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C. Koukouvinos, M. Mitrouli and Jennifer Seberry, in "Growth in Gaussian elimination for weighing matrices, W (n, n — 1)", Linear Algebra and its Appl., 306 (2000), 189-202, conjectured that the growth factor for Gaussian elimination of any completely pivoted weighing matrix of order n and weight n— 1 is n— 1 and that the first and last few pivots are (1,2,2,3 or 4, ..., n–1 or (n–1)/2, , (n–1)/2, n–1) for n > 14. In the present paper we study the growth problem for skew and symmetric conference matrices. An algorithm for extending a k × k matrix with elements 0, ±1 to a skew and symmetric conference matrix of order n is described. By using this algorithm we show the unique W(8, 7) has two pivot structures. We also prove that unique W(10,9) has three pivot patterns.

Keywords

Gaussian elimination, growth, complete pivoting, weighing matrices, AMS Subject Classification: 65F05, 65G05, 20B20.

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On the growth problem for skew and symmetric conference matrices

C. Kravvaritis *, M. Mitrouli * and Jennifer Seberry †

Abstract

C. Koukouvinos, M. Mitrouli and Jennifer Seberry, in "Growth in Gaussian elimination for weighing matrices, $W(n, n-1)$", *Linear Algebra and its Appl.*, 306 (2000), 189-202, conjectured that the growth factor for Gaussian elimination of any completely pivoted weighing matrix of order n and weight $n-1$ is $n-1$ and that the first and last few pivots are $(1, 2, 2, 3 \text{ or } 4, \ldots, n-1 \text{ or } \frac{n-1}{2}, \frac{n-1}{2}, n-1)$ for $n > 14$. In the present paper we study the growth problem for skew and symmetric conference matrices.

An algorithm for extending a $k \times k$ matrix with elements $0, \pm 1$ to a skew and symmetric conference matrix of order n is described. By using this algorithm we show the unique $W(8,7)$ has two pivot structures. We also prove that unique $W(10,9)$ has three pivot patterns.

Key Words and Phrases: Gaussian elimination, growth, complete pivoting, weighing matrices.

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1 Introduction

Let $A \cdot \underline{x} = \underline{b}$, where $A = [a_{ij}] \in \mathcal{R}^{n \times n}$ is nonsingular. The strategy of Gaussian elimination (GE) in order to solve this system is to reduce the full linear system to a triangular system which can be easily solved, using elementary row operations. There are $n-1$ stages, beginning with $A^{(1)} := A, \underline{b}^{(1)} := \underline{b}$ and finishing with the upper triangular system $A^{(n)} \cdot \underline{x} = \underline{b}^{(n)}$. Let $A^{(k)} = [a_{ij}^{(k)}]$ denote the matrix obtained after the first k pivoting operations, so $A^{(n)}$ is the final upper triangular matrix. A diagonal entry of that final matrix will be called a pivot. Matrices with the property that no exchanges are actually needed during GE with complete pivoting are called completely pivoted (CP) or feasible.

Traditionally, backward error analysis for GE is expressed in terms of the *growth factor*

$$g(n, A) = \frac{max_{i,j,k}|a_{ij}^{(k)}|}{max_{i,j}|a_{ij}|}$$

which involves all the elements $a_{ij}^{(k)}$, $k = 1, 2, \ldots, n$ that occur during the elimination. For a CP matrix A let us denote by $g(n) = \sup\{ g(n, A)/A \in \mathcal{R}^{n \times n} \}$. The problem of determining $g(n)$ for various values of n is called the *growth problem.*

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The determination of $g(n)$ remains a mystery. Wilkinson in [8] proved that

$$g(n) \leq [n\, 2\, 3^{1/2} \dots n^{1/n-1}]^{1/2}$$

and that this bound is not attainable and can still be quite large (e.g. it is 3570 for $n = 100$). Wilkinson in [9],[10] noted that there were no known examples of matrices for which $g(n) > n$. In [2] Cryer conjectured that "$g(n, A) \leq n$, with equality iff A is a Hadamard matrix". This conjecture became one of the most famous open problems in numerical analysis and has been investigated by many mathematicians. In 1991 Gould [6] discovered a 13×13 matrix for which the growth factor is 13.0205. Thus the first part of the conjecture was shown to be false. The second part of the conjecture concerning the growth factor of Hadamard matrices still remains open.

An Hadamard matrix H of order $n \times n$ is an orthogonal matrix with elements ± 1 and $HH^T = nI$. If an Hadamard matrix, H, of order n can be written as $H = I + S$ where $S^T = -S$ then H is called *skew–Hadamard.* S is also a conference matrix: we call it a *skew conference matrix.*

Two matrices are said to be *Hadamard equivalent* or *H-equivalent* if one can be obtained from the other by a sequence of operations which permute the rows and/or columns and multiply rows and/or columns by -1.

A $(0, 1, -1)$ matrix $W = W(n, k)$ of order n satisfying $WW^T = kI_n$ is called a *weighing matrix of order n and weight k* or simply a *weighing matrix.* A $W(n, n)$, $n \equiv 0 \,(\mathrm{mod}\, 4)$, is a Hadamard matrix of order n. A $W = W(n, k)$ for which $W^T = -W$ is called a *skew–weighing matrix.* A $W = W(n, n-1)$ satisfying $W^T = W$, $n \equiv 2 \,(\mathrm{mod}\, 4)$, is called a *symmetric conference matrix.* Conference matrices cannot exist unless $n - 1$ is the sum of two squares: thus they cannot exist for orders $22, 34, 58, 70, 78, 94$. For more details and construction of weighing matrices the reader can consult the book of Geramita and Seberry [5].

Wilkinson's initial conjecture seems to be connected with Hadamard matrices. Interesting results in the size of pivots appear when GE is applied to CP weighing matrices of order n and weight $n - 1$. In the present paper we study the growth problem for CP skew and symmetric conference matrices. In these matrices, the growth is also large, and experimentally, we have been led to believe it equals $n - 1$ and special structure appears for the first few and last few pivots. We studied, by computer, the pivots and growth factors for $W(n, n-1)$, $n = 6, 10, 14, 18, 26, 30, 38, 42, 50, 54, 62, 74, 82, 90, 98$ constructed by two circulant matrices and for $n = 8, 12, 16, 20, 28, 36, 44, 52, 60, 68, 76, 84, 92, 100$ constructed by four circulant matrices and obtained the results in Tables 3 and 4. These results give rise to a new conjecture that can be posed for this category of matrices.

The growth conjecture for skew and symmetric conference matrices

Let W be a CP skew and symmetric conference matrix. Reduce W by GE. Then

(i) $g(n, W) = n - 1$.

(ii) The two last pivots are equal to $\frac{n-1}{2}, n - 1$.

(iii) Every pivot before the last has magnitude at most $n - 1$.

(iv) The first four pivots are equal to 1, 2, 2, 3 or 4, for large enough n.

Notation. Write A for a matrix of order n whose initial pivots are derived from matrices with CP structure. Write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper lefthand corner of the matrix A. Throughout this paper -1 will be denoted by $-$. The magnitude of the pivots appearing after the application of GE operations on a CP matrix W is given by

$$p_j = W(j)/W(j-1), \quad j = 1, 2, \ldots, n, \quad W(0) = 1. \tag{1}$$

We use $W(j)$ similarly.

2 The first four pivots

Since pivots are strictly connected with minors we start our study with an effort of computing principal minors of skew and symmetric conference matrices. The following lemma specifies the possible values of determinants of small order. The results for orders 6 and 7 are new.

