} and h_{21} are complex channel gains for a cross-polar scatter, *i.e.*, signals received with different polarization from that at which it was transmitted due to scatter, reflections, and polarization twist between the transmit and receive antennas.

Introducing representation of a quaternion variable $s = z_1 + z_2\mathbf{j}$ as $s = [z_1, z_2]$, transmission of the dual-polarized signal through the channel described by a matrix H can be modelled as a product sH. Hence the received quaternion signal r is given by:

$$\begin{aligned} r &= sH + n \\ &= [z_1, z_2]H + [n_1, n_2] \\ &= [z_1h_{11} + z_2h_{21}, z_1h_{12} + z_2h_{22}] + [n_1, n_2] \\ &= (z_1h_{11} + z_2h_{21}) + (z_1H_{12} + z_2h_{22})\mathbf{j} + n_1 + n_2\mathbf{j} \end{aligned}$$

where n_1, n_2 are complex additive noises being the independent identically distributed (i.i.d.) zero-mean two dimensional Gaussian random variables with identical variance per dimension.

In this paper, we will consider only the case of a single receive antenna but all considerations can be extended to the case of multiple receive antennas. If an OSTPBC Q for N transmit

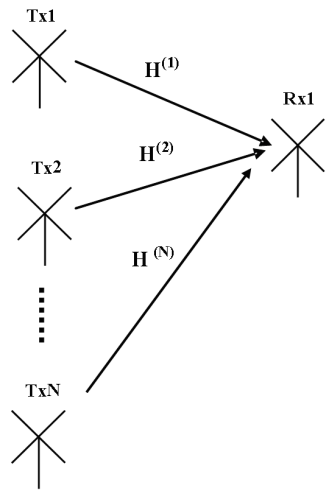

Fig. 1. Considered transmission system utilizing N dual-polarized transmit antennas Tx1, ..., TxN, and a single receive dual-polarized antenna.

antennas is used, the received dual-polarized signal vector R can be considered as a quaternionic vector, and, using the same notation as above, modelled as:

$$R = Q \begin{bmatrix} H^{(1)} \\ \vdots \\ H^{(N)} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ \vdots & \vdots \\ n_{N1} & n_{N2} \end{bmatrix}$$

where n_{ml}; $m = 1, \ldots, N$, $l = 1, 2$ represent complex noises being the independent identically distributed (i.i.d.) zero-mean two dimensional Gaussian random variables with identical variance per dimension.

Assuming perfect channel knowledge at the receiver, *i.e.*, assuming that matrices $H^{(1)}$, ..., $H^{(N)}$ are known and constant for some reasonable time, as in the case of quasi static conditions, the ML decoding rule for any OSTPBC is equivalent to finding a set of signal symbols that minimizes the norm $||R - QH||$, where R is the received signal vector, Q is the code matrix, and H is the matrix of complex channel coefficients. This is equivalent to finding a set of signal symbols minimizing the squared norm, which can be expressed as:

$$\begin{aligned} ||R - QH||^2 &= trace\{(R - QH)^Q (R - QH)\} \\ &= trace\{(R^Q - H^Q Q^Q)(R - QH)\} \\ &= trace\{R^Q R - R^Q QH - H^Q q^Q R + H^Q Q^Q QH\} \\ &= trace\{R^Q R\} - 2Re\{trace[R^Q QH]\} + \lambda trace H^Q H \end{aligned}$$

where $Q^Q Q = \lambda I$. Thus, for any orthogonal Q, the decoding rule can be decoupled.

