University of Wollongong
Research Online

Faculty of Commerce - Papers (Archive)

Faculty of Business

2006

Inverse alpha-Cuts and Interval [a, b)-Cuts

Z. Sun
University of Wollongong, zsun@uow.edu.au

J. Han
Beihang University, Beijing, China, jun_han@buaa.edu.cn

Publication Details

This conference paper was originally published as Sun, Z and Han, J, Inverse a-Cuts and Interval [a, b)-Cuts, in Proceedings of the International Conference on Innocative Computing, Information and Conctrol ICICIC2006, 30 August - 1 September 2006, Beijing, 1, 2006, 441-444. Copyright IEEE. Original conference proceedings here.

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au

Inverse alpha-Cuts and Interval [a, b)-Cuts

Abstract
This paper will propose the symmetric form of alpha-cut: inverse alpha-cut, and then alpha-induced fuzzy set and examine their properties. Further, it introduces interval cut, interval induced fuzzy set and examines their properties. The proposed approach will facilitate research of fuzzy set theory and fuzzy systems.

Keywords
alpha-cut, inverse alpha-cut, interval cut, interval induced fuzzy set, fuzzy system

Disciplines
Business | Social and Behavioral Sciences

Publication Details
This conference paper was originally published as Sun, Z and Han, J, Inverse a-Cuts and Interval [a, b)-Cuts, in Proceedings of the International Conference on Innocative Computing, Information and Conctrol ICICIC2006, 30 August - 1 September 2006, Beijing, 1, 2006, 441-444. Copyright IEEE. Original conference proceedings here.

This conference paper is available at Research Online: http://ro.uow.edu.au/commpapers/176

Zhaohao Sun, Jun Han (2006). Inverse a-Cuts and Interval a-Cuts. In *Proc. Intl Conf on Innocative Computing, Information and Conctrol* (ICICIC2006), 30 August - 1 September, Beijing. IEEE Press, pp.441-444

Inverse α-Cuts and Interval [*a*, *b*)-Cuts

Zhaohao Sun

School of Economics & Information Systems, University of Wollongong, Wollongong NSW 2522, Australia, zsun@uow.edu.au

Jun Han

School of Computer science, Beihang University, Beijing 100083, China
jun_han@buaa.edu.cn

Abstract

This paper will propose the symmetric form of α-cut: inverse α-cut, and then α-induced fuzzy set and examine their properties. Further, it introduces interval cut, interval induced fuzzy set and examines their properties. The proposed approach will facilitate research of fuzzy set theory and fuzzy systems.

1. Introduction

α-*cut* or α -*level set* is one of the most important concepts introduced by Zadeh in 1965 [7] to establish a bridge between fuzzy set theory and traditional set theory. Others used this concept in their books on fuzzy set theory [1][2][3][8]. However, there are no essential progresses looking at this concept since then. This paper will fill in this gap by providing a series of α-related and interval related concepts and corresponding examinations. More specifically, the paper first reviews the concept of α-cut, introduces its symmetric form: inverse α-cut, and also α- induced fuzzy set, and examines their properties. Then it introduces interval cut, interval induced fuzzy set, which are an important extension of α-cut, and examines their properties.

2. α-Cut and Its Properties

Let X is a non-empty set, $F(X)$ denotes the set of all fuzzy sets of X [1] (p. 8).

Let A be a fuzzy set in X, that is, $A \in F(X)$ and . $\alpha \in [0, 1]$ Then the non-fuzzy set (or crisp set)

$$A_\alpha = \{x \in X | \mu_A((x) \geq \alpha)\} \quad (1)$$

is called the α-*cut* or α-*level* set of A [1][8] .

If Eq. (1) is replaced by

$$A_\alpha = \{x \in X | \mu_A((x) > \alpha)\} \quad (2)$$

then A_α is called a strong α-*cut* [1].

Any fuzzy set can be composed from a family of nested crisp sets satisfying Eq. (2), and the problems in the context of fuzzy sets such as decision making could be solved by transforming these fuzzy sets into their families of nested α-cuts and determining solutions to each of them using traditional techniques [3]. Then all the partial results derived in this way are merged reconstructing a solution to the problem in its original fuzzy set based formulation.

Zadeh also used α-cut sets to discuss similarity relations and fuzzy orderings [6]. α-cut has also been used for case base building [5]. However, few in fuzzy sets and systems have paid attention to:

- To which real world problems does α-cut correspond?
- Which social phenomenon is related to α-cut?