Lemma 1 *The maximum determinant of all $n \times n$ matrices with elements ± 1 or 0, where there is at most one zero in each row and column is:*

Order	*Maximum Determinant*	*Possible Determinantal Values*
2×2	2	$0, 1, 2$
3×3	4	$0, 1, 2, 3, 4$
4×4	16	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16$
5×5	48	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 36, 40, 48$
6×6	160	$160, 144, 136, 132, 130, 128, 120, 112, 108, 106, 105, 104, 102, 100, \ldots$
7×7	528	$528, 504, 480, 468, 456, 444, 432, 420, 408, 396, 384, 372, 366, 360, 354, 348, 342, 336, 330, 324, \ldots$

□

Lemma 2 *Let W be a CP skew and symmetric matrix, of order $n \geq 6$ then if GE is performed on W the first two pivots are 1, and 2.*

Proof. We note that in the upper lefthand corner of a CP skew and symmetric conference matrix, of order $n \geq 6$ the following submatrices can always occur

$$\begin{bmatrix} 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$$

Thus, the first two pivots, using equation (1), are

$$p_1 = 1, \quad \text{and} \quad p_2 = 2.$$

□

Lemma 3 *H-equivalence operations can be used to ensure the following submatrices always occur in the upper lefthand corner of a $W(8,7)$ and a $W(10,9)$:*

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix} \quad \text{or} \quad B_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 0 \\ 1 & 1 & - \end{bmatrix},$$

and

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & - & - \end{bmatrix} \quad \text{or} \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & - \\ 1 & - & - & - \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \end{bmatrix}.$$

Proof. We note that each of $W(8,7)$ and $W(10,9)$ is unique upto H-equivalence. Hence it is sufficient to demonstrate that B_1, B_2, A_1 and A_2 exist in each.
Consider the following $W(8,7)$

$$X = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ - & 0 & 1 & 1 & 1 & - & - & - \\ - & - & 0 & 1 & - & 1 & 1 & - \\ - & - & - & 0 & 1 & 1 & - & 1 \\ - & - & 1 & - & 0 & - & 1 & 1 \\ - & 1 & - & - & 1 & 0 & 1 & - \\ - & 1 & - & 1 & - & - & 0 & 1 \\ - & 1 & 1 & - & - & 1 & - & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & 1 & 0 & - & - & 1 & 1 & - \\ 1 & - & - & - & 1 & 1 & 0 & 1 \\ 1 & - & 1 & 1 & 0 & 1 & - & - \\ 1 & 1 & - & 1 & - & 0 & - & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & 1 & - & - & 0 \\ 1 & - & 1 & 0 & - & - & 1 & 1 \\ 1 & 0 & - & 1 & 1 & - & 1 & - \end{bmatrix}.$$

We can see B_1 in the submatrix comprising the first 3 rows and columns 4, 5 and 6 of X. B_2 is in the submatrix comprising the first 3 rows and columns 4, 5 and 2 of X. A_1 appears in the submatrix comprising rows 1, 2, 3 and 7 and columns 4, 8, 5 and 6 of X.

A_2 appears in the top lefthand 4×4 submatrix of Y.
Now consider the following $W(10,9)$

$$W = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - & 1 & 0 \\ 1 & - & - & 1 & 0 & - & - & 1 & 1 & - \\ 1 & 1 & - & - & - & 1 & 0 & - & 1 & - \\ 1 & - & 1 & 1 & - & 1 & 1 & 0 & - & - \\ 1 & 1 & 1 & 1 & - & - & - & - & 0 & 1 \\ 0 & - & 1 & - & - & 1 & - & 1 & 1 & 1 \\ 1 & 0 & - & - & - & - & 1 & 1 & - & 1 \\ 1 & - & - & 0 & 1 & 1 & - & - & - & 1 \\ 1 & 1 & 1 & - & 1 & 0 & - & 1 & - & - \end{bmatrix} \quad Z = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & - & - & 1 & 1 & - & 1 & 0 & - & - \\ 1 & - & 1 & - & - & - & 0 & 1 & - & 1 \\ 1 & 1 & - & - & 1 & - & - & 1 & 1 & 0 \\ 1 & 1 & - & 1 & - & 0 & - & - & - & 1 \\ 1 & - & - & - & 0 & 1 & 1 & - & 1 & 1 \\ 1 & 1 & 1 & 0 & - & - & 1 & - & 1 & - \\ 0 & - & 1 & 1 & 1 & - & - & - & 1 & 1 \\ 1 & - & 0 & 1 & - & 1 & - & 1 & 1 & - \\ 1 & 0 & 1 & - & 1 & 1 & - & - & - & - \end{bmatrix}$$

We can see B_1 in the submatrix comprising the first 3 rows and columns 1, 3 and 4 of Z. B_2 is in the submatrix comprising the first 3 rows and columns 1, 8 and 10 of W. A_1 appears in the submatrix comprising the first four rows and columns 1, 2, 4 and 3 of Z.

A_2 appears by taking columns 1, 3, 9 and the negative of column 4 and then choosing rows 1, 2, 4 and 3. □

Lemma 4 *H-equivalence operations can be used to ensure the following submatrices always occur in a skew and symmetric $W(n, n-1)$:*

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix} \quad \text{or} \quad B_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 0 \\ 1 & 1 & - \end{bmatrix},$$

and

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & - & - \end{bmatrix} \quad \text{or} \quad A_2 = \begin{bmatrix} 1 & 1 & 0 & - \\ 1 & - & - & - \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \end{bmatrix}.$$

Proof. We note that, without loss of generality, the first few rows and columns of any skew and symmetric $W(n, n-1)$ can be written, for large enough n (we considered $n = 8$ and $n = 10$ separately above) as

$$\begin{array}{ccccccccccccccccccccccccccccc}
0 & 1 & 1 & 1 & 1 & .. & 1 & 1 & .. & 1 & 1 & .. & 1 & 1 & .. & 1 & 1 & .. & 1 & 1 & .. & 1 & 1 & .. & 1 & 1 & .. & 1 \\
1 & 0 & a & b & 1 & .. & 1 & 1 & .. & 1 & 1 & .. & 1 & 1 & .. & 1 & - & .. & - & - & .. & - & - & .. & - & - & .. & - \\
1 & \epsilon a & 0 & c & 1 & .. & 1 & 1 & .. & 1 & - & .. & - & - & .. & - & 1 & .. & 1 & 1 & .. & 1 & - & .. & - & - & .. & - \\
1 & \epsilon b & \epsilon c & 0 & 1 & .. & 1 & - & .. & - & 1 & .. & 1 & - & .. & - & 1 & .. & 1 & - & .. & - & 1 & .. & 1 & - & .. & - \\
e & e & e \\
e & e & -e \\
e & -e & e \\
e & -e & -e
\end{array}$$

Tableau I

where a, b, c are ± 1, $\epsilon = (-1)^{\frac{n+2}{2}}$, and e is column of all 1s of suitable length (the length of e may vary in this Tableau).
Clearly we can choose columns (with suitable permutation) that start

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix} \quad \text{or} \quad A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & - & - \end{bmatrix}.$$

We can also choose the three columns (with suitable permutation) that start

$$Y_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \end{bmatrix}.$$

We now extend Y_2 by a third row obtaining Z_2

$$Z_2 = \begin{array}{ccccccccccccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 0 \\
1 & - & 1 & \cdots & 1 & 1 & \cdots & 1 & - & \cdots & - & - & \cdots & - & 0 & u \\
1 & 0 & 1 & \cdots & 1 & - & \cdots & - & 1 & \cdots & 1 & - & \cdots & - & z & w
\end{array}$$

where u, z and w are ± 1. Suppose there are x_1 columns $(1, 1, 1)^T$, x_2 columns $(1, 1, -)^T$, x_3 columns $(1, -, 1)^T$, and x_4 columns $(1, -, -)^T$.

