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Pricing Convertible Bonds Based On A Multi-stage Compound Option Model*

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Abstract In this paper, we introduce the concept of multi-stage compound options to the valuation of convertible bonds. Rather than evaluating a nested high-dimensional integral that has arisen from the valuation of multi-stage compound options, we found that adopting the Finite Difference Method (FDM) to solve the Black-Scholes equation for each stage actually resulted in a better numerical efficiency. By comparing our results with those obtained by solving the Black-Scholes equation directly, we can show that the new approach does provide an approximation approach for the valuation of convertible bonds and demonstrate that it offers a great potential for a further extension to CBs with more complex structures such as those with call and/or put provisions.

Keywords convertible bonds, compound options, Finite Difference Method

1 INTRODUCTION

Convertible bonds are hybrid financial instruments with sophisticated features that provide investors a right to convert the bond into a predetermined number of stocks with a pre-specified price, or to hold the bond till maturity to receive coupons and the principal. The quantitative valuation of convertible bonds in the literature can be traced back to Ingersoll [1, 2], who introduced the contingent claims approach to price

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convertible bonds and derived several closed-form solutions for some special cases. Brennan and Schwartz [3, 4] utilized arbitrage-free arguments to derive a partial differential equation (PDE) and the appropriate boundary conditions for the value of convertible bonds under some quite general conditions and solved them by the finite difference method. In their model, convertible bonds are viewed as the contingent claims on the firm-value. However, the firm value is not directly tradable and unobservable in the market, which makes the estimate of some parameters such as the volatility of underlying variable quite difficult (see, e.g. [5]), models based on the firm value are thus difficult to be used in practice. McConnell and Schwartz [6] took convertible bonds as derivatives of the underlying equity and proposed a single-factor pricing model for zero-coupon CBs. Most of existing pricing models for CBs today are based on this model.

An important feature of convertible bonds is that holders can convert their CBs at any time before expiry. This feature can be viewed as to exercise their “rights” of conversion at any time before expiry. This nature of having rights, but no obligations has made the CBs similar to American options. Many pricing approaches for American options can thus be used to price convertible bonds. One of these approaches is the compound option approach initially proposed by Roll [7] as a natural way to approximate the value of an American option.

A simple compound option is an option on another option, and a multi-stage compound option is a series of sequentially ordered compound options. In financial practice, compound options can be used to model any sequence of rights. So they are most suitable to be applied to problems involving any sequential decision making. For example, R&D projects usually are of multi-stage nature (see, e.g. [8]); each stage has its own objective and funding. Only when the goal of an earlier stage is achieved, can the project be allowed to enter into the next stage. Venture capital investment is another typical example of multi-stage investment. If the given goal during the operation is not achieved, the venture capitalist can cancel the investment of next stage (see, e.g. [9]), resulting in a completely pull out of his initially intended total investment. Sylvia and Trigeogis [10] showed that firms’ strategic decision making can also be viewed as a multi-stage decision process. When senior executives make a managerial decision, they are interested in not only the direct predictable cash flow, but also the potential future investment opportunities that may be created by the current investment, which will potentially generate a considerable amount of cash flow in the future or makes the firm stay in a favorite competition position.

Another important application of compound options is to approximate the value of American options. Based on an early work of Geske [11], Roll [7] derived an analytical valuation formula for American call options on a stock that pays discrete dividends. This formula was further developed by Geske [12], Whaley [13] and Geske and Johnson [14]. These models attracted widely attention and were further extended

later. Shastri and Tandon [15, 16] separately derived a closed-form pricing formula for American options on futures contracts and foreign exchange (FX) options on major currencies. Bunch and Johnson [17] presented a more efficient analytical approach for American puts based on the compound option approach of Geske [12] and Geske and Johnson [14]. Recently Gukhal [18] derived an analytical valuation formula for compound options when the underlying asset follows a jump-diffusion process. His formula can be used to value American call options on stocks that pay discrete dividends.

We have started our research from the simplest form of CBs, i.e., those with earlier conversion privilege only. Instead of solving the Black-Scholes equation for the CB, we approximate a CB by a sequence of multi-stage compound options, the payoff function of each stage is carefully selected so that when the number of stages approaches the infinity the solution of our multi-stage compound option model approaches to that of the original CB.

The lifetime of a CB is firstly divided into a finite number of stages, within each of which the CB is modeled as a European call option. Under the framework of Black-Scholes, the value of the option is governed by the Black-Scholes equation. Then, we solve the Black-Scholes for each stage sequentially to find the value of the option for that stage, subject to the pay-off function that is suitable for the definition of the CB. The optimal conversion price, corresponds to the optimal exercise price for American options, is also found at the end of each stage so that an investor can easily decide if he/she should convert his/her CB to the underlying asset or to continue to hold the CB.

In Section 2, we shall first describe our compound-option model for the valuation of a CB with conversion only. Then, in Section 3, we shall outline the numerical approach we adopt to solve the governing differential equation in each stage. In Section 4, we shall show some examples, demonstrating the validation of our model and finally our conclusions are provided in Section 5.

