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# Modelling of multiplicative structures in a $B_{10}B$ Program

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## Abstract

Multiplication is a key operation in arithmetic. Teachers utilize a range of resources to help children make sense of the conceptual basis of this operation. Despite this, many children experience difficulty in solving multiplicative problems. In recent times, teachers and teacher educators have turned to ICT-based resources in order to provide a more effective learning environment in which to explore multiplication. While this change in teaching strategy has received considerable support, it is based on the assumption that teachers who are already in practice and those who are being trained to become teachers draw on a well-developed knowledge of the multiplicative process, and could exploit the ICT appropriately with the view to helping children construct deeper understandings about multiplication. The aim of the study reported here was to examine the quality of content knowledge of multiplication developed by a group of prospective elementary mathematics teachers in the context of an ICT-based software ($B_{10}B$ Program). Analysis of data showed the existence of gaps in the prospective teachers' knowledge base of multiplication. Specifically, the participants' repertoire of models of multiplicative process was found to be limited. I discuss these results in terms of primary teachers' skills and knowledge and the use of ICT for the construction of appropriate models of multiplication.

## Introduction

Children's informal understandings of whole numbers begin with their pre-school experiences. The development of this understanding is supported in the classroom through various exploratory activities that focus on numeration and operations involving numbers. The four fundamental operations of whole numbers are addition, subtraction, multiplication and division, and these are regarded as being 'central to knowing mathematics' (National Council of Teacher of Mathematics, 2000:41). The importance of understanding these operations lies principally in their utility in solving a multitude of real-life problems.

While the teaching of the above operations of arithmetic, in general, tend to begin with addition and subtraction, it has been suggested that the sequence of learning experiences provided to children must attempt to make explicit the connections among these operations (Putnam, Lambert & Peterson, 1990). These connections provide children with perspectives about what they mean and how they are related to one another.

The noted mathematics educator, Richard Skemp (1976), made a powerful statement about the difference between two forms of mathematical understanding: relational and instrumental. Relational understanding involves understanding structures and connections within concepts, whereas instrumental understanding shows ability to manipulate formulas and carry out operations. Skemp's articulation of mathematical understanding in this manner has been having a significant impact on the strategies that primary teachers adopt in constructing effective learning environment for their children. The modelling of abstract concepts is one such strategy, and this approach in teaching has been suggested as an appropriate way to bring about relational understanding of mathematics among young children (English and Halford, 1995). In this study I address the question of how teachers can utilize an Internet-based software to model multiplication, and explore its links with other operations with their children.

## Theoretical considerations

### Schema, modelling and exploration

Modelling involves the establishment of links among representations of a mathematics concept and its relationship to other concepts. More importantly, a model needs to externalize the links to the learner in ways that would help him or her visualize them. Lets consider the concept of symmetry in the study of geometry of shapes. There are a number of aspects to the understanding of symmetry such as reflection, rotation and being able to cut a figure into two identical parts. When children are introduced to symmetry, they begin by recognising the properties and creating shapes that symbolize symmetry. As their understanding matures, their representation of symmetry will include a network of nodes and relations that involve reflection, and rotation among others. These network of items of information form a *schema* for symmetry. Such a schema may also have information about applications of symmetry and rules/procedures about using symmetry in the solution of problems. If we return to our idea of modelling now, one can see that modelling involves a) the accessing of schemas for symmetry and b) the depiction of the relations that are embedded in that schema graphically or concretely.

Having constructed a model for a concept, teachers could go further and consider exploration of that model. Model exploration could involve activities that help children gain insight into the many interwoven connections that has been established among the relevant knowledge components of the model in question. Such an exploration could reveal the extension of network of schemas to new territories. Children can be expected to access higher levels of prior knowledge and attempt to integrate that knowledge with elements of the model that is being constructed. The modelling process could also contribute to the expansion of schemas associated with mathematics concepts.

