

*Faculty of Informatics*

*Faculty of Informatics - Papers*

---

*University of Wollongong*

*Year 2003*

---

# Orthogonal Designs of Kharaghani Type: I

C. Koukouvinos\*

J. Seberry†

\*National Technical University of Athens, Greece

†University of Wollongong, jennie@uow.edu.au

This article was originally published as Koukouvinos, C and Seberry, J, Orthogonal Designs of Kharaghani Type: I, ARS Combinatoria, 67, 2003, 89-96. Original Charles Babbage Research Centre journal available <a href="http://www.paddle.mb.ca/ArsComb/ArsComb.html">here</a>.

This paper is posted at Research Online.

<http://ro.uow.edu.au/infopapers/283>

# Orthogonal Designs of Kharaghani Type: I

Christos Koukouvinos\* and Jennifer Seberry†

## Abstract

We use an array given in H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Designs*, 8 (2000), 166-173, to obtain infinite families of 8-variable Kharaghani type orthogonal designs,  $OD(8t; k_1, k_1, k_1, k_1, k_2, k_2, k_2, k_2)$ , where  $k_1$  and  $k_2$  must be the sum of two squares. In particular we obtain infinite families of 8-variable Kharaghani type orthogonal designs,  $OD(8t; k, k, k, k, k, k, k, k)$ . For odd  $t$  orthogonal designs of order  $\equiv 8 \pmod{16}$  can have at most eight variables.

*Key words and phrases:* Sequences, zero autocorrelation, orthogonal designs, amicable set of matrices, Kharaghani array, Kharaghani type orthogonal designs.

*AMS Subject Classification:* 05B15, 05B20.

## 1 Preliminaries

An *orthogonal design* of order  $n$  and type  $(s_1, s_2, \dots, s_u)$  denoted  $OD(n; s_1, s_2, \dots, s_u)$  in the variables  $x_1, x_2, \dots, x_u$ , is a matrix  $A$  of order  $n$  with entries in the set  $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$  satisfying

$$AA^T = \sum_{i=1}^u (s_i x_i^2) I_n,$$

where  $I_n$  is the identity matrix of order  $n$ . Let  $B_i$ ,  $i = 1, 2, 3, 4$  be circulant matrices of order  $n$  with entries in  $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$  satisfying

$$\sum_{i=1}^4 B_i B_i^T = \sum_{i=1}^u (s_i x_i^2) I_n.$$

Then the Goethals-Seidel array

$$G = \begin{pmatrix} B_1 & B_2 R & B_3 R & B_4 R \\ -B_2 R & B_1 & B_4^T R & -B_3^T R \\ -B_3 R & -B_4^T R & B_1 & B_2^T R \\ -B_4 R & B_3^T R & -B_2^T R & B_1 \end{pmatrix}$$

---

\*Department of Mathematics, National Technical University of Athens, Zografou 15773, Athens, Greece.

†School of Information Technology and Computer Science, University of Wollongong, Wollongong, NSW, 2522, Australia.

where  $R$  is the back-diagonal identity matrix, is an  $OD(4n; s_1, s_2, \dots, s_u)$ . See page 107 of [1] for details.

Plotkin [5] showed that there is an Hadamard matrix of order  $2t$ , then there is an  $OD(8t; t, t, t, t, t, t, t, t)$ . In the same paper he also constructed an  $OD(24; 3, 3, 3, 3, 3, 3, 3, 3)$ . This  $OD$  has appeared in [1], [6] and in [7]. It is conjectured that there is an  $OD(8n; n, n, n, n, n, n, n, n)$  for each odd integer  $n$ . Until recently, none except the original for  $n = 3$  found by Plotkin, had been constructed in the ensuing twenty eight years. Holzmann and Kharaghani [2] using a new method constructed many new Plotkin  $OD$ s of order 24 and two new Plotkin  $OD$ s of order 40 and 56. Actually their construction provides many new orthogonal designs in 6, 7 and 8 variables which include the Plotkin  $OD$ s of order 40 and 56.

