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# Critical sets for a pair of mutually orthogonal cyclic latin squares of odd order greater than 9

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## Abstract

To date investigations on critical sets for a set of mutually orthogonal latin squares (MOLS) have been carried out only for small orders  $\leq 9$ . In this paper we deal with a pair of cyclic orthogonal latin squares of order  $n$ ,  $n \geq 11$ ,  $n$  odd. Through construction of a uniquely completable set we give an upper bound on the size of the minimal critical set. In particular for  $n = 15$  a critical set achieving this bound is obtained.

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## 1 Introduction

In the last few years, a number of authors have worked on problems concerning *critical* and / or *uniquely completable* (UC) sets for latin squares (cf. Nelder [10], Smetanuik [12], Curran and Van Rees [3], Cooper, Donovan and Seberry [1], Cooper, McDonough and Mavron [2], Donovan, Cooper, Nott and Seberry [4], Donovan and Cooper [5], Fu, Fu and Rodger [7], and Donovan and Howse [6]). There is not a lot known about the critical sets for latin squares in general. However, a class of critical sets is known for a *back circulant* latin square which is a particular latin square having the initial row in the standard form and subsequent rows formed by translating the previous row one element to the left. A generalisation of the problem of finding critical sets of minimum size in a single latin square is that of finding the minimum number of cell entries needed to be prescribed in the members of a set of two or more mutually orthogonal latin squares (MOLS) so that the cell entries in the entire set of squares can thereby be uniquely determined. To date very few results in this direction are known. Keedwell [8] was the first to attempt this problem. He made a preliminary investigation on the size of the minimal critical set for a set consisting of a back circulant latin square of order upto 7 and its  $k$  cyclic orthogonal mates. Subsequently SahaRay, Adhikari and Seberry [11] carried out further investigation for order 7 and characterised a critical set, thereby providing an improved upper bound on the size of the minimal critical set. They also studied the case of order 9. They observed that the critical set obtained in the case of order 9,

when generalised for order  $n$ ,  $n$  odd,  $n \geq 11$  can be easily shown to be UC in general, and provide an upper bound on the size of the minimal critical set. However the construction does not provide a critical set for any  $n$ . With a view to obtain a finer upper bound we further consider the same problem in this paper. In Section 3 through characterisation of a UC set we attain this goal. This UC set is also shown to be a critical set for order 15. Moreover, using computer programming, we obtain UC sets of further smaller size for orders 11 and 13 which is given in section 4, but we omit the details of the completion steps because of very complicated branching arising out of the *weakly completable* nature of the UC set.

Before discussing the main results some background information is needed which is given in the next section.

## 2 Preliminary Definitions and Notations

In this section, we draw the readers' attention to the definitions and known results on critical sets for latin squares of order  $n$  which will be used hereafter to derive the main results. A latin square  $L$  of order  $n$  is an  $n \times n$  array with entries chosen from a set  $N$  of size  $n$  such that each element of  $N$  occurs precisely once in each row and in each column. In what follows  $N$  is assumed to be  $\{1, 2, \dots, n\}$ . For convenience a latin square  $L$  of order  $n$  is sometimes represented by a set of ordered triplets  $\{(i, j, k) \mid \text{element } k \text{ occurs in the position } (i, j), i, j, k \in N\}$ .

A *partial latin square*  $P$  of order  $n$  is an  $n \times n$  array with entries chosen from  $N$  such that each element of  $N$  occurs at most once in each row and in each column of  $P$ . Then  $|P|$  is said to be the size of the partial latin square and the set of positions  $S_P = \{(i, j) \mid (i, j, k) \in P, \exists k \in N\}$  is said to determine the shape of  $P$ . Let  $P$  and  $P'$  be two partial latin squares of the same order, with the same size and shape. Then  $P$  and  $P'$  are said to be *mutually balanced* if the entries in each row (and column) of  $P$  are the same as those in the corresponding row (and column) of  $P'$ . They are said to be *disjoint* if no position in  $P'$  contains the same entry as the corresponding position in  $P$ . A *latin interchange* (also referred to as *latin trade*, cf. Keedwell [9])  $I$  is a partial latin square for which there exists another partial latin square  $I'$  of the same order, size and shape with the property that  $I$  and  $I'$  are disjoint and mutually balanced.

A *Uniquely completable set* (UC set)  $U$  of triplets is such that it yields only one latin square  $L$  of order  $n$  which has element  $k$  in position  $(i, j)$ , for each  $(i, j, k) \in U$ . A set  $C$  is said to be a *critical set* if

1.  $C$  is a UC set, and
2. no proper subset of  $C$  satisfies 1.

A *minimal critical set* is a critical set of the smallest possible size.

