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Some results on self-orthogonal and self-dual codes

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Abstract

We use generator matrices G satisfying $GG^T = aI + bJ$ over \mathbb{Z}_k to obtain linear self-orthogonal and self-dual codes. We give a new family of linear self-orthogonal codes over $GF(3)$ and \mathbb{Z}_4 and a new family of linear self-dual codes over $GF(3)$.

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1 Introduction

A linear code C of length n over \mathbb{Z}_k (or a \mathbb{Z}_k -code of length n) is a \mathbb{Z}_k -submodule of \mathbb{Z}_k^n . If $k = p$ is prime then $\mathbb{Z}_p = GF(p)$ and a linear code of length n is a subspace of $GF(p)$. An element of C is called a codeword. We define the inner product on \mathbb{Z}_k^n by $x \cdot y = x_1y_1 + \cdots + x_ny_n$, where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. The dual code C^\perp of C is defined as $C^\perp = \{v \in \mathbb{Z}_k^n \mid v \cdot w = 0 \text{ for all } w \in C\}$.

$w \in C$. A code C is *self-dual* if $C = C^\perp$. The Hamming weight ($wt(c)$) of a codeword c is the number of non-zero components in the codeword. The *minimum weight* of a code C is the smallest weight among all codeswords of C . The minimum distance of a linear code C is its minimum weight. We say that self-dual codes with the largest minimum weight among self-dual codes of that length are *optimal*. A linear code over $GF(p)$ of length n with k independent rows in its generator matrix will be denoted as $[n, k; p]$. Furthermore, if its minimum distance is d it will be denoted as $[n, k, d; p]$.

Two codes over \mathbb{Z}_k are said to be *equivalent* if one can be obtained from the other by permuting the coordinates and (if necessary) changing the signs of certain coordinates.

There has been a large amount of research recently devoted to self-orthogonal and self-dual codes over the ring \mathbb{Z}_4 , [1, 3, 5, 7]. Patrick Solé's remark that the orthogonality of Hadamard matrices can naturally be interpreted as \mathbb{Z}_4 -orthogonality was investigated in [4]. These self-orthogonal and self-dual codes over \mathbb{Z}_4 were obtained from equivalence classes of Hadamard matrices.

2 The constructions

We give a general theorem which will be used later in the paper.

Theorem 1 *Suppose A and B are two matrices of order n over \mathbb{Z}_k satisfying*

$$AA^T + BB^T = sI + rJ$$

where $s \equiv r \equiv 0 \pmod{k}$. Then

$$G = [A \ B]$$

generates a linear self-orthogonal code over \mathbb{Z}_k , of length $2n$ and with m , $m \leq \frac{n}{2}$ independent rows in its generator matrix. \square

The next corollary is a generalization of a construction given by Georgiou and Koukouvinos [6].

Corollary 1 *Suppose A and B are two matrices of order n over \mathbb{Z}_k satisfying*

$$AA^T = a_1I + a_2J \text{ and } BB^T = b_1I + b_2J$$

where $a_1 + b_1 \equiv a_2 + b_2 \equiv 0 \pmod{k}$. Then

$$G = [A \ B]$$

generates a linear self-orthogonal code of length $2n$ and with m independent rows in its generator matrix, over \mathbb{Z}_k , $m \leq \frac{n}{2}$. \square

Theorem 2 Suppose A and B are two matrices of order n over \mathbb{Z}_k satisfying

$$AA^T = a_1I + a_2J \text{ and } BB^T = b_1I + b_2J$$

where $a_2 + b_2 \equiv 0 \pmod{k}$ and $a_1 + b_1 + a \equiv 0 \pmod{k}$ for some $a \in \mathbb{Z}_k$. Then

$$G_2 = \begin{bmatrix} & A & B \\ aI_{2n} & & \\ & B^T & -A^T \end{bmatrix}$$

generates a linear self-dual code of length $4n$ and with $2n$ independent rows in its generator matrix, over \mathbb{Z}_k . \square