A pair of matrices  $A, B$  is said to be amicable (anti-amicable) if  $AB^T - BA^T = 0$  ( $AB^T + BA^T = 0$ ). Following [3] a set  $\{A_1, A_2, \dots, A_{2n}\}$  of square real matrices is said to be *amicable* if

$$\sum_{i=1}^n (A_{\sigma(2i-1)} A_{\sigma(2i)}^T - A_{\sigma(2i)} A_{\sigma(2i-1)}^T) = 0 \quad (1)$$

for some permutation  $\sigma$  of the set  $\{1, 2, \dots, 2n\}$ . For simplicity, we will always take  $\sigma(i) = i$  unless otherwise specified. So

$$\sum_{i=1}^n (A_{2i-1} A_{2i}^T - A_{2i} A_{2i-1}^T) = 0. \quad (2)$$

Clearly a set of mutually amicable matrices is amicable, but the converse is not true in general. Throughout the paper  $R_k$  denotes the back diagonal identity matrix of order  $k$ .

An amicable set of matrices  $\{B_1, B_2, \dots, B_n\}$  of order  $m$  with entries in  $\{0, \pm x_1, \pm x_2, \dots, \pm x_u\}$  is said to be *amicable plug-in*,  $AP(m; s_1, s_2, \dots, s_u)$ , in the variables  $x_1, x_2, \dots, x_u$  if it satisfies an additive property

$$\sum_{i=1}^n B_i B_i^T = \sum_{i=1}^u (s_i x_i^2) I_m. \quad (3)$$

Let  $\{A_i\}_{i=1}^8$  be an amicable plug-in set of circulant matrices of type  $(s_1, s_2, \dots, s_u)$  of order  $t$ . Then the Kharaghani array

$$H = \begin{pmatrix} A_1 & A_2 & A_4 R_n & A_3 R_n & A_6 R_n & A_5 R_n & A_8 R_n & A_7 R_n \\ -A_2 & A_1 & A_3 R_n & -A_4 R_n & A_5 R_n & -A_6 R_n & A_7 R_n & -A_8 R_n \\ -A_4 R_n & -A_3 R_n & A_1 & A_2 & -A_5^T R_n & A_7^T R_n & A_6^T R_n & -A_5^T R_n \\ -A_3 R_n & A_4 R_n & -A_2 & A_1 & A_7^T R_n & A_8^T R_n & -A_5^T R_n & -A_6^T R_n \\ -A_6 R_n & -A_5 R_n & A_8^T R_n & -A_7^T R_n & A_1 & A_2 & -A_4^T R_n & A_3^T R_n \\ -A_5 R_n & A_6 R_n & -A_7^T R_n & -A_8^T R_n & -A_2 & A_1 & A_3^T R_n & A_4^T R_n \\ -A_8 R_n & -A_7 R_n & -A_6^T R_n & A_5^T R_n & A_4^T R_n & -A_3^T R_n & A_1 & A_2 \\ -A_7 R_n & A_8 R_n & A_5^T R_n & A_6^T R_n & -A_3^T R_n & -A_4^T R_n & -A_2 & A_1 \end{pmatrix}$$

is a Kharaghani type orthogonal design  $OD(8m; s_1, s_2, \dots, s_u)$ .

We use the construction of Holzmann and Kharaghani [2] for an  $OD(56; 7, 7, 7, 7, 7, 7, 7, 7)$  to find an infinite family of 8-variable Kharaghani type orthogonal designs  $OD(8t; k_1, k_1, k_1, k_1, k_2, k_2, k_2, k_2)$ , and  $OD(8t; k, k, k, k, k, k, k, k)$ , where  $k_1, k_2$  and  $k$  must be the sum of two squares.

Given a set of  $\ell$  sequences  $A_j = \{a_{j1}, a_{j2}, \dots, a_{jn}\}$ ,  $j = 1, \dots, \ell$ , of length  $n$  the *non-periodic autocorrelation function*, denoted *NPAF*,  $N_A(s)$  is defined as

$$N_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-s} a_{ji} a_{j,i+s}, \quad s = 0, 1, \dots, n-1, \quad (4)$$

If  $A_j(z) = a_{j1} + a_{j2}z + \dots + a_{jn}z^{n-1}$  is the associated polynomial of the sequence  $A_j$ , then

$$A(z)A(z^{-1}) = \sum_{j=1}^{\ell} \sum_{i=1}^n \sum_{k=1}^n a_{ji} a_{jk} z^{i-k} = N_A(0) + \sum_{j=1}^{\ell} \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), \quad z \neq 0. \quad (5)$$

Given  $A_\ell$ , as above, of length  $n$  the *periodic autocorrelation function*, denoted *PAF*,  $P_A(s)$  is defined, reducing  $i + s$  modulo  $n$ , as

$$P_A(s) = \sum_{j=1}^{\ell} \sum_{i=1}^n a_{ji} a_{j,i+s}, \quad s = 0, 1, \dots, n-1. \quad (6)$$

