University of Wollongong
Research Online

Faculty of Informatics - Papers (Archive)

Faculty of Engineering and Information Sciences

2003

Weighing Matrices and Self-Orthogonal Quaternary Codes

C. Charnes
University of Melbourne

Jennifer Seberry
University of Wollongong, jennifer_seberry@uow.edu.au

Publication Details

This article was originally published as Charnes, C and Seberry, J, Weighing Matrices and Self-Orthogonal Quaternary Codes, Journal of Combinatorial Mathematics and Combinatorial Computing, 44, 2003, 85-89.

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au

Weighing Matrices and Self-Orthogonal Quaternary Codes

Abstract

We consider families of linear self-orthogonal and self-dual codes over the ring Z4 which are generated by weighing matrices W (n, k) $k \equiv 0 \pmod 4$, whose entries are interpreted as elements of the ring Z4. We obtain binary formally self-dual codes of minimal Hamming distance 4 by applying the Gray map to the quaternary codes generated by W (n, 4).

Keywords

weighing matrix, generator matrix, codes, AMS Subject Classification: 05820.

Disciplines

Physical Sciences and Mathematics

Publication Details

This article was originally published as Charnes, C and Seberry, J, Weighing Matrices and Self-Orthogonal Quaternary Codes, Journal of Combinatorial Mathematics and Combinatorial Computing, 44, 2003, 85-89.

This journal article is available at Research Online: http://ro.uow.edu.au/infopapers/285

Weighing Matrices and Self-Orthogonal Quaternary Codes

Chris Charnes and Jennifer Seberry
Department of Computer Science and Software Engineering,
University of Melbourne,
Parkville, Vic, 3052, Australia.
Centre for Computer Security Research,
School of Information Technology and Computer Science,
University of Wollongong,
Wollongong, NSW, 2522, Australia.

Abstract

We consider families of linear self-orthogonal and self-dual codes over the ring Z_4 which are generated by weighing matrices $W(n,k)$ $k \equiv 0 \pmod 4$, whose entries are interpreted as elements of the ring Z_4. We obtain binary formally self-dual codes of minimal Hamming distance 4 by applying the Gray map to the quaternary codes generated by $W(n,4)$.
Key Words and Phrases: weighing matrix, generator matrix, codes *AMS Subject Classification:* 05B20.

1 Introduction

A weighing matrix $W = W(n,k)$ is a square matrix with rational entries 0, ± 1, having k non-zero entries per row and column and inner product of distinct rows zero. Hence W satisfies $WW^T = kI_n$, and W is equivalent to an orthogonal design $OD(n;k)$. The number k is called the *weight* of W.

Weighing matrices have long been studied because of their use in weighing experiments as first studied by Hotelling [11] and later by Raghavarao [12] and others [6, 14].

There has recently been a large amount of research devoted to self-orthogonal and self-dual codes over the ring Z_4, e.g. [1, 5, 13, 2]. Patrick Solé's remark that the orthogonality of Hadamard matrices can be naturally interpreted as Z_4-orthogonality was investigated in [4]. There self-orthogonal and self-dual quaternary codes were obtained from equivalence classes of Hadamard matrices. In this paper we observe that the rational entries of weighing matrices can be similarly replaced by the corresponding elements of Z_4. This procedure produces self-orthogonal and self-dual quaternary codes from weighing matrices (see Lemma 2).

Two constructions of self-orthogonal quaternary codes from weighing matrices are given in Lemma 2. We study in some detail the quaternary codes which arise by one of these constructions from weighing matrices $W(n,4)$. These codes coincide with the *tetrad* codes of Conway and Sloane [5]. Although the structural components of self-orthogonal tetrad codes are known, it

seems that the matrices $W(n,4)$ provide a means of putting together these components so that they form self-dual codes.

The binary image under the Gray map of the quaternary codes generated by the $W(n,4)$ is not always a linear code. But in those cases in which it is linear we obtain $[2n,n,4]$ codes which are formally self-dual (see Section 6). For $n = 6$ and 8, this method yields extremal formally self-dual codes [7].