We now provide a concrete example by implementing the QOD of order 2 originally given in Example 11 as an OSTPBC for two transmit dual-polarized antennas:

$$Q_1 = \begin{bmatrix} z_1 + z_2\mathbf{j} & z_2 + z_1\mathbf{j} \\ -z_2^* + z_1^*\mathbf{j} & z_1^* - z_2^*\mathbf{j} \end{bmatrix}$$

It can be easily noticed that Q_1 comprises two Alamouti codes [1], one used for one polarization and another one, with switched columns, used for orthogonal polarization. Using the same notation

as above, Q_1 can be expressed as:

$$Q_1 = \begin{bmatrix} z_1 & z_2 & z_2 & z_1 \\ -z_2^* & z_1^* & z_1^* & -z_2^* \end{bmatrix}$$

where odd columns represent signals transmitted through one polarization and even columns represent signals transmitted through the polarization orthogonal to the first one.

For a single receive dual-polarized antenna, the channels between dual-polarized transmit antennas Tx1 and Tx2, and the dual-polarized receive antenna Rx1 are described by their own channel gain matrices, $H^{(1)} = [h_{m\ell}^{(1)}]_{2\times 2}$ and $H^{(2)} = [h_{m\ell}^{(2)}]_{2\times 2}$, respectively. The received signal vector R is given by

$$R \;=\; Q_1 \begin{bmatrix} H^{(1)} \\ H^{(2)} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix},$$

where $n_{m\ell}$; $m, \ell = 1, 2$ represent complex noises being the independent identically distributed (i.i.d.) zero-mean two dimensional Gaussian random variables with identical variance per dimension.

Utilizing the principles of quaternionic arithmetic, the decoding rule can be simplified as follows: the decoding statistic for z_1 is $\min_{z_1 \in Z} |z_1|^2(|g_1|^2 + |g_2|^2 + |g_3|^2 + |g_4|^2)$ - 2Re$\{r_1^Q z_1(g_1 + g_2\mathbf{j}) + r_2^Q z_1^*(g_3 + g_4\mathbf{j})\}$, and the decoding statistic for z_2 is $\min_{z_2 \in Z} |z_2|^2(|g_1|^2 + |g_2|^2 + |g_3|^2 + |g_4|^2) - 2\text{Re}\{r_1^Q z_2(g_3 + g_4\mathbf{j}) - r_2^Q z_2^*(g_1 + g_4 2\mathbf{j})\}$, where $g_1 = h_{11}^{(1)} + h_{21}^{(2)}$, $g_2 = h_{12}^{(1)} + h_{22}^{(2)}$, $g_3 = h_{21}^{(1)} + h_{11}^{(2)}$, and $g_4 = h_{22}^{(1)} + h_{12}^{(2)}$.

To assess the performance gain that can be achieved using the code Q_1 and a single receive dual-polarized antenna, the system was implemented using MATLAB. The following conditions were assumed:

- The QPSK $\{0, \pi/2, \pi, 3\pi/2\}$ signal constellation was applied.
- Total transmitted power in both polarizations and through both antennas was equal to 1, and equally distributed per antenna and per polarization. For a fair comparison, the average total received power in all systems before adding noise was set to 1, and we used the same SNR normalization as used by Alamouti [1].
- Channel coefficient matrices, $H^{(1)}$ and $H^{(2)}$ were assumed known at the receiver and kept constant for 1000 data bits.
- The sum of variances of all the channel coefficients was kept equal to 1 and that the variance of real and imaginary part of a particular coefficient was identical. The variances were drawn randomly, every time the new set of coefficients was drawn.
- The channel coefficients were generated as random complex Gaussian i.i.d. variables. The additive noise was assumed to be AWGN added uniformly for each polarization and each real/imaginary component (a quaternion zero-mean Gaussian variable).

The simulation results are provided in Fig. 2. The evident performance improvement compared to the conventional Alamouti scheme has been achieved without a significant increase in the dimensions of the antenna system of either transmitter or receiver. These results are promising, and we expect to explore further the implementation of QODs as OSTPBCs in future work. It remains an open problem to determine if these codes can admit received signal linearization.

