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Year 2005

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Supplementary Difference Sets and Their
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This article was originally published as Koukouvinos, C, Mitrouli, M and Seberry, J, Values of Minors of Some Infinite Families of Matrices Constructed from Supplementary Difference Sets and Their Application to the Growth Problem, *Linear Algebra and its Applications*, 406, 2005, 218-234. Copyright Elsevier 2005. Original journal available <http://www.elsevier.com/wps/find/journaldescription.cws_home/522483/description#description>here.

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Values of Minors of Some Infinite Families of Matrices Constructed from Supplementary Difference Sets and Their Application to the Growth Problem

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October 23, 2004

Abstract

We obtain explicit formulae for the values of the $2v - j$ minors, $j = 0, 1, 2$ of $(1, -1)$ matrices of order $2v$, v odd, where the matrix is constructed using two circulant or type 1 incidence matrices of $2 - \{v; k_1, k_2, \lambda\}$ sds. This allows us to obtain information on the growth problem for families of matrices with moderate growth. Some of our theoretical formulae imply growth close to the order $2v$ but experimentation has not yet supported this result. An open problem remains to establish whether the $(1, -1)$ CP incidence matrices of certain SBIBDs, can have growth greater than $2v$.

Key Words and Phrases: SBIBD, supplementary difference sets, Gaussian elimination, growth, complete pivoting.

AMS Subject Classification: 05B20, 15A15, 65F05, 65G05.

Short running title: Minors of matrices from difference sets and growth problem.

1 Introduction

A set of k residues $D = \{a_1, \dots, a_k\}$ modulo v is called a (v, k, λ) difference set or cyclic difference set, if for every $d \not\equiv 0 \pmod{v}$ there are exactly λ ordered pairs (a_i, a_j) , $a_i, a_j \in D$ such that $a_i - a_j \equiv d \pmod{v}$.

For the purpose of this paper we will define two supplementary difference sets $2 - \{v; k_1, k_2; \lambda\}$, abbreviated as sds, to be two circulant (or type 1) $v \times v$ matrices B_1 and B_2 , with entries 0 or 1, which have exactly k_i entries +1 and $v - k_i$ entries 0, $i = 1, 2$ respectively, in each row and column and for which the inner product of any pair of rows of $[B_1 \ B_2]$ is λ . The $(1, -1)$ incidence matrices of B_i , are obtained by letting $A_i = 2B_i - J$, $i = 1, 2$, where J is the $v \times v$ matrix of all ones. Then we have for all methods below A is of order $2v$ is given by

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[‡]The work of this author was supported by a grant from the University of Athens.

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$$A = \begin{bmatrix} A_1 & A_2 \\ A_2^T & -A_1^T \end{bmatrix} \quad (1)$$

$$A_1 A_1^T + A_2 A_2^T = 4(k_1 + k_2 - \lambda)I + 2(v - 2(k_1 + k_2 - \lambda))J = (2v - \Lambda)I + \Lambda J$$

where $\Lambda = 2v - 4(k_1 + k_2 - \lambda)$, and I is the identity matrix of order v .

We can write

$$AA^T = (2v + 2)I_{2v} - 2I_2 \times J_v.$$

It is easy to use the determinant simplification theorem [8] to see that for methods I and II, which follows, we have

$$\det A = 2^v(2v - 1)(v - 1)^{v-1}.$$

In this paper we study the family $2 - \{v; k_1, k_2; \lambda\}$ sds, and using $\lambda(v - 1) = k_1(k_1 - 1) + k_2(k_2 - 1)$ we can conclude that

$$\det A = 2^v(k_1 + k_2 - \lambda)^{v-1}(v^2 - 2v(k_1 + k_2) + 2k_1^2 + 2k_2^2).$$

Method I.

When $v = 4t + 1$ is a prime power, there exists a supplementary difference set with parameters $2 - \{v; \frac{v-1}{2}, \frac{v-1}{2}; \frac{v-3}{2}\} = 2 - \{4t + 1; 2t, 2t; 2t - 1\}$, see [14, 15].

