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# Some results on Kharaghani type orthogonal designs

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## Abstract

In this paper we give a general theorem which can be used to multiply the length of amicable sequences keeping the amicability property and the type of the sequences. As a consequence we have that if there exist two, four or eight amicable sequences of length  $m$  and type  $(a_1, a_2)$ ,  $(a_1, a_2, a_3, a_4)$  or  $(a_1, a_2, \dots, a_8)$  then there exist amicable sequences of length  $\ell \equiv 0 \pmod{m}$  and of the same type. We also present a theorem that produces a set of  $2v$  amicable sequences from a set of  $v$  (not necessary amicable) sequences and a construction method for amicable sequences of type  $(a_1, a_1, a_2, a_2, \dots, a_v, a_v)$  from  $v$  pairs of disjoint  $(0, \pm 1)$  amicable sequences.

Using these results we can obtain many infinite classes of Kharaghani type orthogonal designs. Actually, if there exists an Kharaghani type orthogonal design of order  $n$  and of type  $(a_1, a_2, \dots, a_v)$ , which is constructed from sequences, then there exists an infinite family of Kharaghani type orthogonal designs of the same type which is constructed from appropriate sequences.

*Key words and phrases:* Sequences, orthogonal designs, Kharaghani type orthogonal designs, amicable sets, Hall polynomial.

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# 1 Introduction

An *orthogonal design* of order  $n$  and type  $(s_1, s_2, \dots, s_u)$  denoted  $OD(n; s_1, s_2, \dots, s_u)$  in the variables  $x_1, x_2, \dots, x_u$ , is a matrix  $A$  of order  $n$  with entries in the set  $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$  satisfying

$$AA^T = \sum_{i=1}^u (s_i x_i^2) I_n,$$

where  $I_n$  is the identity matrix of order  $n$ . Let  $A_1, A_2$  be circulant matrices of order  $n$  with entries in  $\{0, \pm x_1, \pm x_2\}$  satisfying  $A_1 A_1^T + A_2 A_2^T = (s_1 x_1^2 + s_2 x_2^2) I_n$ . Then

$$D = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_1^T \end{pmatrix} \quad \text{or} \quad D = \begin{pmatrix} A_1 & A_2 R \\ -A_2 R & A_1 \end{pmatrix}. \quad (1)$$

is an  $OD(2n; s_1, s_2)$ .

Let  $B_i$ ,  $i = 1, 2, 3, 4$  be circulant matrices of order  $n$  with entries in  $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$  satisfying

$$\sum_{i=1}^4 B_i B_i^T = \sum_{i=1}^u (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2 R & B_3 R & B_4 R \\ -B_2 R & B_1 & B_4^T R & -B_3^T R \\ -B_3 R & -B_4^T R & B_1 & B_2^T R \\ -B_4 R & B_3^T R & -B_2^T R & B_1 \end{pmatrix} \quad (2)$$

where  $R$  is the back-diagonal identity matrix, is an  $OD(4n; s_1, s_2, \dots, s_u)$ . See page 107 of [1] for details.

A pair of matrices  $A, B$  is said to be amicable (anti-amicable) if  $AB^T - BA^T = 0$  ( $AB^T + BA^T = 0$ ). Following [4] a set  $\{A_1, A_2, \dots, A_{2n}\}$  of square real matrices is said to be *amicable* if

$$\sum_{i=1}^n \left( A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T \right) = 0 \quad (3)$$

for some permutation  $\sigma$  of the set  $\{1, 2, \dots, 2n\}$ . For simplicity, we will always take  $\sigma(i) = i$  unless otherwise specified. So

$$\sum_{i=1}^n (A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T) = 0. \quad (4)$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper  $R_k$  denotes the back diagonal identity matrix of order  $k$ .