If  $C$  is a UC or critical set, a triple  $(i, j, k) \in L \setminus C$  will be said to be *forced*, if either  $\forall h \neq i, \exists z$  such that  $(h, j; z)$  or  $(h, z; k) \in C$ , or  $\forall h \neq j, \exists z$  such that  $(z, h; k)$  or  $(i, h; z) \in C$ , or  $\forall h \neq k, \exists z$  such that  $(i, z; h)$  or  $(z, j; h) \in C$ .

A UC set of cell entries for a set of MOLS is called *strong* if the cell entries in the entire set of squares can be successively filled by a sequence of adjunctions of cell entries to individual squares of the set each of which is forced.

A UC set which is not strong is called *weak*.

In this paper, we start with a back circulant latin square  $L_1$  and deal with its cyclic orthogonal mate  $L_2$  having its initial row  $\{p_1, p_2, \dots, p_n\}$  where  $\{p_1, p_2, \dots, p_n\}$  is any permutation of the symbols  $\{1, 2, \dots, n\}$  and try to identify a minimal critical set for the set  $S = \{L_1, L_2\}$ . However, to simplify notations, without loss of generality, we refer to  $p_j$  as  $j$  in our subsequent discussion. It is to be noted that all mathematical operations discussed in this paper are performed modulo  $n$ , however, we use symbol  $n$  instead of 0.

**Definition 2.1** We say a latin square  $L_t$  of order  $n$  is a cyclic latin square if its  $(i, j)$ th cell contains the entry  $1 + (i - 1)t + (j - 1)$ ,  $t = 1, 2, \dots, n - 1$ ,  $i, j \in N$ .

We now quote below the best known lower bound on the size of the critical sets for latin squares of order  $n$ ,  $n$  odd, due to Cooper, Donovan and Seberry [1].



is a UC set of size  $\frac{n^2-1}{4}-6$  for  $L_2$  as a member of a set  $S = \{L_1, L_2\}$  of two pairwise orthogonal cyclic latin squares.

**Proof :** We start with a back circulant latin square  $L_1$  and assume that it has been completed from a partial latin square of size  $\frac{n^2-1}{4}$  given in Lemma 2.2. Not to obscure the essential steps for unique completion of  $L_2$  as a member of  $S = \{L_1, L_2\}$  we breakup  $C$  into two disjoint subsets consisting of odd and even columns. We define

$$C_{UT} = \{(i, j, 2i + j - 2) : i = \frac{n-j}{2} - 2, \frac{n-j}{2} - 3, \dots, 1; j = 1, 3, 5, \dots, n-6\}$$

$$\cup \{(i, j, 2i + j - 2) : i = \frac{n-j}{2}, \frac{n-j}{2} - 1, j = 3, 5, \dots, n-4\}$$

and

$$C_{LT} = \{(i, j, 2i + j - 2) : i = n, n-1, \dots, n - \frac{j}{2} + 1; j = 6, 8, \dots, n-1\}$$

Obviously  $C = C_{UT} \cup C_{LT}$ .

The steps for completion of  $L_2$  differ at some places depending on  $n \equiv 1(mod 4)$  or  $n \equiv 3(mod 4)$ . So we discuss below the steps for unique completion for  $n \equiv 1(mod 4)$  and whenever any deviation arises for  $n \equiv 3(mod 4)$  we mention that within bracket. We first make the following observations:

**Fact 1 :** In  $C_{UT}$ ,

- $x$  is present along the back transversals  $T_{\frac{x+1}{2}}, T_{\frac{x+3}{2}}, \dots, T_x$   
for  $x = 1, 3, \dots, n-6$ ,
- $n-4$  is present along  $T_{\frac{n-1}{2}}, T_{\frac{n+1}{2}}, \dots, T_{n-4}$ , and
- $n-2$  is present along  $T_{\frac{n+1}{2}}, T_{\frac{n+3}{2}}, \dots, T_{n-3}$ .

**Fact 2 :** In  $C_{LT}$ ,

- $x$  is present along  $T_{x+1}, T_{x+2}, \dots, T_{\frac{n+x-1}{2}}$   
for  $x = 4, 6, \dots, n-1$ ,
- 2 is present along  $T_4, T_5, \dots, T_{\frac{n+1}{2}}$ ,
- $n$  is present along  $T_3, T_4, \dots, T_{\frac{n-1}{2}}$ .

**Fact 3 :** In  $L_1$ ,

$i$  occurs along the back transversal  $T_i, i = 1, 2, \dots, n$ , and hence orthogonality of  $L_2$  to  $L_1$  demands that in  $L_2$  all the elements along the back transversal  $T_i$  should be different. Moreover, being a latin square means that all the elements along row  $R_i$  and column  $C_j$  of  $L_2$  should be different,  $i, j \in \{1, 2, \dots, n\}$ .