2 Preliminary Results

Lemma 1 *Suppose $W = W(n,k)$ is a $(0,\pm 1)$ weighing matrix of order n. Then the matrix, $W(4)$, obtained from W by replacing all the -1s by '3' satisfies $W(4)W(4)^T = kI_n \pmod 4$ and has its distinct row vectors orthogonal over Z_4.*

Proof. Suppose $\mathbf{a}$ and $\mathbf{b}$ are two distinct row vectors of W. Now

$$\mathbf{a} \cdot \mathbf{b} = 0 = a(1)^2 + b(-1)^2 + c(1)(-1) + d(-1)(1) + e \cdot 0$$

where a is the number of coordinates in which both $\mathbf{a}$ and $\mathbf{b}$ are '1', b is the number of coordinates in which both $\mathbf{a}$ and $\mathbf{b}$ are '-1', c is the number of coordinates in which $\mathbf{a}$ is '1' and $\mathbf{b}$ is '-1', d is the number of coordinates in which $\mathbf{a}$ is '-1' and $\mathbf{b}$ is '1', and e is the number of coordinates in which at least one of $\mathbf{a}$ and $\mathbf{b}$ is zero. Hence

$$a + b + c + d = n \quad \text{and} \quad a + b - c - d = 0$$

so

$$a + b = c + d.$$

We now replace each '-1' by '3' to get vectors $\mathbf{a}'$ from $\mathbf{a}$ and $\mathbf{b}'$ from $\mathbf{b}$. Then

$$\begin{aligned} \mathbf{a}' \cdot \mathbf{b}' &= a(1)^2 + b(3)^2 + c(1)(3) + d(3)(1) + e \cdot 0 \\ &\equiv a + b + 3c + 3d \pmod 4 \\ &\equiv 4a + 4b \pmod 4 \\ &\equiv 0 \pmod 4 \end{aligned}$$

Thus the row vectors are orthogonal over Z_4.

Lemma 2 *Let $W(4)$ be the matrix obtained from a $(0,\pm 1)$ $W(n,k)$, $k \equiv 0 \pmod 4$ by replacing all the -1s by '3'. Then the following matrices are generator matrices for self-orthogonal codes over Z_4.*

1) $[W(4)]$;

2) $[2I_n \;\; W(4)]$.

Proof. Since $W(4)W(4)^T = kI_n (\text{mod } 4)$ and $k \equiv 0(\text{mod } 4)$, $W(4)W(4)^T = 0(\text{mod } 4)$. The second result follows also as $4II^T \equiv 0(\text{mod } 4)$.

Lemma 3 *For every $n > 3$ every code constructed from a $W(n,4)$ is equivalent to a code containing the all '2's vector.*

Proof. Chan, Rodger and Seberry [3] show that every weighing matrix of weight four is equivalent using the operations permute rows and or columns and/or multiply rows and columns by -1 to

$$\underbrace{\bigoplus}_{a\ copies} W(4,4) \oplus \underbrace{\bigoplus}_{b\ copies} B(8,4) \oplus \underbrace{\bigoplus}_{c\ copies} W(7,4) \oplus \underbrace{\bigoplus}_{i\ copies} (\underbrace{\bigoplus}_{t_i\ copies} C(6+2t_i,4)).$$