A set of matrices  $\{B_1, B_2, \dots, B_n\}$  of order  $m$  with entries in  $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$  is said to satisfy an additive property of type  $(s_1, s_2, \dots, s_u)$  if

$$\sum_{i=1}^n B_i B_i^T = \sum_{i=1}^u (s_i x_i^2) I_m. \quad (5)$$

Let  $\{A_i\}_{i=1}^8$  be an amicable set of circulant matrices (or type 1) of type  $(s_1, s_2, \dots, s_u)$  of order  $t$ . Then the Kharaghani array from [4]

$$H = \begin{pmatrix} A_1 & A_2 & A_4 R_n & A_3 R_n & A_6 R_n & A_5 R_n & A_8 R_n & A_7 R_n \\ -A_2 & A_1 & A_3 R_n & -A_4 R_n & A_5 R_n & -A_6 R_n & A_7 R_n & -A_8 R_n \\ -A_4 R_n & -A_3 R_n & A_1 & A_2 & -A_8^T R_n & A_7^T R_n & A_6^T R_n & -A_5^T R_n \\ -A_3 R_n & A_4 R_n & -A_2 & A_1 & A_7^T R_n & A_8^T R_n & -A_6^T R_n & -A_5^T R_n \\ -A_6 R_n & -A_5 R_n & A_8^T R_n & -A_7^T R_n & A_1 & A_2 & -A_4^T R_n & A_3^T R_n \\ -A_5 R_n & A_6 R_n & -A_7^T R_n & -A_8^T R_n & -A_2 & A_1 & A_3^T R_n & A_4^T R_n \\ -A_8 R_n & -A_7 R_n & -A_6^T R_n & A_5^T R_n & A_4^T R_n & -A_3^T R_n & A_1 & A_2 \\ -A_7 R_n & A_8 R_n & A_5^T R_n & A_6^T R_n & -A_3^T R_n & -A_4^T R_n & -A_2 & A_1 \end{pmatrix} \quad (6)$$

is a Kharaghani type orthogonal design  $OD(8m; s_1, s_2, \dots, s_u)$ .

The Kharaghani array has been used in a number of papers [2, 3, 4, 5, 6] to obtain infinitely many families of Kharaghani type orthogonal designs.

A set  $\{A_i\}_{i=1}^4$  is said to be a *short amicable set* of length  $m$  and type  $(u_1, u_2, u_3, u_4)$  if (4) and (5) are satisfied for  $n = 4$  and  $u \leq 4$ . Short amicable sets can be used in either the Goethals-Seidel array or the *short Kharaghani array*

$$\begin{bmatrix} A & B & CR & DR \\ -B & A & DR & -CR \\ -CR & -DR & A & B \\ -DR & CR & -B & A \end{bmatrix} \quad (7)$$

to form an Goethal-Seidel type orthogonal design  $OD(4m; u_1, u_2, u_3, u_4)$ .

A set of sequences  $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$ ,  $k = 1, 2, \dots, 2v$  where  $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$ ,  $j = 0, 1, \dots, m-1$  and  $k = 1, 2, \dots, 2v$  is said to be a set of  $2v$  *amicable sequences* of length  $m$  and type  $(u_1, u_2, \dots, u_p)$  if the corresponding circulant matrices which are constructed from these sequences satisfy the equations (4) and (5). On the other hand, it is clear that, if we have a set of circulant amicable matrices then their first rows can be considered as a set of amicable sequences. Therefore, throughout this paper we use either circulant amicable matrices or amicable sequences.

Given the sequence  $A = \{a_1, a_2, \dots, a_n\}$  of length  $n$  the *non-periodic autocorrelation function (NPAF)*  $N_A(s)$  is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \quad s = 0, 1, \dots, n-1, \quad (8)$$

Given  $A$  as above of length  $n$  the *periodic autocorrelation function (PAF)*  $P_A(s)$  is defined, reducing  $i + s$  modulo  $n$ , as

$$P_A(s) = \sum_{i=1}^n a_i a_{i+s}, \quad s = 0, 1, \dots, n-1. \quad (9)$$

We define the *NPAF (PAF)* of a set of sequences the sum of the corresponding *NPAF (PAF)* of the individual sequences.