Noting these three points we argue as follows towards completion of a partial latin square  $L'_2$  to  $L_2$ ,  $L'_2$  having entries prescribed by  $C$ . We assume below that  $n = 4t + 1, t \geq 4$  ( $4t + 3, t \geq 3$ ). The cases of  $n = 11$  and  $13$  are dealt with differently, as the general rule does not go through in these two cases.

**Step 1 :** Using Fact 1 and Fact 3 noted above,  $x$  is placed in  $L'_2$  uniquely in  $C_{n-1}, C_{n-3}, \dots, C_2, C_n, C_{n-2}, \dots, C_{x+2}$  in order, for  $x = n-6, n-8, \dots, 2t-1$  ( $2t+1$ ).

**Step 2 :** Using Fact 2 and Fact 3 noted above,  $x$  is placed uniquely in  $C_1, C_3, \dots, C_n, C_2, C_4, \dots, C_x$  in order, for  $x = 4, 6, \dots, 2t-2$ .

**Step 3 :** Now  $n-2$  can be assigned in  $C_{n-1}$  in one of the three available cells viz.

- (a)  $(1, n-1)$ ,
  - (b)  $(\frac{n-1}{2}, n-1)$
- and (c)  $(\frac{n+1}{2}, n-1)$ .

We will argue below that the cases (a) and (b) are infeasible. To this end, we start with the case (a), i.e we assume that  $n-2$  is placed in  $(1, n-1)$ , then  $n-2$  can be placed in  $C_{n-3}$  in one of the two available cells viz.

- (a1)  $(\frac{n+1}{2}, n-3)$
- and (a2)  $(\frac{n+3}{2}, n-3)$ ,

both of which will turn out to be infeasible in the course of our reasoning. Now to establish this,

we start with (a) along with (a1) i.e assignment of  $n - 2$  in  $C_{n-1}$  at  $(1, n - 1)$  and in  $C_{n-3}$  at  $(\frac{n+1}{2}, n - 3)$ . Then  $n - 2$  is placed uniquely in  $C_{n-5}, C_{n-7}, \dots, C_{2t+2}$  ( $C_{2t+4}$ ) in order, proceeding downwards in the process of filling,  $C_{2t}$  ( $C_{2t+2}$ ) is the first column to arise, where two places for  $n - 2$  are available. Now using Fact 1 for  $n - 2$  and the  $T_i$ 's along which  $n - 2$  has been placed so far in alternate columns starting  $C_{n-1}$ , it can be easily seen that there is no place left for  $n - 2$  in  $C_{n-2}$ . Hence (a) with (a1) cannot happen. Now to settle that (a) with (a2) also cannot happen we argue along the same lines. If (a) and (a2) occur together then  $n - 2$  can be placed uniquely in  $C_{n-5}, C_{n-7}, \dots, C_{2t+2}$  ( $C_{2t+4}$ ) leaving no place for  $n - 2$  in  $C_{n-2}$ .

Now we eliminate the possibility (b) i.e placement of  $n - 2$  in  $(\frac{n-1}{2}, n - 1)$ . Suppose that (b) holds, i.e.  $n - 2$  is placed in  $(\frac{n-1}{2}, n - 1)$ . Then using Fact 1 as before,  $n - 2$  can be placed uniquely in  $C_{n-3}, C_{n-5}, \dots, C_{2t+2}$  ( $C_{2t+4}$ ) in order, and then in  $C_{n-2}$  at  $(1, n - 2)$ . Now clearly  $n - 4$  and  $2t - 3$  ( $2t - 1$ ) can be placed sequentially in order, in  $C_{n-1}$  and then in  $C_{n-3}, \dots, C_{2t+4}$  ( $C_{2t+6}$ ) in turn. Considering Fact 1 and  $T_i$ 's along which  $2t - 3$  ( $2t - 1$ ) has been placed sequentially in alternate columns starting with  $C_{n-1}$ , we see that  $2t - 3$  ( $2t - 1$ ) can be placed uniquely in the rest of the columns in the order  $C_{2t+2}$  ( $C_{2t+4}$ ),  $C_{2t}$  ( $C_{2t+2}$ ),  $\dots, C_2, C_n, C_{n-2}, \dots, C_{2t-1}$  ( $C_{2t+1}$ ). Now going back to the placement of  $n - 2$  again, using Fact 1 and the  $T_i$ 's along which  $n - 2$  has been placed so far, it follows that  $n - 2$  can be placed uniquely in  $C_{2t}$  ( $C_{2t+2}$ ),  $C_{2t-2}$  ( $C_{2t}$ )  $\dots, C_2, C_n$  in order. As a result, a contradiction arises since no place is left for  $n - 2$  in  $C_1$ . Hence possibilities (a) and (b) are eliminated and  $n - 2$  is fixed in  $C_{n-1}$  at  $(\frac{n+1}{2}, n - 1)$ .