Example 1 (i) Set $A = B = \text{circ}(1, 1, 1, 1, 0)$. We have that $AA^T = BB^T = I + 3J$. Then

$$G_2 = \begin{bmatrix} & A & B \\ I_{2n} & & \\ & B^T & -A^T \end{bmatrix}$$

generates an $[20, 10, 6; 3]$ extremal self-dual code with weight enumerator

$$W(z) = 1 + 120z^6 + 4260z^9 + 26280z^{12} + 25728z^{15} + 2560z^{18}.$$

(ii) Set $A = \text{circ}(-2, -2, 0, -1, 0)$ and $B = \text{circ}(-1, -1, -1, -1, 1)$. We have that $AA^T = 5I + 4J$ and $BB^T = 4I + J$. Then

$$G_2 = \begin{bmatrix} & A & B \\ I & & \\ & B^T & -A^T \end{bmatrix}$$

generates an $[20, 10, 8; 5]$ extremal self-dual code with weight enumerator

$$W(z) = 1 + 1280z^8 + 3200z^9 + 24848z^{10} + 58560z^{11} + 248480z^{12} + \\ + 464960z^{13} + 1175840z^{14} + 1568000z^{15} + 2267240z^{16} + \\ + 1896720z^{17} + 1398960z^{18} + 541760z^{19} + 115776z^{20}.$$

- (ii) Set $A = circ(-2, -2, 0, -1, 0)$ and $B = circ(-1, -1, -1, -1, 1)$. We have that $AA^T = 5I + 4J$ and $BB^T = 4I + J$. Then

$$G = [A \ B]$$

generates an $[10, 5, 4; 5]$ self-dual code with weight enumerator

$$W(z) = 1 + 40z^4 + 44z^5 + 220z^6 + 760z^7 + 940z^8 + 740z^9 + 380z^{10}.$$

For the SBIBDs we use in the remainder of this paper, we refer the reader to the book of Beth, Jungnickel and Lenz [2]. By $A = SBIBD(v, k, \lambda)$ we denote the $v \times v$ $(0, 1)$ incidence matrix of the $SBIBD(v, k, \lambda)$.

- Example 2** 1. There exist $A=SBIBD(31,10,3)$ and $B=SBIBD(31,15,7)$, so $[A \ B]$ generates a linear self-orthogonal code of length 62 and with k_1 independent rows in its generator matrix, over $GF(5)$ with minimum distance d_1 as

$$AA^T = 7I + 3J \text{ and } BB^T = 8I + 7J.$$

2. There exist $A=SBIBD(71,15,3)$ and $B=SBIBD(71,21,6)$, so $[A \ B]$ generates a linear self-orthogonal code of length 142 and with k_2 independent rows in its generator matrix, over $GF(3)$ with minimum distance d_2 as

$$AA^T = 12I + 3J \text{ and } BB^T = 15I + 6J.$$

3. There exist $A=SBIBD(133,33,8)$ and $B=SBIBD(133,12,1)$, so $[A \ B]$ generates a linear self-orthogonal code of length 266 and with k_3 independent rows in its generator matrix, over $GF(3)$ with minimum distance d_3 as

$$AA^T = 25I + 8J \text{ and } BB^T = 11I + J.$$

□

In the next theorems we use specific families to find linear self-orthogonal codes. We combine skew-Hadamard matrices or conference matrices with incidence matrices of projective planes to construct some linear self-orthogonal codes over \mathbb{Z}_k .

Details on skew-Hadamard matrices and conference matrices required for the next theorem can be found in Seberry and Yamada [9]. Appropriate details of the incidence matrices of projective planes can be found in Ryser [8].