2 THE MULTI-STAGE COMPOUND OPTION MODEL

Suppose that a CB has m discrete coupon payments within its lifetime, as long as the CB is not converted. Let t_k^* ($k = 1, 2, \dots, m$) denote the preset coupon delivery dates of the CB and r_k^* the coupon rate, by which the amount of cash received by the holder of the CB at the time t_k^* is calculated. Denote the preset conversion price as I_c , which is assumed to be constant in the lifetime of convertible bond.

A standard American-style CB can be converted into the underlying stocks anytime before and on the expiry date. By limiting the early conversion privilege to occur only at a set of predetermined instants, a convertible bond can be modeled as a multi-stage compound option. Let t_k ($k = 0, 1, \dots, n; n \geq m$) be the k th decision-making instant at which a decision has to be made on whether the holder

should convert the convertible bond into underlying stocks or continue to hold it. Thus $T = t_n - t_0$ is the lifetime of convertible bond. Without losing generality, we assume all the coupon delivery dates $t_k^* (k = 1, 2, \dots, m)$ are a subset of decision-making instants, i.e. $t_k^* \in \{t_{k'} : k' = 0, 1, \dots, n\}$, while the last coupon delivery date is always the expiry date (i.e., $t_m^* = t_n$). Let

$$r_i = \begin{cases} r_k^* & \text{if } (t_i = t_k^*) \\ 0 & \text{else} \end{cases} \quad (i = 0, 1, \dots, n; k = 1, 2, \dots, m)$$

be the coupon rate at any decision-making instant t_i and V_t be the underlying stock price at time t . Illustrated in Figure 1 is our n -stage compound option model for convertible bonds.

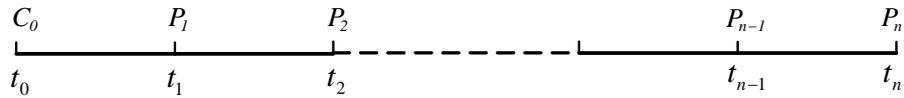


Figure 1. The n -stage compound option model of convertible bonds

In this figure,

$$P_k(V_{t_k}) = \begin{cases} \max \left\{ \frac{K}{I_c} V_{t_k}, Kr_k + C_k(V_{t_k}, t_k) \right\} & k = 0, 1, \dots, n-1 \\ \max \left\{ \frac{K}{I_c} V_{t_n}, K(1 + r_n) \right\} & k = n \end{cases}$$

is the payoff function at $t_k (k = 1, 2, \dots, n)$, where $C_k(V_t, t) (t \in [t_k, t_{k+1}])$, $k = 0, 1, \dots, n-1$ is the value of the convertible bond when the underlying stock price is V_t , and K is the par value of convertible bond. The holder may choose to either convert his convertible bond into the underlying asset or receive the interest income and still keep the right by rolling over to the next stage. The reason that the payoff function is of a different form at $k = n$ is because at expiration the investor must choose between converting the convertible bond into stocks or taking the principal and interest. At any moment, whether the holder carries out his right or not depends on the current profit and the expectation of future profit.

Clearly, for a fixed T , the value of the constructed multi-stage compound options converges to that of the American-style convertible bond as $n \rightarrow \infty$; the holder can actually carry out the conversion at any instant prior to the expiry. Theoretically, without numerical errors, the multi-stage compound option approach can give an approximation to the value of CBs with a conversion of American style with any accuracy when n is sufficiently large.

We start with the assumption that underlying stock price V_t follows a geometric Brownian motion,

$$dV_t = (\alpha - D)V_t dt + \sigma V_t dz$$

where α is the drift rate, σ is the volatility rate, and D is the continuous dividend rate. We also assume that all investors are risk-neutral investors. Then the optimal conversion strategy is to maximize expected wealth. According to the Risk

Neutral Pricing Theory, the expected return rate of any asset (tradable or non-tradable) is exactly the risk-free interest rate. So the present value of any asset is the expected future cash flow under risk neutral measure calculated with the risk-free interest rate:

$$C_k(V_{t_k}, t_k) = e^{-r_f(t_{k+1}-t_k)} \int_0^{+\infty} P_{k+1}(V_{t_{k+1}}) \Psi(V_{t_{k+1}} | V_{t_k}) dV_{t_{k+1}}, \quad (k = 0, 1, \dots, n-1) \quad (1)$$

where $\Psi(\cdot)$ is the probability density function under risk-neutral measure and r_f is the riskfree interest rate.

When closed-form analytical solutions are not available for the compound option model with a complicated payoff function, numerical solutions are usually resorted to. One type of the available numerical approaches is the lattice-based approach such as the binomial method. Trigeorgis [19] presented a numerical method called Log-Transformed Binomial Numerical Analysis Method, to value complex investments with multiple interacting options, including compound options. The method can achieve good consistency, stability, and efficiency. Breen [20] presented the Accelerated Binomial Option Pricing Model based on the binomial and Geske-Johnson models, which is faster than the conventional binomial model and applicable to a wide range of option pricing problems.