In what follows, we will try to examine this question from a new perspective.

Example: 18 students enrolled in the BIS218 subject in Spring 2005 at a university, as shown in table 1. The examination results of the students can be considered as a fuzzy set A ($\mu(i) = k/100$, k is the examination mark of student s_i, see Table 1), as shown in the following figure. $A = \{(s_1, 0.43), (s_2, 0.5), \ldots, (s_{18}, 0.86)\}$, where s_i is the ID of student i.

Let $\alpha = 0.5$ (corresponding to 50 in the examination marks). Then, the α-cut A_α is the set of all the students who at least passed (P) the examination for BIS218 (according to the examination assessment system in Australia) and $A_{0.65}$ is the set of all the students who obtained at least a Credit (C) for BIS218. $A_{0.75}$ is the set of all the students who obtained at least a Distinction (D) for BIS218, and $A_{0.85}$ is all the

students who obtained a High Distinction (HD) for BIS218. However, when we only consider, for example, $A_{0.75}$, then we only emphasize the students who obtained at least a Distinction for BIS218, at the same time, we ignore the students who obtained less marks.

Therefore, the concept of α-cut corresponds to an aspect of the real world problem. At least, this kind of abstraction has ignored the symmetry of abstraction, and lost some information of the real world problem. In order to resolve this drawback, we introduce inverse α-cut in the next section.

3. Inverse α-Cut and Its Properties

This section introduces the concept of inverse α-cut, and examines its properties.

3.1 Inverse α-Cut

Definition 1. Let $A \in F(X)$ and $\alpha \in [0, 1]$. Then the non-fuzzy set

$$A_\alpha^{-1} = \{x \in X | \mu_A(x) < \alpha\} \tag{3}$$

is called an *inverse* α*-cut* or *inverse* α*-level set* of A. If Eq. (3) is replaced by

$$A_\alpha^{-1} = \{x \in X | \mu_A((x) \leq \alpha)\} \tag{4}$$

then we call A_α^{-1} a *weak inverse* α*-cut* of A. Obviously, the weak inverse α-cut corresponds to the strong α-cut, while the inverse α-cut to the α-cut. Furthermore, the weak inverse α-cut has similar properties to the inverse α-cut. Therefore, we do not look at the weak inverse α-cut and strong α-cut in the following.

3.2 Properties of Inverse α-Cut

Theorem 1. Let $A, B \in F(X)$. Then for any $\alpha \in [0, 1]$, the following properties hold:

$$A_1^{-1} = X \text{ (for a weak inverse } \alpha\text{-cut)} \tag{5}$$

$$A \subseteq B \text{ iff } B_\alpha^{-1} \subseteq A_\alpha^{-1} \tag{6}$$

$$A = B \text{ iff } A_\alpha^{-1} = B_\alpha^{-1} \tag{7}$$

Proof. We prove only (6). For all $x \in X$, $A \subseteq B$ iff $\mu_A(x) \leq \mu_B(x)$ iff if $\mu_B(x) \leq \alpha$ then $\mu_A(x) \leq \alpha$ iff if $x \in B_\alpha^{-1}$ then $x \in A_\alpha^{-1}$ iff $B_\alpha^{-1} \subseteq A_\alpha^{-1}$.

Theorem 1 establishes that the property of fuzzy set equality is inverse α-cutworthy, and property of fuzzy inclusion is inverse α-cutworthy reversely, which is parallel to the fact about α-cut [1].

Table 1. Fuzzy set of BIS218 examination results

SID	Exam marks	Normalized	Final Result
s1	43	0.43	Fail
s2	50	0.5	P
s3	53	0.53	P
s4	58	0.58	P
s5	62	0.62	P
s6	63	0.63	P
s7	65	0.65	C
s8	66	0.66	C
s9	68	0.68	C
s10	70	0.7	C
s11	71	0.71	C
s12	73	0.73	C
s13	75	0.75	D
s14	78	0.78	D
s15	79	0.79	D
s16	81	0.81	D
s17	85	0.85	HD
s18	86	0.86	HD

Theorem 2. For any $A \in F(X)$ and any given $\alpha, \beta \in [0, 1]$, the following properties hold:

$$\alpha \leq \beta \Rightarrow A_\alpha^{-1} \subseteq A_\beta^{-1} \text{ (Monotonicity)} \tag{8}$$

$$A = \bigcup_{\alpha \in [0, 1]} \alpha A_\alpha^{-1} \tag{9}$$

Where αA_α^{-1} denotes a non-fuzzy set defined by [6]:

$$\mu_{\alpha A_\alpha^{-1}} = \alpha \mu_{A_\alpha^{-1}}(x), \; x \in X$$

Proof. We prove only (8). For each given $x \in A_{\alpha}^{-1}$ then $\mu_A(x) < \alpha$ because, $\alpha \le \beta$ hence $\mu_A(x) < \beta$ that is, $x \in A_{\beta}^{-1}$.

