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Mohan Chinnappan
University of Wollongong, mohan@uow.edu.au

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A Problem-Based Schema Analysis in Algebra

Mohan Chinnappan
University of Wollongong
<mohan_chinnappan@uow.edu.au>

The development of students' algebraic understanding is generally accepted to be one of the major goals of K-12 mathematics teaching. In this paper I attempt to examine this understanding by characterising a group of high school students' algebraic knowledge and patterns of use of that knowledge during the solution of selected problems. Results show that these students tended to show acceptable levels of proficiency with problems that involve substitution of values for variables, and simplification of equations. However, students experienced difficulties with the solution of equations and the interpretation of variables both in symbolic and graphical modes. These results are interpreted as suggesting that the participating students' understanding was buttressed mainly by schemas that were dominated by procedural knowledge of algebra.

Introduction

Algebra provides conceptual foundation for the understanding of other concepts that students encounter in the school mathematics curriculum. The importance of this area of mathematics has been underlined by the increasing attention the teaching and learning of algebra has received over the past decade from teachers and researchers alike. Children's understanding of algebraic concepts begin in the early years of their school life and continues throughout their mathematics learning experiences in high school and beyond.

The ubiquity of the subject matter of algebra in K-12 mathematics curriculum further attests to its critical role in helping students develop an appreciation of links that exists among other topics in mathematics. Indeed, this issue has been given considerable attention in the agenda of major curricular documents (National Council of Teachers of Mathematics, 1989, 2000). The recently concluded 12th ICMI Study on the theme, 'The Future of the Teaching and Learning of Algebra' further highlights the importance of algebra and brought into focus the many difficulties faced by students in learning algebra.

Despite significant strides that we have made in improving students' confidence and competence in using algebraic skills and concepts, it has been suggested that more work needs to be done in this area as students continue to experience difficulty in going beyond the meaningless manipulation of equations and symbols (Chazan, 1996). In the study reported here I address this issue by exploring the nature of algebraic knowledge that drives students' cognition during problem solving.

Theoretical Considerations

Connections and Mathematical Understanding

The development of mathematical understanding has been analysed from a number of vantage points. Of these, the investigation of connections constructed by students has been the theme of recent debate. It has been suggested that children learn mathematics best when they are encouraged to 'organise their information through making many connections and forming relationships' (Sowder, 2001, p. 4). Hence the analysis of connections seem to

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provide an effective research strategy in the examination of mathematical understanding. The focus on connections has had a long history in psychological literature on concept development and problem solving not only in mathematics but also other domains such as geometry, chess and physics. In their analysis, Carpenter and Lehrer (1999) characterised mathematical understanding as involving problem solving, constructing relationships and reflecting on one's own previous experiences with a particular topic of mathematics. As mathematical understanding is a developmental process connectionist models are appropriate for describing relations between the above activities. The quality of the connections can also be expected to have a major impact on how that knowledge is used in a variety of learning situations (Schoenfeld, 1992).

Research from cognitive psychologists and mathematics educators has advanced several theoretical frameworks about concepts and their growth. In this paper, I adopt the network perspective in making judgments about mathematics knowledge development. According to this view conceptual growth and mathematical understanding can be interpreted in terms of the building of organised knowledge clusters called *schemas*. Schemas can be visualised as knowledge structures or chunks having one or more core concepts which are connected to other concepts and/or schemas by relational statements.

According to this framework of knowledge development, the quality of a schema is a function of two variables: the spread of the network and the strength of the links between the various components of information located within the network (Anderson, 2000). A complex schema can be characterised as having a large number of network of nodes that are built around one or more core concepts. Further, in a mature schema, the links between the various nodes in the network are robust, a feature which contributes to the ready accessing and use of that schema during problem-solving and other learning situations. A well-structured schema can also benefit students by helping them assimilate incoming new mathematical information with less cognitive effort.

The acquisition of mathematical concepts in K-12 can be seen as the construction of schemas each with differing levels of organization and complexity. The difference between a good student and a poor student is that the good student has built up schemas that are more complex, dense and better organised than his low-achieving peer. Chinnappan (1998) used the schema framework to compare the quality of geometry knowledge between high- and low-achieving students. Thus, a useful strategy would be to analyse the schemas of students for gaps in their knowledge, and organisational quality of conceptual nodes and links. According to the schema framework of knowledge and performance, an impoverished schema is not conducive to solving novel problems and describing relations among concepts in mathematics because it does not help students extend their prior knowledge to new boundaries of understanding. Such schemas can be characterised as having a limited number of conceptual points to connect with.