Although numerical methods based on Binomial Option Pricing Model are easy to use, they are known to have poor performance in estimating the Greeks, which market practitioners are often most interested in. On the other hand, those methods based on the entire grid being generated once and for all, such as finite difference method (FDM) and Finite Elements Method (FEM) are applicable for more complex problems and allow us to obtain better estimates of some of the Greeks.

Lin [9] numerically calculated all integrals in (1) to obtain the value of a compound option and the critical exercise price. He used Drezner's improved Gauss quadrature method, the Monte Carlo method and Lattice method to compute multivariate normal integral in order to obtain the value of multi-stage compound options. In his procedure, the critical exercise price needs to be found first, which means that the root of a nonlinear equation needs to be calculated. Lin utilized Newton-Raphson method, Dekker method and Secant method. However, due to the nested high-dimension integrals, when the total number of stages becomes large, it costs too much to compute the nested high dimensional integrals. Fast numerical algorithms for high dimensional integrals are thus required. Besides, it is difficult to handle the accuracy and convergence of numerical solution.

In order to avoid computing nested high dimension integrals arising from the multi-stage compound option model, we solve the problem, stage by stage, with a direct application of the finite difference method to each stage. Such an approach turns out to be much more efficient than integrating nested high dimensional integrals directly.

Although the options embedded in the multi-stage compound option model are sequentially compounded, they are still derivatives written on an underlying asset directly or indirectly. So, under the Black-Scholes framework, the price of the CB within each stage, $C_k(V_t, t)$ ($t \in [t_k, t_{k+1}]$, $k = 0, \dots, n-1$) must satisfy the following governing partial differential equation (PDE)

$$\frac{\partial C_k}{\partial t} + \frac{1}{2} \sigma^2 V_t^2 \frac{\partial^2 C_k}{\partial V_t^2} + (r_f - D)V_t \frac{\partial C_k}{\partial V_t} - rC_k = 0 \quad (2)$$

The boundary and terminal conditions can be set according to the properties of convertible bonds. The computation goes backwards in time. So, we always start with the last stage. At the maturity (or stage n in our model), the holder may choose to convert his convertible bond into stocks, or choose to receive the par and the interest. Thus the terminal condition for $C_{n-1}(\cdot)$ with $t \in [t_{n-1}, t_n]$ in the last stage is

$$C_{n-1}(V_{t_n}, t_n) = P_n(V_{t_n}) = \max \left\{ \frac{K}{I_c} V_{t_n}, K(1 + r_n) \right\}. \quad (3)$$

For any other earlier stage k , with $t \in [t_k, t_{k+1}]$, $k = 1, 2, \dots, n-1$, the terminal condition for the option price C_{k-1} is

$$C_{k-1}(V_{t_k}, t_k) = P_k(V_{t_k}) = \max \left\{ \frac{K}{I_c} V_{t_k}, Kr_k + C_k(V_{t_k}, t_k) \right\}. \quad (4)$$

In fact, the terminal condition (4) reflects exactly the compound relationship between the option in the earlier stage and the one in the next stage. The first part of brackets inside of (3) and (4) corresponds to holder's choice of receiving the par and interest income at expiry or carrying the right into the next stage. The second part, on the other hand, corresponds to holder's choice of exercising his right to convert the bond into the underlying stocks.

From (3) and (4), we can get the optimal conversion price. If we define

$$H(V_{t_k}, t_k) = \begin{cases} Kr_k + C_k(V_{t_k}, t_k) - \frac{K}{I_c} V_{t_k} & k = 1, 2, \dots, n-1 \\ K(1 + r_n) - \frac{K}{I_c} V_{t_n} & k = n \end{cases} \quad (5)$$

then the optimal conversion price, $V_{t_k}^*$, at t_k is defined as

$$H(V_{t_k}^*, t_k) = 0 \quad (6)$$

Once the stock price goes up above the optimal conversion price $V_{t_k}^*$ the holder should convert his convertible bond into stocks. Otherwise, the holder should carry his right over to the next stage.

If we assume that there is no risk of default of the firm, then no matter how small the underlying asset value is, the holder will naturally choose to roll over his right to the next stage, and keeps doing so until he receives the par and the interest at expiry. Therefore, the option component embedded in the convertible bond is trivial and the convertible bond is equivalent to the corresponding pure bond. Consequently, the lower boundary condition should be written as

$$\lim_{V_t \rightarrow 0} C_k(V_t, t) = K e^{-r_f(t_{k+1}-t)} \left(\sum_{m=0}^{n-k-1} r_{k+m+1} e^{-r_f(t_{k+m+1}-t_{k+1})} + e^{-r_f(t_n-t_{k+1})} \right) \quad (7)$$

where $t \in [t_k, t_{k+1}]$, $k = (0, 1, \dots, n-1)$.

On the other hand, when underlying stock price is very high, it is almost sure that convertible bond will be converted into stocks. As stated before, the underlying stock value, at which the conversion should optimally take place, forms a moving boundary. However, in the next section when we need to compute the CB's value, using an approach suggested in [22], as a comparison to our compound-option model, we need to set up an upper boundary to start the iteration. This can be set by using a similar approach to the one that an upper boundary condition is set for European options (see, e.g. [21]), i.e., the convertible bond is equivalent to the underlying stock but without its dividend income. Thus the upper boundary condition is

$$\lim_{V_t \rightarrow +\infty} C_k(V_t, t) = \frac{K}{I_c} V_t e^{-D(t_{k+1}-t)} \quad (8)$$

where $t \in [t_k, t_{k+1}]$, $k = (0, 1, \dots, n-1)$.