Suppose  $C = \text{circ}(c_0, c_1, \dots, c_{n-1})$  is a circulant matrix of order  $n$ .

Let

$$T_n = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

of order  $n$ , be the shift matrix. Then we can write  $C = c_0 I + c_1 T_n + \dots + c_{n-1} T_n^{n-1}$ . Note that  $T_n^n = I$  the identity matrix of order  $n$ . We say the Hall polynomial of  $C$  is  $\sum_{i=0}^{n-1} c_i x^i$ . The Hall polynomial of  $C^T$  is  $\sum_{i=0}^{n-1} c_i x^{n-i}$ .

## 2 Multiplication of the length of amicable sets of sequences

**Theorem 1** Let  $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$ ,  $k = 1, 2, \dots, 2v$  where  $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$ ,  $j = 0, 1, \dots, m-1$  and  $k = 1, 2, \dots, 2v$  be a set of  $2v$  amicable sequences of length  $m$  and type  $(u_1, u_2, \dots, u_p)$ . Then there exist a set of  $2v$  amicable sequences of length  $\ell \equiv 0 \pmod{m} = mi$  for all  $i = 1, 2, \dots$  and type  $(u_1, u_2, \dots, u_p)$ .

**Proof.** Let  $i$  be a constant integer. We use the map  $T_m^k$  to define sequences  $A_k$  and the map  $S_\ell^k = T_m^k$  to define sequences  $B_k$

$$B_k = \sum_{j=0}^{m-1} a_{k,j} S_\ell^j, \quad k = 1, 2, \dots, 2v$$

Now

$$\sum_{k=1}^{2v} A_k A_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} (a_{k,j} a_{k,x} T_m^{j-x}) = \left( \sum_{k=1}^p u_k x_k^2 \right) I_m.$$

Thus we have that

- (i) If  $m$  is odd then the coefficients of  $T_m^\sigma$ ,  $\sigma = -(m-1), \dots, -1, 1, \dots, m-1$  is zero, and the coefficient of  $T_m^0$  is  $\sum_{k=1}^p u_k x_k^2$ . That means

$$\sum_{j=0}^{m-1} j, x = 0j - x = \sigma \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^p u_k x_k^2 \quad (10)$$

- (ii) If  $m$  is even,  $m = 2n$  then we have that  $T_m^n = T_m^{-n}$  and so the coefficients of  $T_m^\sigma$ ,  $\sigma = -(2n-1), \dots, -(n+1), -(n-1), \dots, -1, 1, \dots, n-1, n+1, \dots, 2n-1$  are zero, the coefficient of  $T_m^n$  plus the coefficient of  $T_m^{-n}$  is zero and the coefficient of  $T_m^0$  is  $\sum_{k=1}^p u_k x_k^2$ . That means

$$\sum_{j=0}^{m-1} j, x = 0j - x = \sigma \neq \pm n \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0, \quad \sum_{j=0}^{m-1} j, x = 0j - x = \pm n \sum_{k=1}^{2v} a_{k,j} a_{k,x} = 0 \quad \text{and} \quad \sum_{j=0}^{m-1} \sum_{k=1}^{2v} a_{k,j}^2 = \sum_{k=1}^p u_k x_k^2 \quad (11)$$

Now

$$\sum_{k=1}^{2v} B_k B_k^T = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^{2v} \left( a_{k,j} a_{k,x} S_\ell^{j-x} \right)$$

We have that the coefficients of  $S_\ell^\sigma$  are equal to the coefficients of  $T_m^\sigma$  for all  $\sigma = -(m-1), \dots, m-1$ , and so using equations (10) or (11) we obtain

$$\sum_{k=1}^{2v} B_k B_k^T = \left( \sum_{k=1}^p u_k x_k^2 \right) I_{2mi} = \left( \sum_{k=1}^p u_k x_k^2 \right) I_\ell \quad (12)$$

