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# Curvature contraction of convex hypersurfaces by nonsmooth speeds

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## **Abstract**

We consider contraction of convex hypersurfaces by convex speeds, homogeneous of degree one in the principal curvatures, that are not necessarily smooth. We show how to approximate such a speed by a sequence of smooth speeds for which behaviour is well known. By obtaining speed and curvature pinching estimates for the flows by the approximating speeds, independent of the smoothing parameter, we may pass to the limit to deduce that the flow by the nonsmooth speed converges to a point in finite time that, under a suitable rescaling, is round in the  $C^2$  sense, with the convergence being exponential.

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## **Authors**

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# Curvature contraction of convex hypersurfaces by nonsmooth speeds

By *Ben Andrews* at Canberra, *Andrew Holder* at Wollongong, *James McCoy* at Wollongong,  
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**Abstract.** We consider contraction of convex hypersurfaces by convex speeds, homogeneous of degree one in the principal curvatures, that are not necessarily smooth. We show how to approximate such a speed by a sequence of smooth speeds for which behaviour is well known. By obtaining speed and curvature pinching estimates for the flows by the approximating speeds, independent of the smoothing parameter, we may pass to the limit to deduce that the flow by the nonsmooth speed converges to a point in finite time that, under a suitable rescaling, is round in the  $C^2$  sense, with the convergence being exponential.

## 1. Introduction

Let  $M_0$  be a compact, strictly convex hypersurface of dimension  $n \geq 2$ , without boundary, embedded in  $\mathbb{R}^{n+1}$  and represented by  $X_0 : S^n \rightarrow X_0(S^n) = M_0 \subset \mathbb{R}^{n+1}$ . In this paper we require  $M_0$  to be of class  $C^2$ , that is,  $M_0$  can be represented locally as the graph of a  $C^2$  function  $u$ . We consider the family of maps  $X_t = X(\cdot, t)$  evolving according to

$$(1.1) \quad \begin{aligned} \frac{\partial}{\partial t} X(x, t) &= -F(\mathcal{W}(x, t))\nu(x, t), \quad x \in S^n, \quad 0 < t \leq T < \infty, \\ X(\cdot, 0) &= X_0, \end{aligned}$$

where  $\mathcal{W}(x, t)$  is the matrix of the Weingarten map of  $M_t = X_t(S^n)$  at the point  $X_t(x)$  and  $\nu(x, t)$  is the outer unit normal to  $M_t$  at  $X_t(x)$ .

**Conditions 1.1.** The function  $F$  is assumed to satisfy:

- (i)  $F(\mathcal{W}) = f(\kappa(\mathcal{W}))$ , where  $\kappa(\mathcal{W})$  gives the eigenvalues of  $\mathcal{W}$  and  $f$  is a symmetric function defined on the positive cone

$$\Gamma^+ = \{\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n : \kappa_i > 0 \text{ for all } i = 1, 2, \dots, n\},$$

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- (ii)  $f$  is strictly increasing in each argument: at every point  $\kappa \in \Gamma^+$ , for each  $i = 1, \dots, n$  and every  $\delta > 0$ ,

$$f(\kappa + \delta e_i) > f(\kappa),$$

where  $e_i$  is the standard  $i$ th basis direction in  $\Gamma^+ \subset \mathbb{R}^n$ ,

- (iii)  $f$  is homogeneous of degree one:  $f(k\kappa) = kf(\kappa)$  for any  $k > 0$ ,  
 (iv)  $f$  is strictly positive on  $\Gamma^+$  and  $f(1, \dots, 1) = 1$ ,  
 (v)  $f$  is convex.

**Remarks.** (1) Since  $f$  is convex,  $f$  is locally Lipschitz continuous. There are many proofs of this; we refer the reader to [19], for example. Further,  $f$  is almost everywhere twice differentiable [1].

(2) If  $f$  is actually smooth, relations between the functions  $F$  and  $f$  and their derivatives are reasonably well known (see, for examples [3, 11, 20]). In the case that  $F$  is only continuous, the relationship between  $F$  and  $f$  in (i) follows from results of Ball [7], and both functions are continuous.

Following Huisken's celebrated result for the mean curvature flow [14], convex hypersurfaces flowing by classes of smooth  $F$  with properties similar to the above have been considered before [2, 3, 5, 6, 13]. The difference here is that we assume no smoothness of  $f$  other than that implied by convexity. Consequently, Condition 1.1 (ii) above is written for a continuous function  $f$  that is not necessarily differentiable. We remark that Condition 1.1 (v) does not cover all cases considered in previous work with smooth speeds. Convexity is used here in various places; perhaps the most critical consequence is that the evolution of the Weingarten map permits the application of the maximum principle for tensors only when the speed is convex.

Nonsmooth speed functions are of interest in image processing applications of curvature flow (see, for example, [17]). To construct a simple example for surfaces, that satisfies Conditions 1.1 and is not smooth, write

$$\kappa_{\max} = \frac{1}{2}(\kappa_1 + \kappa_2 + |\kappa_1 - \kappa_2|) \quad \text{and} \quad \kappa_{\min} = \frac{1}{2}(\kappa_1 + \kappa_2 - |\kappa_1 - \kappa_2|),$$

and set

$$(1.2) \quad f(\kappa) = \alpha \kappa_{\min} + \beta \kappa_{\max},$$

where  $\alpha, \beta \in (0, 1)$ ,  $\alpha + \beta = 1$ ,  $\alpha \leq \beta$ . When  $\alpha = \beta = \frac{1}{2}$ , this corresponds to mean curvature flow but otherwise,  $f$  is not smooth due to the presence of the absolute value.

As a generalisation for higher dimensions, let  $C$  be a positive definite symmetric matrix with trace  $C = 1$ . Arrange the eigenvalues of  $C$  in increasing order  $(c_1, \dots, c_n)$ . Then

$$f(x) = \sum_i c_i x_i,$$

where  $x_i$  are also arranged in increasing order, is also not differentiable while satisfying Conditions 1.1.

Bellmann-type operators, of interest from a control theory perspective, naturally generalise these weighted sums to a large class. Let  $\mathcal{C}$  be a compact subset of the space of positive

definite symmetric matrices  $C$  with trace  $C = 1$  for  $C \in \mathcal{C}$ . Set

$$F(A) = \sup_{C \in \mathcal{C}} \sup_{\mathbb{O} \in O(n)} \text{trace}((\mathbb{O}^T C \mathbb{O})A).$$

This is by construction homogeneous of degree one, symmetric, convex, and satisfies the required monotonicity conditions. When written as a function of the eigenvalues this becomes

$$f(x) = \sup_{c \in \mathcal{A}} \sum_i c_i x_i,$$

where  $\mathcal{A}$  is the set of  $n$ -tuples  $(c_1, \dots, c_n)$  of eigenvalues of elements of  $\mathcal{C}$  (arranged in increasing order) and  $x = (x_1, \dots, x_n)$  is also arranged in increasing order. This is in general nonsmooth.

Other nonsmooth speed functions may be obtained by taking suitable maxima of convex functions, for example

$$f(\kappa) = \max\left(\frac{H}{n}, a|A|\right), \quad \frac{1}{n} < a < \frac{1}{\sqrt{n}}.$$

Our main result is the following:

**Theorem 1.2.** *Let  $M_0$  be a strictly convex hypersurface of class  $C^2$  and suppose  $F$  satisfies Conditions 1.1. Then the evolution equation (1.1) has a  $C^{2,\alpha}$  solution  $M_t$  on a finite maximal time interval  $0 \leq t < T$ . The hypersurfaces  $M_t$  converge to a point  $p \in \mathbb{R}^n$  as  $t \rightarrow T$ . The rescaled hypersurfaces  $\tilde{M}_t$  given by*

$$(1.3) \quad \tilde{X}(x, t) = \frac{X(x, t) - p}{\sqrt{2(T-t)}}$$

*converge in the  $C^{2,\alpha'}$  topology as  $t \rightarrow T$  to an embedding  $\tilde{X}(\cdot, T)$  whose image is equal to the unit sphere centred at the origin, where  $0 < \alpha' < \alpha$ . The convergence of the rescaled curvatures to 1 is exponential with respect to the natural time parameter. If the speed  $F$  is more regular, then the solution and its exponential convergence to the sphere is correspondingly more regular.*

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## 2. Notation

We will use similar notation as in [3, 6, 14]. In particular,  $g = \{g_{ij}\}$ ,  $A = \{h_{ij}\}$  and  $\mathcal{W} = \{h^i_j\}$  denote respectively the metric, second fundamental form and Weingarten map of  $M_t$ . The mean curvature of  $M_t$  is

$$H = g^{ij} h_{ij} = h^i_i$$

and the norm of the second fundamental form is

$$|A|^2 = g^{ij} g^{lm} h_{il} h_{jm} = h^j_l h^l_j,$$

where  $g^{ij}$  is the  $(i, j)$ -entry of the inverse of the matrix  $(g_{ij})$ . Throughout this paper we sum over repeated indices from 1 to  $n$  unless otherwise indicated. Raised indices indicate contraction with the metric.

In dealing with a nonsmooth speed function  $F$  we will introduce smooth approximating speeds which we will denote by  $F^\varepsilon$ . When we are dealing with quantities associated with the approximating flows we will also indicate this with a super- or sub-script  $\varepsilon$ . Similarly, a tilde  $\sim$  will be used to denote quantities associated with the rescaled flows (1.3).