Then $x_1 + x_2 + x_3 + x_4 = n - 4$ (by counting), $x_1 + x_2 - x_3 - x_4 = 0$ (by inner product of the first and second rows), $x_1 - x_2 + x_3 - x_4 = -1 - z$ (by inner product of the first and third rows), and $x_1 - x_2 - x_3 + x_4 = -1 - uw$ (by inner product of the second and third rows). From these four equations we obtain $4x_2 = n - 2 + z + uw$. So, since the minumum and maximum of $+z + uw$ is -2 and $+2$ respectively, $n - 4 \leq 4x_2 \leq n$. Hence $x_2 \geq 1$ for $n \geq 8$. So we can choose the first two columns of Z_2 plus a column from the x_2 columns $(1, 1, -)$ to see that B_2^* always exists where

$$B_2^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 0 & - \end{bmatrix}.$$

This can be rearranged to give B_2. A similar counting argument, given that $n \geq 12$ allows us to see that A_1 always appears. It remains to establish that A_2 will always occur. We discriminate two cases:

Case I: For $n \equiv 0(mod\ 4)$

In this case the matrix is skew and thus the upper 4×4 block of the above Tableau I will be:

$$\begin{matrix} 0 & 1 & 1 & 1 \\ - & 0 & a & b \\ - & -a & 0 & c \\ - & -b & -c & 0 \end{matrix}.$$

Since we showed that the matrix B_2 always occur we can set $a = 1$. By setting all the possible four choices for b, c, we see that always, for each choice, appears in the 4×4 block a column (or a equivalent one) of the form $\begin{bmatrix} 1 & 0 & - & - \end{bmatrix}^T$. Thus we can choose the columns of A_2 directly from Tableau I.

Case II: For $n \equiv 2(mod\ 4)$

In this case the matrix is symmetric and thus the upper 4×4 block of the above Tableau I will be:

$$\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{matrix}.$$

Since we showed that the matrix B_2 always occur we can set $a = -1$. By setting all the possible four choices for b, c, we see that always, for each choice, appears in the 4×4 block a column (or a equivalent one) of the form $\begin{bmatrix} 1 & 0 & - & - \end{bmatrix}^T$. Thus we can choose the columns of A_2 directly from Tableau I. $\square$

Lemma 5 *Let W be a CP skew and symmetric conference matrix, of order $n \geq 12$ then if GE is performed on W the third pivot is* 2.

Proof. Since in the 2×2 upper lefthand corner of a CP skew and symmetric conference matrix, the following submatrix will always occur:

$$\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$$

we try to extend it to all the possible 3×3 matrices. It is interesting to specify all possible 3×3 matrices with elements ± 1 that contain this 2×2 part and also have the maximum possible value of the determinant which for the 3×3 matrices is 4. Thus we extend this matrix to the all possible 3×3 matrices M with elements ± 1 i.e.

$$M = \begin{bmatrix} 1 & 1 & * \\ 1 & - & * \\ * & * & * \end{bmatrix}$$

where * can take the values 1 or -1 and 0 with the restriction that each row and column will contain at most one zero.

Next, we required the determinant of the matrix to be 4 and the matrix to be normalised i.e. the elements in the positions $(3,1)$ and $(1,3)$ to be 1. Under these restrictions we found six matrices which are equivalent to the following two CP matrices:

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix} \quad \text{or} \quad B_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 0 \\ 1 & 1 & - \end{bmatrix}.$$

Since in Lemma 4 was shown that the matrices B_1 and B_2 always occur in a skew and symmetric weighing matrix, in the upper left 3×3 corner of a CP skew and symmetric $W(n, n-1)$ the matrix B_1 or B_2 will occur, and hence the third pivot, using equation (1), is

$$p_3 = 2.$$

□

Proposition 1 *Let W be a CP skew and symmetric conference matrix, of order $n \geq 12$ then if GE is performed on W the fourth pivot is 3 or 4.*

Proof. Since in the 3×3 upper lefthand corner of a CP skew and symmetric conference matrix, the matrix $B1$ or $B2$ will always occur we try to extend it to all the possible 4×4 matrices. It is interesting to specify all possible 4×4 matrices M with elements $0, \pm 1$ that contain these 3×3 matrices and also have the maximum possible values of the determinant which for the 4×4 matrices are 16 and 12.

First Case

$$M = \begin{bmatrix} 1 & 1 & 1 & * \\ 1 & - & 1 & * \\ 1 & 1 & - & * \\ * & * & * & * \end{bmatrix}$$

where * can take the values 1 or -1 and 0 with the restriction that each row and column will contain at most one zero.

Next, we required the determinant of the matrix to be 16 and the matrix to be normalised i.e. the elements in the positions $(4,1)$ and $(1,4)$ to be 1. Under these restrictions we found one matrix which is equivalent to the following one:

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & - & - \end{bmatrix}$$

Second Case

$$M = \begin{bmatrix} 1 & 1 & 1 & * \\ 1 & - & 0 & * \\ 1 & 1 & - & * \\ * & * & * & * \end{bmatrix}$$

where * can take the values 1 or -1 and 0 with the restriction that each row and column will contain at most one zero.

Next, we required the determinant of the matrix to be 12 (the closest to the maximum value of minor since the value of 16 did not appear) and the matrix to be normalised i.e. the elements in the positions $(4,1)$ and $(1,4)$ to be 1. Under these restrictions we found one matrix which is equivalent to the following one:

$$A_2 = \begin{bmatrix} 1 & 1 & 0 & - \\ 1 & - & - & - \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \end{bmatrix}.$$

Since in Lemma 4 was shown that the matrices A_1 and A_2 always occur in a skew and symmetric weighing matrix, in the upper left 4×4 corner of a CP skew and symmetric $W(n, n-1)$ the matrix A_1 or A_2 will occur, and hence the fourth pivot for $n \geq 12$, using equation (1), can take the value

$$p_4 = 4 \quad \text{or } 3.$$

□

Next, we tried to extend the 4×4 matrices to the all possible 5×5 matrices. It is interesting to specify all possible 5×5 matrices M with elements 0, ± 1 that contain the matrices A_1 or A_2 and also have the maximum possible values of the determinant which for the 5×5 matrices are given in Lemma 2. We found the following results:

Extension of matrix A_1

det	18	20	22	24	26	28	30	32	36	40	48
matrices	0	30	0	42	0	42	0	81	21	18	3

Table 1

Extension of matrix A_2

det	14	16	18	20	22	24	26	28	30	32	36	40	48
matrices	48	108	48	0	10	61	4	18	10	12	11	3	0

Table 2

For odd values of determinants there weren't any matrices found.