Thus, model exploration activities can be expected to extend links that have already been built, and help children identify the various representations that reside within the model such as concepts, relations, patterns, and translations. Modelling activities must also have an inbuilt flexibility to help children externalize constituents of a model. These activities need to be grounded within the experiences of children. Exploration would also reveal children's ability to use that model to conjecture about other situations and solve problems.

### Modelling and the development of multiplicative structures

The above analysis of models and the processes underlying modelling has direct implications to the elucidation of knowledge that underlies children's understanding of multiplication. The framework of models suggests that understanding of multiplication and its applications are based on the quality of multiplicative structures or schemas that anchor a model in memory. It would seem that the teaching of multiplication must assist children explore its different meanings and properties as a means to building and expanding useful multiplicative structures. This outcome can be achieved by adopting a strategy in which teachers employ resources to model multiplication in different ways. The complex nature of multiplication is reflected in the number of models that one can construct.

Two models of multiplication are repeated addition and area/rectangular array. These macromodels are built on submodels which in turn are built on schemas of multiples and factors, grouping, properties of multiplication (commutative, associative, distributive) and multiplication algorithm. Repeated addition shows, for example, that 7 x 5 is equivalent to 7 + 7 + 7 + 7 + 7. It is important for children to see the relationship between addition and multiplication. That is, multiplying 7 by 5 is the same thing as adding seven fives together. Modelling should aim to help children discover that adding seven fives together will give you the same result as adding five sevens (commutativity). The use of rectangular arrays provides an effective way to help children visualize multiplication, but this strategy should be grounded in symbol manipulation, i.e. writing out 7 + 7 + 7 + 7 + 7 and seeing that it equals 35. Figure 1 shows modelling of 12 x 3.

An important conceptual structure underlying these cognitions is counting in multiples. Children must also be able to coordinate two composite units in the context of multiplication. For example, in a task involving multiplication of 6 x 3, children must visualize six groups of three. The understanding of place value is also a key requirement in performing multiplication operations involving whole numbers, as children ought to recognize that the magnitude of the product is always larger than any of the factors.

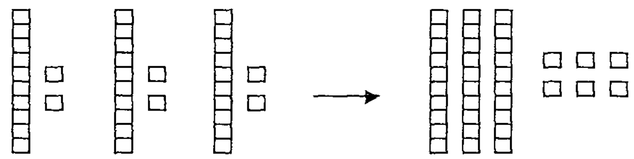

Figure 1: Repeated addition and multiplication

Children's grasp of multiplication as an operation could also be enhanced via engagement with a variety of real-life situations. One essential component of what it means to understand an operation is recognizing conditions in real-world situations that indicate that the operation would be useful in those situations (Vergnaud, 1988). Thus, modelling also needs to examine ways to make explicit the connections among numbers that are involved in multiplication and elements in real-life problems. The above analysis of the modelling process in the context of learning about multiplication raises an important issue about teacher knowledge that could influence not only modelling but also the appropriate use resources in order to model multiplication situations. Kaput (1986) argued that Information and Communication Technology (ICT)-related tools provided a dynamic learning environment to model and extend concepts and skills in the mathematics classroom. While this view has received considerable support from the teaching community, less is known about what motivates teachers in designing lessons that draws on these resources.

Shulman (1986) and Leinhardt (1987) expressed the view that research needed to investigate both teachers' subject-matter and pedagogical content knowledge that could drive teachers' actions. This issue also featured prominently in arguments advanced by Brown and Borko (1992) that there is a need to examine the development of knowledge base of prospective teachers of mathematics. I address this issue in the present study in two ways. Firstly, the study aims to identify the quality of pre-service teachers' subject-matter knowledge of multiplication, an area in K-6 mathematics that had proved to be difficult for this group (Clarkson, 1998; Tirosh & Graeber, 1989). Secondly, the present study attempts to generate data that is relevant to the debate about the relationship between the quality of teachers' subject-matter knowledge and the use of that knowledge to model multiplication within an ICT environment. Teachers who have built up a richer store of subject-matter (multiplication) and pedagogical content knowledge (modeling of multiplication) can be expected to exploit ICT more effectively than those with a weaker knowledge base.