Their incidence $(1, -1)$ matrices A_1 and A_2 satisfying the relation

$$A_1 A_1^T + A_2 A_2^T = (2v + 2)I - 2J.$$

Method II.

As described in Baumert [1, p.119], the best known n^{th} power residue difference sets are the quadratic residue sets of Paley [11].

When $v = 4t - 1$ is a prime power, the quadratic residues modulo v form a difference set with parameters $(v, k, \lambda) = (4t - 1, 2t - 1, t - 1)$. Thus, if we take this difference set twice we form a $2 - \{v; \frac{v-1}{2}, \frac{v-1}{2}; \frac{v-3}{2}\} = 2 - \{4t - 1; 2t - 1, 2t - 1; 2t - 1\}$ sds.

Their incidence $(1, -1)$ matrices are identical in this case, say $P = A_1 = A_2$, and then we have

$$2PP^T = (2v + 2)I - 2J.$$

We can write

$$AA^T = (2v + 2)I_{2v} - 2I_2 \times J_v.$$

It is easy to use the determinant simplification theorem [8] to see that for methods I and II we have

$$\det A = 2^v(2v - 1)(v - 1)^{v-1}.$$

We also examine some other methods of constructing sds given in [18].

Method III.

When $q = 25 + 4b^2 \equiv 13 \pmod{16}$ is a prime power, we consider the following $2 - \{q; \frac{q-1}{4}, \frac{q-1}{2}; \frac{5q-17}{16}\}$ sds. In this case we have

$$A_1 A_1^T + A_2 A_2^T = \left(\frac{7q + 5}{4}\right)I + \left(\frac{q - 5}{4}\right)J.$$

As an example for $q = 29$ we obtain a $2 - \{29; 7, 14; 8\}$ sds, which gives

$$A_1 A_1^T + A_2 A_2^T = 52I + 6J.$$

Method IV.

When $q = 49 + 4b^2 \equiv 5 \pmod{16}$ is a prime power, we consider the following $2 - \{q; \frac{q-1}{2}, \frac{3(q-1)}{4}, \frac{13q-33}{16}\}$ sds. In this case we have

$$A_1 A_1^T + A_2 A_2^T = \left(\frac{7q+17}{4}\right)I + \left(\frac{q-17}{4}\right)J.$$

As an example for $q = 53$ we obtain a $2 - \{53; 26, 39; 41\}$ sds, which gives

$$A_1 A_1^T + A_2 A_2^T = 97I + 9J.$$

Method V.

When $q = 1 + 4b^2 \equiv 5 \pmod{16}$ is a prime power, we consider the following $2 - \{q; \frac{q-1}{2}, \frac{3(q-1)}{4}, \frac{13q-33}{16}\}$ sds. In this case we have

$$A_1 A_1^T + A_2 A_2^T = \left(\frac{7q+17}{4}\right)I + \left(\frac{q-17}{4}\right)J.$$

As an example for $q = 37$ we obtain a $2 - \{37; 18, 27; 28\}$ sds, which gives

$$A_1 A_1^T + A_2 A_2^T = 69I + 5J.$$

In the present paper we get values for the pivots [17] of $2 - \{v; k_1, k_2; \lambda\}$ sds, and $(1, -1)$ matrices of order $2v$ made from them. Calculations have given moderate values of growth for such matrices. An open problem concerning the possibility of finding $(1, -1)$ $2v \times 2v$ CP matrices having growth greater than $2v$ is posed.

Notation 1. Write A for a matrix of order n whose initial pivots are derived from matrices with CP structure. Write $A(j)$ for the absolute value of the determinant of the $j \times j$ principal submatrix in the upper lefthand corner of the matrix A and $A[j]$ for the absolute value of the determinant of the $(n-j) \times (n-j)$ principal submatrix in the bottom righthand corner of the matrix A . Throughout this paper when we have used i pivots we then find all possible values of the $A(n-i)$ minors. Hence, if any minor is CP it must have one of these values. The magnitude of the pivots appearing after the application of GE operations on a CP matrix W is given by

$$p_j = A(j)/A(j-1), \quad j = 1, 2, \dots, n, \quad A(0) = 1. \quad (2)$$

In particular for a CP A , constructed from the \pm incidence matrices of 2-sds,

$$p_v = A(v)/A(v-1), \quad p_{v-1} = A(v-1)/A(v-2). \quad (3)$$

We use the notation M_j to denote the $j \times j$ minor of A .