The governing equation (2) together with the terminal conditions (3) and (4) and the boundary conditions (7) and (8) constitute a PDE system. The solution of this system can be numerically solved in many different ways. We decided to adopt the finite difference method because the solution domain is a regular half-zonal domain, $(t, V_t) \in \{[t_k, t_{k+1}], [0, +\infty)\}$, and the terminal conditions and boundary conditions are both of Dirichlet type.

3 NUMERICAL SOLUTION

The computation was carried out in a log-transformed domain with the logarithm of the compound option price, $z_t = \ln(V_t) \in (-\infty, +\infty)$, as one of the independent variables and the current time as the other. To avoid dealing with infinite values of z_t , we chose a small value of V_t, \underline{V} , different from zero so that the range of z_t values starts from a value small enough, $\underline{z} = \ln(\underline{V})$. Similarly, a finite upper bond is chosen for z_t so the largest value of z_t is $\bar{z} = \ln(\bar{V})$, which is sufficiently large but still finite. Thus, the truncated log-transformed domain of computation is $t \in [t_k, t_{k+1}]$, $z_t \in [\underline{z}, \bar{z}]$. Now, on an equally-spaced grid,

$$t_{k+1} - t_k = M \Delta t, \quad \bar{z} - \underline{z} = J \Delta z,$$

where M and J are the number of equal-length intervals in t and z directions respectively, the Crank-Nicolson finite difference scheme representation of the governing equation can be written as:

$$C_k \approx \frac{1}{2} (C_j^{m+1} + C_j^m); \quad \frac{\partial C_k}{\partial t} \approx \frac{C_{k,j}^{m+1} - C_{k,j}^m}{\Delta t}; \quad \frac{\partial C_k}{\partial z_k} \approx \frac{1}{2} \left(\frac{C_{k,j+1}^{m+1} - C_{k,j-1}^{m+1}}{2\Delta z_t} + \frac{C_{k,j+1}^m - C_{k,j-1}^m}{2\Delta z_t} \right);$$

Upon imposing the terminal conditions (3) and (4)

$$C_{n-1,j}^M = \max \left\{ K(1+r_n), \frac{K}{I_c} e^{z+j\Delta z} \right\}, (j=0, \dots, J) \quad (11)$$

$$C_{k-1,j}^M = \max \left\{ Kr_k + C_{k,j}^0, \frac{K}{I_c} e^{z+j\Delta z} \right\}, (k=1, 2, \dots, n-1; j=0, \dots, J) \quad (12)$$

and the boundary conditions (7) and (8)

$$C_{k,0}^m = Ke^{-r_f(t_{k+1}-t_k-m\Delta t)} \left(\sum_{l=0}^{n-k-1} r_{k+l+1} e^{-r_f(t_{k+l+1}-t_{k+1})} + e^{-r_f(t_n-t_{k+1})} \right), \quad (13)$$

$$(m=0, \dots, M; k=0, 1, \dots, n-1)$$

$$C_{k,J}^m = \frac{K}{I_c} e^{\bar{z}} e^{-D(t_{k+1}-t_k-m\Delta t)}, (m=0, \dots, M; k=0, 1, \dots, n-1), \quad (14)$$

the unknown option values $C_{k,j}^m = C_k(z + j\Delta z, t_k + m\Delta t)$ can be calculated step by step until the expiry time.

Thus equations (11)~(14) define the value of our multi-stage compound option model along all the edges of the grid. Substituting the scheme into (5) gives

$$G_{k,j} = \begin{cases} Kr_k + C_{k,j}^0 - \frac{K}{I_c} e^{z+j\Delta z} & k=1, 2, \dots, n-1 \\ K(1+r_n) - \frac{K}{I_c} e^{z+j\Delta z} & k=n \end{cases} \quad (15)$$

From (15) we can get the optimal conversion price $V_k^* = e^{z+j^*(k)\Delta z}$, where

$$j^*(k) = \min_{j \in [0, J]} |G_{k,j}|$$

Computing the linear equation system (10) stage by stage backward, we can obtain the value of multistage compound option model in every grid point (t, z_i) , and the optimal conversion price $V_{t_k}^*$.

The adopted finite difference scheme has good consistence, stability and convergence properties. The consistence of the Crank-Nicolson scheme for the Black-Scholes can be shown as easily as that for the standard diffusion equation; the scheme is of a second order accuracy in both space and time. Thus, we shall only show the numerical stability of scheme here using Fourier method of stability analysis.