**Growth of knowledge of multiplication**

Providing a visual representation enriches the development of children's understanding of multiplication in a number of ways. For example, the grouping notion of multiplication could be illustrated by using a combination of blocks. This representation of the concept of multiplication as addition of groups of objects was argued to provide analogs for better conceptualization (English and Halford, 1995). Young children are exposed to whole numbers and multiplication at an early age in a variety of real-life situations. Depending on individual experiences, each child could be expected to develop different meaning about multiplication. These experiences allow young children to develop a personal knowledge of multiplication, which matures through the learning situations they encounter in the classroom. Thus, the growth of understanding of multiplication could be characterized as involving a progressive change in the mixture of personal and formal knowledge. The models that teachers construct must aim to assist children bridge these two understandings.

The multifaceted nature of multiplicative process has made the task of describing its growth difficult. Several attempts have been made to capture the complexity of multiplication and children's construction of appropriate conceptual structures (Greer, 1992; Marshall, 1995). Their analysis showed that multiplication knowledge consists of many interwoven strands. An important outcome of this framework is the specification of cognitive structures (subconstructs) that provide support for the maturing of multiplicative structures among young children.

The above analysis indicates that multiplication is a complex operation and that prospective teachers need a rich knowledge base in this area of arithmetic in order to assist young children understand the many facets of the operation. The principal aim of the present study is to examine the quality of this knowledge base of prospective teachers. Data generated in this study were expected to inform both teacher educators and student teachers about potential knowledge gaps that might exist, and suggest appropriate diagnostic programs.

## $B_{10}B$ (Base 10 Blocks) Program

The $B_{10}B$ is Java-based Applet program developed by Bulaevsky (1999). The program consists of a panel as shown in Figure 2. On the left-hand side of the panel there are three different blocks each representing a unit, 10 units (*long*) and hundred units (*flat*) that can be dragged into the working panel. Children can then move, rotate, break, and glue the blocks to explore base-10 place values. Clicking on the base-ten chart allows children redefine the largest block to be a unit thus permitting explorations of decimal fractions. On the top row, there are eight icons. Icons 1-6 are useful for the performance of arithmetic operations. The hammer allows children to break a *long* into *units*. The lasso helps children to group and move pieces within any part of the panel. The second icon on the row permits children rotate any of the three blocks. The glue helps children group and create a shape with smaller blocks. The recycle bin is useful because it allows children to hide or thrash blocks that are not needed for an operation. This reduces clutter and help children focus on task at hand. For example, if children break a block into 23 units and only require some part of this, the other part can be put in the recycle bin.

In this manipulative system children can break apart the virtual blocks to decompose them into smaller blocks or glue groups of smaller blocks to make larger blocks. $B_{10}B$ encourages flexibility in children's approach to creating numbers. For example, if a child wants to make 89, she can pull out 8 *longs* (80) and 9 *units* (9) or she can pull out a *flat* (100), break it up so that she can use 90, and next break a *long* (from the 90) so that she can just use 9 *units*. These actions are based on understandings of groupings and regroupings that are consistent with the base-10 numeration system.

The facilities provided in the program confers an advantage over physical blocks in that with the latter children have to trade a collection of blocks for another block or vice versa. The system also imposes a limitation on the gluing and breaking of blocks, so that children cannot make incorrect regroupings. The system can be used for individual work or group work. Alternatively, teachers could lead a class discussion with $B_{10}B$.