2 Minors of Size $(2v - 1)$

We denote by $\Delta(h, i, j, k, m)$ the following matrix of order $2v$:

$$\Delta(h, i, j, k, m) = \left[\begin{array}{c|c|c|c} \overbrace{\begin{matrix} mI + (\Lambda - 1)(J - I) \\ (\Lambda + 1)J \\ -J \\ J \end{matrix}}^h & \overbrace{\begin{matrix} (\Lambda + 1)J \\ mI + (\Lambda - 1)(J - I) \\ J \\ -J \end{matrix}}^i & \overbrace{\begin{matrix} -J \\ J \\ mI + (\Lambda - 1)(J - I) \\ -J \end{matrix}}^j & \overbrace{\begin{matrix} J \\ -J \\ -J \\ mI + (\Lambda - 1)(J - I) \end{matrix}}^k \end{array} \right].$$

where $m = 2v = h + i + j + k$. Then by the Determinant Simplification Theorem [8]

$$\det \Delta(h, i, j, k, m) = (m-1)^{m-4} \begin{vmatrix} m + (h-1)(\Lambda-1) & (\Lambda+1)h & -h & h \\ (\Lambda+1)i & m + (i-1)(\Lambda-1) & i & -i \\ -j & j & m + (j-1)(\Lambda-1) & (\Lambda+1)j \\ k & -k & (\Lambda+1)k & m + (k-1)(\Lambda-1) \end{vmatrix}$$

To find the $(2v - 1) \times (2v - 1)$ minors we remove the first row and column of A to get D . The matrix DD^T is obtained from $\Delta(h, i, j, k, m)$ by removing a row and the corresponding column and performing on it appropriate permutations of rows and columns. Removing a row is equivalent by taking $m - 1$ rows and removing a column is equivalent by considering $h - 1$ or $i - 1$ or $j - 1$ or $k - 1$ columns. Thus $\det DD^T$ is $\det \Delta(h - 1, i, j, k, m - 1)$ or $\det \Delta(h, i - 1, j, k, m - 1)$ or $\det \Delta(h, i, j - 1, k, m - 1)$ or $\det \Delta(h, i, j, k - 1, m - 1)$.

In the sequel, we study the values of the $(2v - 1) \times (2v - 1)$ minors for the methods I-V mentioned in the Introduction.

Lemma 1 *The $(2v - 1) \times (2v - 1)$ minors of the matrix A given in (1), and constructed using Method II are:*

Proof. Here we use the $(1, -1)$ incidence matrices of the $2 - \{4t - 1; 2t - 1, 2t - 1; 2t - 1\}$ sds. PLEASE SPECIFY THESE VALUES

By the reasoning above, with $v = 4t - 1$, $h =$, $i =$, $j =$, $k =$, $m =$, substituted into $\det \Delta(h - 1, i, j, k, m - 1)$, $\det \Delta(h, i, j - 1, k, m - 1)$, $\det \Delta(h, i - 1, j, k, m - 1)$, and $\det \Delta(h, i, j, k - 1, m - 1)$ we obtain the result.

Specifically the $(2v - 1) \times (2v - 1)$ minor is the square root of the determinant and are given by one of

- 1) $\det \Delta(h - 1, i, j, k, m - 1) =$
- 2) $\det \Delta(h, i - 1, j, k, m - 1) =$
- 3) $\det \Delta(h, i, j - 1, k, m - 1) =$
- 4) $\det \Delta(h, i, j, k - 1, m - 1) =$

□

3 Minors of size $(2v - 2)$

We recall the partitioned matrix A of the design matrix, C , is composed from $2 - \{v; k_1, k_2; \lambda\}$ supplementary difference sets. Using the formula for the inner product of the rows of the $(1, -1)$ incidence matrix formed from these sds we see that the inner product is $2v - 4(k_1 + k_2 - \lambda) = \Lambda$.