3 Extention of specific matrices with elements $0, \pm 1$ to $W(n, n-1)$ matrices

Algorithm for extending a $k \times k$ matrix with elements $0, \pm 1$ to $W(n, n-1)$

For a $k \times k$ matrix $A = [\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_k]^T$ the following algorithm specifies its extension, if it exists, to a $W(n, n-1)$.

```
Algorithm Extend
Step 1
read the $k \times k$ matrix $A$
Step 2
complete the first row of the matrix without loss of generality: it has exactly one 0
complete the first column of the matrix without loss of generality: it has exactly one 0
Step 3
complete(almost) the second row of the matrix without loss of generality:
            $\underline{r}_2 \cdot \underline{r}_1^T = 0$
            every row and column has exactly one zero
complete(almost) the second column of the matrix without loss of generality:
            it is orthogonal to the first column
            every row and column has exactly one zero
Step 4
Procedure Extend Rows
find all possible entries $a_{3,k+1}, a_{3,k+2}, \ldots, a_{3,n}$:
      $\underline{r}_3 \cdot \underline{r}_1^T = 0$ and $\underline{r}_3 \cdot \underline{r}_2^T = 0$
      every row and column has exactly one zero
store the results in a new matrix $B_3$ whose rows are all the possible entries
for $i = 4, \ldots, k$
    for every possible extension of the rows $\underline{r}_j$,   $j = 3, \ldots, i-1$
        find all possible entries $a_{i,k+1}, a_{i,k+2}, \ldots, a_{i,n}$:
            $\underline{r}_i$ is orthogonal with all the previous rows
            every row and column has exactly one zero
        store the results in a new matrix $B_i$ whose rows are all the possible entries
    end
end
extend the $k$-th row of $A$ with the first row of $B_k$
extend the $k-1, \ldots, 2$ rows of $A$ with the corresponding rows of
         the appropriate matrices $B_i$,   $i = k-1, \ldots, 3$
end {of Procedure Extend Rows}
Step 5
extend columns 3 to $k$ following a similar procedure as the one used to the rows.
Step 6
for $i = k+1, \ldots, n$
    find all possible entries $a_{i,k+1}, a_{i,k+2}, \ldots, a_{i,n}$:
         $\underline{r}_i$ is orthogonal with all the previous rows
```

```
        every row and column has exactly one zero
end
complete rows $k+1$ to $n$.
if columns $k+1$ to $n$ are orthogonal with all the previous columns
    $A$ is extended to $W(n, n-1)$.
```

Comment: In Step 3 by writing "complete almost" we mean that the second row can be completed in at most two ways upto permutation of columns. If the first row in the $k \times k$ part of the matrix contains a zero, then we complete the second row in a unique way without loss of generality. If the first row in the $k \times k$ part of the matrix doesn't contain a zero, then we complete the second row in two ways by setting the element below the 0 of the first row to 1 or -1 respectively. The same is done with the columns.

Implementation of the Algorithm Extend
We apply the algorithm for k=5, n=10 .
Steps of the algorithm

1. We start with

$$A = \begin{bmatrix} 1 & - & 0 & 1 & 1 \\ - & - & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & - \\ - & 1 & - & 1 & 1 \\ 1 & - & - & 0 & - \end{bmatrix};$$

2. The first row and column is completed, without loss of generality, so that the property of a $W(10, 9)$ having exactly one zero in each row and column is preserved. The software package fills with zeros the rest of the entries of the required 10×10 matrix;

$$A = \begin{bmatrix} 1 & - & 0 & 1 & 1 & 1 & - & - & 1 & 1 \\ - & - & 1 & 1 & 0 & 0 & & \cdots & & 0 \\ 1 & 1 & 1 & 1 & - & \vdots & & & & \vdots \\ - & 1 & - & 1 & 1 & & & & & \\ 1 & - & - & 0 & - & 0 & & \cdots & & 0 \\ - & 0 & \cdots & & & & \ddots & & & 0 \\ 0 & \vdots & & & & & & & & \vdots \\ - & & & & & & & & & \\ 1 & & & & & & & & & \\ 1 & 0 & \cdots & & & & & & & 0 \end{bmatrix}$$

3. As before, the algorithm completes the second row in a unique way and the second column in two ways, because the element a beside the 0 of the first column below can take both values ± 1;

$$A = \left[\begin{array}{cccccccccc}
1 & - & 0 & 1 & 1 & 1 & - & - & 1 & 1 \\
- & - & 1 & 1 & 0 & - & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & - & 0 & & \cdots & & 0 \\
- & 1 & - & 1 & 1 & 0 & & \cdots & & 0 \\
1 & - & - & 0 & - & 0 & & \cdots & & 0 \\
- & - & 0 & \cdots & & & & & & 0 \\
0 & a & \vdots & & & \ddots & & & & \vdots \\
- & 1 & & & & & & & & \\
1 & 0 & & & & & & & & \\
1 & 1 & 0 & \cdots & & & & & & 0
\end{array}\right]$$

4. The algorithm takes as input this matrix A and finds all possible completions for rows 3-5 (columns 6-10), so that every row has exactly one zero, every column has at most one zero and the inner product of every two distinct rows is zero. If many ways have been found to complete rows 3-5, the algorithm keeps as a result the first solution found;

$$A = \left[\begin{array}{cccccccccc}
1 & - & 0 & 1 & 1 & 1 & - & - & 1 & 1 \\
- & - & 1 & 1 & 0 & - & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & - & 1 & 0 & 1 & 1 & - \\
- & 1 & - & 1 & 1 & - & - & 1 & 1 & 0 \\
1 & - & - & 0 & - & - & 1 & 1 & 1 & 1 \\
- & - & 0 & \cdots & & & & & & 0 \\
0 & a & \vdots & & & \ddots & & & & \vdots \\
- & 1 & & & & & & & & \\
1 & 0 & & & & & & & & \\
1 & 1 & 0 & \cdots & & & & & & 0
\end{array}\right]$$

5. The algorithm finds all possible completions for columns 3-5 (rows 6-10) in the same way it has done with the rows 3-5;

$$A = \left[\begin{array}{cccccccccc}
1 & - & 0 & 1 & 1 & 1 & - & - & 1 & 1 \\
- & - & 1 & 1 & 0 & - & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & - & 1 & 0 & 1 & 1 & - \\
- & 1 & - & 1 & 1 & - & - & 1 & 1 & 0 \\
1 & - & - & 0 & - & - & 1 & 1 & 1 & 1 \\
- & - & - & - & - & 0 & \cdots & & & 0 \\
0 & - & - & 1 & 1 & \vdots & \ddots & & & \vdots \\
- & 1 & - & 1 & - & & & & & \\
1 & 0 & - & 1 & - & & & & & \\
1 & 1 & - & - & 1 & 0 & & & & 0
\end{array}\right]$$

6. The algorithm tries to complete,if possible, the rows 6-10(columns 6-10) in the same way as before;

$$A=\begin{bmatrix} 1 & - & 0 & 1 & 1 & 1 & - & - & 1 & 1 \\ - & - & 1 & 1 & 0 & - & 1 & - & 1 & - \\ 1 & 1 & 1 & 1 & - & 1 & 0 & 1 & 1 & - \\ - & 1 & - & 1 & 1 & - & - & 1 & 1 & 0 \\ 1 & - & - & 0 & - & - & 1 & 1 & 1 & 1 \\ - & - & - & - & - & 1 & - & 0 & 1 & - \\ 0 & - & - & 1 & 1 & 1 & 1 & 1 & - & - \\ - & 1 & - & 1 & - & 1 & 1 & - & 0 & 1 \\ 1 & 0 & - & 1 & - & - & - & - & - & - \\ 1 & 1 & - & - & 1 & 0 & 1 & - & 1 & - \end{bmatrix}$$