We will denote by  $(\dot{F}_\varepsilon^{kl})$  the matrix of first partial derivatives of  $F^\varepsilon$  with respect to the components of its argument:

$$\left. \frac{\partial}{\partial s} F^\varepsilon(A + sB) \right|_{s=0} = \dot{F}_\varepsilon^{kl}(A) B_{kl}.$$

Similarly for the second partial derivatives of  $F^\varepsilon$  we write

$$\left. \frac{\partial^2}{\partial s^2} F^\varepsilon(A + sB) \right|_{s=0} = \ddot{F}_\varepsilon^{kl,rs}(A) B_{kl} B_{rs}.$$

We will also use the notation

$$f_\varepsilon^i(\kappa^\varepsilon) = \frac{\partial f^\varepsilon}{\partial \kappa_i}(\kappa^\varepsilon) \quad \text{and} \quad \ddot{f}_\varepsilon^{ij}(\kappa^\varepsilon) = \frac{\partial^2 f^\varepsilon}{\partial \kappa_i \partial \kappa_j}(\kappa^\varepsilon),$$

where  $\kappa^\varepsilon$  denotes the curvatures associated with  $M_t^\varepsilon$ , the hypersurface evolving with normal speed  $F^\varepsilon$ . Unless otherwise indicated, throughout this paper we will always evaluate partial derivatives of  $F^\varepsilon$  at  $\mathcal{W}^\varepsilon$  and partial derivatives of  $f^\varepsilon$  at  $\kappa(\mathcal{W}^\varepsilon)$  where  $\mathcal{W}^\varepsilon$  is the Weingarten map associated with  $M_t^\varepsilon$ .

Several function spaces and associated norms on  $\mathbb{S}^n$  and on  $\mathbb{S}^n \times [0, T)$  will be needed. These are as used, for example, by Urbas in [24, 25] and by the third author in [18]. For  $k \in \mathbb{N}$ ,  $C^k(\mathbb{S}^n)$  is the Banach space of real valued functions on  $\mathbb{S}^n$  that are  $k$ -times continuously differentiable, equipped with the norm

$$\|u\|_{C^k(\mathbb{S}^n)} = \sum_{|\beta| \leq k} \sup_{\mathbb{S}^n} |\bar{\nabla}^\beta u|.$$

Here  $\beta$  is a standard multi-index for partial derivatives and  $\bar{\nabla}$  is the derivative on  $\mathbb{S}^n$ . We further define, for  $\alpha \in (0, 1]$ ,  $C^{k,\alpha}$  to be the space of functions  $u \in C^k(\mathbb{S}^n)$  such that the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{S}^n)} = \|u\|_{C^k(\mathbb{S}^n)} + \sup_{\substack{|\beta|=k \\ x,y \in \mathbb{S}^n \\ x \neq y}} \frac{|\bar{\nabla}^\beta u(x) - \bar{\nabla}^\beta u(y)|}{|x - y|^\alpha}$$

is finite. Here  $|x - y|$  is the distance between  $x$  and  $y$  in  $\mathbb{S}^n$ .

On the space-time  $\mathbb{S}^n \times I$ ,  $I = [a, b] \subset \mathbb{R}$ , we denote by  $C^k(\mathbb{S}^n \times I)$  the space of real valued functions  $u$  that are  $k$ -times continuously differentiable with respect to  $x$  and  $\lfloor \frac{k}{2} \rfloor$ -times continuously differentiable with respect to  $t$  such that the norm

$$\|u\|_{C^k(\mathbb{S}^n \times I)} = \sum_{|\beta| + 2r \leq k} \sup_{\mathbb{S}^n \times I} |\bar{\nabla}^\beta D_t^r u|$$

is finite. Here  $\lfloor \frac{k}{2} \rfloor$  is the largest integer not greater than  $\frac{k}{2}$ . We also denote by  $C^{k,\alpha}(\mathbb{S}^n \times I)$  the space of functions in  $C^k(\mathbb{S}^n \times I)$  such that the norm

$$\|u\|_{C^{k,\alpha}(\mathbb{S}^n \times I)} = \|u\|_{C^k(\mathbb{S}^n \times I)} + \sup_{|\beta|+2r=k} \sup_{\substack{(x,s),(y,t) \in \mathbb{S}^n \times I \\ (x,s) \neq (y,t)}} \frac{|\bar{\nabla}^\beta D_t^r u(x,s) - \bar{\nabla}^\beta D_t^r u(y,t)|}{(|x-y|^2 + |s-t|)^{\frac{\alpha}{2}}}$$

is finite.

### 3. Approximating speeds

We wish to construct a family of smooth functions  $f^\varepsilon$ , each of which satisfy Conditions 1.1. For general  $f$  locally integrable on  $\Gamma^+$  we may use mollifiers, as discussed, for example, in [10]. We summarise the main ideas here for the convenience of the reader.

For two functions  $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$  we define the *convolution* of  $f$  and  $g$  by

$$f \star g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

There are various classes of functions for which the above is well defined. If, for example,  $f \in L^p(\mathbb{R}^n)$  and  $g \in L^q(\mathbb{R}^n)$  where  $\frac{1}{p} + \frac{1}{q} = 1$ , then the integral is bounded, by Hölder's inequality.

A change of variables in the above integral shows

$$(3.1) \quad f \star g(x) = g \star f(x).$$

Let  $j$  be a non-negative function in  $C^\infty(\mathbb{R}^n)$  that vanishes outside the unit ball  $B_1(0)$  and satisfies  $\int_{\mathbb{R}^n} j(x)dx = 1$ . One suitable function  $j$  is

$$j(x) = \begin{cases} c_n e^{|x|^2-1} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where  $c_n$  is a normalisation constant such that  $\int_{\mathbb{R}^n} j(x)dx = 1$ . For  $\varepsilon > 0$  we now define

$$(3.2) \quad j_\varepsilon(x) = \varepsilon^{-n} j\left(\frac{x}{\varepsilon}\right).$$

Clearly,  $j_\varepsilon$  has support  $B_\varepsilon(0)$  and it is straightforward to check that  $\int_{\mathbb{R}^n} j_\varepsilon(x)dx = 1$ .

We will use the following steps to construct a sequence of smooth functions approximating the nonsmooth speed function  $f$ .

- (1) Set  $\tilde{f}_\varepsilon = j_\varepsilon \star f$ , where  $j_\varepsilon$  is given by (3.2).
- (2) Set  $g_\varepsilon(x) = \tilde{f}_\varepsilon\left(\frac{x}{H}\right)$ , where  $H = x_1 + x_2 + \dots + x_n > 0$ .
- (3) Set  $\hat{f}_\varepsilon(x) = Hg_\varepsilon(x)$ .
- (4) Set  $f_\varepsilon(x) = \frac{\hat{f}_\varepsilon(x)}{\hat{f}_\varepsilon(1,\dots,1)}$ .

Note that since the initial data for our flow problem is strictly convex, we can take  $\varepsilon \leq \varepsilon_0$  small enough such that in the above process, we do not evaluate  $f$  outside  $\Gamma^+$ . In Section 5 we will quantify this in terms of a curvature condition on the initial hypersurface, specifically, see

Lemmas 5.2 and 5.3. However, for the purposes of showing monotonicity of the  $f_\varepsilon$  it will be useful to have  $f$  extended to be convex on  $\mathbb{R}^n$ . This can be done using the convex Bellman extension (see, for example, [18]).

**Remarks.** (1) We have used the above process to ensure that each  $f_\varepsilon$  is homogeneous of degree one. Together with Conditions 1.1 this means that for each  $f_\varepsilon$  as a flow speed, we already know that arbitrary smooth, convex hypersurfaces  $M_0$  flow in a finite maximal time to round points [2]. In particular, a curvature pinching condition on  $M_0$  is not required since the  $f_\varepsilon$  are homogeneous of degree one.

(2) From the above process, we can write explicitly

$$(3.3) \quad \hat{f}_\varepsilon(x) = H \int_{\mathbb{R}^n} j_\varepsilon(y) f\left(\frac{x}{H} - y\right) dy = \int_{\mathbb{R}^n} j_\varepsilon(y) f(x - Hy) dy.$$

In the proof of Theorem 3.1 below we will see that  $1 - n\varepsilon \leq \hat{f}_\varepsilon(1, \dots, 1) < 1 + n\varepsilon$ , ensuring that the division in step 4 above makes sense.

(3) In other applications when working on cones larger than  $\Gamma^+$ , it could be beneficial to use a different degree one homogeneous function in place of  $H$  in steps 3 and 4 above. For example, if working in a cone larger than the  $\{H > 0\}$  half space, the function

$$|A| = \sqrt{x_1^2 + \dots + x_n^2}$$

could be more appropriate.

**Theorem 3.1.** *Given a function  $f$  satisfying Conditions 1.1, the  $f_\varepsilon$  constructed by the above process satisfy the following:*

(1) *For each  $\varepsilon \in (0, \min(\frac{1}{n}, \varepsilon_0))$ , the function  $f_\varepsilon$  satisfies the same statements from Conditions 1.1 as the function  $f$ . In particular, division by  $\hat{f}_\varepsilon(1, \dots, 1)$  is well-posed, with*

$$(3.4) \quad 1 - n\varepsilon \leq \hat{f}_\varepsilon(1, \dots, 1) < 1 + n\varepsilon.$$

(2) *For each  $\varepsilon > 0$ , the function  $f_\varepsilon$  is smooth.*

(3)  *$f_\varepsilon \rightarrow f$  uniformly on compact subsets of  $\mathbb{R}^n$ .*

(4)  *$f(x) - \varepsilon H \leq f_\varepsilon(x) \leq f(x) + \varepsilon H$  where, as before,  $H = \sum_i x_i$ .*

*Proof.* (1) (i) Since  $f$  is a symmetric function of the eigenvalues  $\kappa$  of the Weingarten map, so is  $f_\varepsilon$ .