Theorem 2 demonstrates that the inverse α-cuts preserve a kind of monotonicity. Eq. (9) implies that the inverse α-cuts can also be considered as a bridge between fuzzy sets and crisp sets. It also demonstrates that, like the α-cuts, the principal role of the inverse α-cuts is their capability to represent fuzzy sets Klir GJ, Yuan B. Fuzzy Sets and Fuzzy Logic: Theory and Applications. Prentice Hall, Upper Saddle River, NJ, 1995 (1995, p. 39), that is, every fuzzy set can uniquely be represented by the family of all its inverse α-*cut*s. This representation allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts.

3.3 Inverse α-Cut Decomposition

Using α-cuts to decompose any fuzzy set is not a new topic [1]. However, such a decomposition is not a partition of a set, because the intersection of any two α-cuts might not be empty. In what follows, we will propose a partition of every fuzzy set in $F(X)$ using α-cuts and inverse α-cuts.

Theorem 3. For every $A \in F(X)$ and any particular $\alpha \in [0, 1]$, the following property holds:

$$X = A_{\alpha} \cup A_{\alpha}^{-1},\ A_{\alpha} \cap A_{\alpha}^{-1} = \varnothing \tag{10}$$

This theorem establishes that set X can be decomposed by an α-cut and its inverse α-cut. Further, it also shows that the introduction of inverse α-cut concept is of significance in fuzzy set theory. However, this is not a decomposition of the fuzzy set A but that of X. How to find a decomposition of the fuzzy set A is an interesting topic. We will examine it in the next section.

4. 4. α-Induced Fuzzy Set and Its Inverse

Definition 2. Let $A \in F(X)$ and $\alpha \in [0, 1]$. Then the fuzzy set A_{α}^{f} is called α- induced fuzzy set, if

$$A_{\alpha}^{f} = \{(x, \mu_{A_{\alpha}^{f}}(x)) \mid x \in X\} \text{ and}$$

$$\mu_{A_{\alpha}^{f}}(x) = \begin{cases} \mu_A(x), & if\ \mu_A(x) \ge \alpha \\ 0, & \text{otherwise} \end{cases}$$

For example, in the fuzzy set of BIS 218, $A_{0.85}^{f} = \{(s_1, 0), (s_{16}, 0), (s_{17}, 0.85), (s_{18}, 0.86)\}$. Hence, $A_{0.85}^{f}$ only emphasizes the students who obtained a HD and their examination marks. Further, we can get the precise information of those who has obtained a HD and how many marks the corresponding students have obtained from α-induced fuzzy set $A_{0.85}^{f}$, whereas we can only obtain the information about who have obtained a HD from α-cut $A_{0.85}$. Therefore, α- induced fuzzy set preserves some important information comparing with the α-cut. This is significant for any fuzzy or incomplete information reasoning, because any informa- tion loss will, sometimes, lead to invalid consequence.

Definition 3. Let $A \in F(X)$ and $\alpha \in [0, 1]$. Then the fuzzy set A_{α}^{f-1} is called *inverse* α-induced fuzzy set, if

$$A_{\alpha}^{f-1} = (x, \mu_{A_{\alpha}^{f-1}}(x)) \mid x \in X \text{ and}$$

$$\mu_{A_{\alpha}^{f-1}}(x) = \begin{cases} \mu_A(x), & if\ \mu_A(x) < \alpha \\ 0, & \text{otherwise} \end{cases}$$

For example, in the fuzzy set of BIS 218, $A_{0.5}^{f-1} = \{(s_1, 0.43), (s_2, 0), \ldots, (s_{18}, 0)\}$. Hence, $A_{0.5}^{f-1}$ only emphasizes the students who did not pass the examination and their examination marks. Further, we can get the precise information about who did not pass the examination and how many marks the corresponding students obtained from inverse α-induced fuzzy set $A_{0.5}^{f-1}$, whereas we can only obtain the information about who did not pass the examination from the inverse α- cut $A_{0.5}^{-1}$. Therefore, inverse α-induced fuzzy set also preserves some important information comparing with the inverse α-cut.