7. Finally, if matrix A could be extended, the algorithm gives the completed matrix $W(10,9)$ and verifies whether the relationship $AA^T = 9I_{10}$ is valid. □

Using the above algorithm we can prove the following propositions:

Proposition 2 $W(5) = 28$ *for a* $W(8,7)$

Proof. We must show that from all the matrices in Tables 1 and 2, only the ones with determinant 28 can be extended to a $W(8,7)$. By using Algorithm Extend for $k = 5$, $n = 8$ and by testing all 5×5 matrices that have been found in Tables 1 and 2, we found that only the following matrices with determinant 28 can be extended to a $W(8,7)$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & - \\ 1 & - & - & 1 & 0 \\ 1 & 1 & - & - & 1 \\ 1 & 0 & - & 1 & - \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & - \\ 1 & - & - & 1 & 0 \\ 1 & 1 & - & - & 1 \\ 1 & 1 & 1 & 0 & - \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 & - & 1 \\ 1 & - & - & - & 0 \\ 1 & - & 1 & 1 & 1 \\ 1 & 1 & - & 1 & 1 \\ 1 & 0 & 1 & 1 & - \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & - & 1 \\ 1 & - & - & - & - \\ 1 & - & 1 & 1 & - \\ 1 & 1 & - & 1 & 0 \\ 1 & 1 & 1 & - & - \end{bmatrix}$$

The result follows obviously. □

Proposition 3 $W(5) = 48,\ 36$ *or* 30 *for a* $W(10,9)$

Proof. We applied Algorithm Extend for $k = 5$, $n = 10$ for all the matrices in Tables 1 and 2 and we found that only some 5×5 matrices with determinants 48, 36 or 30 can be extended to a $W(10,9)$. This means that $W(5) = 48,\ 36$ *or* 30 for a $W(10,9)$. □

Proposition 4 $W(6) = 144$ *or* 108 *for a* $W(10,9)$

Proof. We tried to extend the 5×5 matrices with determinants $48, 36$ and 30, which can be extended to a $W(10,9)$, to 6×6 matrices with all possible determinant values. Next, we used Algorithm Extend for $k = 6$, $n = 10$ and we found that only some 6×6 matrices with determinants 144 or 108 can be extended to a $W(10,9)$ This means that $W(6) = 144$ *or* 108 for a $W(10,9)$. $\square$

Proposition 5 *$W(7) = 432$ or 324 for a $W(10,9)$*

Proof. We tried to extend the 6×6 matrices with determinants 144 and 108, which can be extended to a $W(10,9)$, to 7×7 matrices with all possible determinant values. Next, we used Algorithm Extend for $k = 7$, $n = 10$ and we found that only some 7×7 matrices with determinants 432 or 324 can be extended to a $W(10,9)$ This means that $W(7) = 432$ or 324 for a $W(10,9)$. $\square$

4 Exact Calculations

We assume that row and column permutations have been carried out so we have a CP skew and symmetric conference matrix W in the initial steps from which we can calculate the maximum minors $W(n)$, $W(n-1)$ and $W(n-2)$. We explore the use of a variation of a clever proof used by combinatorialists to find the determinant of a matrix satisfying $AA^T = (k-\lambda)I + \lambda J$, where I is the $v \times v$ identity matrix, J is the $v \times v$ matrix of ones and k, λ are integers to simplify our proofs. The determinant is $k + (v-1)\lambda(k-\lambda)^{v-1}$.

For the conference matrix $W(n, n-1)$ since $WW^T = (n-1)I$ we have that $det(W) = (n-1)^{\frac{n}{2}}$.

Proposition 6 *Let W be a CP skew and symmetric or conference matrix of order n. Then the $(n-1) \times (n-1)$ minors are: $W(n-1) = (n-1)^{\frac{n}{2}-1}$.*

Proof: Since we have that matrix W is CP let us suppose that it can written in the following form:

$$W = \left[\begin{array}{c|cccc} 1 & 0 & 1 & \dots & 1 \\ \hline 0 & & & & \\ 1 & & & & \\ \vdots & & \mathrm{B} & & \\ 1 & & & & \end{array} \right]$$

The $(n-1) \times (n-1)$ matrix BB^T has the form

$$BB^T = \begin{bmatrix} n-1 & 0 & 0 & \cdots & 0 \\ 0 & n-2 & -1 & \cdots & -1 \\ 0 & -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & & & \vdots \\ 0 & -1 & -1 & \cdots & n-2 \end{bmatrix}$$

Then, $det\ BB^T = (n-1)(n-2-(n-3))(n-2+1)^{n-3} = (n-1)^{n-2}$. So $det\ B = (n-1)^{\frac{n}{2}-1}$.
□

Proposition 7 *Let W be a CP skew and symmetric conference matrix of order n. Then the $(n-2)\times(n-2)$ minors are $W(n-2) = 2(n-1)^{\frac{n}{2}-2}$.*

Proof: Since we have that matrix W is CP let us suppose that it can be written in the following form:

$$W = \left[\begin{array}{cc|cccc} 1 & 1 & 0 & 1 & \overbrace{1,\ldots,1}^{u} & \overbrace{1,\ldots,1}^{v} \\ 1 & -1 & \pm 1 & 0 & 1,\ldots,1 & -1,\ldots,-1 \\ \hline 0 & \pm 1 & & & & \\ 1 & 0 & & & & \\ 1 & 1 & & & & \\ \vdots & \vdots & & & & \\ 1 & 1 & & & \text{C} & \\ 1 & -1 & & & & \\ \vdots & \vdots & & & & \\ 1 & -1 & & & & \end{array}\right]$$

The $(n-2)\times(n-2)$ matrix CC^T has the form

$$CC^T = \begin{bmatrix} C_1 & C_2 & C_3 \\ C_2^T & C_4 & 0 \\ C_3 & 0 & C_4 \end{bmatrix}$$

where $C_1 = diag\{n-2, n-2\}$, C_4 is a $(\frac{n-4}{2}\times\frac{n-4}{2})$ of the form

$$C_4 = \begin{bmatrix} n-3 & -2 & \ldots & -2 \\ -2 & n-3 & \ldots & -2 \\ \vdots & \vdots & & \vdots \\ -2 & -2 & \ldots & n-3 \end{bmatrix}$$

C_2 is a $(2\times\frac{n-4}{2})$ matrix having 1's in its first row and -1's in its second row, and finally C_3 is a $(2\times\frac{n-4}{2})$ matrix of -1's. Set $C_5 = diag\{C_4, C_4\}$, $C_6 = [C_2C_3]$ and $C_7 = [C_2^TC_3]^T$. Then, $det\ CC^T = det\ C_1 \cdot det\ (C_5 - C_7C_1^{-1}C_6)$ This formula after the appropriate computations gives us the value $2(n-1)^{\frac{n}{2}-2}$. □

In [7] it was proved the following:

Proposition 8 *Let W be a skew and symmetric conference matrix of order n. Then the $(n-3)\times(n-3)$ minors are $W(n-3) = 0$, $2(n-1)^{\frac{n}{2}-3}$, or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 0(mod\ 4)$ and $2(n-1)^{\frac{n}{2}-3}$, or $4(n-1)^{\frac{n}{2}-3}$ for $n \equiv 2(mod\ 4)$.*

□

Theorem 1 *When Gaussian Elimination is applied on a CP skew and symmetric conference matrix W of order n the last two pivots are $n-1$, and $\frac{n-1}{2}$.*

Proof. The last two pivots are given by

$$p_n = \frac{W(n)}{W(n-1)} \quad p_{n-1} = \frac{W(n-1)}{W(n-2)}.$$

Since

$$\begin{array}{ll} W(n) & = (n-1)^{\frac{n}{2}} \\ W(n-1) & = (n-1)^{\frac{n}{2}-1} \\ W(n-2) & = 2(n-1)^{\frac{n}{2}-2}\ . \end{array}$$

the values of the two last pivots are $n-1$, and $\frac{n-1}{2}$ respectively. □

5 Specification of pivot patterns

We proceed our study by trying to specify the pivot structure of some small weighing matrices. In [7] the unique pivot structure of the $W(6,5)$ was specified. It is $\{1,\ 2,\ 2,\ \frac{5}{2},\ \frac{5}{2},\ 5.\}$ Next we will determine the pivot structure of the $W(8,7)$.