We firstly assume that the numerical scheme admits a solution of the form: $C_j^m = A^m(k)e^{ikj\Delta z}$, where k is wave number and $i = \sqrt{-1}$. Substituting the solution into (9) we obtain the amplification factor

$$\begin{aligned} G(k) &= \frac{A^m(k)}{A^{m+1}(k)} = \frac{b_{-1}e^{-ik\Delta z} + b_0 + b_1e^{ik\Delta z}}{a_{-1}e^{-ik\Delta z} + a_0 + a_1e^{ik\Delta z}} \\ &= \frac{(b_{-1} + b_1)\cos(k\Delta z) + b_0 + i(b_1 \sin(k\Delta z) - b_{-1} \sin(k\Delta z))}{(a_{-1} + a_1)\cos(k\Delta z) + a_0 + i(a_1 \sin(k\Delta z) - a_{-1} \sin(k\Delta z))} \\ &= \frac{\left(\frac{\sigma^2\Delta t}{\Delta z^2} \sin^2 \frac{k\Delta z}{2} + \frac{r_f\Delta t}{2} - 1\right) - i \frac{(r_f - D - \frac{1}{2}\sigma^2)\Delta t}{2\Delta z}}{\left(-\frac{\sigma^2\Delta t}{\Delta z^2} \sin^2 \frac{k\Delta z}{2} - \frac{r_f\Delta t}{2} - 1\right) + i \frac{(r_f - D - \frac{1}{2}\sigma^2)\Delta t}{2\Delta z}} \end{aligned}$$

Clearly, the strict von Neumann stability condition is satisfied because

$$|G|^2 = \frac{\left(\frac{\sigma^2 \Delta t}{\Delta z^2} \sin^2 \frac{k \Delta z}{2} + \frac{r_f \Delta t}{2} - 1\right)^2 + \left(\frac{(r_f - D - \frac{1}{2} \sigma^2) \Delta t}{2 \Delta z}\right)^2}{\left(\frac{\sigma^2 \Delta t}{\Delta z^2} \sin^2 \frac{k \Delta z}{2} + \frac{r_f \Delta t}{2} + 1\right)^2 + \left(\frac{(r_f - D - \frac{1}{2} \sigma^2) \Delta t}{2 \Delta z}\right)^2} \leq 1$$

Therefore, the scheme is stable. From Lax equivalence theorem, the numerical solution will converge to that of original PDE in the limit when M and J approach to infinity. In other words, when M and J are large enough, reasonable accuracy should be expected.

To test the numerical efficiency and accuracy, we have implemented our scheme in Matlab to obtain numerical results for a couple of numerical examples. These results are discussed in the next section.

4 NUMERICAL EXAMPLES

To validate our numerical approach, we firstly degenerate our general n -stage model to a single stage one. In this way, we can make sure that the adopted grid resolution produces results with satisfactory accuracy. This is achieved by comparing our results with those produced with the Decomposition Method.

Decomposition Method is an approach, widely used in industry as a simple way of calculating the value of convertible bonds. This approach splits the value of convertible bond as the sum of a bond with all the coupon payments converted to a single total coupon payment at the end of the term and a European option, which can be also easily calculated from the Black-Scholes formula. That is, we first calculate the payoff at the expiry date as

$$\begin{aligned} & \max \left\{ \frac{K}{I_c} V_{t_n}, K \left(1 + \sum_{k=1}^m r_k^* e^{r_f (t_n - t_k^*)} \right) \right\} \\ &= \frac{K}{I_c} \max \left\{ V_{t_n} - I_c \left(1 + \sum_{k=1}^m r_k^* e^{r_f (t_n - t_k^*)} \right), 0 \right\} + K \left(1 + \sum_{k=1}^m r_k^* e^{r_f (t_n - t_k^*)} \right). \end{aligned}$$

Then the value of the convertible bond is calculated by

$$\begin{aligned} C_0(V_{t_0}, t_0) &= \frac{K}{I_c} BSC \left(V_{t_0}, I_c \left(1 + \sum_{k=1}^m r_k^* e^{r_f (t_n - t_k^*)} \right), r_f, \sigma, (t_n - t_0), D \right) \\ &+ K \left(e^{-r_f (t_n - t_0)} + \sum_{k=1}^n r_k^* e^{-r_f (t_k^* - t_0)} \right) \end{aligned}$$

where $BSC(\cdot)$ denotes the Black-Scholes formula for European style option value. In general, this approach is however not correct because an investor who bought a CB of American style should be allowed to convert at anytime before the expiry. But, in the Decomposition Method, the investor's right of converting before expiry is striped and thus the valuation using this approach would be lower than what it should be. However, if we reduce the multi-stage compound option into 1 stage, the option

embedded in convertible bond becomes a European style option. Then this approach would be correct for this particular case, as an investor cannot convert prior to the expiry now. Therefore, we use this property to validate our code, as the value of our finite difference solution with a single stage should be exactly equal to the corresponding value obtained through the Decomposition Method.

The example we used to validate our new compound-option approach is a 5-year CB with coupon payment at the end of each year. The volatility of the underlying stock is assumed to be 30% and the stock also has a continuous dividend payment with yield rate $D = 1\%$. The issuing company plans to finance for one project by issuing a convertible bond with par of $K = 100$ Yan and the coupon rate is set to be $r_k^* = 1.5\%$ ($k = 1, 2, \dots, 5$) of the par value of the CB. Let's also assume that the company sets up a constant conversion price, $I_c = 6$. Furthermore, we assume that the risk-free interest rate r_f remains a constant for 5 years (this assumption can be easily relaxed and a risk-free interest rate associated with a much short period can be easily set) at the current deposit-interest rate, 2.53%, of Chinese banks.