We now observe that $W(4,4)$ and W(7,4) can be written as

$$\begin{bmatrix} - & 1 & 1 & 1 \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \\ 1 & 1 & 1 & - \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} - & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & - & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & - & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & - & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & - & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & - & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & - \end{bmatrix},$$

both of which have three '1's and one '-1' in each row and column. The $B(8,4)$ can be written as

$$\begin{bmatrix} - & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & - & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & - & 0 \\ 1 & 0 & 0 & 1 & - & 0 & 1 & 0 \\ 0 & 1 & 0 & - & - & 0 & 0 & - \\ 0 & - & 1 & 0 & 0 & - & 0 & - \\ 0 & 0 & - & 1 & 0 & 0 & - & - \\ 0 & 0 & 0 & 0 & - & - & - & 1 \end{bmatrix},$$

which has either one or three elements '$-$' in each row and column. The $C(4m,4)$ and $C(4m+2,4)$ can be written as

$$
\left[\begin{array}{cccccccccccc}
- & 1 & 1 & 1 & & & & & & & & \\
1 & - & 1 & 1 & & & & & & & & \\
1 & 1 & 0 & 0 & - & 1 & & & & & & \\
1 & 1 & 0 & 0 & 1 & - & & & & & & \\
 & & - & 1 & & & 1 & 1 & & & & \\
 & & 1 & - & & & 1 & 1 & & & & \\
 & & & & 1 & 1 & & & & & & \\
 & & & & 1 & 1 & & & & & & \\
 & & & & . & & & . & & & & \\
 & & & & & . & & & . & & & \\
 & & & & & & . & & & . & & \\
 & & & & & & & . & & & 1 & 1 \\
 & & & & & & & & . & & 1 & 1 \\
 & & & & & & & & 1 & 1 & 1 & - \\
 & & & & & & & & 1 & 1 & - & 1
\end{array}\right],
$$

or

$$
\left[\begin{array}{cccccccccccc}
- & 1 & 1 & 1 & & & & & & & & \\
1 & - & 1 & 1 & & & & & & & & \\
1 & 1 & 0 & 0 & - & 1 & & & & & & \\
1 & 1 & 0 & 0 & 1 & - & & & & & & \\
 & & - & 1 & & & 1 & 1 & & & & \\
 & & 1 & - & & & 1 & 1 & & & & \\
 & & & & 1 & 1 & & & - & 1 & & \\
 & & & & 1 & 1 & & & 1 & - & & \\
 & & & & & & - & 1 & & & & \\
 & & & & & & 1 & - & & & & \\
 & & & & & & . & & & . & & \\
 & & & & & & & . & & & . & \\
 & & & & & & & & . & & & . \\
 & & & & & & & & . & & - & 1 \\
 & & & & & & & & & . & 1 & - \\
 & & & & & & & & - & 1 & 1 & 1 \\
 & & & & & & & & 1 & - & 1 & 1
\end{array}\right],
$$

so it has the same property.

We now see if we make a code over Z_4 by writing 3 in place of '$-$' the column (row) sum of each column (row) is 6 or 10. So the all two's vector is in any code based on these four basic blocks.

Example 1 The quaternary code with generator matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 3 \\ 1 & 0 & 1 & 3 & 0 & 1 \\ 1 & 1 & 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 & 3 & 3 \\ 1 & 0 & 3 & 3 & 0 & 3 \\ 3 & 1 & 0 & 3 & 3 & 0 \end{bmatrix}$$

is a self-dual code of *type* 42^4. Its Gray image is a linear binary code (see Section 5). It is a *formally self-dual* $[12, 6, 4]$ code with Hamming weight enumerator

$$W(x, y) = x^{12} + 15x^8y^4 + 32x^6y^6 + 15x^4y^8 + y^{12}$$

and generator matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The binary image is an *isodual* code – equivalent to its dual. It is an extremal formally self-dual (f.s.d.) code as it meets the bound

$$d_{f.s.d} \leq 2\left\lfloor \frac{n}{8} \right\rfloor + 2$$

(see Fields, Gaborit, Huffman and Pless [7]). It appears not have been previously noticed that this code attains the upper bound for the minimum *Lee distance* of self-dual quaternary codes

$$d_L \leq 2\Big(\left\lfloor \frac{n}{4} \right\rfloor + 1\Big).$$

The only other quaternary code which is known to attain this bound is the octacode $\mathcal{O}_8$ (see [2]).