Lemma 6 *The pivot patterns of the $W(8,7)$ are $\{1,\ 2,\ 2,\ 4,\ \frac{7}{4},\ \frac{7}{2},\ \frac{7}{2},\ 7\}$ or $\{1,\ 2,\ 2,\ 3,\ \frac{7}{3},\ \frac{7}{2},\ \frac{7}{2},\ 7\}$.*

Proof. From Lemma 2 and Proposition 5 we have that

$$p_1 = 1, \quad p_2 = 2, \quad p_3 = 2, \quad p_4 = 4 \quad \text{or} \quad 3.$$

From Theorem 1 we also have that

$$p_8 = 7, \quad p_7 = \frac{7}{2}.$$

Since $W(4) = 16$ *or* 12 for every $W(n, n-1)$ and $W(5) = 28$ for $W(8,7)$ we have $p_5 = \frac{W(5)}{W(4)} = \frac{28}{16}$ *or* $\frac{28}{12} \Rightarrow p_5 = \frac{7}{4}$ *or* $\frac{7}{3}$.

Also $p_6 = \frac{det(W(8,7))}{\prod_{i=1_{i\neq 6}}^{8} p_i} = \frac{7^4}{1\cdot 2\cdot 2\cdot 4\cdot \frac{7}{4}\cdot \frac{7}{2}\cdot 7}$ *or* $\frac{7^4}{1\cdot 2\cdot 2\cdot 3\cdot \frac{7}{3}\cdot \frac{7}{2}\cdot 7}$

$\Rightarrow \quad p_6 = \frac{7}{2}$ □

Remark 1 *The following matrices have pivot patterns $\{1,\ 2,\ 2,\ 4,\ \frac{7}{4},\ \frac{7}{2},\ \frac{7}{2},\ 7\}$ and $\{1,\ 2,\ 2,\ 3,\ \frac{7}{3},\ \frac{7}{2},\ \frac{7}{2},\ 7\}$ respectively.*

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & - & 1 & 1 & - & 0 & - \\ 1 & - & 1 & - & - & 0 & 1 & - \\ 1 & 1 & - & - & 0 & 1 & - & - \\ 1 & - & 0 & 1 & - & 1 & - & 1 \\ 1 & 1 & - & 0 & - & - & 1 & 1 \\ 0 & 1 & 1 & 1 & - & - & - & - \\ 1 & 0 & 1 & - & 1 & - & - & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 0 & - & - & 1 & 1 & - \\ 1 & - & - & - & 1 & 1 & 0 & 1 \\ 1 & - & 1 & 1 & 0 & 1 & - & - \\ 1 & 1 & - & 1 & - & 0 & - & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & 1 & - & - & 0 \\ 1 & - & 1 & 0 & - & - & 1 & 1 \\ 1 & 0 & - & 1 & 1 & - & 1 & - \end{bmatrix}.$$

$\square$

Lemma 7 *The pivot patterns of the* $W(10,9)$ *are* $\{1,2,2,3,3,4,\frac{9}{4},\frac{9}{2},\frac{9}{2},9\}$ *or* $\{1,2,2,4,3,3,\frac{9}{4},\frac{9}{2},\frac{9}{2},9\}$ *or* $\{1,2,2,3,\frac{10}{4},\frac{18}{5},\frac{9}{3},\frac{9}{2},\frac{9}{2},9\}$.

Proof. We have shown that for every $W(10,9)$, $n \geq 8$, the first four pivots are $1,2,2,3$ or 4. From Theorem 1 we also have that

$$p_{10} = 9, \quad p_9 = \frac{9}{2}.$$

We have

$$w(5) = 48 \text{ or } 36 \text{ or } 30 \text{ for } W(10,9)$$

The 5×5 matrices with determinant 48 contain in the upper left corner the 4×4 matrix A_1 with determinant 16. The 5×5 matrices with determinant 36 contain in the upper left corner the 4×4 matrix A_2 with determinant 12. The 5×5 matrices with determinant 30 contain in the upper left corner the 4×4 matrix A_2 with determinant 12. So, the fifth pivot of $W(10,9)$ can be calculated using relationship (1):

$$p_5 = \tfrac{w(5)}{w(4)} \Rightarrow p_5 = \tfrac{48}{16} \text{ or } \tfrac{36}{12} \text{ or } \tfrac{30}{12} \Rightarrow p_5 = 3 \text{ or } \tfrac{10}{4}.$$

With the same logic, we go on to the sixth pivot: we have

$$w(6) = 144 \text{ or } 108 \text{ for } W(10,9)$$

The 6×6 matrices with determinant 144 contain in the upper left corner the 5×5 matrices with determinants 36 and 48. The 6×6 matrices with determinant 108 contain in the upper left corner the 5×5 matrices with determinants 48, 36 and 30. So, the sixth pivot of $W(10,9)$ can be calculated using relationship (1):

$$p_6 = \tfrac{w(6)}{w(5)} \Rightarrow p_6 = \tfrac{144}{36} \text{ or } \tfrac{144}{48} \text{ or } \tfrac{108}{48} \text{ or } \tfrac{108}{36} \text{ or } \tfrac{108}{30} \Rightarrow p_6 = 4 \text{ or } 3 \text{ or } \tfrac{18}{5}.$$

About the seventh pivot: we have

$$w(7) = 432 \text{ or } 324 \text{ for } W(10,9)$$

The 7×7 matrices with determinant 432 contain in the upper left corner the 6×6 matrix with determinant 144. The 7×7 matrices with determinant 324 contain in the upper left corner the 6×6 matrices with determinants 144 and 108. So, the seventh pivot of $W(10,9)$ can be calculated using relationship (1):

$$p_7 = \tfrac{w(7)}{w(6)} \Rightarrow p_7 = \tfrac{432}{144} \text{ or } \tfrac{324}{144} \text{ or } \tfrac{324}{108} \Rightarrow p_7 = 3 \text{ or } \tfrac{9}{4}.$$

$$p_8 = \frac{det(W(10,9))}{\prod_{i=1_{i \neq 8}}^{10} p_i} = \frac{9^5}{1 \cdot 2 \cdot 2 \cdot 4 \cdot 3 \cdot 3 \cdot \frac{9}{4} \cdot \frac{9}{2} \cdot 9} \quad or \quad \frac{9^5}{1 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot \frac{9}{4} \cdot \frac{9}{2} \cdot 9} \quad or \quad \frac{9^5}{1 \cdot 2 \cdot 2 \cdot 3 \cdot \frac{5}{2} \cdot \frac{18}{5} \cdot 3 \cdot \frac{9}{2} \cdot 9} \quad \Rightarrow \quad p_8 = \frac{9}{2} \quad \square$$