(ii) Fix  $x \in \Gamma^+$  and  $i \in \{1, 2, \dots, n\}$ . We wish to show that for any  $\delta > 0$ ,

$$f_\varepsilon(x + \delta e_i) > f_\varepsilon(x).$$

Using (3.3), we calculate using a supporting hyperplane for  $f$  at  $x$

$$\begin{aligned} f_\varepsilon(x + \delta e_i) - f_\varepsilon(x) &= \int_{\mathbb{R}^n} j_\varepsilon(y) [f(x - Hy + \delta(e_i - y)) - f(x - Hy)] dy \\ &\geq \int_{\mathbb{R}^n} j_\varepsilon(y) \delta \langle r(x), e_i - y \rangle dy \\ &= \delta \langle r(x), e_i \rangle. \end{aligned}$$



In the case that  $f$  is differentiable at  $x$ ,  $r(x) = Df(x)$ . More generally, since  $f$  is increasing in each variable, the above inner product is positive. The other term disappears as the integral of an odd function over  $\mathbb{R}^n$ .

(iii) For each  $\varepsilon > 0$ ,  $f_\varepsilon$  is clearly homogeneous of degree one by construction.

(iv) Clearly  $\hat{f}_\varepsilon > 0$  on  $\Gamma^+$  since  $j_\varepsilon \geq 0$  and  $f > 0$  on  $\Gamma^+$ . We compute

$$\begin{aligned} \hat{f}_\varepsilon(1, \dots, 1) &= n \tilde{f}_\varepsilon\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \\ &= n \int_{|(\frac{1}{n}, \dots, \frac{1}{n}) - y| \leq \varepsilon} j_\varepsilon\left(\frac{1}{n} - y_1, \dots, \frac{1}{n} - y_n\right) f(y) dy. \end{aligned}$$

Now,  $f$  attains a minimum and a maximum on the ball  $|(\frac{1}{n}, \dots, \frac{1}{n}) - y| \leq \varepsilon$ , depending on the particular  $f$ . However, enclosing the ball with a cube and using the monotonicity and degree one homogeneity of  $f$  we can estimate everywhere on the ball

$$\frac{1}{n} - \varepsilon < f(y) < \frac{1}{n} + \varepsilon.$$

Consequently,

$$1 - n \varepsilon < \hat{f}_\varepsilon(1, \dots, 1) < 1 + n \varepsilon$$

and so for  $\varepsilon < \min(\frac{1}{n}, \varepsilon_0)$  we have

$$f_\varepsilon(1, \dots, 1) = \frac{\hat{f}_\varepsilon(1, \dots, 1)}{\hat{f}_\varepsilon(1, \dots, 1)} = 1,$$

as required.

(v) Let  $a, b \in \Gamma^+$  and write  $A = \sum_i a_i$  and  $B = \sum_i b_i$ . Since  $f$  is convex, for  $\lambda \in [0, 1]$  we compute using the definition of  $\tilde{f}_\varepsilon$  and the homogeneity and convexity of  $f$

$$\begin{aligned} f_\varepsilon(\lambda a + (1 - \lambda)b) &= \left[ \sum_i (\lambda a_i + (1 - \lambda)b_i) \right] \tilde{f}_\varepsilon\left(\frac{\lambda a + (1 - \lambda)b}{\sum_i (\lambda a_i + (1 - \lambda)b_i)}\right) \\ &= \left[ \sum_i (\lambda a_i + (1 - \lambda)b_i) \right] \int_{\mathbb{R}^n} j_\varepsilon(y) f\left(\frac{\lambda a + (1 - \lambda)b}{\sum_i (\lambda a_i + (1 - \lambda)b_i)} - y\right) dy \\ &= \int_{\mathbb{R}^n} j_\varepsilon(y) f\left(\lambda a + (1 - \lambda)b - \sum_i [\lambda a_i + (1 - \lambda)b_i] y\right) dy \\ &= \int_{\mathbb{R}^n} j_\varepsilon(y) f(\lambda[a - Ay] + (1 - \lambda)[b - By]) dy \\ &\leq \int_{\mathbb{R}^n} j_\varepsilon(y) [\lambda f(a - Ay) + (1 - \lambda) f(b - By)] dy \\ &= \int_{\mathbb{R}^n} j_\varepsilon(y) \left[ \lambda A f\left(\frac{a}{A} - y\right) + (1 - \lambda) B f\left(\frac{b}{B} - y\right) \right] dy \\ &= \lambda A \tilde{f}_\varepsilon\left(\frac{a}{A}\right) + (1 - \lambda) B \tilde{f}_\varepsilon\left(\frac{b}{B}\right) \\ &= \lambda f_\varepsilon(a) + (1 - \lambda) f_\varepsilon(b) \end{aligned}$$

as required.

(2) This is a standard property of mollification when the mollifier  $j$  is a smooth function.

(3) This follows by an argument similar to that in the proof of [10, Lemma 7.1] using, in particular, since  $f$  is homogeneous of degree one,

$$f(x) = Hf\left(\frac{x}{H}\right).$$

(4) Using (3.3) and (3.1) we have for any  $x \in \Gamma^+$ ,

$$f_\varepsilon(x) = H \int_{|\frac{x}{H}-y|\leq\varepsilon} j_\varepsilon\left(\frac{x}{H}-y\right) f(y) dy$$

so

$$H \min_{y \in B_\varepsilon(\frac{x}{H})} f(y) \leq f_\varepsilon(x) \leq H \max_{y \in B_\varepsilon(\frac{x}{H})} f(y).$$

Replacing  $B_\varepsilon(\frac{x}{H})$  by a cube and using the monotonicity of  $f$  we can in turn estimate

$$Hf\left(\frac{x}{H} - \varepsilon(1, \dots, 1)\right) \leq f_\varepsilon(x) \leq Hf\left(\frac{x}{H} + \varepsilon(1, \dots, 1)\right)$$

which, in view of homogeneity of  $f$ , becomes  $f_\varepsilon(x) \leq f(x + \varepsilon H(1, \dots, 1))$ . The result follows using the normalisation  $f(1, \dots, 1) = 1$  and the lemma to follow.  $\square$

Our next result is an easy consequence of convexity and homogeneity. Because it is straightforward, we provide it for general homogeneity of degree  $\alpha$ .

**Lemma 3.2.** *Let  $x, y \in \mathbb{R}^n$ . Suppose  $f$  is convex and homogeneous of degree  $\alpha$ . Then for any  $t > 0$ ,*

$$f(x + ty) \leq (t + 1)^{\alpha-1} [f(x) + tf(y)]$$

and, provided  $x - ty \in \Gamma^+$ ,

$$(3.5) \quad f(x - ty) \geq (t + 1)^{1-\alpha} f(x) - tf(y).$$

*Proof.* Since  $f$  is convex, we have for any  $\lambda \in [0, 1]$  and any  $k > 0$ ,

$$k^\alpha f(\lambda x + (1 - \lambda)y) \leq k^\alpha \lambda f(x) + k^\alpha (1 - \lambda) f(y).$$

Taking  $k = \frac{1}{\lambda}$  and using the homogeneity of  $f$  on the left, this becomes

$$f\left(x + \left(\frac{1}{\lambda} - 1\right)y\right) \leq \lambda^{1-\alpha} f(x) + \lambda^{-\alpha} (1 - \lambda) f(y).$$

Setting  $t = \frac{1}{\lambda} - 1$  gives the first statement, and replacing  $x$  by  $x - ty$  gives the second statement.  $\square$

#### 4. Evolution equations

Using the method of Section 3 we approximate the speed function  $F$  in (1.1) by a family of smooth functions  $f_\varepsilon$ , where  $0 < \varepsilon < \varepsilon_0$  and  $\varepsilon_0$  is chosen small enough, depending on  $M_0$ , such that the argument of  $f$  in the integrand of the convolution (3.3) remains within  $\Gamma^+$ . We will label all quantities associated with the flows by the approximating speeds with an  $\varepsilon$ . We write  $F^\varepsilon(\mathcal{W}^\varepsilon) = f^\varepsilon(\kappa(\mathcal{W}^\varepsilon))$ .

For each  $\varepsilon \in (0, \min(\varepsilon_0, \frac{1}{n}))$  we have a flow,

$$(4.1) \quad \begin{aligned} \frac{\partial}{\partial t} X^\varepsilon(x, t) &= -F^\varepsilon(\mathcal{W}^\varepsilon(x, t))\nu^\varepsilon(x, t), \quad x \in \mathbb{S}^n, \quad 0 < t \leq T_\varepsilon < \infty, \\ X(\cdot, 0) &= X_0. \end{aligned}$$

Note that we use the same initial data for each flow. The speed of each flow is smooth and itself satisfies Conditions 1.1, by Theorem 3.1. We label each evolving hypersurface  $M_t^\varepsilon$ . It is known that each  $M_t^\varepsilon$  shrinks to a round point in finite time [2]; here we want to establish estimates independent of  $\varepsilon$  to deduce the behaviour of the ‘limit flow’ (1.1).

We have the following evolution equations for various geometric quantities associated with  $M_t^\varepsilon$ . These equations are easily derived, similarly as in [14] and [2]. We use  $\nabla^\varepsilon$  to denote the gradient on the evolving hypersurface  $M_t^\varepsilon$ .