Theorem 4. For every $A \in F(X)$ and $\alpha \in [0, 1]$, the following property holds:

$$A = A_{\alpha}^{f} \cup A_{\alpha}^{f-1},\ A_{\alpha}^{f} \cap A_{\alpha}^{f-1} = \varnothing \tag{11}$$

Proof. For each particular $x \in X$ and any given $\alpha \in [0,1]$, let $\beta = \mu_A(x)$. If $\beta \geq \alpha$ then,

$$\mu_{A_\alpha^f}(x) = \begin{cases} \mu_A(x) & \mu_A(x) \geq \alpha \\ 0 & \mu_A(x) < \alpha \end{cases}$$

$$= \mu_{A_\alpha^f}(x) = \mu_A(x);$$

$$\mu_{A_\alpha^{f-1}}(x) = \begin{cases} \mu_A(x) & \mu_A(x) < \alpha \\ 0 & \mu_A(x) \geq \alpha \end{cases}$$

$$= \mu_{A_\alpha^{f-1}}(x) = 0.$$

If $\beta < \alpha$, then

$$\mu_{A_\alpha^f}(x) = \begin{cases} \mu_A(x) & \mu_A(x) \geq \alpha \\ 0 & \mu_A(x) < \alpha \end{cases}$$

$$= \mu_{A_\alpha^{f-1}}(x) = \mu_A(x),$$

$$\mu_{A_\alpha^{f-1}}(x) = \begin{cases} \mu_A(x) & \mu_A(x) < \alpha \\ 0 & \mu_A(x) \geq \alpha \end{cases}$$

$$= \mu_{A_\alpha^f}(x) = 0.$$

Therefore, the theorem is proved.

The Eq. (11) can therefore be expressed as

$$A_\alpha^{f-1} = A - A_\alpha^f \tag{12}$$

That is, the inverse of the α-induced fuzzy set can be considered as the complement of itself. It is a new complement for any fuzzy subset.

So far, we have examined the extended forms of α-cut: inverse α-cut, α-induced fuzzy set and its inverse and their properties. However, these concepts and properties are only related to a special value or a "point" α. In other words, we can call them point concepts and point properties. In fact, there are many real world problems that require interval related concepts and properties. For example, the educational manager has more interest in the following questions:

- How many students have obtained a Credit in the BIS218 examination?
- Who has obtained a Credit in the BUS218 examination?

Obviously, we cannot answer these questions only based on α-cuts or inverse α-cuts. Because a Credit will be granted to a student who obtained not less than 65 (i.e. 0.65 in the corresponding fuzzy set) marks and less than 75 (0.75) marks, we introduce interval–cut related concepts to model this class of real world problems in the following section.

4. Interval $[a,b)$-Cut and Its Properties

Definition 4. Let A be a fuzzy set in X, that is, $A \in F(X)$, and $a, b \in [0,1]$, $a < b$. Then the non-fuzzy set $A_{[a,b)}$ is called interval $[a,b)$-cut, if

$$A_{[a,b)} = \{x \mid \mu_{A_{[a,b)}}(x) = 1\} \text{ and}$$

$$\mu_{A_{[a,b)}}(x) = \begin{cases} 1 & a \leq \mu_A(x) < b \\ 0 & otherwise \end{cases}$$

Because there are still other two interval forms for any given a and b: $(a,b]$, $[a, b]$, one can introduce similar interval cut concepts in order to model the corresponding real world problems. However, we would not go into them any more in this paper.

Definition 5. Let A be a fuzzy set in X, that is, $A \in F(X)$, and $a, b \in [0,1]$, $a < b$. Then the fuzzy set $A^f_{[a,b)}$ is called interval $[a,b)$-induced fuzzy set, if

$$A^f_{[a,b)} = \{x \mid \mu_{A^f_{[a,b)}}(x) = 1\} \text{ and}$$

$$\mu_{A^f_{[a,b)}}(x) = \begin{cases} \mu_A(x) & a \leq \mu_A(x) < b \\ 0 & otherwise \end{cases}$$

For example, in the fuzzy set of BIS 218 examination, $A^f_{[0.65,0.75)} = \{(0.65/0.65), (0.70/0.70), (0.75/0.75)\}$, which emphasizes who have not passed the examination of BIS218 and their corresponding marks.