3 Tetrad codes

Conway and Sloane [5] classified the self-orthogonal quaternary codes which are generated by tetrads. From now on we shall consider the quaternary codes generated by $W(n, 4)$ (cf. $W(4)$ Lemma 2). A tetrad is a vector which contains four components congruent to 1 or 3 $\pmod 4$, the other components being congruent to 0 $\pmod 4$. Evidently the row vectors of the weighing matrices $W(n, 4)$ are tetrads hence we obtain

Proposition 1 *The self-orthogonal quaternary code generated by a weighing matrix $W(n, 4)$ (n even) is a tetrad code.*

A tetrad code is equivalent to a direct sum of codes taken from a list of seven quaternary codes:

$$\mathcal{D}_{2m}, \mathcal{D}_{2m}^{\circ}, \mathcal{D}_{2m}^{+}, \mathcal{D}_{2m}^{\oplus}(m = 1, 2, \ldots), \mathcal{E}_7, \mathcal{E}_7^{+}, \mathcal{E}_8$$

(see Theorem 1 of [5]). Thus the building blocks of the quaternary codes generated by weighing matrices $W(n, 4)$ are known. Consulting Fig. 2 of [5] we conclude that the quaternary code in Example 1 is equivalent to the code $\mathcal{D}_4^{\oplus}\mathcal{A}_1^2$.

Proposition 2 *If the tetrad code $\mathcal{C}$ generated by a weighing matrix $W(n, 4)$ ($n \geq 6$, even) has type $4^{(n-4)/2}2^4$ then it is a self-dual code.*

Proof. By Lemma 2 $\mathcal{C}$ is a self-orthogonal code ($\mathcal{C} \subset \mathcal{C}^{\perp}$). Since $\mathcal{C}^{\perp}$ has type $4^{n-(n-4)/2-4}2^4$ (see [5]) we conclude that $\mathcal{C}^{\perp}$ contains the same number of code vectors as $\mathcal{C}$. Thus $\mathcal{C} = \mathcal{C}^{\perp}$.

4 Euclidean weights

Recall that the *Euclidean weights* of the elements 0,1,2 and 3 of Z_4 are: 0,1,4 and 1 respectively. The Euclidean weight of a quaternary code vector is the rational sum of the Euclidean weights of the components of the code vector. The minimum Euclidean weight d_E of a quaternary code is the smallest Euclidean weight of all the nonzero code vectors of the code.

The Euclidean weights of row vectors of a generator matrix of a self-dual tetrad code $W(n, 4)$ are equal to 4 and therefore are not divisible by 8. In the terminology of Bannai, Dougherty, Harada and Oura [1] these are Type I codes.

The concept of a *shadow* for Type I codes over rings Z_{2k} was introduced in [1]. This is defined in terms of the *$4k$-weight subcode* $\mathcal{C}_0$, and for quaternary ($k = 2$) Type I codes this is the subcode consisting of all the code vectors whose Euclidean weights are divisible by 8.

Proposition 3 *If the tetrad code $\mathcal{C}$ generated by a weighing matrix $W(n, 4)$ (n even) is a self-dual code then the Euclidean weights of all the code vectors of are divisible by 4. $\mathcal{C}$ has as many code vectors with Euclidean weights divisible by 4 as code vectors with weights divisible by 8.*

Proof. Let t (of shape $(\pm 1^4, 0^{n-4})$) be any generating tetrad of $\mathcal{C}$ and $\mathcal{C}_0$ the 8-weight subcode. By Lemma 2.3 of [1] $\mathcal{C}_0$ is a linear subcode of index 2 in $\mathcal{C}$. Thus we obtain the coset decomposition

$$\mathcal{C} = \mathcal{C}_0 \cup \{\mathcal{C}_0 + t\}. \tag{1}$$

Now by Lemma 2.2 [1] $wt_E(c + t) \equiv wt_E(c) + wt_E(t) \pmod 8$, $c \in \mathcal{C}_0$. Since $wt_E(t) = 4$ it follows that $wt_E(c + t)$ is divisible by 4. The last assertion is a consequence of the coset decomposition (1).