**Lemma 4.1.** *Under the flow (4.1),*

$$\begin{aligned} \text{(i)} \quad & \frac{\partial}{\partial t} g_{ij}^\varepsilon = -2F^\varepsilon h_{ij}^\varepsilon, \\ \text{(ii)} \quad & \frac{\partial}{\partial t} g_\varepsilon^{ij} = 2F h_\varepsilon^{ij}, \\ \text{(iii)} \quad & \frac{\partial}{\partial t} \mu^\varepsilon = -F^\varepsilon H^\varepsilon \mu^\varepsilon, \\ \text{(iv)} \quad & \frac{\partial}{\partial t} \nu^\varepsilon = \nabla^\varepsilon F^\varepsilon, \\ \text{(v)} \quad & \frac{\partial}{\partial t} h_{ij}^\varepsilon = \nabla_i^\varepsilon \nabla_j^\varepsilon F^\varepsilon - F^\varepsilon h_{im}^\varepsilon h_\varepsilon^m{}_j, \\ \text{(vi)} \quad & \frac{\partial}{\partial t} h_\varepsilon^i{}_j = \nabla_\varepsilon^i \nabla_j^\varepsilon F^\varepsilon + F^\varepsilon h_\varepsilon^i{}_m h^m{}_j. \end{aligned}$$

**Remark.** Notice that the evolution equations (i), (ii) and (iii) above do not involve differentiating the speed. Such equations continue to hold for the ‘limit flow’ (1.1). Since  $F$  is almost everywhere twice differentiable, the remaining equations hold weakly for the limit flow, but we will not use them.

We will denote by  $\mathcal{L}^\varepsilon$  the operator given by  $\mathcal{L}^\varepsilon \psi = \dot{F}_\varepsilon^{kl} \nabla_k^\varepsilon \nabla_l^\varepsilon \psi$ . Condition 1.1(ii) ensures  $\mathcal{L}^\varepsilon$  is an elliptic operator.

The following evolution equations are also easily computed, similarly as, for example, in [2]. We use the Codazzi equations, the Gauss equations and interchange second covariant derivatives in parts (ii) and (iii).

**Lemma 4.2.** *Under the flow (4.1),*

$$\begin{aligned} \text{(i)} \quad & \frac{\partial}{\partial t} F^\varepsilon = \mathcal{L}^\varepsilon F^\varepsilon + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_\varepsilon^m{}_l F^\varepsilon, \\ \text{(ii)} \quad & \frac{\partial}{\partial t} h_{ij}^\varepsilon = \mathcal{L}^\varepsilon h_{ij}^\varepsilon + \ddot{F}_\varepsilon^{kl,rs} \nabla_i^\varepsilon h_{kl}^\varepsilon \nabla_j^\varepsilon h_{rs}^\varepsilon + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_\varepsilon^m{}_l h_{ij}^\varepsilon - 2F^\varepsilon h_{im}^\varepsilon h_\varepsilon^m{}_j, \\ \text{(iii)} \quad & \frac{\partial}{\partial t} h_\varepsilon^i{}_j = \mathcal{L}^\varepsilon h_\varepsilon^i{}_j + \ddot{F}_\varepsilon^{kl,rs} \nabla_\varepsilon^i h_{kl}^\varepsilon \nabla_j^\varepsilon h_{rs}^\varepsilon + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_\varepsilon^m{}_l h_\varepsilon^i{}_j, \\ \text{(iv)} \quad & \frac{\partial}{\partial t} H^\varepsilon = \mathcal{L}^\varepsilon H^\varepsilon + \ddot{F}_\varepsilon^{kl,rs} \nabla_\varepsilon^i h_{kl}^\varepsilon \nabla_i^\varepsilon h_{rs}^\varepsilon + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_\varepsilon^m{}_l H^\varepsilon. \end{aligned}$$

We will also need to use the support function of  $M_t^\varepsilon$ ,  $s^\varepsilon : \mathbb{S}^n \times [0, T_\varepsilon) \rightarrow \mathbb{R}$ , given by

$$s^\varepsilon(x, t) = \langle X^\varepsilon(x, t), \nu^\varepsilon(x, t) \rangle.$$

The support function gives the perpendicular distance to the origin of the tangent plane to  $M_t^\varepsilon$  at  $X^\varepsilon(x, t)$ . Observe that  $s^\varepsilon(x, 0) = s_0$ , the support function of  $M_0$ , is independent of  $\varepsilon$ . We complete this section with the evolution equation for  $s^\varepsilon$ .

**Lemma 4.3.** *Under the flow (1.1) with speed  $F^\varepsilon$ , the support function of  $M_t^\varepsilon$  evolves according to*

$$(4.2) \quad \frac{\partial}{\partial t} s^\varepsilon = \mathcal{L}^\varepsilon s^\varepsilon + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_{\varepsilon l}^m s^\varepsilon - 2F^\varepsilon.$$

The proof of Lemma 4.3 is a calculation similar to the proof of [18, Lemma 4.5].

### 5. A priori estimates independent of $\varepsilon$

First we obtain estimates on the maximal time  $T$  of solutions to (1.1). These estimates depend only on the inner and outer radii of the initial hypersurface  $M_0$ .

**Lemma 5.1.** *The maximal time  $T_\varepsilon$  of existence of a solution to (4.1) satisfies*

$$\frac{\rho_-^2}{2} \leq T_\varepsilon \leq \frac{\rho_+^2}{2},$$

where  $\rho_-$  and  $\rho_+$  denote the inradius and outradius respectively of the initial hypersurface  $M_0$ .

*Proof.* We will work with the solutions to (4.1) as at this stage we have not yet shown that the curvatures remain within a compact region of  $\Gamma$  under (1.1), and we have only that  $f$  is locally Lipschitz.

The radius  $r^\varepsilon(t)$  of a sphere evolving under (4.1) satisfies

$$\frac{d}{dt} r^\varepsilon(t) = -f^\varepsilon\left(\frac{1}{r^\varepsilon}, \dots, \frac{1}{r^\varepsilon}\right) = -f^\varepsilon(1, \dots, 1) \frac{1}{r^\varepsilon} = \frac{1}{r^\varepsilon},$$

where we have used that the  $f^\varepsilon$  are normalised. With condition  $r^\varepsilon(t_0) = r_0$  (independent of  $\varepsilon$ ), the above ordinary differential equation has solution

$$r^\varepsilon(t) = \sqrt{r_0^2 - 2(t - t_0)}.$$

The sphere shrinks to a point at time  $t = t_0 + \frac{r_0^2}{2}$ . Our initial convex hypersurface  $M_0$  encloses a ball  $B_{\rho_-}$  and is enclosed by a ball  $B_{\rho_+}$ , so that using the comparison principle as in [2, proof of Theorem 6.2], the maximal time of existence  $T_\varepsilon$  of solutions to (4.1) satisfies

$$\frac{\rho_-^2}{2} \leq T_\varepsilon \leq \frac{\rho_+^2}{2}. \quad \square$$

Next we use the maximum principle to obtain absolute lower bounds on the mean curvature and the speed, depending only on  $n$  and  $M_0$ .

**Lemma 5.2.** *Under the flows (4.1), the mean curvature  $H^\varepsilon$  and the speed  $F^\varepsilon$  are bounded below, by constants depending only on  $n$  and  $M_0$ .*

*Proof.* Since the  $F^\varepsilon$  are convex, applying the maximum principle to Lemma 4.2 (iv) shows that the mean curvature of  $M_t^\varepsilon$  satisfies

$$H^\varepsilon(x, t) \geq \min_{M_0} H,$$

since in suitable coordinates at an assumed minimum,

$$\dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_{\varepsilon l}^m = \frac{\partial f^\varepsilon}{\partial \kappa_i} (\kappa_i^\varepsilon)^2 \geq 0.$$

For any convex  $F^\varepsilon$  satisfying Conditions 1.1 we have

$$(5.1) \quad F^\varepsilon \geq \frac{1}{n} H^\varepsilon,$$

so the uniform lower bounds for  $F^\varepsilon$  follow. For a proof of (5.1) in the case of smooth  $F$ , we refer the reader to [25]. We note that the estimate (5.1) also holds in the case where  $F$  satisfies Conditions 1.1 but is not smooth; a proof in this setting may be found in [8].  $\square$

Our next step is to show that the flow (1.1) preserves a pointwise curvature pinching ratio. Similar arguments were used for smooth speeds in [2] and [18].

**Lemma 5.3.** *Under the flow (4.1), the following matrix inequality is preserved:*

$$h_{\varepsilon j}^i - \eta F^\varepsilon \delta_j^i \geq 0,$$

where  $\eta$  is chosen appropriately depending only on  $M_0$ .

*Proof.* Using Lemma 4.2 (i) and (ii) we have for any constant  $\eta$ ,

$$\frac{\partial}{\partial t} (h_{\varepsilon j}^i - \eta F^\varepsilon \delta_j^i) = \mathcal{L}^\varepsilon (h_{\varepsilon j}^i - \eta F^\varepsilon \delta_j^i) + \ddot{F}_\varepsilon^{kl,rs} \nabla_\varepsilon^i h_{kl}^\varepsilon \nabla_j^\varepsilon h_{rs}^\varepsilon + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_{\varepsilon l}^m (h_{\varepsilon j}^i - \eta F^\varepsilon \delta_j^i),$$

where  $\delta_j^i$  denotes the Kronecker delta. Hamilton's maximum principle for tensors [12] implies that the inequality is preserved, that is, for each  $\varepsilon > 0$ , if the inequality

$$(5.2) \quad h_{\varepsilon j}^i - \eta F^\varepsilon \delta_j^i \geq 0$$

holds initially, then it is preserved under the flow (4.1).