Table 1. Price of a 1-stage CB

Stock price	Compound Option Pricing Approach	Decomposition Approach
0.1	95.074	95.074
1	95.103	95.103
2	95.826	95.826
3	98.481	98.481
4	103.55	103.55
5	110.87	110.87
6	120.08	120.08
7	130.78	130.78
8	142.63	142.63
9	155.36	155.36
10	168.77	168.78
20	318.69	318.69
30	475.98	475.99

Shown in Table 1 are the data generated from the 1-stage option and the decomposition method, respectively. Clearly, from this table, we can see that the results obtained from our FDM calculation are indeed in an excellent agreement with those obtained from the Decomposition Method; the relative error is $O(10^{-5})$. This verifies that the code written for our multi-stage compound option model is correct, at least for this very special case.

Having tested our code against the Decomposition Method, we then proceeded to a full model test. Since there is no analytic solution available at the moment, we chose to compare our results with those obtained from the American style contingent claim pricing approach, i.e., to deal with the presence of the free boundary directly. The approach we adopted is to solve the original differential system directly with a finite difference discretization based on the Projected Successive-Over-Relaxation (PSOR) method.

If we let $d_{k,j}^{m+1} = b_{-1}C_{k,j-1}^{m+1} + b_0C_{k,j}^{m+1} + b_1C_{k,j+1}^{m+1}$, the corresponding iterative system can be written as

$$C_{k,j}^{m(p)} = \max \left\{ C_{k,j}^{m(p-1)} + \frac{\omega}{a_0} \left(d_{k,j}^{m+1} - a_{-1}C_{k,j-1}^{m(p)} - a_0C_{k,j}^{m(p-1)} - a_1C_{k,j+1}^{m(p-1)} \right), \frac{Par}{I_{k+1}} e^{\bar{z} + j\Delta z} \right\}$$

$$(m = 0, \dots, M-1; j = 1, \dots, J-1; k = 0, 1, \dots, n-1)$$

where $C_{k,j}^{m(p)}$ is the p th iteration of $C_{k,j}^m$, and $\omega \in (0, 2)$ is the relaxation parameter.

The starting point of each iteration must be chosen carefully. Notice the fact that since the conversion right can now be exercised at any time, the value of convertible bond is nothing but the value of stocks generated from conversion when the underlying stock price is high enough. Thus

$$C_{n-1,J}^m = \frac{Par}{I_n} e^{\bar{z}}, (m = 0, \dots, M),$$

and

$$C_{k,J}^m = \frac{Par}{I_{k+1}} e^{\bar{z}}, (m = 0, \dots, M).$$

On the other hand, to satisfy the moving boundary conditions, we followed the iteration scheme outlined in Kwok [22] and carried out the iteration until

$$\sqrt{\sum_{j=1}^{J-1} \left(C_{k,j}^{m(p)} - C_{k,j}^{m(p-1)} \right)^2} < \varepsilon \quad (16)$$

In Equation (16), ε is a preset small tolerance value that controls the accuracy of the solution. For the example presented here, ε was set to 10^{-6} when the satisfactory results were obtained.

In the final model test, we let stage number vary from 5 to 10, 20 and eventually to 60. When $n = 5$, the decision dates coincide with the dividend dates. On the other hand, when the stage number is greater than 5, there are more decision-making dates than the dividend payment dates in this example. So, only some of decision-making dates coincide with the dividend payment dates. Other parameters used in the numerical procedure are $\bar{V} = 10^4$, $\underline{V} = 10^{-2}$, $M = 50$, $J = 200$.

Shown in Figure 1 are the results of using the American Contingent Claim (ACC) Pricing Approach and the Multi-stage Compound Option (MCO) Pricing Approach, respectively. Clearly, as the number of stages is increased from 5 to 60, the calculated CB's prices with the Multi-stage Compound Option Pricing Approach indeed asymptotically approach those produced with the American Contingent Claim Pricing Approach. A small window is placed at the center of the figure to zoom up a small region of the CB price vs. the asset price in order to demonstrate such a convergence (otherwise, all the curves appear to be only a single line as shown outside of this small

window). This convergence test has ultimately validated our approach based on the compound option approach.