**Step 4 :** Now  $n - 2$  is placed uniquely in  $C_{n-3}, C_{n-5}, \dots, C_{2t+2}$  ( $C_{2t+4}$ ) and then in  $C_{n-2}$  at  $(1, n - 2)$ .

**Step 5 :** Now  $n - 4$  and  $2t - 3$  ( $2t - 1$ ) are fixed sequentially in order in  $C_{n-1}$  and then in  $C_{n-3}, \dots, C_{2t+4}$  ( $C_{2t+6}$ ) in turn.

**Step 6 :** Now using the  $T_i$ 's along which  $2t - 3$  ( $2t - 1$ ) is present in  $C_{UT}$  and also in the alternate columns starting with  $C_{n-1}$ , place  $2t - 3$  ( $2t - 1$ ) uniquely in  $C_{2t+2}$  ( $C_{2t+4}$ ),  $C_{2t}$  ( $C_{2t+2}$ ),  $\dots, C_2, C_n, C_{n-2}, \dots, C_{2t-1}$  ( $C_{2t+1}$ ).

**Step 7 :** Now fix  $n - 2$  uniquely in  $C_{2t}$  ( $C_{2t+2}$ ),  $C_{2t-2}$  ( $C_{2t}$ ),  $\dots, C_2, C_n, C_1$ .

**Step 8 :** Now  $n - 4$  is fixed in  $C_{2t+2}$  ( $C_{2t+4}$ ).

**Step 9 :** Now fix  $2t - 5$  ( $2t - 3$ ) in  $C_{n-1}, C_{n-3}, \dots, C_2, C_n, C_{n-2}, \dots, C_{2t-3}$  ( $C_{2t-1}$ ).

**Step 10 :** Now  $n - 4$  is placed uniquely in  $C_{2t}$  ( $C_{2t+2}$ ),  $C_{2t-2}$  ( $C_{2t}$ ),  $\dots, C_2, C_n, C_{n-2}$  and finally in  $C_1$ .

**Step 11 :** Now  $x$  is placed uniquely in  $C_{n-1}, C_{n-3}, \dots, C_2, C_n, C_{n-2}, \dots, C_{x+2}$  where  $x$  is odd,  $x \neq 1$ ,  $x = 2t - 7$  ( $2t - 5$ ),  $2t - 5$  ( $2t - 3$ ),  $\dots, 3$ .

**Step 12 :** Now  $C_{n-1}, C_{n-3}, \dots, C_6$  can be completed uniquely in order starting from  $R_1$  and moving downwards.

**Step 13 :** Now we try to place 1 in  $C_4$ . Clearly, there are three places in  $C_4$  available for 1, viz.  $(\frac{n-1}{2}, 4)$ ,  $(n - 1, 4)$  and  $(n, 4)$ . If 1 is placed in  $(n - 1, 4)$  then no place is left for 1 in  $R_n$ . Similarly if 1 is placed at  $(n, 4)$ , 1 can not be placed in  $R_{n-1}$ . Hence 1 is placed in  $C_4$  at  $(\frac{n-1}{2}, 4)$  and henceforth in  $C_2, C_n, C_{n-2}, \dots, C_3$ .

**Step 14 :** Now using the fact that  $2t$  is present in  $C_{LT}$  along  $T_{\frac{n+1}{2}}, \dots, T_{\frac{n+2t-1}{2}}$ ,  $2t$  is uniquely placed in  $C_1, C_3, \dots, C_n, C_2, C_4$ .

**Step 15 :** We now try to position 2. In  $C_2$  only two places are available for 2 viz.  $(1, 2)$  and  $(n, 2)$  as the other empty cells in  $C_2$  fall along the transversals  $T_{t+2}, T_{t+3}, \dots, T_{\frac{n+1}{2}}$ ,  $t \geq 3$  and 2 is already present in  $C_{LT}$  along  $T_4, T_5, \dots, T_{\frac{n+1}{2}}$ . If 2 is in  $(n, 2)$  then in  $C_1$  the only place available for 2 is  $(\frac{n+3}{2}, 1)$  as 2 is already along  $T_{\frac{n+1}{2}}$  and in the other rows of  $C_1$ . Now clearly no place for 2 is available in  $C_4$  and hence 2 is fixed in  $(1, 2)$  and consequently  $n$  is fixed in  $(1, n)$ . Hence in  $R_2$ ,  $n$  is placed uniquely in  $(2, n - 2)$  and consequently 2 is placed in  $(2, n)$ . Thus in  $R_n$ ,  $n$  is placed uniquely in  $(n, 2)$  because of presence of  $n$  along  $T_n$  and  $T_3$ .