Moreover

$$\sum_{k=1}^v (A_{2k-1} A_{2k}^T - A_{2k} A_{2k-1}^T) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^v ((a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) T_m^{j-x}) = 0$$

and from these we have that

- (i) if  $m$  odd, then the coefficients of  $T_m^\sigma$ ,  $\sigma = -(m-1), \dots, m-1$  are zero. That means

$$\sum_{j=0}^{m-1} j, x = 0j - x = \sigma \sum_{k=1}^v (a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) = 0 \quad (13)$$

(ii) if  $m$  is even,  $m = 2n$  then the coefficients of  $T_m^\sigma$ ,  $\sigma = -(2n - 1), \dots, -(n+1), -(n-1), \dots, n-1, n+1, \dots, 2n-1$  are zero and the coefficient of  $T_m^n$  plus the the coefficient of  $T_m^{-n}$  is zero. That means

$$\sum_{\substack{j-x=0 \\ \sigma \neq \pm n}}^{m-1} \sum_{k=1}^v (a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) = 0 \text{ and } \sum_{k=1}^{m-1} j, x = 0 \quad j-x = \pm n \sum_{k=1}^v (a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) = 0$$

(14)

Now

$$\sum_{k=1}^v (B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T) = \sum_{j=0}^{m-1} \sum_{x=0}^{m-1} \sum_{k=1}^v ((a_{2k-1,j} a_{2k,x} - a_{2k,j} a_{2k-1,x}) S_\ell^{j-x})$$

We have that the coefficients of  $S_\ell^\sigma$  are equal to the coefficients of  $T_m^\sigma$  for all  $\sigma = -(m-1), \dots, m-1$  and so using equations (13) or equations (2) we obtain

$$\sum_{k=1}^v (B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T) = 0 \quad (15)$$

Equations (12) and (15) show that  $\{B_k\}_{k=1}^{2v}$  is an amicable set of matrices (sequences) of length  $\ell \equiv 0 \pmod{m}$ ,  $\ell = mi$ ,  $i = 1, 2, \dots$  and type  $(u_1, u_2, \dots, u_p)$ .  $\square$

**Corollary 1** Let  $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$ ,  $k = 1, 2$  where  $a_{k,j} \in \{0, \pm x_1, \pm x_2\}$ ,  $j = 0, 1, \dots, m-1$  and  $k = 1, 2$  be a set of two amicable sequences of length  $m$  and type  $(u_1, u_2)$ . Then there exist a set of two amicable sequences of length  $\ell \equiv 0 \pmod{m} = mi$  and type  $(u_1, u_2)$ .

**Proof.** Use Theorem 1 with  $2v = 2$  and  $p = 2$ .

**Example 1** We have that  $A_1 = 0T_4^0 + aT_4^1 + bT_4^2 - aT_4^3$  and  $A_2 = 0T_4^0 + aT_4^1 + 0T_4^2 + aT_4^3$  is a set of two amicable matrices (sequences) of length  $m = 4$  and type  $(1, 4)$ . Corollary 1 gives a set of two amicable sequences of length  $m = 4i$  and type  $(1, 4)$  for all  $i = 1, 2, \dots$ .

**Corollary 2** Let  $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$ ,  $k = 1, 2, 3, 4$  where  $a_{k,j} \in \{0, \pm x_1, \pm x_2, \pm x_3, \pm x_4\}$ ,  $j = 0, 1, \dots, m-1$  and  $k = 1, 2, 3, 4$  be a set of four amicable sequences of length  $m$  and type  $(u_1, u_2, u_3, u_4)$ . Then there exist a set of four amicable sequences of length  $\ell \equiv 0 \pmod{m} = mi$  and type  $(u_1, u_2, u_3, u_4)$ .