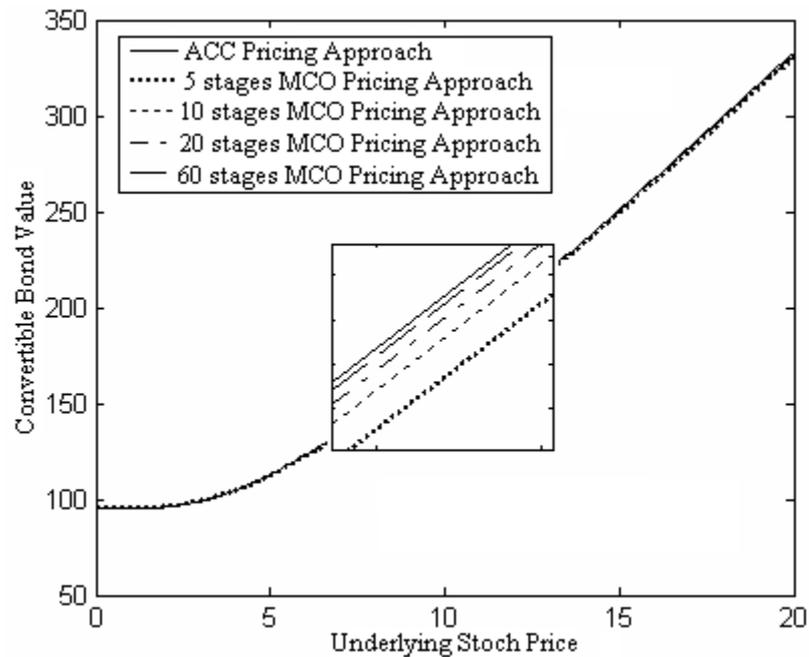


Figure 1. The Values of CB calculated from the MCO pricing approach and from the ACC pricing approach.

We have also compared the results obtained from the Multi-stage Compound Option (MCO) Pricing Approach with those obtained from the Decomposition (DEC) Pricing Approach, as shown in Table 2. Clearly, much smaller option price values were obtained with Decomposition (DEC) Pricing Approach (also shown in Fig. 2, in which data from three approaches are plotted over a sub-region of the asset price between 4 Yan and 7 Yan). This is as expected, because the DEC prices the option part of a CB as a European style option and thus ignores the early-conversion right of the holder, which is a flexibility that has value too. On the other hand, the multi-stage compound option approach allows the holder exercise his right at some limited instants, while American style contingent claim approach consider that the holder can exercise his right at any time before maturity. The more exercisable instants are in the multi-stage compound option approach, the more flexibility the holder has. In the limit when the number of exercisable instants becomes infinite, the multi-stage compound option is equivalent to corresponding American style contingent claim as demonstrated in Fig. 1 already, since the holder can actually now continuously exercise his right of conversion anytime before the expiry.

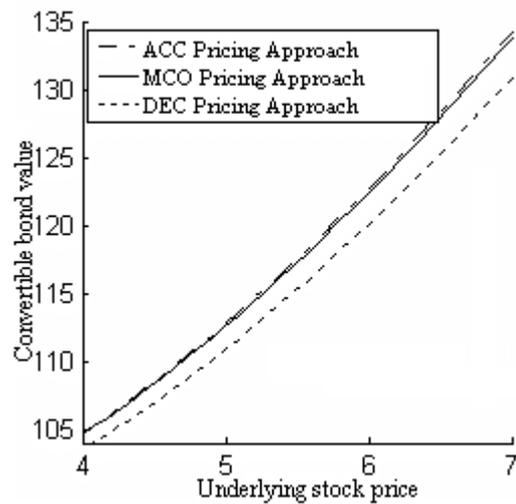


Figure 2. The Values of CB vs. the underlying asset price

Table 2. The numerical results from three pricing approaches

Underlying Stock Price	ACC Pricing Approach	MCO Pricing Approach	DEC Pricing Approach
0.1	95.07	95.07	95.07
1	95.11	95.11	95.10
2	95.99	95.98	95.83
3	99.05	99.01	98.48
4	104.73	104.62	103.55
5	112.76	112.56	110.87
6	122.71	122.41	120.08
7	134.16	133.74	130.78
8	146.75	146.21	142.63
9	160.22	159.55	155.36
10	174.37	173.58	168.78
20	333.33	330.55	318.69
30	500.00	495.05	475.99

The optimal conversion prices, which are equivalent to the optimal exercise price in American options, are plotted in Fig. 3 for this case. There are four figures in Fig. 3, each showing the optimal conversion price for a particular stage number. It can be observed that as the number of stages is increased, dash lines are moving closer to the

solid lines. This has once again showed that the numerical results of optimal conversion price from the MCO pricing approach to the results from the ACC pricing approach, when n becomes large. In order to make two curves really coincide with each other, we carried out the calculation with 500 stage numbers (i.e., $n=500$) and the results are shown in Fig. 4. The corresponding option values produced from the MCO pricing approach with $n=500$ are compared with those produced by the ACC pricing approach in Table 3. This time, the maximum difference between the two is less than 0.01%.