V. Conclusions

We have established the existence of quaternion orthogonal designs (QODs) on real variables, on complex variables, and on quaternion variables. We offered construction techniques that utilize existing real and complex orthogonal designs to build QODs. These constructions provide insight into the combinatorial structure of QODs, and they are useful for building a library of QOD

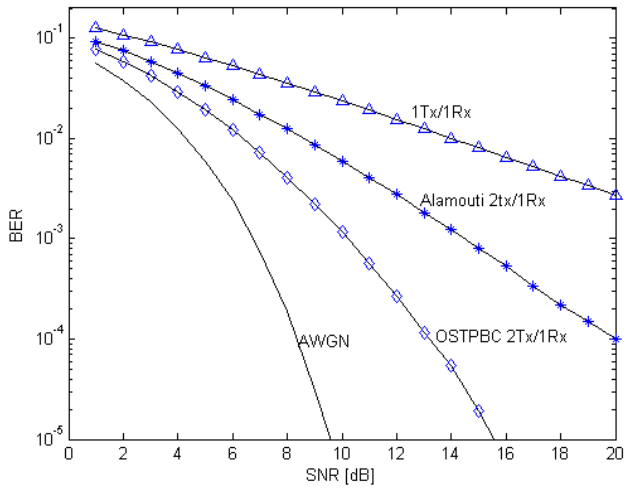

Fig. 2. Bit-error-rate (BER) performance of the developed scheme (OSTPBC) using the code Q_1 combined with QPSK modulation in a slow flat Rayleigh fading channel experiencing random cross-polar scatter compared with the Alamouti scheme and a single transmit/single receive antenna system.

examples. However, by depending on existing ODs and CODs, the resulting QODs may be limited in their rate and delay. We propose that in order to generate optimal QODs, we must develop construction techniques that do not limit the QODs through imposing structure based on ODs or CODs. To work towards the goal of developing QODs (specifically QODs on quaternion variables) without reliance on ODs or CODs, we defined and explored the concepts of quaternion-commutivity and quaternion amicability. We hope these concepts will stimulate further work in this area. Our work towards building 2×2 QODs on two quaternion variables illuminates some challenges in constructing these designs. Handling QODs on multiple quaternion variables is more difficult than handling ODs, CODs, or even QODs on real and complex variables. The non-commutivity of the quaternions, as well as the increase to four real dimensions in each quaternion variable ($\mathbf{a} = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$), contributes to this difficulty.

Our definition of symmetric-paired designs and our treatment of amicable designs lead to several open questions. Necessary and sufficient conditions must be determined for when CODs A and B can be constructed so that $A^H B$ is symmetric. Similarly, it must be determined when QODs A and B can be constructed so that $A^Q B = B^Q A$ or $BA^Q = AB^Q$. We expect that an analysis of symmetric-paired and amicable designs will be useful in future constructions of QODs

Our preliminary simulations involving the application of QODs as quaternion orthogonal space-time-polarization block codes, particular those involving the QOD presented in Example 11, are promising. The simulations suggest that certain QODs will be capable of achieving the same performance as complex orthogonal space-time-block codes that require twice the number of transmit and receive antennas. It remains to determine which QODs are optimally suited for application in space-time-polarization block coding. This determination will require further simulations and the further development of the theory of QODs. For example, the maximum rate and minimum decoding delay of QODs must be determined. In our future work, we plan to both advance the theory of these novel designs and further investigate their application as space-time-polarization block codes.

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REFERENCES

[1] S. M. Alamouti. A simple transmit diversity technique for wireless communications. *IEEE Journal on Selected Areas in Communications*, 16(8):1451–1458, 1998.

[2] S. L. Altmann. *Rotations, Quaternions, and Double Groups.* Oxford, England: Clarendo Press, 1986.

[3] A. R. Calderbank, S. Das, N. Al-Dhahir, and S. N. Diggavi. Construction and analysis of a new quaternionic space-time code for 4 transmit antennas. *Communications in Information and Systems*, 5(1):1-26, 2005.

[4] B. S. Collins. Polarization-diversity antennas for compact base stations. *Microwave Journal*, 43(1):76–88, January 2000.