We now return to A with two rows and columns removed to find the generic matrix. In expanded form this gives for C , simplified by the Determinant Simplification Theorem [8], the matrix D given by

$$\begin{bmatrix} N & \Lambda u_2 & \Lambda u_3 & (\Lambda + 2)u_4 & -2u_5 & 0 & 0 & 2u_8 \\ \Lambda u_1 & N & (\Lambda + 2)u_3 & \Lambda u_4 & 0 & -2u_6 & 2u_7 & 0 \\ \Lambda u_1 & (\Lambda + 2)u_2 & N & \Lambda u_4 & 0 & 2u_6 & -2u_7 & 0 \\ (\Lambda + 2)u_1 & \Lambda u_2 & \Lambda u_3 & N & 2u_5 & 0 & 0 & -2u_8 \\ -2u_1 & 0 & 0 & 2u_4 & N & 2u_6 & 2u_7 & (\Lambda + 2)u_8 \\ 0 & -2u_2 & 2u_3 & 0 & \Lambda u_5 & N & (\Lambda + 2)u_7 & \Lambda u_8 \\ 0 & 2u_2 & -2u_3 & 0 & \Lambda u_5 & (\Lambda + 2)u_6 & N & \Lambda u_8 \\ 2u_1 & 0 & 0 & -2u_4 & (\Lambda + 2)u_5 & \Lambda u_6 & \Lambda u_7 & N \end{bmatrix}.$$

where $N = 2v - 2u_i + (u_i - 1)\Lambda$ on the diagonal.

This gives the determinant of A with two rows and columns removed, as $(2v - \Lambda)^{v-5} \sqrt{\det D}$. Diagrammatically, we have used the matrix form

$$\begin{bmatrix} A_1 & A_2 \\ A_2^T & -A_1^T \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

For case I both rows and columns are removed from B_1 ; for case II one row is from B_1 and one from B_3 but both columns are from A_1 ; for case III one row is from B_1 and one from B_3 and one column is from B_1 and one column is from B_2 .

$\begin{array}{ccc} 1 & 1 & \\ \vdots & \vdots & \lambda_1 \\ 1 & 1 & \\ 1 & - & \\ \vdots & \vdots & k_1 - \lambda_1 \\ 1 & - & \\ - & 1 & \\ \vdots & \vdots & k_1 - \lambda_1 \\ - & 1 & \\ - & - & \\ \vdots & \vdots & v - 2k_1 + \lambda_1 \\ - & - & \end{array}$	$v - 2$ rows which have inner product Δ with rows one and two
$\begin{array}{ccc} 1 & 1 & \\ \vdots & \vdots & \lambda_2 \\ 1 & 1 & \\ 1 & - & \\ \vdots & \vdots & k_2 - \lambda_2 \\ 1 & - & \\ - & 1 & \\ \vdots & \vdots & k_2 - \lambda_2 \\ - & 1 & \\ - & - & \\ \vdots & \vdots & v - 2k_2 + \lambda_2 \\ - & - & \end{array}$	$\lambda = \lambda_1 + \lambda_2$ v rows which have inner product 0 with rows one and two