In a linear quaternary code the *torsion* code $\mathcal{E}$ is the subcode consisting of code vectors whose components are even.

Proposition 4 *In a self-dual tetrad code $\mathcal{C}$ generated by a weighing matrix $W(n, 4)$ (n even) the intersection of the 8-weight subcode and the torsion code $\mathcal{C}_0 \bigcap \mathcal{E}$ has index 2 in $\mathcal{E}$.*

Proof. We first show that $\mathcal{C}_0 \not\subseteq \mathcal{E}$. Let x and y be two tetrad row vectors of a weighing matrix $W(n,4)$. Then by Lemma 2.1 of [1] the Euclidean weight of the sum of these two vectors satisfies $wt_E(x+y) \equiv wt_E(x) + wt_E(y) \pmod 8$. Since $wt_E(x) = wt_E(y) = 4$, $x+y$ belongs to $\mathcal{C}_0$. The components of $x+y$ are either: all odd, or are a mixture of even and odd components. In either case $x+y$ does not belong to $\mathcal{E}$.

We conclude that

$$\mathcal{C} = \mathcal{C}_0 + \mathcal{E}. \tag{2}$$

The isomorphism $\frac{\mathcal{C}}{\mathcal{C}_0} \cong \frac{\mathcal{E}}{\mathcal{C}_0 \bigcap \mathcal{E}}$ and Lemma 2.3 of [1] now establish our assertion.

Proposition 5 *The Euclidean weights of code vectors of the self-orthogonal quaternary code generated by a weighing matrix $W(n,8)$ (n even) are divisible by 8.*

Proof. The row vectors of $W(n,8)$ are mutually orthogonal and their Euclidean weights are equal to 8. Hence by Lemma 2.2 of [1], the Euclidean weights of all the code vectors of the quaternary code generated by $W(n,8)$ are divisible by 8.

We conclude from Proposition 5 that if $W(n,8)$ generates a self-dual quaternary code, then this is a Type II code. By [2] this can only occur only when the length n of the code is a multiple of eight.

5 Linearity of the Gray image

We now study binary images under the Gray map of the tetrad codes. The Gray map $\phi : Z_4^n \to Z_2^{2n}$ maps a quaternary code $\mathcal{C}$ of length n onto a binary (but not necessarily linear) code $\phi(\mathcal{C})$ of length $2n$ as: $0 \to (0,0)$, $1 \to (0,1)$, $2 \to (1,1)$, $3 \to (1,0)$. We also require the map $\alpha : Z_4 \to Z_2$ defined by $\alpha(c) = 0$ if $c = 0$, 2 and $\alpha(c) = 1$ if $c = 1$, 3. This map extends to quaternary vectors (see Hammons *et al* [9]).

Lemma 4 *Let $\mathcal{C}$ be a quaternary code generated by a weighing matrix $W(n,4)$ and $\mathcal{E}$ the torsion subcode. If $e \in \mathcal{E}$ and t is a tetrad row vector of $W(n,4)$ then $\alpha(e+t) = \alpha(t)$.*

Proof. Whenever a component of the vector e is equal to 2 and coincides with and odd component (1 or 3) of t, the corresponding component of $e+t$ is: $c = 2+1$ or $2+3$. Hence $\alpha(c) = \alpha(2) + \alpha(1) = \alpha(2) + \alpha(3)$. Similarly, α is additive in the case that a 2 component coincides with a 0 component of t.

In order to prove that the binary image $\phi(\mathcal{C})$ is a linear code, we need to show by Theorem 5 of [9] that

$$a, b \in \mathcal{C} \Rightarrow 2\alpha(a) * \alpha(b) \in \mathcal{C} \tag{3}$$

($*$ denotes the component wise product of vectors).