Theorem 3 *Let $p + 1$ be the order of a skew-Hadamard matrix or a conference matrix. Suppose $p = q^2 + q + 1$ for some prime power q . Then there exists a self-orthogonal code over \mathbb{Z}_k of length $2p$, with m independent rows in its generator matrix and minimum distance d whenever $p + q = (q + 1)^2 \equiv 0 \pmod{k}$.*

Proof. Write the skew-Hadamard matrix $S + I$, minus its diagonal entries, or conference matrix as

$$\begin{bmatrix} 0 & e \\ \pm e^T & P \end{bmatrix}$$

where e is the $1 \times p$ matrix of ones. Then P is a $p \times p$ matrix satisfying

$$PP^T = pI - J.$$

Write Q for an incidence matrix of the projective plane over $GF(q)$. Then Q , of order $p = q^2 + q + 1$, is circulant and satisfies

$$QQ^T = qI + J.$$

Now $G_1 = [P \ Q]$ generates the required self-orthogonal code over \mathbb{Z}_k of length $2p$ and with m , $m \leq p$ independent rows in its generator matrix as $G_1 G_1^T = (p + q)I = (q + 1)^2 I \equiv 0$. \square

Corollary 2 *Let $p + 1$ be the order of a skew-Hadamard matrix or a conference matrix. Suppose $p = q^2 + q + 1$ for some prime power q , and $q \equiv 2 \pmod{3}$. Then there exists a self-orthogonal $[2p, m, d]$ ternary code with $m \leq p - 1$. Note that $m = p$ iff $q \equiv 1 \pmod{3}$ and thus $G_1 = [P \ Q]$ is the generator matrix of a self-dual code.*

Proof. Use theorem 3. □

Example 3 Let $q = 2$, $p = 7$, $P = \text{circ}(0, 1, 1, -1, 1, -1, -1)$ and $Q = \text{circ}(1, 1, 0, 1, 0, 0, 0)$. We consider the matrix $[P \ Q]$ and we remove its first row. Then the derived matrix is the generator matrix of a $[14, 6, 6; 3]$ code with weight enumerator

$$W(z) = 1 + 84z^6 + 476z^9 + 168z^{12}.$$

Theorem 4 The codes over $GF(3)$ and \mathbb{Z}_4 we obtain using G_1 are

(i) $[2p, p, d]$ for $q \equiv 1(\text{mod } 3)$

(ii) $[2p, p - 1, d]$ for $q \equiv 0, 2(\text{mod } 3)$ and $q \equiv 0, 1, 2, 3(\text{mod } 4)$.

Proof. Consider the matrix P of order $p = q^2 + q + 1$. Now $PP^T = (q^2 + q + 1)I - J$ and $\det PP^T \equiv 0(\text{mod } 3)$ and $0(\text{mod } 4)$. Now consider P' with one row of P removed. Then the matrix P' has size $(q^2 + q) \times (q^2 + q + 1)$ and so $P'P'^T$ is of order $q^2 + q$ and has the following form:

$$P'P'^T = \begin{bmatrix} q^2 + q & -1 & -1 & \cdots & -1 \\ -1 & q^2 + q & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & q^2 + q \end{bmatrix}$$

and $\det P'P'^T = (1)(q^2 + q + 1)^{q^2+q-1} \not\equiv 0$ for $q \equiv 0, 2(\text{mod } 3)$ and $q \equiv 0, 1, 2, 3(\text{mod } 4)$. Hence the rank of the matrix P' is $p - 1$ for these cases.

Now the matrix Q satisfies $QQ^T = qI + J$ and $\det QQ^T = (q + 1)^2(q)^{q^2+q} \not\equiv 0(\text{mod } 3)$ for $q \equiv 1(\text{mod } 3)$. Hence the rank of the matrix Q is p for this case. □

Remark 1 We recall that a self-orthogonal code, C , of length $2p$, with p independent rows in its generator matrix and distance d_1 with C^\perp a self-orthogonal code of length $2p$ and p independent rows in its generator matrix with distance d_2 we have that $C = C^\perp$ and so C is in fact self-dual.