We may obtain an  $\eta > 0$  independent of  $\varepsilon$  as follows. Choose  $\eta > 0$  small enough such that the curvatures of  $M_0$  everywhere satisfy

$$h_{\varepsilon j}^i \geq \eta \left( f + \frac{1}{n} H \right) \delta_j^i.$$

Then in view of Theorem 3.1, 4, we also have initially

$$h_{\varepsilon j}^i \geq \eta f_\varepsilon \delta_j^i,$$

and by the above argument, this inequality is preserved.  $\square$

**Remarks.** (1) In view of (5.1) and (5.2) the following holds under the flows (4.1):

$$h_{\varepsilon}^i \delta_j \geq \frac{\eta}{n} H^{\varepsilon} \delta_j^i.$$

Taking the trace of this inequality shows that necessarily  $\eta \leq 1$ . If we take  $\varepsilon_0 = \frac{\eta}{n}$ , the argument of  $f$  in the convolution (3.3) remains inside  $\Gamma^+$  for any  $\varepsilon < \varepsilon_0$ . From now on we will assume  $\varepsilon < \varepsilon_0$ .

(2) Since our flows preserve convexity, an estimate on the extinction times  $T_{\varepsilon}$  and  $T$  can be given in terms of the width of  $M_0$ . Such an estimate was developed in [9] for the mean curvature flow. A similar estimate was shown in [8] for the class of fully nonlinear flows of convex hypersurfaces [2]. We remark that the class in [2] has been broadened [3–6] and a width estimate also holds in these cases, since uniform parabolicity of these flows implies  $F$  may be estimated from below by  $H$ , important in the proof of the width estimate.

As in [2, 5, 6, 13] and elsewhere, we next obtain an upper bound on the speed, while the inradius of the evolving hypersurfaces remains positive, using an idea of Tso [23].

**Lemma 5.4.** *Let  $\varepsilon \leq \min\{\varepsilon_0, \frac{1}{n}\}$ . Under the flow (4.1), for  $t_0 \in [0, T_{\varepsilon})$  and  $t \in [0, t_0]$  we have the upper speed bound*

$$F^{\varepsilon}(x, t) \leq 2\rho_+ \max \left\{ \max_{M_0} \frac{F + \frac{1}{n}H}{R_-}, \frac{2}{R_-^2} \right\},$$

where  $\rho_+$  is the outer radius of  $M_0$  and  $R_-$  is the inradius of  $M_{t_0}^{\varepsilon}$ .

*Proof.* Choose the origin as the centre of a sphere of radius  $R_-$  that is enclosed by  $M_{t_0}^{\varepsilon}$ . Then  $s^{\varepsilon}(x, t) \geq R_-$  for all  $x \in \mathbb{S}^n$  and  $t \in [0, t_0]$ . Using Lemma 4.2 (i) and Lemma 4.3 we have on this time interval that the function  $Q^{\varepsilon} = \frac{F^{\varepsilon}}{2s^{\varepsilon} - R_-}$  evolves according to

$$\frac{\partial}{\partial t} Q^{\varepsilon} = \mathcal{L}^{\varepsilon} Q^{\varepsilon} + 4\dot{F}_{\varepsilon}^{kl} \frac{\nabla_k^{\varepsilon} s^{\varepsilon}}{2s^{\varepsilon} - R_-} \nabla_l^{\varepsilon} Q^{\varepsilon} + Q_{\varepsilon}^2 \left( 2 - R_- \frac{\dot{F}_{\varepsilon}^{kl} h_{km}^{\varepsilon} h_{\varepsilon l}^m}{F^{\varepsilon}} \right).$$

We may estimate the last term using [6, Lemma 5] (since  $F^{\varepsilon}$  convex implies  $F^{\varepsilon}$  is inverse concave):

$$\dot{F}_{\varepsilon}^{kl} h_{km}^{\varepsilon} h_{\varepsilon l}^m \geq F_{\varepsilon}^2.$$

Noting also that  $2s^{\varepsilon} - R_- \geq R_-$  we have

$$\frac{\partial}{\partial t} Q^{\varepsilon} \leq \mathcal{L}^{\varepsilon} Q^{\varepsilon} + 4\dot{F}_{\varepsilon}^{kl} \frac{\nabla_k^{\varepsilon} s^{\varepsilon}}{2s^{\varepsilon} - R_-} \nabla_l^{\varepsilon} Q^{\varepsilon} + Q_{\varepsilon}^2 (2 - R_-^2 Q^{\varepsilon}).$$

If a new maximum of  $Q^{\varepsilon}$  is attained, then the left-hand side is non-negative, while the first term on the right-hand side is non-positive and the second term is equal to zero, so the bracketed term must be non-negative. Therefore, for  $(x, t) \in \mathbb{S}^n \times [0, t_0]$ ,

$$Q^{\varepsilon}(x, t) \leq \max \left\{ \max_{M_0} Q^{\varepsilon}, \frac{2}{R_-^2} \right\} \leq \max \left\{ \max_{M_0} \frac{F + \frac{1}{n}H}{R_-}, \frac{2}{R_-^2} \right\},$$

where we have used Theorem 3.1 and the fact that  $\varepsilon \leq \frac{1}{n}$ . Since  $s \leq \rho_+$ , the outer radius of  $M_0$  (independent of  $\varepsilon$ ), we have

$$F^\varepsilon(x, t) \leq 2\rho_+ \max \left\{ \max_{M_0} \frac{F + \frac{1}{n}H}{R_-}, \frac{2}{R_-^2} \right\}$$

for  $(x, t) \in \mathbb{S}^n \times [0, t_0]$ . □

For working with the rescaled hypersurfaces we will need an estimate first used by Smoczyk [21] for the mean curvature flow, later used for fully nonlinear speeds in [5, 6].

**Lemma 5.5.** *Under the flow (4.1), if  $s + 2(t - t_1)F^\varepsilon \geq 0$  everywhere on  $M_{t_1}^\varepsilon$ , the inequality continues to hold everywhere on  $M_t^\varepsilon$  for  $t > t_1$  as long as the solution exists.*

*Proof.* From Lemma 4.2 (i) and Lemma 4.3 we have the evolution equation

$$\frac{\partial}{\partial t} [s^\varepsilon + 2(t - t_1)F^\varepsilon] = \mathcal{L}^\varepsilon [s^\varepsilon + 2(t - t_1)F^\varepsilon] + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_{\varepsilon l}^m [s^\varepsilon + 2(t - t_1)F^\varepsilon],$$

from which it follows by the maximum principle that the minimum does not decrease. Specifically, for  $T_\varepsilon > t > t_1$ ,

$$s^\varepsilon(x, t) + 2(t - t_1)F^\varepsilon(x, t) \geq \min_{M_t^\varepsilon} [s^\varepsilon + 2(t - t_1)F^\varepsilon] \geq \min_{M_{t_1}^\varepsilon} s^\varepsilon(x, t_1) \geq 0.$$

This completes the proof. □

### 6. Existence and convergence to a point

In the previous section, we established various a priori estimates for solutions to (4.1) independent of the approximating parameter  $\varepsilon$  in the speed. In this section, these uniform estimates are used to prove existence, regularity and convergence results for solutions to (1.1).

With  $F$  satisfying Conditions 1.1, equation (1.1) represents a degenerate parabolic system due to the invariance of solutions with respect to diffeomorphisms. However, we may obtain short-time existence of image solution hypersurfaces  $M_t = X(\mathbb{S}^n, t)$  in the following manner. Note that writing the hypersurfaces as graphs over the initial hypersurface  $M_0$  removes the parametrisation invariance. Indeed, there is a one-to-one correspondence between solutions  $u : \mathbb{S}^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  to

$$(6.1) \quad \frac{\partial u}{\partial t} = -F(W)$$

and solutions  $X : \mathbb{S}^n \times [0, T) \rightarrow \mathbb{R}^{n+1}$  to (1.1). This procedure is well known: we refer the reader to [2, Lemma 3.2] for details. Performing this for the mollified flows, we find the corresponding scalar equations

$$(6.2) \quad \frac{\partial u^\varepsilon}{\partial t} = -F^\varepsilon(W^\varepsilon).$$

While the inradius of the evolving hypersurfaces remains positive, space-time  $C^{2,\alpha}$  estimates for the solutions  $u^\varepsilon$  follow by what is now standard theory. The general idea is that  $C^{2,\alpha}$  regularity of solutions relies upon uniform ellipticity of the operator, which is understood

via the ellipticity constants: we call  $\underline{C}$  and  $\overline{C}$  ellipticity constants for the flow (6.2) if, for each  $i = 1, \dots, n$  and every  $\delta > 0$ ,

$$(6.3) \quad \underline{C}\delta \leq f^\varepsilon(\kappa + \delta e_i) - f^\varepsilon(\kappa) \leq \overline{C}\delta,$$

where  $f^\varepsilon$  is related to  $F^\varepsilon$  as in Conditions 1.1 (i).

In particular, [22, Theorem 1.1] can be used to obtain the following theorem.

**Theorem 6.1.** *Any solution  $u^\varepsilon : \mathbb{S}^n \times [0, T^\varepsilon) \rightarrow \mathbb{R}^{n+1}$  to (6.2) satisfies*

$$\|u^\varepsilon\|_{C^{2,\alpha}(\mathbb{S}^n)} \leq C \|u^\varepsilon\|_{L^\infty(\mathbb{S}^n)},$$

where  $\alpha$  depends only on  $M_0$  and the ellipticity constants of  $F^\varepsilon$ , and  $C$  depends only on  $n$ ,  $M_0$ , and the ellipticity constants of  $F^\varepsilon$ .