Proposition 6 *If condition (3) holds for the generating tetrads of $W(n,4)$ then $\phi(\mathcal{C})$ is a linear code.*

Proof. The subcode generated by $\mathcal{E}$ and the $t_1, \ldots, t_n$ (the tetrad generators) is the whole code $\mathcal{C}$. So by Lemma 4 condition (3) needs only to be checked for the tetrad generators.

We now show that binary images $\phi(\mathcal{C})$ of Type I codes are formally self-dual codes (see Fields *et al* [7]).

Theorem 1 *If the binary image $\phi(\mathcal{C})$ of a self-dual quaternary code $\mathcal{C}$ of type $4^{(n-4)/2}2^4$ ($n \geq 6$, even) generated by a weighing matrix $W(n,4)$ is a linear code, then this is a formally self-dual $[2n, n, 4]$ code.*

Proof. By hypothesis, $\mathcal{C}$ has type $4^{(n-4)/2}2^4 = 2^n$ and $\phi(\mathcal{C})$ is a linear code. Hence $\phi(\mathcal{C})$ is a $[2n, n, d]$ code. Since $\mathcal{C}$ is a self-dual quaternary code we have that $\phi(\mathcal{C}) = \phi(\mathcal{C}^{\perp})$. This implies that the weight distributions of these codes are the MacWilliams transforms of one another (see [9]). Hence $\phi(\mathcal{C})$ is a formally self-dual code.

We still have to prove that the minimal Hamming weight d of $\phi(\mathcal{C})$ is 4. For any two tetrad row vectors x and y of $W(n,4)$ (the generators of $\mathcal{C}$), we have that $wt_E(x+y) \equiv wt_E(x)+wt_E(y)$ (mod 8); cf. Lemma 2.2 [1]. Since Euclidean weights of these generators are equal to 4, it follows that the Euclidean weight of any nonzero code vector a of $\mathcal{C}$ other than a generator is greater than 4. Hence the Hamming weight of $\phi(a)$ is greater than 4. However the Hamming weight of $\phi(t)$ where t is a generator of $\mathcal{C}$ is 4, hence $d = 4$.

6 Examples of codes

In this section we consider examples of quaternary codes generated by the $W(n,4)$ for $n = 8$ up to 20 and their Gray images.

The quaternary codes generated by the weighing matrices given below have types $4^{(n-4)/2}2^4$ hence by Proposition 2 they are self-dual codes.

There are three equivalence classes of $W(8,4)$. Type A, Type B and Type C are the matrices:

$$\begin{bmatrix} 3&1&1&1&0&0&0&0\\ 1&3&1&1&0&0&0&0\\ 1&1&3&1&0&0&0&0\\ 1&1&1&3&0&0&0&0\\ 0&0&0&0&3&1&1&1\\ 0&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&3&1\\ 0&0&0&0&1&1&1&3 \end{bmatrix} \begin{bmatrix} 3&1&1&1&0&0&0&0\\ 1&1&0&0&3&1&0&0\\ 1&0&1&0&1&0&3&0\\ 1&0&0&1&0&3&1&0\\ 0&1&3&0&1&0&0&1\\ 0&3&0&1&0&1&0&1\\ 0&0&1&3&0&0&1&1\\ 0&0&0&0&1&1&1&3 \end{bmatrix} \begin{bmatrix} 1&1&3&1&0&0&0&0\\ 1&1&1&3&0&0&0&0\\ 3&1&0&0&1&1&0&0\\ 1&3&0&0&1&1&0&0\\ 0&0&1&1&0&0&3&1\\ 0&0&1&1&0&0&1&3\\ 0&0&0&0&3&1&1&1\\ 0&0&0&0&1&3&1&1 \end{bmatrix}$$