Theorem 5 *Let $p + 1$ be the order of a skew-Hadamard matrix or a conference matrix. Suppose $p = q^2 + q + 1$ for some prime power q . Then there exists a self-orthogonal \mathbb{Z}_k -code of length $2p$, with m independent rows in its generator matrix and minimum distance d , whenever $p + q \equiv 0 \pmod{k}$.*

Proof. Construct the matrices P and Q as in the proof of theorem 3. Set

$$G_3 = \begin{bmatrix} P & Q \\ Q^T & -P^T \end{bmatrix}.$$

We have that

$$G_3 G_3^T = \begin{bmatrix} P & Q \\ Q^T & -P^T \end{bmatrix} \begin{bmatrix} P^T & Q \\ Q^T & -P \end{bmatrix} = \begin{bmatrix} PP^T + QQ^T & PQ - QP \\ Q^T P^T - P^T Q^T & Q^T Q + P^T P \end{bmatrix}$$

If $PQ = QP$ (for example, this is true if P is circulant, in which case p is prime) then this matrix generates the required self-orthogonal code of length $2p$ with m independent rows in its generator matrix, as $G_3 G_3^T = (q + 1)^2 I_m \equiv 0 \pmod{k}$. □

Theorem 6 *Let $p + 1$ be the order of a skew-Hadamard matrix or a conference matrix. Suppose $p = q^2 + q + 1$ for some prime power q . Then there exists a self-dual \mathbb{Z}_k -code of length $4p$, with $2p$ independent rows in its generator matrix and minimum distance d , whenever $p + q + a \equiv 0 \pmod{k}$ for some $a \in \mathbb{Z}_k$.*

Proof. Construct the matrices P, Q and G_3 as in the proof of theorem 5. Set $G_4 = [I_{2p} \ G_3]$. If $PQ = QP$ (for example, this is true if P is circulant, in which case p is prime) then the matrix G_4 generates the required self-dual code of length $4p$ with $2p$ independent rows in its generator matrix, as $G_4 G_4^T = (q + p + a) I_{2p}$. □

We are able to use the considerable literature on the minimum distance of codes generated by skew-Hadamard matrices, $I + S$, minus its diagonal entries, to obtain lower bounds for the minimum distance of codes with generator matrix $[P \ Q]$, where P and Q are given in the proof of Theorem 3 via the following lemma:

Lemma 1 Suppose A and B are two matrices of order n with elements from \mathbb{Z}_k and $\det(A) \neq 0$. We denote the minimum weights among all linear combinations of their rows (over \mathbb{Z}_k) by d_A and d_B respectively. Then the code, C , with generator matrix $[A \ B]$ has minimum Hamming distance $d_C \geq d_A + d_B$.

Remark 2 There are many pairs (p, q) which satisfy the conditions of Theorem 3. The first few pairs are $(7, 2)$, $(13, 3)$, $(31, 5)$, $(73, 8)$, $(91, 9)$, $(183, 13)$, $(307, 17)$, $(757, 27)$, $(1723, 41)$.

Example 4 1. Let $q = 3$, $p = 13$, $P = \text{circ}(0, 1, -1, 1, 1, -1, -1, -1, -1, 1, 1, -1, 1)$ and $Q = \text{circ}(1, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0)$. We consider the matrix $[P \ Q]$ and we remove its first row. Then the derived matrix is the generator matrix of a self-orthogonal \mathbb{Z}_4 -code of length 26, with 12 independent rows in its generator matrix and minimum distance 8 with weight enumerator

$$\begin{aligned} W(z) = & 1 + 390z^8 + 1716z^{10} + 40092z^{12} + 17056z^{13} + 226720z^{14} + \\ & + 422656z^{15} + 541593z^{16} + 2348320z^{17} + 1012440z^{18} + \\ & + 4010240z^{19} + 2425436z^{20} + 2384096z^{21} + 2247648z^{22} + \\ & + 559104z^{23} + 472680z^{24} + 56160z^{25} + 10868z^{26}. \end{aligned}$$

2. Let $q = 5$, $p = 31$, $P = \text{circ}(0, -1, -1, 1, -1, -1, 1, -1, -1, -1, -1, 1, 1, -1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1)$ and $Q = \text{circ}(1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. We consider the matrix $[P \ Q]$ and we remove its first row. Then the derived matrix is the generator matrix of a self-orthogonal code over $GF(3)$ of length 62, with 30 independent rows in its generator matrix and minimum distance 12. Thus we can obtain a $[62, m, d; 3]$ code for all $m \leq 30$ and with $d(m) \geq 12$ by removing rows. \square

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