Clearly, comparison with a sphere of radius  $\rho_+$  (cf. the proof of Lemma 5.1) gives immediately the uniform-in- $\varepsilon$  bound

$$\|u\|_{L^\infty(\mathbb{S}^n)} \leq \rho_+,$$

and so Theorem 6.1 yields uniform  $C^{2,\alpha}$  bounds dependent solely on the ellipticity constants of  $F^\varepsilon$ . We now turn our attention to these.

**Lemma 6.2.** *While the inradius of the evolving hypersurface  $M_t^\varepsilon$  remains positive, the flow (6.2) remains uniformly parabolic in the sense that (6.3) is satisfied; equivalently*

$$(6.4) \quad \underline{C} \leq \frac{\partial f^\varepsilon}{\partial \kappa_i} \leq \overline{C},$$

where the constants  $\underline{C}$  and  $\overline{C}$  are independent of  $\varepsilon$ .

*Proof.* The upper bound is easy to establish using Conditions 1.1 (ii), Lemma 3.2 (i) and Theorem 3.1 (iv): as in [6, Lemma 1],  $f^\varepsilon$  may be continuously extended to  $\overline{\Gamma}_+$  and we have, for any  $\kappa \in \Gamma_+$ ,

$$\begin{aligned} f^\varepsilon(\kappa + \delta e_i) - f^\varepsilon(\kappa) &\leq f^\varepsilon(\kappa) + \delta f^\varepsilon(e_i) - f^\varepsilon(\kappa) \\ &= \delta f^\varepsilon(e_i) \\ &\leq \delta [f(e_i) + \varepsilon] \\ &\leq \delta \left(1 + \frac{1}{n}\right). \end{aligned}$$

Setting  $\overline{C} = (1 + \frac{1}{n})$  gives the required upper bound.

For the lower bound, we proceed as follows. From Lemma 5.2, the mean curvature  $H^\varepsilon$  remains positively bounded below. From Lemma 5.4, in view of (5.1),  $H^\varepsilon$  remains bounded above. (Recall that the inradius is assumed to be strictly positive, and so we may apply Lemma 5.4.) Together with preserved curvature pinching, Lemma 5.3 (ii), these results imply that the curvatures of  $M_t^\varepsilon$ , during the time interval where the inradius is assumed to be positive, are contained within a compact set  $K \subset \Gamma^+$ , this set is independent of  $\varepsilon$ .

As in the proof of Theorem 3.1 (3), since  $f^\varepsilon$  is convex, we have using a supporting hyperplane for  $f$  at  $\kappa$  that

$$f^\varepsilon(\kappa + \delta e_i) - f^\varepsilon(\kappa) \geq \delta \langle r(\kappa), e_i \rangle,$$



where  $r(\kappa)$  is a unit normal to the supporting hyperplane and is independent of  $\varepsilon$ . In view of Conditions 1.1 (ii), the normal can be chosen such that the inner product is strictly positive and the function  $\langle r(\kappa), e_i \rangle$  is continuous. This function attains a positive minimum, say  $\underline{C}_i$  over  $K$  and taking  $\underline{C} = \min_i \underline{C}_i$  gives the required lower bound.  $\square$

**Remark.** Since  $\dot{F}^\varepsilon$  is homogeneous of degree zero, the ellipticity bounds continue to hold under rescaling and so the rescaled flows of the next section are also uniformly parabolic.

By the comparison principle each of the flows (4.1) have inradius bounded away from zero on the time interval  $[0, \delta]$  where  $\delta$  depends only on  $M_0$ . Since each of the flows (4.1) are uniformly parabolic independent of  $\varepsilon$ , we have by Theorem 6.1  $C^{2,\alpha}$  control independent of  $\varepsilon$ ; that is,

$$(6.5) \quad \|X^\varepsilon\|_{C^{2,\alpha}(\mathbb{S}^n)} \leq C(M_0, n, F)$$

Taking the limit  $\varepsilon \rightarrow 0$  gives the limit map  $X$  is  $C^2$  so gives a classical solution of (1.1) for which the estimates independent of  $\varepsilon$  in Section 5 continue to hold. Short time existence and uniqueness of the solution to (6.1), now follows, we state this as a result analogous to [16, Theorem 14.5].

**Theorem 6.3.** *For any  $s_0 \in C^{1,\alpha}(\mathbb{S}^n)$  and  $F$  satisfying Conditions 1.1 there exists a  $\delta > 0$  and unique classical solution to (6.1) where  $s \in C^{1,\beta}(\mathbb{S}^n \times (0, \delta)) \cap C(\mathbb{S}^n \times [0, \delta])$ .*

**Remarks.** (1) The lower bound  $\min_{M_0} H^\varepsilon$  of Lemma 5.2 is independent of  $\varepsilon$ . In light of the estimate (6.5), upon taking the limit  $\varepsilon \rightarrow 0$  we find that the mean curvature of  $M_t$  evolving under (1.1) satisfies

$$H(x, t) \geq \min_{M_0} H.$$

Upon applying (5.1), we conclude Lemma 5.2 for the flow (1.1).

(2) In view of the upper speed bound, Lemma 5.4 and curvature pinching Lemma 5.3 (ii), we have that the curvature associated to (6.1) is bounded while the inradius remains bounded below. Consequently we may take  $\beta = 1$  above.

(3) The regularity of  $s_0$  in the theorem is particularly useful so that the estimates of the previous section facilitate convergence to a point of the evolving  $M_t$ . In contrast to most previous work, we cannot obtain higher regularity of solutions to (6.1) as  $f$  is only almost everywhere twice differentiable as a function of the principal curvatures, so we need less regular initial data in the contradiction argument for convergence to a point.

(4) The arguments of [16, Chapter 14] are for bounded space-time domains  $\Omega \subset \mathbb{R}^{n+1}$  however they modify easily for the domain  $\mathbb{S}^n \times [0, \delta]$ .

(5) In [16], mollification is used in showing short-time existence by the method of continuity of solutions to

$$u_t = F(D^2u),$$

given known short-time existence of solutions to the heat equation and previously established a priori estimates. An approximation argument also allows the ellipticity condition (6.3) in place of the usual condition on  $\frac{\partial F}{\partial r_{ij}}$ .

(6) Theorem 14.5 of [16] is stated with various structure conditions on the speed function in addition to uniform ellipticity. These are not required in our setting due to (6.5) and the a priori estimates of the previous section. Specifically:

- The support function  $s$  is bounded in terms of the initial data since the hypersurface  $M_t$  is contracting.
- The gradient  $|\bar{\nabla}s|$  is uniformly bounded since convexity of  $M_t$  is preserved.
- The gradient  $\bar{\nabla}s$  is Hölder continuous in view of the uniform curvature bound.

Further, as in the proof of Lemma 6.2, the principal radii of curvature remain within a compact subset of  $\Gamma^+$ , so  $f$  is Lipschitz continuous.

Our a priori estimates together with Theorem 6.3 allow us to show that the solution to (1.1) contracts to a point in finite time.

**Theorem 6.4.** *The maximal time of existence  $T$  of the solution to (1.1) is finite and the solution converges to a point  $p \in \mathbb{R}^{n+1}$  as  $t \rightarrow T$ .*

*Proof.* It was shown in Lemma 5.1 that  $T$  is finite. To show contraction to the final point, we follow the argument as in [5], which is similar to other arguments such as that in [23]. Suppose on the contrary the inradius  $R_-$  of  $M_t$  does not approach zero as  $t \rightarrow T$ . We then have from Lemmas 5.2 and 5.4 uniform bounds on the speed  $F$  under the flow and these imply via the curvature pinching of Lemma 5.3 uniform bounds above and below on the principal curvatures. Therefore we have convergence to a  $C^{1,1}$  hypersurface  $M_T$  at time  $T$ . (In fact,  $M_T$  is a  $C^2$  hypersurface in view of Theorem 6.1.) This hypersurface could then be used as an initial surface in the short-time existence theorem, Theorem 6.3, contradicting the maximality of  $T$ . Therefore the inradius must approach zero at  $t \rightarrow T$ . Because of curvature pinching, Lemma 5.3, the circumradius also approaches zero ([2, Lemma 5.4]).  $\square$

## 7. Convergence to a sphere

It remains to show that under appropriate rescaling the solution hypersurfaces approach the unit sphere. As in [2], for example, we use the natural rescaling

$$\tilde{X}(x, t) = \frac{1}{\sqrt{2(T-t)}}(X(x, t) - p),$$

where  $T$  is the final time at which the solution to (1.1) has contracted to the point  $p$  of Theorem 6.4. We further define the new time parameter as

$$\tau = -\frac{1}{2} \ln\left(1 - \frac{t}{T}\right)$$

and observe  $t \in [0, T)$  corresponds to  $\tau \in [0, \infty)$ . The rescaled immersions  $\tilde{M}_\tau$  evolve according to

$$(7.1) \quad \frac{\partial}{\partial \tau} \tilde{X}(x, \tau) = -F(\tilde{W}(x, \tau))\tilde{\nu}(x, \tau) + \tilde{X}(x, \tau),$$

with initial condition

$$\tilde{X}(x, 0) = \frac{1}{\sqrt{2T}}(X_0 - p).$$

Similarly as in [5], the rescaled flow has a uniform lower bound on  $\tilde{F} = F(\tilde{W})$  using Lemma 5.5 and a uniform upper bound on  $\tilde{F}$ , applying the Tso estimate as in Lemma 5.4 but to the rescaled equation (see, for example, [2]). Curvature pinching also holds; together with the speed bounds this shows that the rescaled curvatures are uniformly bounded above and below. Since  $F$  is convex and (7.1) is uniformly parabolic,  $C^{2,\alpha}$  regularity follows using results of Krylov [15] (e.g. [16, Corollary 14.9]), similarly as in [6, proof of Lemma 5 (after the proof of Lemma 13)]. If  $F$  happens to be more regular, then standard arguments involving differentiating the evolution equations and using Schauder estimates (e.g. [16, Theorem 4.9]) show that the solution is correspondingly more regular.