We note NPAF sequences imply PAF sequences exist, the NPAF sequences being padded at the end with sufficient zeros to make longer lengths. Hence NPAF sequences can give more general results. If two NPAF sequences have differing lengths then sufficient zeros are added to the end of each to make all the sequences the same length. In all cases NPAF and PAF sequences can be used to make circulant matrices satisfying the additive property (see [2, 3]); if NPAF sequences of lengths  $n_1$  and  $n_2$  are used, then by padding, circulant matrices for all orders  $n \geq \max(n_1, n_2)$  will exist; if PAF sequences of lengths  $n$  are used, then circulant matrices of order  $n$  exist.

## 2 Using sequences with zero NPAF to make ODs

We now consider the use of sequences with zero non-periodic autocorrelation function to make an amicable set of eight matrices. We refer the reader to [6, 7] for any undefined terms.

**Theorem 1** *Let  $X_i, Y_i$  be two pairs of disjoint  $(0, \pm 1)$  sequences with zero non-periodic autocorrelation function of length  $n_i$  and weight  $k_i$ ,  $i = 1, 2$ . We pad the end of these sequences with zeros to obtain sequences of length  $n \geq \max(n_1, n_2)$ . Let  $a, b, c, d, e, f, g, h$  be commuting variables and write  $aV_i, bW_i$  for the circulant (type 1 or group generated also suffice) matrices of order  $n$  formed by using the first rows with the elements of  $X_i$  multiplied by  $a$  and the elements of  $Y_i$  multiplied by  $b$  respectively.*

*Let  $A_i$  be the circulant matrices of order  $n$  given by*

$$\begin{aligned} A_1 &= aV_1 + bW_1 & A_3 &= dV_1 - cW_1 & A_5 &= eV_2 + fW_2 & A_7 &= hV_2 - gW_2 \\ A_2 &= cV_1 + dW_1 & A_4 &= bV_1 - aW_1 & A_6 &= gV_2 + hW_2 & A_8 &= fV_2 - eW_2 \end{aligned} \quad (7)$$

then  $\{A_i\}_{i=1}^8$  is an amicable plug-in set satisfying

$$\sum_{i=1}^4 (A_{2i-1}A_{2i}^T - A_{2i}A_{2i-1}^T) = 0, \quad (8)$$

and the additive property

$$\sum_{i=1}^8 (A_iA_i^T) = (k_1(a^2 + b^2 + c^2 + d^2) + k_2(e^2 + f^2 + g^2 + h^2))I_n. \quad (9)$$

**Proof:** Now  $A_1 = aV_1 + bW_1$ , where  $V_1, W_1$  are disjoint  $(0, \pm 1)$  circulant (type 1) matrices of order  $n$  which satisfy  $V_1V_1^T + W_1W_1^T = k_1I_n$ , and similarly for the other  $A_j, j = 2, \dots, 8$ .

Then

$$A_1A_1^T = (aV_1 + bW_1)(aV_1^T + bW_1^T) = a^2V_1V_1^T + b^2W_1W_1^T + ab(V_1W_1^T + W_1V_1^T).$$

Hence

$$\begin{aligned} \sum_{i=1}^4 (A_iA_i^T) &= (a^2 + b^2 + c^2 + d^2)(V_1V_1^T + W_1W_1^T) \\ &= k_1(a^2 + b^2 + c^2 + d^2)I_n. \end{aligned}$$

Similarly

$$\begin{aligned} \sum_{i=5}^8 (A_iA_i^T) &= (e^2 + f^2 + g^2 + h^2)(V_2V_2^T + W_2W_2^T) \\ &= k_2(e^2 + f^2 + g^2 + h^2)I_n. \end{aligned}$$

Now

$$\begin{aligned} A_1A_2^T - A_2A_1^T &= (aV_1 + bW_1)(cV_1^T + dW_1^T) - (bV_1 - aW_1)(aV_1^T + bW_1^T) \\ &= (ad - bc)V_1W_1^T + (-ad + bc)W_1V_1^T. \end{aligned}$$

We also see that

$$\begin{aligned} A_3A_4^T - A_4A_3^T &= (dV_1 - cW_1)(bV_1^T - aW_1^T) - (bV_1 - aW_1)(dV_1^T - cW_1^T) \\ &= (-ad + bc)V_1W_1^T + (ad - bc)W_1V_1^T. \end{aligned}$$

Thus summing over the eight  $A_i$  we obtain equations (8) and (9).  $\square$

**Corollary 1** Let  $X_i, Y_i$  be two disjoint  $(0, \pm 1)$  sequences with zero non-periodic autocorrelation function of length  $n_i$  and weight  $k_i, i = 1, 2$  and  $n \geq \max(n_1, n_2)$ . Then there exists a Kharaghani type orthogonal design  $OD(8s; k_1, k_1, k_1, k_1, k_2, k_2, k_2, k_2), s \geq n$ .