**Step 16 :** Now  $n$  is fixed in  $R_{n-1}$  at  $(n - 1, 4)$ .

**Step 17 :** Now considering the possibilities of 2 in  $R_n$  we find that 2 can be at  $(n, 1)$  or  $(n, 4)$ . To eliminate the placement of 2 in  $(n, 1)$  we argue as follows. Suppose that 2 is placed in  $(n, 1)$ . Now we attempt to assign 2 in  $R_{\frac{n+3}{2}}$ . It is seen at this stage that in  $R_{\frac{n+3}{2}}$  the empty cells are

$(\frac{n+3}{2}, 1), (\frac{n+3}{2}, 2t+1), (\frac{n+3}{2}, 2t+3), \dots, (\frac{n+3}{2}, n-2)$  which are along  $C_1, T_1, T_3, T_5, \dots, T_{2t-1}$  respectively, whenever  $n = 4t+1$ . Thus we see that for  $n = 4t+1$ , 2 can be placed uniquely in  $R_{\frac{n+3}{2}}$  at  $(\frac{n+3}{2}, 2t+3)$  along  $T_3$ . Now because of the presence of 2 along  $T_n$ , 2 is placed in  $R_3$  uniquely in  $(3, n-4)$  and henceforth in  $R_4, R_5, \dots, R_{t-1}$  uniquely, leading to a contradiction in placement of 2 in  $R_{t-1}$ . (Whenever  $n = 4t+3$  the empty cells in  $R_{\frac{n+3}{2}}$  are along  $C_1, T_n, T_2, T_4, \dots, T_{2t}$  and thus it readily leads to a contradiction in the placement of 2 in  $R_{\frac{n+3}{2}}$  as 2 is already in  $C_1$  and along these transversals.) Hence in  $R_n$ , 2 should be placed at  $(n, 4)$ .

$L'_2$  with the entries fixed so far can now be permuted to  $\tilde{L}'_2$  which has the standard form of Cooper, Donovan and Seberry [1] and Smetanuik [12] where we now have a critical set in the back circulant latin square as given in Lemma 2.2. This allows us to uniquely complete  $\tilde{L}'_2$  and reversing the permutations it gives back  $L_2$  completed from  $L'_2$ .

Now we deal with the case  $\mathbf{n} = 13$ .

Place  $x$  uniquely in  $C_{12}, C_{10}, \dots, C_2, C_{13}, C_{11}, \dots, C_{x+2}$  in order, for  $x = 7, 5$ . Then position 4 in  $C_1, C_3, \dots, C_{13}, C_2, C_4$  sequentially. In  $C_{12}$ , 11 can now occur in three places viz. (1,12), (6,12), (7,12). Now 11 in (1,12) leads to two possibilities for 11 in  $C_{10}$  viz. (7,10) and (8,10). Placement of 11 in (7,10) is followed by unique placement of 11 in  $C_8, C_{11}, C_6$  leading to a contradiction of placement of 11 in  $C_{13}$ . Again placement of 11 in (8,10) is followed by placement of 11 in  $C_8$  leading to a contradiction in placement of 11 in  $C_{11}$ . Thus choice of 11 in  $C_{12}$  at (1,12) is eliminated. Similarly choice of 11 in  $C_{12}$  at (6,12) is infeasible because it leads to the unique placement of 9 in  $C_{12}$  and  $C_{10}$ , then 3 in  $C_{12}$  and  $C_{10}$ , 11 in  $C_{10}$ , 3 and 11 simultaneously in order in  $C_8, C_6, C_4, C_2$  sequentially, then 11 in  $C_{13}$  leaving no place for 11 in  $C_{11}$ . Thus 11 in  $C_{12}$  is fixed at (7,12). Now, in turn, 11 is fixed in  $C_{10}$  and  $C_8$ , 9 in  $C_{12}$  and  $C_{10}$ , 3 in  $C_{12}, C_{10}, C_8$ , 9 in  $C_8$ , 3 in  $C_6$ , 11 in  $C_{11}, C_6, C_4, C_2, C_{13}, C_1$ , 1 in  $C_{12}, C_{10}, C_8, C_6$ , 9 in  $C_6, C_4, C_2, C_{13}, C_{11}$  and finally in  $C_1$ . Then we follow step 11 onwards as discussed above for  $n = 4t+1$  to complete  $L_2$  uniquely.