Now we show that the image of the solution  $\tilde{X}$  to (7.1) converges to that of the sphere in the  $C^{2,\beta}$  topology (for  $\beta < \alpha$ ). Since the solution  $\tilde{X}$  is itself of class  $C^{2,\alpha}$ , if we have convergence of  $\tilde{X}$  to a sphere in any lower order topology ( $C^0$  for example), then standard interpolation implies convergence in the  $C^{2,\beta}$  topology. We will obtain an estimate independent of  $\varepsilon$  on the solutions  $\tilde{X}^\varepsilon$  to equation

$$(7.2) \quad \frac{\partial}{\partial \tau} \tilde{X}^\varepsilon(x, \tau_\varepsilon) = -F^\varepsilon(\tilde{W}^\varepsilon(x, \tau_\varepsilon))\tilde{v}^\varepsilon(x, \tau_\varepsilon) + \tilde{X}^\varepsilon(x, \tau_\varepsilon),$$

with initial condition

$$\tilde{X}^\varepsilon(x, 0) = \frac{1}{\sqrt{2T_\varepsilon}}(X_0 - p_\varepsilon).$$

Here the rescaled time parameter is

$$\tau_\varepsilon = -\frac{1}{2} \ln\left(1 - \frac{t}{T_\varepsilon}\right),$$

$\tau_\varepsilon \in [0, \infty)$  for all  $\varepsilon$ . The estimate, independent of  $\varepsilon$ , will allow us to conclude that all limiting hypersurfaces  $\tilde{M}_\infty^\varepsilon$  including  $\tilde{M}_\infty$  are spheres.

The basic idea behind the estimate is to prove that the principal curvatures become closer to those of a sphere through application of the weak Harnack inequality for supersolutions (see, for example, [16, Chapter 7]). We begin by developing an appropriate auxiliary function.

**Lemma 7.1.** *For any  $\bar{\alpha}$  there is an absolute constant  $k_0 = k_0(\bar{\alpha})$  such that the following holds. For  $\alpha \leq \bar{\alpha}$  and any  $k \geq k_0 > 0$  define  $\tilde{Q}_\alpha^\varepsilon := \tilde{G}^\varepsilon - \alpha \tilde{F}^\varepsilon$ , where  $\tilde{G}^\varepsilon := \tilde{G}_0^\varepsilon + k \tilde{H}^\varepsilon$  and  $\tilde{G}^\varepsilon$  is smooth, positive, increasing, concave, degree one homogeneous and normalised. Under the rescaled flow,  $\tilde{Q}_\alpha^\varepsilon$  evolves according to*

$$(7.3) \quad \frac{\partial}{\partial \tau_\varepsilon} \tilde{Q}_\alpha^\varepsilon \geq \tilde{\mathcal{L}}^\varepsilon \tilde{Q}_\alpha^\varepsilon + (\dot{\tilde{F}}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon,m} \tilde{h}_{ml}^\varepsilon - 1) \tilde{Q}_\alpha^\varepsilon.$$

*Proof.* We require  $\tilde{G}_0^\varepsilon(\tilde{W}^\varepsilon) = g_0^\varepsilon(\kappa(\tilde{W}^\varepsilon))$  to be a smooth, positive, concave, degree one homogeneous function of the principal curvatures, where  $g_0^\varepsilon$  is strictly monotone increasing in each variable and normalised such that  $g_0^\varepsilon(1, \dots, 1) = 1$ . Existence of such a function is not an issue, since the  $n$ -th root of the Gauss curvature or the normalised square root of the scalar curvature are candidates.

The function  $g_0^\varepsilon$  depends on  $\varepsilon$  only through the dependence of the curvatures  $\kappa_i^\varepsilon$  on  $\varepsilon$ . Since the curvatures  $\kappa_i^\varepsilon$  of the rescaled flows remain within a bounded region of  $\Gamma^+$ , the region being independent of  $\varepsilon$ ,  $g_0^\varepsilon$  may be estimated above and below, independent of  $\varepsilon$ .

A direct computation shows that the quantity  $\tilde{Q}_\alpha$  evolves under (7.2) according to

$$(7.4) \quad \frac{\partial}{\partial \tau_\varepsilon} \tilde{Q}_\alpha^\varepsilon = \tilde{\mathcal{L}}^\varepsilon \tilde{Q}_\alpha^\varepsilon + (\dot{\tilde{Q}}_\alpha^{\varepsilon,ij} \ddot{F}_\varepsilon^{kl,mn} - \dot{F}_\varepsilon^{ij} \ddot{\tilde{Q}}_\alpha^{\varepsilon,kl,mn}) \tilde{\nabla}_i^\varepsilon \tilde{h}_{kl}^\varepsilon \tilde{\nabla}_j^\varepsilon \tilde{h}_{mn}^\varepsilon + (\dot{\tilde{F}}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon,m} \tilde{h}_{ml}^\varepsilon - 1) \tilde{Q}_\alpha^\varepsilon.$$

Note that the  $-\tilde{Q}_\alpha^\varepsilon$  term arises due to the zero order term in (7.2).

We wish to now choose  $k > 0$  such that the gradient term in (7.4) is non-negative. Observe that  $\dot{\tilde{F}}_\varepsilon^{ij} \ddot{\tilde{Q}}_\alpha^{\varepsilon,ij,kl}$  is non-positive in view of Condition 1.1 (ii) and concavity of  $\tilde{Q}_\alpha^\varepsilon$ . Since  $F$  is convex, in order for the entire gradient term in (7.4) to be non-negative, we require  $\dot{\tilde{Q}}_\alpha^\varepsilon$  to be a non-negative matrix. Let us use lowercase letters to denote the version of a matrix function that operates with eigenvalues as arguments. We have in coordinates that diagonalise the Weingarten map

$$\frac{\partial \tilde{q}_\alpha^\varepsilon}{\partial \tilde{\kappa}_i} = \frac{\partial \tilde{g}_0^\varepsilon}{\partial \tilde{\kappa}_i} + k - \alpha \frac{\partial \tilde{f}^\varepsilon}{\partial \tilde{\kappa}_i} > k_0 - \bar{\alpha} \bar{C},$$

where the inequality follows from the strict monotonicity of  $\tilde{g}_0^\varepsilon$  and the estimate (6.4), which continues to hold under rescaling since  $\dot{F}^\varepsilon$  is homogeneous of degree zero. Taking  $k_0 = \bar{\alpha} \bar{C}$  we ensure that  $\dot{\tilde{Q}}_\alpha^\varepsilon > 0$  and so (7.3) follows from (7.4).  $\square$

We have the following upper and lower bounds for  $\tilde{Q}_\alpha^\varepsilon$ .

**Lemma 7.2.** *There exists an absolute constant  $\hat{C} > 0$  such that*

$$(7.5) \quad (\hat{C} - \alpha) \tilde{F}^\varepsilon \leq \tilde{Q}_\alpha^\varepsilon \leq (1 - \alpha) \tilde{F}^\varepsilon.$$

*Proof.* Since the speeds  $\tilde{F}^\varepsilon$  have a uniform positive lower bound and since  $\tilde{G}^\varepsilon / \tilde{F}^\varepsilon$  is homogeneous of degree zero, there are constants  $\hat{d} > \hat{C} > 0$  such that

$$\hat{C} \leq \frac{\tilde{G}^\varepsilon}{\tilde{F}^\varepsilon} \leq \hat{d},$$

in particular,

$$\hat{C} \tilde{F}^\varepsilon \leq \tilde{G}^\varepsilon$$

and the first inequality of the lemma follows in view of the definition of  $\tilde{Q}_\alpha^\varepsilon$ .

Since  $\tilde{G}^\varepsilon$  is concave and  $\tilde{F}^\varepsilon$  is convex, any critical point of  $\tilde{G}^\varepsilon / \tilde{F}^\varepsilon$  is a local maximum, and therefore there can be only one local maximum on  $\Gamma^+ \cap \{|\tilde{A}^\varepsilon| = 1\}$ . By direct computation, the point  $(1, \dots, 1)$  is a critical point for  $\tilde{g}^\varepsilon / \tilde{f}^\varepsilon$ , in view of symmetry and normalisation. Therefore we have the upper bound

$$\frac{\tilde{G}^\varepsilon}{\tilde{F}^\varepsilon} \leq \frac{\tilde{g}^\varepsilon(1, \dots, 1)}{\tilde{f}^\varepsilon(1, \dots, 1)} = 1.$$

This clearly implies the upper bound in (7.5), and finishes the proof.  $\square$

*Completion of the proof of Theorem 1.2.* Consider  $\tilde{Q}_{\alpha_m}^\varepsilon$  on the time interval  $[m, m+1]$  for  $m \geq 1$ . Every such interval is contained within  $[0, \infty)$ , the time interval of existence for all the flows (7.2) and (7.1). Let us fix  $\bar{\alpha} = 1$  and choose  $\alpha = \alpha_0$  such that

$$\min_{M_0} \tilde{Q}_{\alpha_0}^\varepsilon = 0.$$

Note that the lower bound in (7.5) implies  $\alpha_0 > 0$ , moreover there is an upper bound on  $\alpha_0$  beyond which  $\min_{M_t, t \in [m, m+1]} \tilde{Q}_{\alpha_m}^\varepsilon < 0$ . The sequence  $\{\alpha_m\}_{m \in \mathbb{N}_0}$  is generated as follows, analogous to how  $\alpha_0$  was chosen. We choose  $\alpha = \alpha_m$  such that

$$\min_{M_m} \tilde{Q}_{\alpha_m}^\varepsilon = 0.$$

Note that for all  $m$  we have  $\alpha_m \leq 1$  since otherwise  $\max_{M_t, t \in [m, m+1]} \tilde{Q}_{\alpha_m}^\varepsilon < 0$  by (7.5).