**Proof:** Use the sequences as in the theorem to form an amicable plug-in set with the additive property. Then use this set in the Kharaghani array to obtain the required Kharaghani type orthogonal design.  $\square$

**Example 1** We use the sequences of length  $n \geq 6 = \max(6, 4)$  and weights 5 and 4,  $X_1 = \{1, 0, 1, 0, 0, 0\}$  and  $Y_1 = \{0, 0, 0, 1, 1, -1\}$ ,  $X_2 = \{1, 1, 0, 0\}$  and  $Y_2 = \{0, 0, 1, -1\}$ , and the sequence  $0_s$  of  $s$  zeros, to form the circulant matrices with first rows

$$\begin{aligned} A_1 &= (a \ 0 \ a \ b \ b \ -b \ 0_s) & A_3 &= (d \ 0 \ d \ -c \ -c \ c \ 0_s) \\ A_2 &= (c \ 0 \ c \ d \ d \ -d \ 0_s) & A_4 &= (b \ 0 \ b \ -a \ -a \ a \ 0_s) \\ A_5 &= (e \ e \ f \ -f \ 0 \ 0 \ 0_s) & A_7 &= (h \ h \ -g \ g \ 0 \ 0 \ 0_s) \\ A_6 &= (g \ g \ h \ -h \ 0 \ 0 \ 0_s) & A_8 &= (f \ f \ -e \ e \ 0 \ 0 \ 0_s). \end{aligned}$$

By the theorem these form an amicable plug-in set of eight matrices of order  $s+6$  and weights 5 and 4 which can be used in the Kharaghani array to give a Kharaghani type orthogonal design  $OD(8s+48; 4, 4, 4, 4, 5, 5, 5, 5)$  for every order  $s \geq 0$ .  $\square$

Let  $P, Q$  be two  $(0, \pm 1)$  sequences with zero non-periodic autocorrelation function of length  $n$  and weight  $k$ . Then the sequences  $X = \{P, 0_n\}$  and  $Y = \{0_n, Q\}$  are two  $(0, \pm 1)$  disjoint sequences with zero non-periodic autocorrelation function of length  $2n$  and weight  $k$ .

Let  $\alpha, \beta, \gamma, \delta, \epsilon, \phi, \psi, \mu, \nu$  non-negative integers. Koukouvinos and Seberry [4, p. 160] show that we have two  $(0, \pm 1)$  disjoint sequences with zero non-periodic autocorrelation function of lengths  $\geq n$ ,  $n \in N = \{2 \times 2^\alpha 6^\beta 10^\gamma 9^\delta 14^\epsilon 18^\phi 26^\psi 24^\mu 34^\nu\}$  and weights  $2^\alpha 5^\beta 10^\gamma 13^\delta 17^\epsilon 25^\phi 26^\psi 34^\mu 50^\nu$ .

**Corollary 2** Let  $Z = \{z_1, z_2, \dots, z_n\}$ ,  $W = \{w_1, w_2, \dots, w_n\}$  be two disjoint  $(0, \pm 1)$  sequences with zero periodic autocorrelation function of length  $n$  and weight  $k$ . Then there exists a Kharaghani type  $OD(8s; k, k, k, k, k, k, k, k)$  for all  $s = mn$ ,  $m = 1, 2, \dots$ .

**Proof:** Set  $X = \{z_1, 0_{m-1}, z_2, 0_{m-1}, \dots, z_n, 0_{m-1}\}$  and  $Y = \{0_{m-1}, w_1, 0_{m-1}, w_2, \dots, 0_{m-1}, w_n\}$ . These are two disjoint sequences of length  $s = mn$  and weight  $k$  and can be used in Theorem 1 to form an amicable set of eight matrices with the additive property. Then we can use these eight matrices in the Kharaghani array to obtain the required orthogonal design of Kharaghani type.  $\square$

We give some examples of the Kharaghani type orthogonal designs we obtain in the Table:

$OD$	Lengths $n$	$OD$	Lengths $n$
$OD(8n; 1, 1, 1, 1, 1, 1, 1, 1)$	$\geq 1$	$OD(8n; 1, 1, 1, 1, 2, 2, 2, 2)$	$\geq 2$
$OD(8n; 1, 1, 1, 1, 4, 4, 4, 4)$	$\geq 4$	$OD(8n; 1, 1, 1, 1, 5, 5, 5, 5)$	$\geq 6$
$OD(8n; 2, 2, 2, 2, 4, 4, 4, 4)$	$\geq 4$	$OD(8n; 2, 2, 2, 2, 5, 5, 5, 5)$	$\geq 6$
$OD(8n; 4, 4, 4, 4, 5, 5, 5, 5)$	$\geq 6$	$OD(8n; 4, 4, 4, 4, 8, 8, 8, 8)$	$\geq 8$
$OD(8n; 5, 5, 5, 5, 5, 5, 5, 5)$	$\geq 6$	$OD(8n; 5, 5, 5, 5, 8, 8, 8, 8)$	$\geq 8$
$OD(8n; 5, 5, 5, 5, 10, 10, 10, 10)$	$\geq 10$	$OD(8n; 5, 5, 5, 5, 13, 13, 13, 13)$	$\geq 18$
$OD(8n; 5, 5, 5, 5, 17, 17, 17, 17)$	$\geq 26$	$OD(8n; 13, 13, 13, 13, 17, 17, 17, 17)$	$\geq 26$
$OD(8n; 20, 20, 20, 20, 25, 25, 25, 25)$	$\geq 36$	$OD(8n; 25, 25, 25, 25, 25, 25, 25, 25)$	$\geq 36$

### 3 Using sequences with zero PAF to make ODs

Provided the sequences used in the theorem all have the same length the corollary can be extended to include sequences with zero PAF.

**Corollary 3** *Let  $X_i, Y_i$  be two pairs of disjoint  $(0, \pm 1)$  sequences with zero periodic or non-periodic autocorrelation function of length  $n$  and weight  $k_i, i = 1, 2$ . Then there exists a Kharaghani type orthogonal design  $OD(8s; k_1, k_1, k_1, k_1, k_2, k_2, k_2, k_2)$ ,  $s \geq n$ .*

**Example 2** We use the sequences of length  $n = 11$ , and weights 4 and 9,

$$\begin{aligned} X_1 &= \text{circ}\{1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \text{ and } Y_1 = \text{circ}\{0, 0, 1, -, 0, 0, 0, 0, 0, 0, 0\} \\ X_2 &= \text{circ}\{0, 1, 0, 1, 1, 0, 0, 1, 0, -, 0\} \text{ and } Y_2 = \text{circ}\{0, 0, 0, 0, 0, 1, -, 0, -, 0, 1\} \end{aligned}$$

to form the circulant matrices with first rows which can be used in the theorem to give an amicable plug-in set of matrices of order 11 and weights 4 and 9 which can be used in the Kharaghani array to obtain Kharaghani type orthogonal designs  $OD(88; 4, 4, 4, 4, 4, 4, 4, 4)$ ,  $OD(88; 4, 4, 4, 4, 9, 9, 9, 9)$  and  $OD(88; 9, 9, 9, 9, 9, 9, 9, 9)$ .

### References

- [1] A.V.Geramita, and J.Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.
- [2] W.H. Holzmann, and H. Kharaghani, On the Plotkin arrays, *Australas. J. Combin.*, 22 (2000), 287-299.
- [3] H. Kharaghani, Arrays for orthogonal designs, *J. Combin. Designs*, 8 (2000), 166-173.
- [4] C. Koukouvinos and J. Seberry, New weighing matrices and orthogonal designs constructed using two sequences with zero autocorrelation function - a review, *J. Statist. Plann. Inference*, 81 (1999), 153-182.
- [5] M. Plotkin, Decomposition of Hadamard matrices, *J. Combin. Theory, Ser. A*, 13 (1972), 127-130.
- [6] J. Seberry and R. Craigen, Orthogonal designs, in *CRC Handbook of Combinatorial Designs*, C.J. Colbourn and J.H. Dinitz (Eds.), CRC Press, (1996), 400-406.
- [7] J. Seberry and M. Yamada, Hadamard matrices, sequences and block designs, in *Contemporary Design Theory: A Collection of Surveys*, J.H. Dinitz and D.R. Stinson (Eds.), J. Wiley and Sons, New York, (1992), 431-560.