**Proof.** Use Theorem 1 with  $2v = 4$  and  $p = 4$ .

**Example 2** We have that  $A_1 = aT_3^0 - bT_3^1 + aT_3^2$ ,  $A_2 = bT_3^0 + aT_3^1 + bT_3^2$  and  $A_3 = aT_3^0 + aT_3^1 - aT_3^2$ ,  $A_4 = bT_3^0 + bT_3^1 + bT_3^2$  is a set of four amicable matrices (sequences) of length  $m = 3$  and type  $(6, 6)$ . Corollary 2 gives a set of four amicable sequences of length  $m = 3i$  and type  $(6, 6)$  for all  $i = 1, 2, \dots$ .

**Corollary 3** Let  $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$ ,  $k = 1, 2, \dots, 8$  where  $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_8\}$ ,  $j = 0, 1, \dots, m-1$  and  $k = 1, 2, \dots, 8$  be a set of eight amicable sequences of length  $m$  and type  $(u_1, u_2, \dots, u_8)$ . Then there exist a set of eight amicable sequences of length  $\ell \equiv 0 \pmod{m} = mi$  and type  $(u_1, u_2, \dots, u_8)$ .

**Proof.** Use Theorem 1 with  $2v = 8$  and  $p = 8$ .

**Example 3** We have that  $A_1 = -aT_7^0 + aT_7^1 + aT_7^2 + gT_7^3 + aT_7^4 + eT_7^5 + cT_7^6$ ,  $A_2 = -fT_7^0 + fT_7^1 + fT_7^2 - hT_7^3 + fT_7^4 + bT_7^5 - dT_7^6$ ,  $A_3 = -gT_7^0 + gT_7^1 + gT_7^2 - aT_7^3 + gT_7^4 + cT_7^5 - eT_7^6$ ,  $A_4 = -hT_7^0 + hT_7^1 + hT_7^2 + fT_7^3 + hT_7^4 + dT_7^5 + bT_7^6$ ,  $A_5 = -eT_7^0 + eT_7^1 + eT_7^2 - cT_7^3 + eT_7^4 - aT_7^5 + gT_7^6$ ,  $A_6 = -dT_7^0 + dT_7^1 + dT_7^2 - bT_7^3 + dT_7^4 - hT_7^5 + fT_7^6$ ,  $A_7 = -bT_7^0 + bT_7^1 + bT_7^2 + dT_7^3 + bT_7^4 - fT_7^5 - hT_7^6$  and  $A_8 = -cT_7^0 + cT_7^1 + cT_7^2 + eT_7^3 + cT_7^4 - gT_7^5 - aT_7^6$  is a set of eight amicable matrices (sequences) of length  $m = 7$  and type  $(7, 7, 7, 7, 7, 7, 7, 7)$ . Corollary 3 gives a set of eight amicable sequences of length  $m = 7i$  and type  $(7, 7, 7, 7, 7, 7, 7, 7)$  for all  $i = 1, 2, \dots$ .

**Remark 1** Using Corollaries 1, 2 and 3 as indicated by the examples and using array (1), (2) or (7) and (6) respectively we obtain many infinite classes of orthogonal designs.

### 3 Construction of amicable sets of sequences from non amicable sets of sequences

**Lemma 1** Let  $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$ ,  $k = 1, 2, \dots, v_1$ , where  $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$ ,  $j = 0, 1, \dots, m-1$  and  $k = 1, 2, \dots, v_1$  be a set of  $v_1$  amicable sequences of length  $m$  and type  $(u_1, u_2, \dots, u_p)$  and  $B_r = \{b_{r,0}, b_{r,1}, \dots, b_{r,m-1}\}$ ,  $r = 1, 2, \dots, v_2$ , where  $b_{r,s} \in \{0, \pm y_1, \pm y_2, \dots, \pm y_q\}$ ,  $s = 0, 1, \dots, m-1$  and  $r = 1, 2, \dots, v_2$  be a set of  $v_2$  amicable sequences of length  $m$  and type  $(t_1, t_2, \dots, t_q)$ .

Then there exist a set of  $v_1 + v_2$  amicable sequences of length  $m$  and type  $(u_1, u_2, \dots, u_p, t_1, t_2, \dots, t_q)$ .