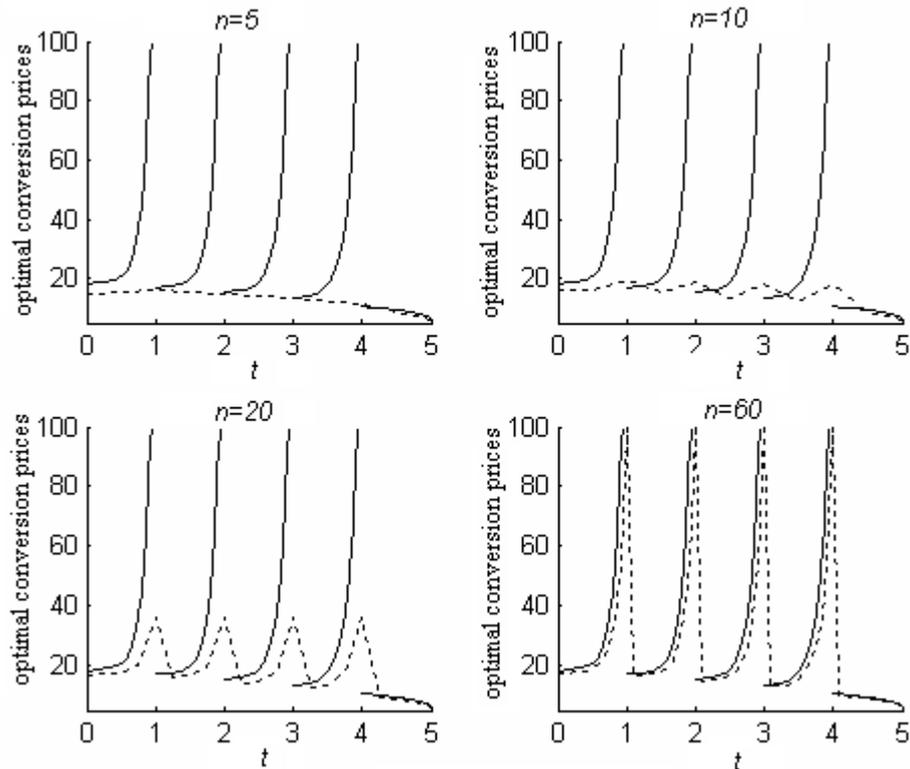


Figure 3. Optimal conversion prices calculated with the MCO pricing approach (dotted lines) and with the ACC pricing approach (solid lines).

One should also observe that every time when it is getting close to coupon delivery date, the optimal conversion stock prices shoot sky high with a sharp up slope. Computationally, we observed that at the delivery date, optimal conversion stock price actually approaches positive infinity. This can be explained from the fact that the holder of the CB will always try to find the optimal conversion stock price to convert his CB. Assume the optimal conversion stock price at the instant t_i^- right before an ex-coupon date t_i ($t_i = t_i^- + dt$) is $S^* < \infty$. This means that the holder should convert this CB. If he does, he will get $n_c S^*$, with n_c being the conversion ratio. Now, on the other hand, if the holder delays his exercise till the time immediately after the ex-coupon date, t_i , he would receive the coupon payment. In addition, because when $dt \rightarrow 0$, $E(dS) = 0$, the underlying stock price is expected to be unchanged during dt . That is to say, he would get a total of $n_c S^* + c_i$, which is greater than $n_c S^*$. This implies that the original assumption was not correct. Or

alternatively, we should conclude that it is never optimal to convert right before the ex-coupon date t_i . Mathematically, this conclusion is reflected by the fact that optimal conversion stock price is never equal to a finite value as demonstrated in Fig. 3 and Fig. 4.

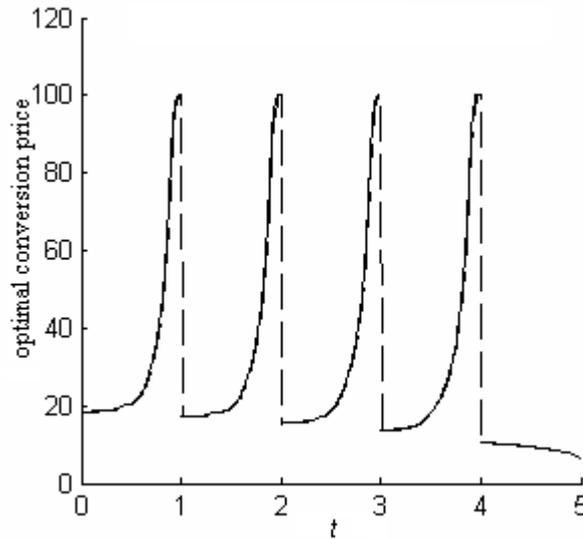


Figure 4. the result of optimal conversion price from 500 stages MCO pricing approach and ACC pricing approach. The solid line is the result from ACC pricing approach, and the dotted line is the result from MCO pricing approach.

Table 3. The convertible bond value when $n=500$

Underlying stock price	ACC pricing approach	500 stages MCO pricing approach
0.1	95.07	95.07
1	95.11	95.11
5	112.76	112.76
10	174.36	174.36
50	833.33	833.25
100	1666.70	1666.50

5 CONCLUSIONS

The concept of multi-stage compound options is introduced to the valuation of convertible bonds in this paper. This can be viewed as an approximation to the original American style conversion when the number of stage is finite. Different from the integral approach of Lin [9], we solved the multi-stage compound option with the Crank-Nicolson scheme, which provides a much better numerical efficiency than integrating the nested high-dimensional integral. The high order accuracy and

convergence of the finite difference scheme is demonstrated through numerical examples.

For CBs with more complex structures such as those with call and/or put provisions, it is envisaged that the current approach can be applied. Such an extension is however non-trivial and is being currently worked out. We shall present our results elsewhere in the future.

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