Structure of Algebraic Schema

While schema provides a broad theoretical framework for analysing organisational features of students' algebraic knowledge there is a need to disentangle components of the schema that underlies algebraic understanding. Literature on the development of algebraic understanding has advanced two constructs: procedural and structural conceptions. Broadly speaking, students who have attained procedural understanding can be expected to perform operations involving algebraic expressions such as simplifying equations. Conceptual understanding, on the other hand refers to the elucidation of relations between algebraic expressions and components that make up a particular algebraic statement. Sfard (1991) used a *process-object* model to articulate the relationship between procedural and conceptual elements of algebra.

Students' experiences with algebra begin with the acquisition of knowledge about procedures or operations that are used in dealing with algebraic situations. These procedures include strategies and rules for simplifying, factoring and solving equations. Also included in this set of skills is an understanding of conventions and symbols that are used to represent algebraic expressions. One such convention could be the use of letters to represent variables or $f(x)$ to represent function of x. As students' experiences with algebra matures they are able to transfer knowledge of procedures to conceptual characteristics of relations. Kieren (1992) referred to this advancement as the evolution from the 'procedural to structural' (p. 413). She argued that most students learn procedural skills but do not make the transition to structural understanding. Tall and Thomas (1991) also alluded to this link in their analysis of the nature of difficulties faced by students in learning algebra. Students who have developed multiple representations of an algebraic relationship can be expected to show high levels of structural understanding of variables that are embedded in that relationship.

The above analysis suggests that, among other things, algebraic schema consists of networks of nodes that are procedural and conceptual in nature. Accordingly, students who have built up a better connected and organised algebraic schema can be expected to make a smooth transition from the procedural to the conceptual aspects of the knowledge structure. For instance, in order for students to develop a sophisticated schema, say, about solution of quadratic equations, they need to make multiple connections among variables, families of equations and unknowns. As their schema becomes elaborated further one might expect to see information about how to use strategies in order to construct equations to model a problem situation. In this sense the maturation of schemas can be seen as progressing along the procedural-conceptual continuum. Thus, the construction of algebra schemas that are loaded with procedural information can be argued to be less complex than one that has more conceptual information. In a problem situation both components of the knowledge base are necessary but conceptual knowledge is more useful in generating powerful representations of a given problem. In this sense, one could argue that the conceptual part of the schema in indicative of deeper learning of algebra.

The purpose of the present study was to describe the quality of algebraic schema developed by a group of Year 10 students. In particular, I was interested to examine the level of procedural and conceptual knowledge that students could access in a number of problem contexts, and the integration of these knowledge components during problem representation. The research questions for this study are:

- What is the nature of procedural knowledge that Year 10 students activate during the solution of algebra problems?
- What is the nature of conceptual knowledge that Year 10 students activate during the solution of algebra problems?
- Is there evidence of transition from procedural to conceptual understanding of algebra problems among Year 10 students?

Method

Participants

The participants in this study consisted of 58 Year 10 (Form 4) students (28 males and 30 females) in a New Zealand secondary college. The students came from two classes representing 'average to above average' ability levels of the college's Year 10 cohort. All students had completed algebra topic. Students in the study also reflected the socio-cultural composition of the local community.

Instrument

The *Algebra Schema and Access Instrument* (ASAI) was developed for the study. The instrument contained 16 problems all of which required the accessing and use of algebraic knowledge. Problems 9, 14 and 15 consisted of two (9a, 9b), three (14a, 14b, 14c) and two parts (15a, 15b) respectively. An important consideration in the development of the instrument was the identification of algebraic schemas that teachers would expect from their Year 10 students. It is important to point out that I have used a problem-based schema identification approach, and that schemas activated by the students were necessarily limited by the problem contexts. It is possible that a non-problem based strategy could be expected to generate a different set of schemas. However, a problem-driven schema activation and use by the students could be argued to provide a more complete picture about the quality of schemas that students have built up because it has the potential to reveal more complex connections that exist not only within schemas but among schemas. The latter complex of connections among schemas has been argued to exert a major influence in the construction of problem representations (Sweller and Cooper, 1985). This line of reasoning was used in classifying the 16 problems into six representations (Table 1). A selected set of the problems from each category is provided in the Appendix.