In the Appendix, we present the details through a tree structure for  $\mathbf{n} = 11$  to allow easy comprehension for completion as the weakly completable nature of the UC set causes considerable minor details. ■

Now Lemma 2.2 and Theorem 3.1 provide an upper bound for the size of the minimal critical set for  $S = \{L_1, L_2\}$ .

**Theorem 3.2** *Let  $L_1$  be the back circulant latin square of order  $n$ ,  $n$  odd,  $n \geq 11$  and  $L_2$  be its cyclic orthogonal mate translated two elements to the left. Then the size of the minimal critical set for  $S = \{L_1, L_2\}$  is atmost  $\frac{n^2-1}{2} - 6$ .*

**Proof:** From Lemma 2.2 it follows that the back circulant latin square  $L_1$  has a critical set of size  $\frac{n^2-1}{4}$ . It follows from Theorem 3.1 that the size of the critical set for  $L_2$  as a member of  $S$  is atmost  $\frac{n^2-1}{4} - 6$  as a critical set is embedded in a UC set. Hence for joint completion of  $L_1$  and  $L_2$  the size of the minimal critical set is atmost  $\frac{n^2-1}{2} - 6$ . ■

Now we show that the UC set  $C$  given in Theorem 3.1 is indeed a critical set for  $L_2$  for  $n = 15$ .

**Theorem 3.3** *For  $n = 15$  the set  $C$  given in Theorem 3.1 is a critical set for  $L_2$  as a member of  $S = \{L_1, L_2\}$ .*

**Proof:** Now to prove that the UC set  $C$  in Fig. 1 is a critical set for  $L_2$ , we show that for each  $(i, j, k) \in C$ , there exist two latin interchanges  $L_{2t}, t = 1, 2$  in  $L_2$  satisfying

$$C \cap L_{2t} = \{(i, j, k)\}, \quad t = 1, 2.$$

It is easy to verify that for any  $(i, j, k) \in C$ ,  $L_2$  contains a partial latin square  $L_{21}$  of the form

$$L_{21} = \{(i, j, k), (i, j + \alpha, k + \alpha), (i + \alpha, j + \alpha, k), (i + \alpha, j + 2\alpha, k + \alpha), (i + 2\alpha, j + 2\alpha, k), (i + 2\alpha, j, k + \alpha)\},$$

which can be replaced by another partial latin square  $L_{22}$  of the form

$$L_{22} = \{(i, j, k + \alpha), (i, j + \alpha, k), (i + \alpha, j + \alpha, k + \alpha), (i + \alpha, j + 2\alpha, k), \\ (i + 2\alpha, j + 2\alpha, k + \alpha), (i + 2\alpha, j, k)\},$$

yielding a different latin square  $\tilde{L}_2$  orthogonal to  $L_1$  where,

$$\alpha = 5 \quad \text{for } (i, j, k) \in C_{UT} \quad \setminus \{(1, 11, 11), (2, 11, 13), (6, 3, 13)\} \\ \cup \{(9, 14, 15), (10, 12, 15), (10, 14, 2)\}$$

and

$$\alpha = 10 \quad \text{for } (i, j, k) \in C_{LT}. \quad \setminus \{(9, 14, 15), (10, 12, 15), (10, 14, 2)\} \\ \cup \{(1, 11, 11), (2, 11, 13), (6, 3, 13)\}.$$

Orthogonality follows from the fact that  $(i, j, k)$  and  $(i + \alpha, j + 2\alpha, k + \alpha)$  fall along  $T_{i+j-1}$ , similarly  $(i, j + \alpha, k + \alpha)$  and  $(i + 2\alpha, j + 2\alpha, k)$  fall along  $T_{i+j+\alpha-1}$  and  $(i + \alpha, j + \alpha, k)$  and  $(i + 2\alpha, j, k + \alpha)$  fall along  $T_{i+j+2\alpha-1}$  where  $k$  and  $k + \alpha$  can be interchanged.

Thus if we remove any element from  $C$  then we can complete the subset to at least two latin squares orthogonal to  $L_1$  each of which has one of the partial latin squares given above. So  $C$  with size 50 turns out to be a critical set for  $L_2$ . ■

**Remark 2:** Note that using orthogonality with  $L_1$  the size of the critical set for  $L_2$  can be reduced by 6 from 56, the size of the critical set of smallest size known so far, as given in Lemma 2.2 due to Cooper, Donovan and Seberry [1]. It is to be noted that there does not exist any cyclic latin square orthogonal to both  $L_1$  and  $L_2$ .

## 4 UC set of further smaller size for $n = 11$ and 13

It is interesting to note that for  $n = 11$  and 13 we obtain through computer verification UC sets of further smaller size, reduced three more elements in each case. As these sets are weakly completable, due to various branching at different stages we omit the details of the steps for completion, however we present below the computer output. Whether these sets are critical are yet to be verified.