**Proof.** These are the sequences  $A_k$ ,  $k = 1, 2, \dots, v_1$  and  $B_k$ ,  $k = 1, 2, \dots, v_2$  together.  $\square$

**Corollary 4** Let  $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m_1-1}\}$ ,  $k = 1, 2, \dots, v_1$ , where  $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$ ,  $j = 0, 1, \dots, m_1-1$  and  $k = 1, 2, \dots, v_1$  be a set of  $v_1$  amicable sequences of length  $m_1$  and type  $(u_1, u_2, \dots, u_p)$  and  $B_r = \{b_{r,0}, b_{r,1}, \dots, b_{r,m_2-1}\}$ ,  $r = 1, 2, \dots, v_2$ , where  $b_{r,s} \in \{0, \pm y_1, \pm y_2, \dots, \pm y_q\}$ ,  $s = 0, 1, \dots, m_2-1$  and  $r = 1, 2, \dots, v_2$  be a set of  $v_2$  amicable sequences of length  $m_2$  and type  $(t_1, t_2, \dots, t_q)$ .

Then there exist a set of  $v_1 + v_2$  amicable sequences of length  $\ell \cdot i$  where  $\ell = [m_1, m_2]$  is the least common multiple (l.c.m.) of  $m_1$  and  $m_2$  and type  $(u_1, u_2, \dots, u_p, t_1, t_2, \dots, t_q)$ .

**Proof.** Since  $\ell$  is the least common multiple of  $m_1$  and  $m_2$  then  $\ell = m_1 \cdot i_1 = m_2 \cdot i_2$ . Using theorem 1 we can construct a set of  $v_1$  amicable sequences of length  $\ell$  and type  $(u_1, u_2, \dots, u_p)$  and a set of  $v_2$  amicable sequences of length  $\ell$  and type  $(t_1, t_2, \dots, t_q)$ . Now using Lemma 1 we obtain a set of  $v_1 + v_2$  amicable sequences of length  $\ell$  and type  $(u_1, u_2, \dots, u_p, t_1, t_2, \dots, t_q)$ .

Using theorem 1 again in the derived sequences we have the result.  $\square$

**Example 4** We have that  $A_1 = \{e, f\}$ ,  $A_2 = \{e, -f\}$ ,  $A_3 = \{e, 0\}$ ,  $A_4 = \{f, 0\}$  is a short amicable set of length 2 and type (3, 3). We also have that  $A_1 = \{a, a, b, -b\}$ ,  $A_2 = \{c, c, d, -d\}$ ,  $A_3 = \{d, d, -c, c\}$ ,  $A_4 = \{b, b, -a, a\}$  is a short amicable set of length 4 and type (4, 4, 4, 4). Now  $\ell = [4, 2] = 4$  and thus from corollary 4 we obtain eight amicable sequences of length  $\ell \cdot i$  and type (3, 3, 4, 4, 4, 4) for all  $i = 1, 2, \dots$ .

**Theorem 2 (Doubling the number of sequences )** Let  $A_k = \{a_{k,0}, a_{k,1}, \dots, a_{k,m-1}\}$ ,  $k = 1, 2, \dots, v$  where  $a_{k,j} \in \{0, \pm x_1, \pm x_2, \dots, \pm x_p\}$ ,  $j = 0, 1, \dots, m-1$  and  $k = 1, 2, \dots, v$  be  $v$  sequences with  $PAF=0$  (or  $NPAF=0$ ) of length  $m$  and type  $(u_1, u_2, \dots, u_p)$ . Then there exist a set of  $2v$  amicable sequences of length  $m$  and type  $(2u_1, 2u_2, \dots, 2u_p)$  with  $PAF=0$  (or  $NPAF=0$ ).