Remark 2 *The following matrices have pivot patterns* $\{1, 2, 2, 3, 3, 4, \frac{9}{4}, \frac{9}{2}, \frac{9}{2}, 9\}$, $\{1, 2, 2, 4, 3, 3, \frac{9}{4}, \frac{9}{2}, \frac{9}{2}, 9\}$ *and* $\{1, 2, 2, 3, \frac{10}{4}, \frac{18}{5}, 3, \frac{9}{2}, \frac{9}{2}, 9\}$ *respectively.*

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - & 1 & 0 \\ 1 & - & - & 1 & 0 & - & - & 1 & 1 & - \\ 1 & 1 & - & - & - & 1 & 0 & - & 1 & - \\ 1 & - & 1 & 1 & - & 1 & 1 & 0 & - & - \\ 1 & 1 & 1 & 1 & - & - & - & - & 0 & 1 \\ 0 & - & 1 & - & - & 1 & - & 1 & 1 & 1 \\ 1 & 0 & - & - & - & - & 1 & 1 & - & 1 \\ 1 & - & - & 0 & 1 & 1 & - & - & - & 1 \\ 1 & 1 & 1 & - & 1 & 0 & - & 1 & - & - \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & - & - & 1 & 1 & - & 1 & 0 & - & - \\ 1 & - & 1 & - & - & - & 0 & 1 & - & 1 \\ 1 & 1 & - & - & 1 & - & - & 1 & 1 & 0 \\ 1 & 1 & - & 1 & - & 0 & - & - & - & 1 \\ 1 & - & - & - & 0 & 1 & 1 & - & 1 & 1 \\ 1 & 1 & 1 & 0 & - & - & 1 & - & 1 & - \\ 0 & - & 1 & 1 & 1 & - & - & - & 1 & 1 \\ 1 & - & 0 & 1 & - & 1 & - & 1 & 1 & - \\ 1 & 0 & 1 & - & 1 & 1 & - & - & - & - \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & - & 0 & 1 & 1 & 1 & - & -1 & 1 & 1 \\ - & - & 1 & 1 & 0 & - & 1 & - & 1 & - \\ 1 & 1 & 1 & 1 & - & 1 & 0 & 1 & 1 & - \\ - & 1 & - & 1 & 1 & - & - & 1 & 1 & 0 \\ 1 & - & - & 0 & - & - & 1 & 1 & 1 & 1 \\ - & - & - & - & - & 1 & - & 0 & 1 & - \\ 0 & - & - & 1 & 1 & 1 & 1 & 1 & - & - \\ - & 1 & - & 1 & - & 1 & 1 & - & 0 & 1 \\ 1 & 0 & - & 1 & - & - & - & - & - & - \\ 1 & 1 & - & - & 1 & 0 & 1 & - & 1 & - \end{bmatrix}$$

□

Tables 3 and 4 give us some of the pivot patterns calculated by computer for the first few $W(n, n-1)$ for both $n \equiv 2(mod\ 4)$ and $n \equiv 0(mod\ 4)$. For each value of n were tested $50000 - 1000000$ H-equivalent matrices and the corresponding pivot patterns were found. The last column shows the number of different pivot patterns that appeared.

n	**growth**	**Pivot Pattern**	number
6	5	$(1,2,2,\frac{5}{2},\frac{5}{2},5)$	1
10	9	$(1,2,2,3,3,4,\frac{9}{4},\frac{9}{2},\frac{9}{2},9)$ or $(1,2,2,4,3,3,\frac{9}{4},\frac{9}{2},\frac{9}{2},9)$ or $\{1,\ 2,\ 2,\ 3,\ \frac{10}{4},\ \frac{18}{5},\ 3,\ \frac{9}{2},\ \frac{9}{2},\ 9\}$	3
14	13	$(1,2,2,3,\frac{10}{3},\frac{17}{5},3.2941,3.9464,3.8235,\frac{13}{5/2},\frac{13}{4},\frac{13}{2},\frac{13}{2},13)$ or $(1,2,2,4,\frac{5}{2},\frac{17}{5},3.2941,3.9464,3.8235,\frac{13}{5/2},\frac{13}{4},\frac{13}{2},\frac{13}{2},13)$	10
18	17	$(1,2,2,4,3,\frac{10}{3},\frac{17}{5},\ldots,5.3125,\frac{17}{10/3},\frac{17}{3},\frac{17}{4},\frac{17}{2},\frac{17}{2},17)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{11}{3},\ldots,4.5156,5,\frac{17}{5/2},\frac{17}{4},\frac{17}{2},\frac{17}{2},17)$ or $(1,2,2,4,\frac{5}{2},\frac{18}{5},\frac{11}{3},\ldots,4.5156,5,\frac{17}{5/2},\frac{17}{4},\frac{17}{2},\frac{17}{2},17)$	19
26	25	$(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{25}{4},\frac{25}{4},\frac{25}{2},\frac{25}{4},\frac{25}{2},\frac{25}{2},25)$ or $(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,6.8182,\frac{25}{3},\frac{25}{3},\frac{25}{4},\frac{25}{2},\frac{25}{2},25)$ or $(1,2,2,4,2,4,4,\ldots,6.6406,7.3529,\frac{25}{5/2},\frac{25}{4},\frac{25}{2},\frac{25}{2},25)$	89
30	29	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,9.0625,\frac{29}{10/3},\frac{29}{3},\frac{29}{4},\frac{29}{2},\frac{29}{2},29)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{34}{9},\ldots,\frac{29}{4},\frac{29}{4},\frac{29}{2},\frac{29}{4},\frac{29}{2},\frac{29}{2},29)$	62
38	37	$(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,11.5625,11.1,\frac{37}{3},\frac{37}{4},\frac{37}{2},\frac{37}{2},37)$ or $(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,10.0909,\frac{37}{3},\frac{37}{3},\frac{37}{4},\frac{37}{2},\frac{37}{2},37)$	44
42	41	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{41}{4},\frac{41}{4},\frac{41}{2},\frac{41}{4},\frac{41}{2},\frac{41}{2},41)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{34}{9},\ldots,12.0588,\frac{41}{10/3},\frac{41}{3},\frac{41}{4},\frac{41}{2},\frac{41}{2},41)$	43
50	49	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{49}{17/5},\frac{49}{10/3},\frac{49}{3},\frac{49}{4},\frac{49}{2},\frac{49}{2},49)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{34}{9},\ldots,\frac{49}{4},\frac{49}{4},\frac{49}{2},\frac{49}{4},\frac{49}{2},\frac{49}{2},49)$	36
54	53	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{53}{4},\frac{53}{4},\frac{53}{2},\frac{53}{4},\frac{53}{2},\frac{53}{2},53)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,15.5882,\frac{53}{10/3},\frac{53}{3},\frac{53}{4},\frac{53}{2},\frac{53}{2},53)$	34
62	61	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{61}{4},\frac{61}{4},\frac{61}{2},\frac{61}{4},\frac{61}{2},\frac{61}{2},61)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{61}{18/5},\frac{61}{10/3},\frac{61}{3},\frac{61}{4},\frac{61}{2},\frac{61}{2},61)$	33
74	73	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{73}{4},\frac{73}{4},\frac{73}{2},\frac{73}{4},\frac{73}{2},\frac{73}{2},73)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{34}{9},\ldots,\frac{73}{4},\frac{73}{4},\frac{73}{2},\frac{73}{4},\frac{73}{2},\frac{73}{2},73)$	31
82	81	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{81}{4},\frac{81}{4},\frac{81}{2},\frac{81}{4},\frac{81}{2},\frac{81}{2},81)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,23.8235,\frac{81}{10/3},27,\frac{81}{4},\frac{81}{2},\frac{81}{2},81)$	28
90	89	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{89}{4},\frac{89}{4},\frac{89}{2},\frac{89}{4},\frac{89}{2},\frac{89}{2},89)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{89}{18/5},\frac{89}{10/3},\frac{89}{3},\frac{89}{4},\frac{89}{2},\frac{89}{2},89)$	32
98	97	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{97}{4},\frac{97}{4},\frac{97}{2},\frac{97}{4},\frac{97}{2},\frac{97}{2},97)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{97}{4},\frac{97}{4},\frac{97}{2},\frac{97}{4},\frac{97}{2},\frac{97}{2},97)$	27