1		3		5					
3		5		7		9			
5		7							
									11
							11	2	
					11	2	4		
					2	4	6		
					4	6	8		

UC set for  $n = 11$ , Fig.2

1		3		5		7		9				
3		5		7		9		11				
5		7		9								
7		9										
												13
									13		2	
							13		2		4	
					13		2		4		6	
					2		4		6		8	
					4		6		8		10	

*UC set for  $n = 13$ , Fig.3*

## 5 Concluding Remarks

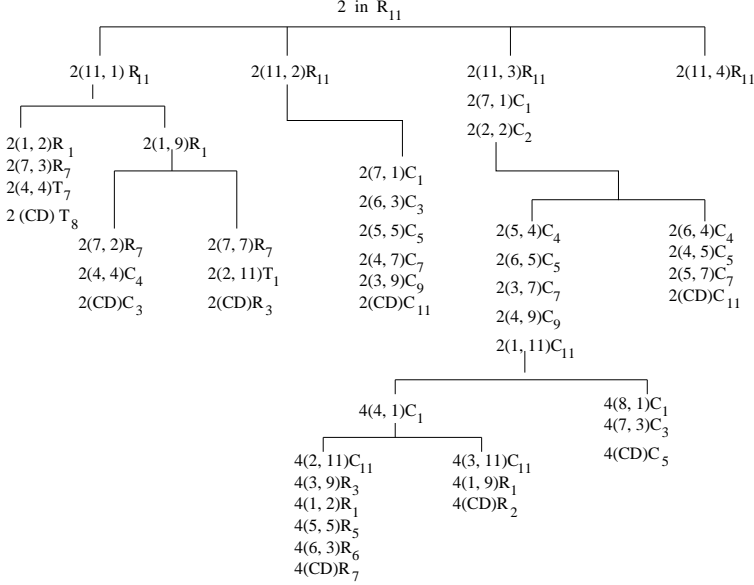
We strongly believe that as  $n$  grows ,  $n$  prime,  $n > 15$  there is a UC set of smaller size but to write down the steps for unique completion in general, becomes very much more complicated. Investigation is going on as to whether we can arrive at a reduction in the required size of the UC set depending on  $n$ .

## Appendix:

We now present the steps for completion of  $L_2$  for  $n = 11$ . We use the notation  $i(j, k)Q$  where  $Q = R_j$  or  $C_k$  or  $T_l$ ,  $i, j, k, l \in \{1, 2, \dots, 11\}$  to denote that  $i$  is placed in  $R_j$  or  $C_k$  or along  $T_l$  in  $(j, k)$  respectively. Let  $i(CD)Q$  denote that a contradiction arises in the placement of  $i$  in  $Q$ .