The binary image of the matrix of Type A generates an isodual $[16, 8, 4]$ code with weight enumerator

$$W(x,y) = x^{16} + 28x^{12}y^4 + 198x^8y^8 + 28x^4y^{12} + y^{16}.$$

The binary image of the quaternary code corresponding to a Type B matrix generates a $[16, 11, 4]$ code (the dual of the first order Reed-Muller code). Hence the binary image is <u>not</u> linear. The binary image of the quaternary code corresponding to a Type C matrix is linear. It generates an isodual $[16, 8, 4]$ code with weight enumerator

$$W(x,y) = x^{16} + 20x^{12}y^4 + 32x^{10}y^6 + 150x^8y^8 + 32x^6y^{10} + 20x^4y^{12} + y^{16}.$$

For n even $n = 10, \ldots, 20$ the quaternary codes generated by weighing matrices $C(n,4)$ also have linear Gray images and hence are f.s.d $[2n, n, 4]$ codes.

7 Conclusion

To summarize, we have obtained f.s.d $[2n, n, 4]$ codes as linear binary images of quaternary codes generated by weighing matrices $W(n, 4)$. The f.s.d $[12, 6, 4]$ and $[16, 8, 4]$ codes have the highest possible minimal Hamming weights–they are extremal (see Fields, Gaborit, Huffman and Pless [7]).

Acknowledgement

We would like to thank Patrick Solé and Joe Fields for helpful remarks.

References

[1] E. Bannai, S. T. Doughert, M. Harada and M. Oura, Type II codes, even unimodular lattices, and invariant rings, IEEE Trans. Inform. Theory, **45**, (1999), 1194-2204.

[2] A. Bonnecaze, P. Solé, C. Bahoc and B. Mourrain, Type II codes over Z_4, IEEE Trans. Inform. Theory, **43**, (1997), 969-976.

[3] H. C. Chan, C. A. Rodger and J. Seberry, On inequivalent weighing matrices, *Ars Combinatoria*, 21-A (1986), 299-333.

[4] C. Charnes, Hadamard matrices, codes over the integers modulo 4 and their Gray images, Proceedings of SETA'98 *C. Ding, T. Helleseth and H. Niederreiter (Eds), Discrete Mathematics and Theoretical Computer Science*, Springer-Verlag, Berlin, 1999, pp. 171-183.

[5] J. H. Conway and N. J. A. Sloane, Self-dual codes over the integers modulo 4, Journ. Comb. Theory A **62** (1993), 30-45.

[6] R. Craigen, Weighing matrices and conference matrices, in *The CRC Handbook of Combinatorial Designs*, ed C. J. Colbourn and J. H. Dinitz, CRC Press, Boca Raton, Fla., 1996, pp. 496-504.

[7] J. E. Fields, P. Gaborit, W. C. Huffman and V. Pless, On the classification of formally self-dual codes, *Proceedings of the 36th Allerton Conference on Communication, Control and Computing*, UIUC, Oct. 1998, 566-575.

[8] A. V. Geramita and J. Seberry, *Orthogonal Designs: Quadratic Forms and Hadamard Matrices*, Marcel Dekker, New York-Basel, 1979.

[9] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane and P. Solé, The Z_4-linearity of Kerdock, Preparata, Goethals, and related codes, IEEE Trans. Inform. Theory, **40**, (1994), 301-319.

[10] M. Harada, Weighing matrices and self-dual codes, *Ars Combinatoria*, 47 (1997), 65-73.

[11] H. Hotelling, Some improvements in weighing and other experimental techniques, *Ann. Math. Stat.*, 16:294-300, 1944.

[12] D. Raghavarao, *Constructions and Combinatorial Problems in Design of Experiments*, Wiley Series in Probability and Statistics, John Wiley and Sons, New York-Sydney-London, 1971.

[13] E. M. Rains and N. J. A. Sloane, Self-Dual Codes, in *Handbook of Coding Theory*, ed. V. Pless and W. C. Huffman, Amsterdam, Elsevier, 1998.

[14] J. Seberry and M. Yamada, Hadamard matrices, sequences and block designs, in *Contemporary Design Theory - a Collection of Surveys*, eds J. Dinitz and D.J. Stinson, John Wiley and Sons, New York, pp. 431-560, 1992.