Table 3

n	**growth**	**Pivot Pattern**	number
8	7	$(1,2,2,4,\frac{7}{4},\frac{7}{2},\frac{7}{2},7)$ or $(1,2,2,3,\frac{7}{3},\frac{7}{2},\frac{7}{2},7)$	2
12	11	$(1,2,2,3,\frac{10}{3},\frac{17}{5},\frac{11}{17/5},\frac{11}{5/2},\frac{11}{4},\frac{11}{2},\frac{11}{2},11)$ or $(1,2,2,4,3,\frac{10}{3},\frac{11}{10/3},\frac{18}{5},\frac{11}{4},\frac{11}{2},\frac{11}{2},11)$ or $(1,2,2,3,3,4,\frac{11}{3},\frac{11}{3},\frac{11}{4},\frac{11}{2},\frac{11}{2},11)$	3
16	15	$(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{34}{9},4.4418,\ldots,4.5,\frac{15}{3},\frac{15}{4},\frac{15}{2},\frac{15}{2},15)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{11}{3},\ldots,4.0909,5,\frac{15}{3},\frac{15}{4},\frac{15}{2},\frac{15}{2},15)$	108
20	19	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,5.2778,\frac{19}{10/3},\frac{19}{3},\frac{19}{4},\frac{19}{2},\frac{19}{2},19)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{34}{9},\ldots,5.2778,\frac{19}{10/3},\frac{19}{3},\frac{19}{4},\frac{19}{2},\frac{19}{2},19)$ or $(1,2,2,4,\frac{5}{2},\frac{18}{5},\frac{34}{9},\ldots,5.3438,\frac{19}{3},\frac{19}{3},\frac{19}{4},\frac{19}{2},\frac{19}{2},19)$	309
28	27	$(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{27}{4},\frac{27}{4},\frac{27}{2},\frac{27}{4},\frac{27}{2},\frac{27}{2},27)$ or $(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{27}{4},\frac{27}{4},\frac{27}{5/2},\frac{27}{4},\frac{27}{2},\frac{27}{2},27)$ or $(1,2,2,4,\frac{5}{2},\frac{18}{5},\frac{34}{9},\ldots,7.1719,7.9412,\frac{27}{2},\frac{27}{4},\frac{27}{2},\frac{27}{2},27)$	129
36	35	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{35}{4},\frac{35}{4},\frac{35}{2},\frac{35}{4},\frac{35}{2},\frac{35}{2},35)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{34}{9},\ldots,9.7222,\frac{35}{10/3},\frac{35}{3},\frac{35}{4},\frac{35}{2},\frac{35}{2},35)$	74
44	43	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{43}{4},\frac{43}{4},\frac{43}{2},\frac{43}{4},\frac{43}{2},\frac{43}{2},43)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},\frac{34}{9},\ldots,\frac{43}{4},\frac{43}{4},\frac{43}{2},\frac{43}{4},\frac{43}{2},\frac{43}{2},43)$	46
52	51	$(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{51}{4},19.1250,17,\frac{51}{4},\frac{51}{2},\frac{51}{2},51)$ or $(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{51}{4},19.1250,17,\frac{51}{4},\frac{51}{2},\frac{51}{2},51)$	42
60	59	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{59}{4},22.1250,\frac{59}{3},\frac{59}{4},\frac{59}{2},\frac{59}{2},59)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,18.4375,\frac{59}{10/3},\frac{59}{3},\frac{59}{4},\frac{59}{2},\frac{59}{2},59)$	44
68	67	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{67}{4},\frac{67}{4},\frac{67}{2},\frac{67}{4},\frac{67}{2},\frac{67}{2},67)$ or $(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{67}{4},\frac{67}{4},\frac{67}{2},\frac{67}{4},\frac{67}{2},\frac{67}{2},67)$	35
76	75	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{75}{4},28.1250,25,\frac{75}{4},\frac{75}{2},\frac{75}{2},75)$	34
		$(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{75}{4},28.1250,25,\frac{75}{4},\frac{75}{2},\frac{75}{2},75)$	
84	83	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{83}{4},\frac{83}{4},\frac{83}{2},\frac{83}{4},\frac{83}{2},\frac{83}{2},83)$	31
		$(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{83}{4},\frac{89}{8/3},\frac{83}{3},\frac{83}{4},\frac{83}{2},\frac{83}{2},83)$	
92	91	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{91}{4},\frac{91}{4},\frac{91}{2},\frac{91}{4},\frac{91}{2},\frac{91}{2},91)$	30
		$(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,23.8,26.7647,\frac{91}{5/2},\frac{91}{4},\frac{91}{2},\frac{91}{2},91)$	
100	99	$(1,2,2,4,3,\frac{10}{3},\frac{18}{5},\ldots,\frac{99}{4},\frac{99}{4},\frac{99}{2},\frac{99}{4},\frac{99}{2},\frac{99}{2},99)$	27
		$(1,2,2,3,\frac{10}{3},\frac{18}{5},4,\ldots,\frac{99}{2},30.9375,\frac{99}{5/2},\frac{99}{4},\frac{99}{2},\frac{99}{2},99)$	

Table 4

In the following table we present all the values appearing for the first six and last six pivots after applying Gaussian Elimination with complete pivoting on skew and symmetric conference matrices of order $n \geq 6$.

p_1	p_2	p_3	p_4	p_5	p_6	p_{n-5}	p_{n-4}	p_{n-3}	p_{n-2}	p_{n-1}	p_n
				$\frac{8}{4}$	$\frac{8}{2}$	$\frac{n-1}{8/2}$	$\frac{n-1}{8/3}$, $\frac{n-1}{8/4}$				
			3	$\frac{9}{3}$, $\frac{9}{4}$	$\frac{9}{3}, \frac{10}{3}$	$\frac{n-1}{8/3}, \frac{n-1}{9/3}$, $\frac{n-1}{10/3}$	$\frac{n-1}{9/3}$, $\frac{n-1}{9/4}$	$\frac{n-1}{3}$			
1	2	2		$\frac{10}{3}$, $\frac{10}{4}$			$\frac{n-1}{10/3}$, $\frac{n-1}{10/4}$		$\frac{n-1}{2}$	$\frac{n-1}{2}$	$n-1$
			4		$\frac{34}{9}$	$\frac{n-1}{32/9}$		$\frac{n-1}{4}$			
					$\frac{32}{10}, \frac{34}{10}$, $\frac{36}{10}$	$\frac{n-1}{32/10}, \frac{n-1}{33/10}$, $\frac{n-1}{34/10}$					

Table 5

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