Table1
Problem Categorisation

Problems	Category	Representations
1 and 2	A	Factorisation
3 and 4	B	Evaluation
5, 6, 7, 8, 9 and 10	C	Solution of equations
11, 12 and 13	D	Word problems
14a, 14b and 14c	E	Pattern generation
15a, 15b and 16	F	Graphical interpretation

Procedure

The class teachers administered the instrument to students during normal class periods. The study was conducted in the fourth term of the college's academic year. Students were given 60-90 minutes to complete the problems. Students were encouraged to attempt all the 16 problems. They were also asked to write down every step in their solution attempts even if they did not arrive at the 'correct' answer. Students were permitted to use calculators if required.

The following scoring scheme was developed to code students' solution attempts. There were two major considerations in this scheme: solution approach and generation of relevant values. The former was concerned with problem representation and the latter factor provided information on the use of schemas to generate values that were relevant to the problem representation. While solution outcome was important it was not the sole factor in the scoring scheme. The scheme was trialled with two independent coders who were mathematics teachers and researchers in order to resolve potential differences in the interpretation of the codes. The final scoring system was used to code students' solution transcripts.

0 - No attempt was made to solve the problem
1 - Solution was attempted but both the approach and values generated were incorrect
2- - Solution was attempted with a correct approach but none of the values generated were correct or relevant; incorrect solution outcome.
3 - Solution was attempted with a correct approach; one correct value was generated; incorrect solution outcome
4 - Solution was attempted with a correct approach; two correct values were generated; incorrect solution outcome
5 - Solution was attempted with a correct approach; three or more correct values were generated; correct solution outcome.

Results and Discussion

Results of analysis of students' solution attempts is presented in Table 2. The results show that students attempted all the 16 problems with varying degrees of success. The means and standard deviations indicate that some problems were more difficult (Problems 6-10, 13-16) than others (Problems 1, 2, 3, 11 and 12). From the representational angle the solution attempts reveal a number of patterns. Firstly, students constructed correct representations for most of the problems in category A and B, and a few problems in category C. In category D, students experienced more success with problems 11 and 12 in comparison with 13. In general, students experienced difficulty with the solution of all the problems in categories E (pattern generation) and F (graphical representation). In more than 50% of the problems presented students scored mean values of 1 or less indicating failure to attempt or construction of incorrect problem representations.

Table 2
Descriptive Statistics of Solution Scores

Problems	Mean	SD
1	2.55	1.59
2	3.59	1.60
3	4.64	1.18
4	2.19	2.08
5	2.12	1.76
6	1.69	1.85
7	1.71	1.36
8	1.48	1.53
9a	2.17	2.15
9b	0.66	1.19
10	1.09	1.42
11	3.48	1.75
12	4.48	1.41
13	0.93	0.72
14a	2.40	2.25
14b	1.64	2.13
14c	1.21	1.87
15a	1.48	1.75
15b	1.21	1.63
16	1.02	1.10

The present study attempted to answer three questions that are related to high school students' knowledge and understanding of algebraic concepts by examining schema activation and problem representation. The first question aimed at describing the quality of procedural knowledge of their schemas. Analysis of solution attempts to problems that mainly involved application of procedural skills (Category A and B) indicate that students had acquired a reasonable level of procedural knowledge of algebra. This was evidenced by the high proportion of success with problems that required substitution of numerical values into a given expression. Students also showed that they could expand and simplify algebraic expressions. However, students tended to experience difficulty in finding solutions to a number of equations that had a complex structure. Taken together, these results suggest that while students have built up a repertoire of process skills there were also knowledge gaps in their procedural knowledge. It is also possible that the solution of more complex equations in Category C required an understanding which needs to be supported by schemas that are more conceptually loaded. This suggests that students' algebraic schemas had more procedural than conceptual information.

Analysis of solution attempts relevant to research questions two and three focused on a) students' conceptual understanding of algebraic expressions, and b) evidence of establishing links between conceptual and procedural elements in the given problems. The mean scores indicate a high proportion of the students could not generate equations to model a given situation, and solve that equation. In addition, students experienced difficulty in representing and interpreting graphical forms of given equations (Category F). An equation is a symbolic form of a relationship that can be expressed in a graphical mode. A large number of students who participated in this study failed to translate the symbolic to the graphical form of a given equation and vice versa. The present results are consistent with findings of a number of other recent investigations of problem solving that involves modelling of problems in terms of algebraic expressions (Nathan, Kintch, & Young, 1992; Schoenfeld & Arcavi, 1988).