Theorem 5. For every $A \in F(X)$ and any interval decomposition of $[0,1]: [a_i, b_i)$, where $a_1 = 0$, $a_i < b_i$, $a_{i+1} < b_i$, $b_n = 1$, $i = 1, \ldots, n$, then

$$A = \bigvee_i A^f_{[a_i,b_i)}, \tag{13}$$

$$A^f_{[a_i,b_i)} \wedge A^f_{[a_j,b_j)} = \varnothing \text{ if } a_i \neq a_j. \tag{14}$$

Proof. For each particular $x \in X$, let $\beta = \mu_A(x)$. Then there exists only one interval, e.g.

$[a_j, b_j)$ such that $\beta \in [a_j, b_j)$,

$$\mu_{A^f_{[a_i,b_i)}}(x) = \min_i [\mu_{A^f_{[a_i,b_i)}}(x)]$$

$$= \mu_{A^f_{[a_j,b_j)}}(x) = \mu_A(x).$$

Further, if $a_i \neq a_j$, then

$$[a_i, b_i) \cap [a_j, b_j) = \emptyset,$$

Hence,

$$\mu_{A^f_{[a_i,b_i)} \cap A^f_{[a_j,b_j)}}(x) = \min(\mu_{A^f_{[a_i,b_i)}}(x), \mu_{A^f_{[a_j,b_j)}}(x))$$

$$= \min(0, \mu_A(x)) = 0.$$

Therefore, the theorem is proved.

This theorem demonstrates that the fuzzy set A can be decomposed into a family of interval-induced fuzzy sets satisfying Eq. (13) and Eq. (14). The $A^f_{[a_i,b_i)}$ in the theorem is called *interval induced decomposition* of the fuzzy set A.

For example, in the fuzzy set of BUS218 examination results, let

$$[a_1, b_1) = [0, 0.5), \quad [a_2, b_2) = [0.5, 0.6), \quad [a_3, b_3) = [0.6, 0.7), \quad [a_4, b_4) = [0.7, 0.8), \quad [a_5, b_5) = [0.8, 1.0), \quad 1$$ [1]

then the corresponding interval induced decomposition of the mentioned fuzzy set is as follows (we ignored the elements with $\mu_A(x) = 0$ for brevity):

$$A^f_{[0,0.5)} = \{0, 0.4, 0.3\},$$

$$A^f_{[0.5,0.6)} = \{0, 0.4, 0.3, 0.6, 0.7, 0.8, 0.9\},$$

$$A^f_{[0.6,0.7)} = \{0, 0.4, 0.3, 0.6, 0.7, 0.8, 0.9\},$$

$$A^f_{[0.7,0.8)} = \{0, 0.4, 0.3, 0.6, 0.7, 0.8, 0.9, 1\},$$

$$A^f_{[0.8,1.0)} = \{0, 0.4, 0.3, 0.6, 0.7, 0.8, 0.9, 1\}.$$

These fuzzy subsets answered the questions that the educational manager concerned at the end of the previous section.

1. In practice, 1.001 is used for the interval including 1.

6. Conclusions

This paper introduced the concept of inverse α-*cut*, α-induced fuzzy set, interval cut, interval induced fuzzy set, and examined their properties. The proposed approach will facilitate research and development of fuzzy set theory and fuzzy systems with applications.

In the future work, we will apply the proposed approach in e-commerce, experience management, and computer science.

7. References

[1] Klir GJ, Yuan B. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice Hall, Upper Saddle River, NJ, 1995

[2] Kruse R, Gebhardt J and Klawonn. *Foundations of Fuzzy Systems*. John Wiley & Sons, Chichester. New York, 1994

[3] Pedrycz W. *Fuzzy Sets Engineering*. CRC Press, Boca Raton and London, 1995

[4] Sun Z, Finnie G, Weber K. Case base building based on similarity relations. *Infor. Scie.*165 (2004) 21-43

[5] Sun Z, Finnie G. *Intelligent Techniques: A Case based Reasoning Perspective*, Heidelberg: Springer 2004

[6] Zadeh, LA. Similarity relations and fuzzy orderings. *Infor. Scie*. 3 (1971) 177-200

[7] Zaheh, L.A. Fuzzy sets. *Information and Control*. 8 (1965) 338-353

[8] Zimmermann H.-J. *Fuzzy Set Theory and Its Applications* (3rd Edn), Dordrecht: Kluwer Academic Publishers, 1996.