[5] B. S. Collins. The effect of imperfect antenna cross-polar performance on the diversity gain of a polarization-diversity system. *Microwave Journal*, 43(4):84–94, April 2000.

[6] C. B. Dietrich, K. Dietze, J. R. Nealy, and W. L. Stulzman. Spatial polarization and pattern diversity for wireless handheld terminals. *IEEE Trans. Antennas Propagat.*, 49:1271–1281, 2001.

[7] K. Finlayson, J. Seberry, T. Wysocki and T. Xia. Orthogonal designs with quaternion elements. ISCTA'05, *Proceedings of 8th International Symposium on Communication Theory and Applications,* Ambleside, UK, 17-22 July 2005, HW Communications, (2005) 270-272.

[8] G. Ganesan and P. Stoica. Space-time diversity using orthogonal and amicable orthogonal designs. *IEEE International Conference on Acoustics, Speech, and signal Processing (ICASSP)*, 5:2561–2564, 2000.

[9] G. Ganesan and P. Stoica. Space-time block codes: A maximum SNR approach. *IEEE Trans. on Inform. Theory*, 47(4):1650–1656, 2001.

[10] A. V. Geramita, J. M. Geramita, and J. Seberry Wallis. Orthogonal designs. *Linear and Multilinear Algebra* 3 (1976), 281–306.

[11] A. V. Geramita and J. M. Geramita. Complex orthogonal designs. *J Combinatorial Theory*, Ser A, 3 (1978), 211–225.

[12] O. M. Isaeva, and V. A. Sarytchev. Quaternion presentations polarization state. *Proc. 2nd IEEE Topical Symposium of Combined Optical-Microwave Earth and Atmosphere Sensing*, Atlanta, GA USA, 3–6 April 1995, 195–196.

[13] X.-B. Liang. Orthogonal designs with maximal rates. *IEEE Trans. Inform. Theory*, 49(10):2468–2503, 2003. Special issue on space-time transmission, reception, coding and signal processing.

[14] R. U. Nabar, H. Bölcskei, V Ereg, D. Gesbert, A. J. Paulraj. Performance of multiantenna signalling techniques in the presence of polarization diversity. *IEEE Trans. Signal Processing*, 50:2553–2562, 2002.

[15] P. J. Robinson. Amicable orthogonal designs. *Bull. Austral. Math. Soc.*, 14(2):303–314, 1976.

[16] P. J. Robinson and J. Seberry. On the structure and existence of some amicable orthogonal designs *J. Austral. Math. Soc. (Ser. A)*, 25, 118–128, 1978.

[17] J. Seberry, S. A. Spence, T. Wysocki. A construction technique for generalized complex orthogonal designs and applications to wireless communications. *Linear Algebra Appl.*, 405:163–176, 2005.

[18] V. Tarokh, H. Jafarkhani, and A. R. Calderbank. Space-time block codes from orthogonal designs. *IEEE Trans. Inform. Theory*, 45(5):1456–1467, 1999.

[19] J. S. Wallis. Constructions for amicable orthogonal designs. *Bull. Austral. Math. Soc.*, 12:179–182, 1975.

[20] W. Wolfe. Amicable orthogonal designs—existence. *Canad. J. Math.*, 28(5):1006–1020, 1976.

[21] C. Yuen, Y. L. Guan, and T. T. Tjhung. Orthogonal space-time block code from amicable complex orthogonal design. *IEEE International Conference on Acoustics, Speech, and signal Processing (ICASSP)*, 4:469–472, 2004.

[22] Z. Zhao, S. Stapleton, and J. K. Cavers. Analysis of polarization diversity scheme with channel codes. *Proc. IEEE VTC'99*, 1337–1381, 1999.

[23] Z. Chen, G. Zhu, J. Shen, and Y. Liu. Differential space-time block codes from amicable orthogonal designs. *IEEE Wireless Communications and Networking*, 2:768–772, 2003.