In the present study students performed poorly in a task that involved the establishment of a relation between two variables (M=1.21;SD=1.87). This problem can also be seen as exploring a pattern that exists between two sets of numbers. While a number of students could determine the value of one variable given the other, these students did not describe the overall relationship in any meaningful manner. These results reflect those obtained by Stacey (1989) who found that students had difficulty in reasoning that led to generalising a pattern among variables.

Students also did not seem to understand the notion of ordered pairs (x, y), and that the is a rule that connects values x with values of y. This misunderstanding was evident in the solution attempt of Problem 10 where students were required to decide if a point was a solution to the given equation. It would seem that these students had yet to establish a link between the coordinates of the point and the rule that was expressed in the equation. The mean score for this problem was 1.09 suggesting most students did not attempt or used an incorrect representation. Solution attempts to Problem 10 again provides further support to the claim that students' algebra schema lacked appropriate conceptual information.

On a more general level, the results of the present study are relevant to the debate over the causal relations between procedural and conceptual knowledge not only in the solution of algebra problems but mathematical problems. The present result is consistent with earlier research by Rittle-Johnson and Alibali (1999) who found that conceptual knowledge has a greater influence not only on the understanding of problems but also on the further development of procedural knowledge.

While it is too early to speculate, the present findings do suggest that teaching needs to focus on the development of both procedural and structural or conceptual aspects of algebra. It seems that higher levels of procedural skill development is a necessary but not a sufficient condition for students to solve problems that involve generation and manipulation of variables in an equation. Thus, classroom learning experiences need to make explicit the connections between these two aspects of algebraic knowledge.

This study represents a modest attempt at exploring the construction of representations for algebraic problems and the nature of schemas that support that construction. While there is some support here for the claim that teaching and learning algebra needs to focus on facilitating the building of more conceptually based schemas there is a need for a fine-grained analysis of algebraic schemas that drive problem representation.

References

Anderson, J. R. (2000). *Cognitive psychology and its implications*. Fifth edition. New York: W. H. Freeman & Company.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classroom that promote understanding* (pp. 19-32). Marwah, NJ: Lawrence Erlbaum.

Chazan, D. (1996). Algebra for all students? *Journal of Mathematical Behavior*; *15*(4), 455-477.

Chinnappan, M. (1998). Schemas and mental models in geometry problem solving. *Educational Studies in*

Mathematics, 36(3), 201-217.
Kieren, C (1992). The learning and teaching of school algebra. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). NY: Macmillan.
Nathan, M. J., Kintch, W., & Young, E. (1992). A theory of algebra-word problem comprehension and its implications for the design of learning environments. *Cognition and Instruction, 9*, 329-389.
National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, Va.:The Council
National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, Va.:The Council
Rittle-Johnson, B., & and Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*(1), 175-189.
Sard, A. (1991). On the dual nature of mathematical conceptions: Reflections on the processes and objects as the different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.
Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grows (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). NY: Macmillan.
Schoenfeld, A. H., & Arcavi, A. (1988). On the meaning of variable. *Mathematics Teacher, 81*, 420-442.
Sowder, J. (2001). Connecting mathematics education research to practice. In J. Bobis, B. Perry and M. Mitchelmore (Eds.), *Numeracy and beyond* (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia, Vol 1, pp. 1-8), Sydney: MERGA.
Stacey, K. (1989). Finding and using patterns in linear generalising problems. *Educational Studies in Mathematics, 20*, 147-164.
Sweller, J., & Cooper, G. A. (1985). The use of worked examples as a substitute for problem solving in algebra. *Cognition and Instruction, 2*(1), 59-89.
Tall, D. O., & Thomas, M. O. J. (1991). Encouraging versatile thinking in algebra using the computer, *Educational Studies in Mathematics, 22*, 125–147.

Appendix

1. Factorise the expression $3x^2 + 6x - 9$

4. If $f(x) = 3x^2 - 7x$, what is the value of $f(2.5)$?

7. Solve the equation $x + \frac{6}{x} = 5$

11. A photograph is 3cm longer that it is wide. Its area is 40 cm^2.
Find its length and width.

14.

a) What is P when $Q = 3$?
b) What is P when $Q = n$?
c) Describe the relationship between P and Q.

Q	P
1	1
3	?
4	10
6	16
n	?

15(a). Graph the equation $5y = -15 + 3x$ in Figure A. (Figure A is a grid with x and y coordinates)

16 If you start with the light line, what would you do to get to the other line (dark)? (The light and dark lines were provided in a grid with x and y coordinates)