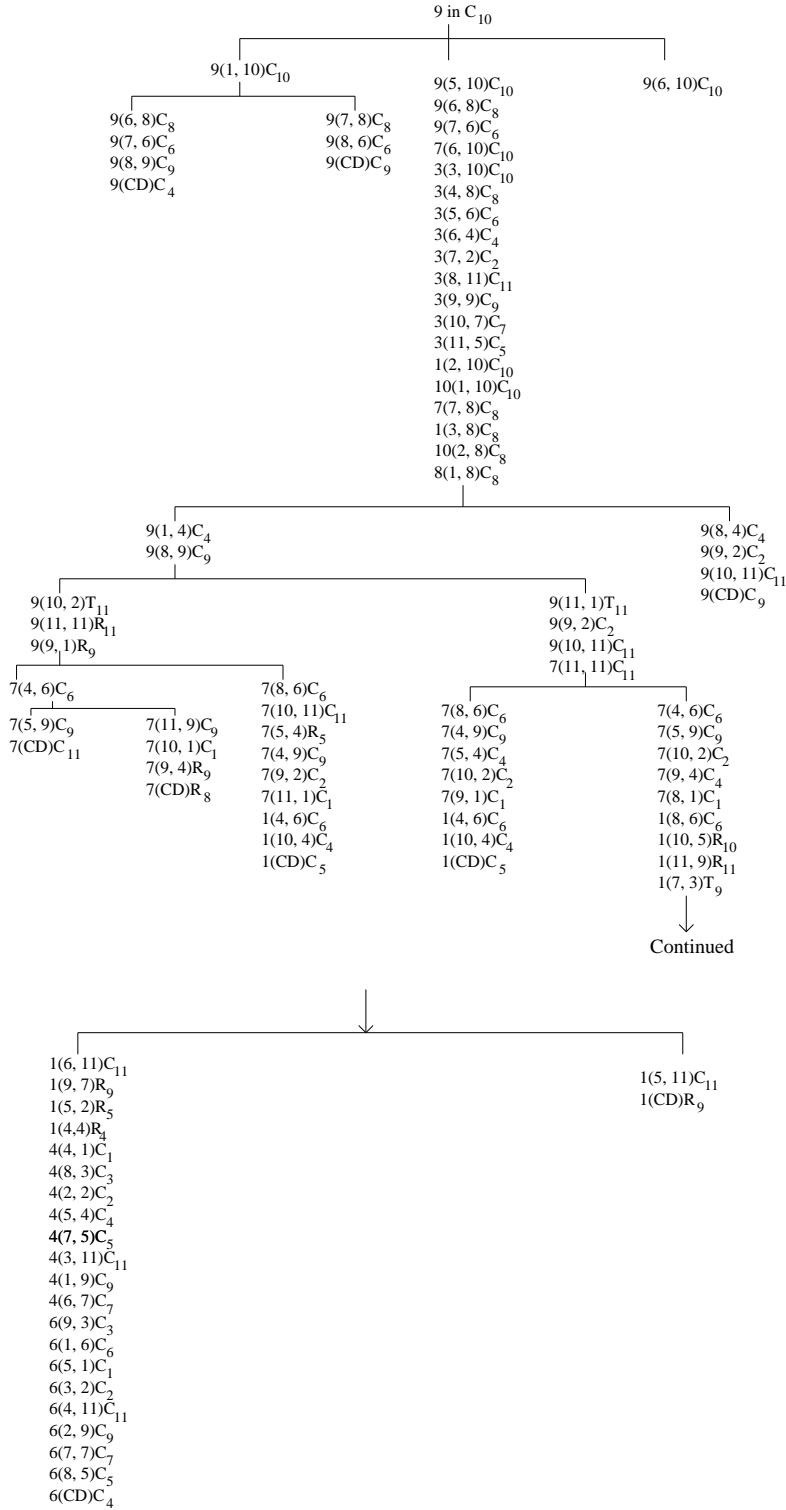
**Step 1:** 5 is placed in  $L_2$  uniquely in  $C_{10}, C_8, \dots, C_2, C_{11}, C_9, C_7$  in order.

**Step 2:** Now 2 can be assigned in  $R_{11}$  in one of the four places viz.  $(11,1), (11,2), (11,3)$  and  $(11,4)$ . We argue below through a tree structure of completion steps that the only feasible place for 2 in  $R_{11}$  is  $(11,4)$ .



Hence 2 is fixed at  $(11,4)$  in  $R_{11}$  and subsequently in  $C_1, C_3, \dots, C_{11}$  and  $C_2$  in order.

**Step 3:** Now we try to place 9 in  $C_{10}$ . Clearly there are three places in  $C_{10}$  available for 9, viz.  $(1,10), (5,10)$ , and  $(6,10)$ . The following tree of completion steps eliminate the possibility of placement of 9 in  $(1,10)$  and  $(5,10)$ .



- Hence 9 is fixed at (6,10) in  $C_{10}$  and subsequently in  $C_8$  at (7,8).
- Step 4:** Now 3 is fixed uniquely in  $C_{10}$ ,  $C_8$ , ...,  $C_2$ ,  $C_{11}$ ,  $C_9$ , ...,  $C_5$  in order.
- Step 5:** Now 9 is placed uniquely in  $C_6$ ,  $C_9$ ,  $C_4$ ,  $C_2$ ,  $C_{11}$ ,  $C_1$  in order.
- Step 6:** Now 7 is uniquely placed in  $C_{10}$  and  $C_8$  in order.
- Step 7:** Position 1 uniquely in  $C_{10}$ ,  $C_8$  and  $C_6$  in order.

**Step 8:** Now 7 is placed uniquely in  $C_6, C_4, C_2, C_{11}, C_9, C_1$  in order.

**Step 9:** Complete  $C_{10}, C_8$  and  $C_6$  sequentially.

**Step 10:** Now 4 is positioned uniquely in  $C_1, C_3, \dots, C_{11}, C_2, C_4$  in order. So  $R_1$  is fixed containing 11 at (1,11).

**Step 11:** 11 is placed uniquely in  $R_2$  at (2,9).

**Step 12:** 1 is placed uniquely in  $R_{11}$  at (11,3).

**Step 13:** 11 is fixed at (11,2) in  $R_{11}$  and hence is placed at (10,4) in  $R_{10}$ .

Now as before, following the arguments given after step 17 in the proof of Theorem 3.1,  $L'_2$  can be uniquely completed to  $L_2$ .

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