The evolution equation (7.3) implies that on the interval  $[m, m+1]$  the quantity  $\tilde{Q}_{\alpha_m}^\varepsilon$  is non-negative. To obtain more useful information than this we need to apply a Harnack inequality (see, e.g., [16, Chapter 7]). In order to do this we first rewrite (7.3) in a local coordinate system around  $B_\rho(x)$  (for any  $x \in M$ ), with corresponding Christoffel symbols  $\Gamma_{jk}^i$ :

$$\frac{\partial}{\partial \tau_\varepsilon} \tilde{Q}_{\alpha_m}^\varepsilon \geq \dot{F}_\varepsilon^{ij} \left( \frac{\partial^2}{\partial x_i \partial x_j} \tilde{Q}_{\alpha_m}^\varepsilon - \Gamma_{ij}^k \frac{\partial}{\partial x_k} \tilde{Q}_{\alpha_m}^\varepsilon \right) + (\dot{F}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon, m} \tilde{h}_{ml}^\varepsilon - 1) \tilde{Q}_{\alpha_m}^\varepsilon.$$

We shall make two transformations to bring this evolution equation into a suitable form. First, from the above, note that

$$\begin{aligned} \frac{\partial}{\partial \tau_\varepsilon} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} &\geq \frac{1}{2\sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}} \left[ \dot{F}_\varepsilon^{ij} \left( \frac{\partial^2}{\partial x_i \partial x_j} \tilde{Q}_{\alpha_m}^\varepsilon - \Gamma_{ij}^k \frac{\partial}{\partial x_k} \tilde{Q}_{\alpha_m}^\varepsilon \right) + (\dot{F}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon, m} \tilde{h}_{ml}^\varepsilon - 1) \tilde{Q}_{\alpha_m}^\varepsilon \right] \\ &= \frac{1}{2\sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}} \dot{F}_\varepsilon^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tilde{Q}_{\alpha_m}^\varepsilon - \dot{F}_\varepsilon^{ij} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} \\ &\quad + \frac{1}{2} (\dot{F}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon, m} \tilde{h}_{ml}^\varepsilon - 1) \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} \\ &= \dot{F}_\varepsilon^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} + \frac{\dot{F}_\varepsilon^{ij}}{\sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}} \frac{\partial}{\partial x_i} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} \frac{\partial}{\partial x_j} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} - \dot{F}_\varepsilon^{ij} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} \\ &\quad + \frac{1}{2} (\dot{F}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon, m} \tilde{h}_{ml}^\varepsilon - 1) \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} \\ &\geq \dot{F}_\varepsilon^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} + \frac{\underline{C}}{\sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}} |D \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}|^2 - \bar{C} |\Gamma| |D \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}| \\ &\quad + \frac{1}{2} (\dot{F}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon, m} \tilde{h}_{ml}^\varepsilon - 1) \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}, \end{aligned}$$

where  $D$  denotes the gradient in  $B_\rho(x)$  and we have used the ellipticity constants (6.4) that continue to hold for the rescaled flow since  $\tilde{f}^\varepsilon$  is homogeneous of degree zero. We now estimate

$$\bar{C} |\Gamma| |D \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}| \leq \frac{\underline{C}}{\sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}} |D \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}|^2 + \frac{\sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}}{4\underline{C}} \bar{C}^2 |\Gamma|^2,$$

from which it follows that

$$\frac{\partial}{\partial \tau_\varepsilon} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} \geq \dot{F}_\varepsilon^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon} + \frac{1}{2} \left( \dot{F}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon,m} \tilde{h}_{ml}^\varepsilon - \frac{1}{2} \frac{\bar{C}^2}{\underline{C}} |\Gamma|^2 - 1 \right) \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}.$$

To obtain an expression without a zero order term consider  $e^{\lambda t} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}$ , for constant  $\lambda$  to be chosen:

$$\begin{aligned} \frac{\partial}{\partial \tau_\varepsilon} (e^{\lambda t} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}) &\geq \dot{F}_\varepsilon^{ij} \frac{\partial^2}{\partial x_i \partial x_j} (e^{\lambda t} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}) \\ &\quad + \frac{1}{2} \left( \dot{F}_\varepsilon^{kl} \tilde{h}_k^{\varepsilon,m} \tilde{h}_{ml}^\varepsilon - \frac{1}{2} \frac{\bar{C}^2}{\underline{C}} |\Gamma|^2 - 1 + \lambda \right) (e^{\lambda t} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon}). \end{aligned}$$

Since the rescaled curvatures are bounded, there is a positive  $\lambda = \lambda_0$  such that for

$$\mathcal{Z}_{\alpha_m} = (e^{\lambda_0 t} \sqrt{\tilde{Q}_{\alpha_m}^\varepsilon})$$

we have

$$(7.6) \quad \left( \frac{\partial}{\partial \tau_\varepsilon} - \dot{F}_\varepsilon^{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right) \mathcal{Z}_{\alpha_m} \geq 0$$

in the local coordinate chart. We may now apply the weak Harnack inequality from [16, Chapter 7] to conclude for each  $x \in M$ ,

$$(7.7) \quad \min_{B_{\rho/2}(x) \times [m+1, m+2]} \mathcal{Z}_{\alpha_m} \geq c \int_{m-1}^m \left( \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |\mathcal{Z}_{\alpha_m}|^\sigma dx \right)^{\frac{1}{\sigma}} d\tau,$$

for absolute positive  $\sigma$  and bounded  $c$  independent of  $x$ . Since the original solution is of regularity class  $C^{2,\alpha}$ , the associated function  $\mathcal{Z}_{\alpha_m}$  is uniformly  $\alpha$ -Hölder continuous, and so

$$\min_{B_{\rho/2}(x) \times [m+1, m+2]} \mathcal{Z}_{\alpha_m} \geq c \max_{B_\rho(x) \times [m-1, m]} \mathcal{Z}_{\alpha_m},$$

where  $c$  here depends on the absolute constants  $\sigma$  and  $\alpha$ . Chaining together space-time cylinders we conclude the global estimate

$$\min_{M \times [m+1, m+2]} \mathcal{Z}_{\alpha_m} \geq c \max_{M \times [m-1, m]} \mathcal{Z}_{\alpha_m}.$$

Absorbing the exponential factor in  $\mathcal{Z}_{\alpha_m}$  and squaring both sides gives

$$(7.8) \quad \min_{M \times [m+1, m+2]} \tilde{Q}_{\alpha_m}^\varepsilon \geq c \max_{M \times [m-1, m]} \tilde{Q}_{\alpha_m}^\varepsilon,$$

where  $c > 0$  is an absolute constant.

Keeping in mind that at any time the evolving surface has at least one umbilic point where the function  $\tilde{G}^\varepsilon / \tilde{F}^\varepsilon$  achieves its maximum, we apply the estimate (7.8) to obtain

$$\alpha_{m+1} - \alpha_m = \min \left. \frac{\tilde{G}^\varepsilon}{\tilde{F}^\varepsilon} \right|_{t=m+1} - \alpha_m \geq c(1 - \alpha_m)$$

so

$$(7.9) \quad 1 - \alpha_{m+1} \leq -c(1 - \alpha_m) + (1 - \alpha_m) \leq (1 - c)(1 - \alpha_m).$$

Since  $\alpha_m < 1$ , this implies that  $1 - c > 0$  and so  $1 - c \in (0, 1)$ . Iterating the recurrence relation (7.9) and using this fact we have for  $C = (1 - \alpha_0) \in (0, 1)$  and  $\gamma = -\log(1 - c) \in (0, \infty)$ ,

$$0 < 1 - \alpha_{m+1} \leq (1 - c)^m (1 - \alpha_0) = (1 - \alpha_0) e^{m \log(1-c)} \leq C e^{-\gamma m}.$$

The above estimate holds for  $t \in [m, m + 1]$ . Therefore

$$0 < 1 - \alpha_{m+1} \leq C e^{-\gamma(m+1)} e^\gamma \leq \hat{C} e^{-\gamma t},$$

which holds for all  $t \geq 0$ .

Now on a compact subset of the positive curvature cone in which the principal curvatures are contained during the flow we have the estimate

$$\left(1 - \frac{\tilde{G}^\varepsilon}{\tilde{F}^\varepsilon}\right) \geq \tilde{c} \left(\frac{\tilde{\kappa}_{\max}^\varepsilon}{\tilde{\kappa}_{\min}^\varepsilon} - 1\right)^2$$

so that

$$\left(\frac{\tilde{\kappa}_{\max}^\varepsilon}{\tilde{\kappa}_{\min}^\varepsilon} - 1\right)^2 \leq \tilde{C} e^{-\gamma t}.$$

This gives exponential improvement in pinching for the rescaled immersions independent of  $\varepsilon$ . It further implies that the limit is umbilic, in particular, that it is a sphere. This finishes the proof.  $\square$

**Remark.** If the speed function  $F$  is more regular than locally Lipschitz continuous, such that we can obtain uniform curvature derivative estimates for the rescaled flow, then we can show higher derivatives of the support function also converge to zero and, by interpolation, we may obtain exponential convergence to zero of more of the derivatives of the radial graph function.

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