**Proof.** Set  $B_{2k-1} = B_{2k} = \text{circ}(A_k)$ ,  $k = 1, 2, \dots, v$ . Then

$$\sum_{k=1}^{2v} B_k B_k^T = 2 \cdot \sum_{k=1}^v A_k A_k^T = \left( \sum_{i=1}^p 2u_i x_i^2 \right) I_m$$

and

$$B_{2k-1} B_{2k}^T - B_{2k} B_{2k-1}^T = A_k A_k^T - A_k A_k^T = 0, \quad k = 1, 2, \dots, v.$$

Thus  $\{B_k\}_{k=1}^{2v}$  is a set of  $2v$  amicable matrices (sequences) of length  $m$  and type  $(2u_1, 2u_2, \dots, 2u_p)$ .  $\square$

## 4 More Constructions

**Theorem 3** *Let  $(X_k, Y_k)$ ,  $k = 1, 2, \dots, v$  be  $v$  pairs of sequences of lengths  $m_k$  with the properties*

$$Z_k Z_k^T + W_k W_k^T = p_k I_{m_k} \quad (16)$$

$$Z_k W_k^T - W_k Z_k^T = 0 \quad (17)$$

$$Z_k * W_k = 0 \quad (18)$$

for all  $k = 1, 2, \dots, v$ , where  $Z_k = \text{circ}(X_k)$  and  $W_k = \text{circ}(Y_k)$ . Then there exist a set of  $2v$  amicable sequences of length  $\ell \equiv 0 \pmod{[m_1, m_2, \dots, m_v]}$ , where  $[m_1, m_2, \dots, m_v]$  is the least common multiple (l.c.m.) of  $m_1, m_2, \dots, m_v$  and of type  $(p_1, p_1, p_2, p_2, \dots, p_v, p_v)$  on the set  $\{a_1, a_2, \dots, a_{2v}\}$  of commuting variables.

**Proof.** Set

$$B_k = a_{2k} X_k + a_{2k-1} Y_k, \quad \text{and} \quad C_k = -a_{2k-1} X_k + a_{2k} Y_k, \quad k = 1, 2, \dots, v$$

Condition (18) gives that  $B_k$ ,  $k = 1, 2, \dots, v$  and  $C_k$ ,  $k = 1, 2, \dots, v$  are sequences of lengths  $m_k$ ,  $k = 1, 2, \dots, v$  and type  $(p_1, p_1, p_2, p_2, \dots, p_v, p_v)$ .

For any  $k$  and by simple calculations using conditions (16) and (17) we have that

$$B_k B_k^T + C_k C_k^T = (p_k a_{2k-1}^2 + p_k a_{2k}^2) I_{m_k} \quad \text{and} \quad B_k C_k^T - C_k B_k^T = 0$$

Now from theorem 1, there are sequences  $D_k$  and  $E_k$  of length  $\ell \equiv 0 \pmod{[m_1, m_2, \dots, m_v]}$ ,  $k = 1, 2, \dots, v$ , with the desirable properties. By lemma 1 we have the result.  $\square$

**Example 5** Set  $Z_1 = \{1\}$ ,  $W_1 = \{0\}$ ,  $Z_2 = \{1, 0\}$ ,  $W_2 = \{0, 1\}$ ,  $Z_3 = \{1, 1, 1, -1\}$ ,  $W_3 = \{0, 0, 0, 0\}$ ,  $Z_4 = \{0, 1, 0, -1, 0, 1\}$  and  $W_4 = \{0, 0, 1, 0, 1, 0\}$ . These are four pair of sequences of lengths 1, 2, 4 and 6 satisfying conditions (16), (17) and (18) with  $p_1 = 1$ ,  $p_2 = 2$ ,  $p_3 = 4$  and  $p_4 = 5$ . We have that  $[1, 2, 4, 6] = 12$  and from theorem 3 we obtain eight sequences of length  $\ell \equiv 0 \pmod{12}$  and of type  $(1, 1, 2, 2, 4, 4, 5, 5)$  on the set  $\{a_1, a_2, \dots, a_8\}$  of commuting variables which can be used in the Kharaghani array (6) to obtain an infinite class of Kharaghani type orthogonal designs  $OD(8\ell; 1, 1, 2, 2, 4, 4, 5, 5)$ .

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