Number of Rows of Each Type I and II

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
$\lambda_1 - 2$	$k_1 - \lambda_1$	$k_1 - \lambda_1$	$v - 2k_1 + \lambda_1$	λ_2	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
$\lambda_1 - 1$	$k_1 - \lambda_1 - 1$	$k_1 - \lambda_1$	$v - 2k_1 + \lambda_1$	λ_2	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
$\lambda_1 - 1$	$k_1 - \lambda_1$	$k_1 - \lambda_1 - 2$	$v - 2k_1 + \lambda_1 - 1$	λ_2	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
$\lambda_1 - 1$	$k_1 - \lambda_1$	$k_1 - \lambda_1 - 2$	$v - 2k_1 + \lambda_1$	$\lambda_2 - 1$	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
$\lambda_1 - 1$	$k_1 - \lambda_1$	$k_1 - \lambda_1 - 2$	$v - 2k_1 + \lambda_1 - 1$	λ_2	$k_2 - \lambda_2 - 1$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
$\lambda_1 - 1$	$k_1 - \lambda_1 - 2$	$k_1 - \lambda_1 - 2$	$v - 2k_1 + \lambda_1 - 1$	λ_2	$k_2 - \lambda_1$	$k_2 - \lambda_1$	$v - 2k_2 + \lambda_2$
λ_1	$k_1 - \lambda_1 - 1$	$k_1 - \lambda_1 - 1$	$v - 2k_1 + \lambda_1$	$\lambda_2 - 1$	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
λ_1	$k_1 - \lambda_1 - 1$	$k_1 - \lambda_1$	$v - 2k_1 + \lambda_1 - 1$	$\lambda_2 - 1$	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
λ_1	$k_1 - \lambda_1 - 1$	$k_1 - \lambda_1$	$v - 2k_1 + \lambda_1$	$\lambda_2 - 1$	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
λ_1	$k_1 - \lambda_1 - 1$	$k_1 - \lambda_1 - 1$	$v - 2k_1 + \lambda_1$	$\lambda_2 - 1$	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
λ_1	$k_1 - \lambda_1 - 1$	$k_1 - \lambda_1 - 1$	$v - 2k_1 + \lambda_1$	$\lambda_2 - 1$	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
λ_1	$k_1 - \lambda_1 - 1$	$k_1 - \lambda_1 - 1$	$v - 2k_1 + \lambda_1$	$\lambda_2 - 1$	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$
λ_1	$k_1 - \lambda_1 - 1$	$k_1 - \lambda_1 - 1$	$v - 2k_1 + \lambda_1$	$\lambda_2 - 1$	$k_2 - \lambda_2$	$k_2 - \lambda_2$	$v - 2k_2 + \lambda_2$

Table 1

Case III

To help understand Case III we recall that in this case one column removed comes from the columns with $k_1 + k_2$ ones per column and the other from the columns with $v - k_2 + k_1$ ones per column in the original design. This means the generic form of these two columns is

Let A be an $2v \times 2v$ CP D -optimal design of KKSSS family which is constructed from $2 - \{s^2 + s + 1; \frac{s(s-1)}{2}, \frac{s(s+1)}{2}; \frac{s(s-1)}{2}\}$ sds. Reduce A by GE. Set $\mathcal{P} = 2s^2 + 2s + 1$. Then we conjecture

- (i) $g(v, A) = \frac{s+1}{s}\mathcal{P}$, or $\frac{s}{s+1}\mathcal{P}$, or $\frac{s(s+1)}{s^2+s+1}\mathcal{P}$, or \mathcal{P} ;
- (ii) The last pivot is equal to $\frac{s+1}{s}\mathcal{P}$, or $\frac{s}{s+1}\mathcal{P}$, or $\frac{s(s+1)}{s^2+s+1}\mathcal{P}$, or \mathcal{P} ;
- (iii) The second last pivot can take the values given in Table 8.
- (iv) Every pivot before the last has magnitude at most $2v$;
- (v) The first four pivots are equal to $1, 2, 2, 4$;
- (vi) The fifth pivot may be 2 or 3.

We prove (ii) and (iii) in this paper. (v) and (vi) were proved for Brouwer's SBIBD($2s^2 + 2s + 1, s^2, \frac{1}{2}s(s-1)$) in [7] and we also show they hold for the KKSSS family.

v	growth	Pivot Pattern
7	8	(1, 2, 2, 4, 2, 4, 4, 2, 4, 4, 8, 4, 8, 8)
7	8	(1, 2, 2, 4, 3, $\frac{10}{3}$, 3.2, 4, 4, 4, 4, 4, 8, 8)
22	12	(1, 2, 2, 4, 3, $\frac{10}{3}$, $\frac{18}{5}$, 4, 3, 6, 6, 2, 4, 4, 8, 6, $\frac{20}{3}$, 7, 2, 8, 6, 12, 12)

Table 8: Growth Factors and Pivots Patterns for small CP KKSSS designs

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