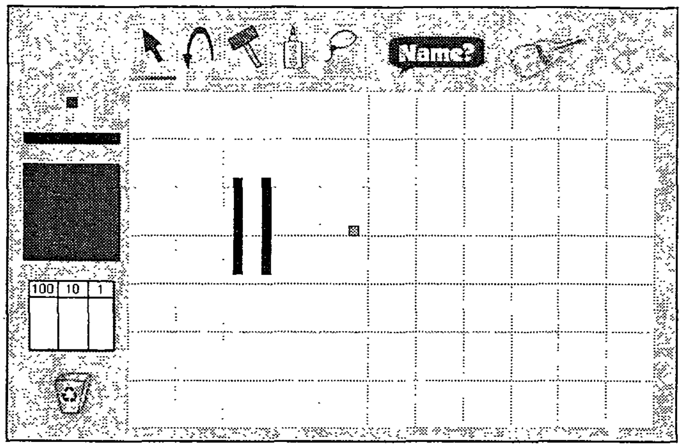


Figure 2: $B_{10}B$ Working Panel

## Method

### Participants

The participants in the present study were 15 preservice teachers enrolled in the third year of their BEd (Primary) program. Prior to the study, these student teachers have completed two mathematics methods subjects all of which emphasized constructivist principles in primary and early childhood mathematics teaching and learning. Before this study, they were involved in six weeks of teaching practice. During the two years prior to the study, the participants had also completed mathematics discipline requirements for the BEd (Primary), which included number, geometry and algebra. All of them had made use of computers during their courses, and exhibited reasonable levels of facility with ICT.

### Material and Procedure

The investigator met the group on two occasions. During the first meeting, which lasted about sixty minutes, the participants were informed about the project and asked to revise previous work that examined teaching arithmetic skills to K-6 children. At this meeting, participants were encouraged to engage in a discussion about concepts that are relevant to teaching numbers and operations involving whole numbers. A number of previous tutorial activities in which the student teachers had explored the teaching of numbers were also revisited, including the appropriate use of concrete material to help young children grasp numbers and operations.

During the second session, the participating student teachers downloaded $B_{10}B$ from the Internet and explored its use as a teaching and learning tool in the classroom. The investigator helped the participants interact with $B_{10}B$. Participants were given about 60 minutes to explore the menu and functions embedded in $B_{10}B$, and encouraged to raise questions. When the student teachers had indicated that they were happy and felt comfortable with $B_{10}B$ the investigator asked them to respond to two focus questions. Firstly, they were required to talk about ways in which the program could be used to teach multiplication involving one-digit and 2-digit numbers to third and fourth graders. The second question asked the participants to show how the program could be utilized to illustrate properties of multiplication in general. Participants' were asked to record their responses on the paper sheets that were provided. This second part of session 2 lasted between 60-80 minutes.

The transcripts were then analyzed for evidence of two groups of knowledge: content knowledge about multiplication and teaching this operation in $B_{10}B$. The former also included student teachers' articulation of properties of multiplication. The latter knowledge component examined two subcomponents: the modelling of multiplicative process, and exploitation of $B_{10}B$ for that purpose. Taken together, these two components could be regarded as providing insight into subject-matter and pedagogical content knowledge of student teachers. An important feature of this analysis was the many links that participating student teachers made among the above components of their knowledge base. The links were considered to be an additional index of richness of participants' knowledge.

Participants' responses were scored for activation of concepts, and the modelling of the concepts via $B_{10}B$. The activation of concepts was scored as follows: 0 – not activated, 1 – incorrect use or interpretation of the concept, 2 – partly correct use or interpretation of the concept and 3 – correct use or interpretation of the concept. Instances of modelling with $B_{10}B$ were measured by using a similar system: 0 – no evidence of modelling, 1 – incorrect modelling of the concept, 2 – partly correct modelling of the concept and 3 – correct modelling of the concept.

## Results

Table 1 shows results of analysis of knowledge components involving multiplication. Most of the prospective student teachers were successful in using $B_{10}B$ program for the purposes of displaying place value concepts involving whole numbers. In so doing, the participants not only displayed the concept of place value in the context of multiplication but also showed skills in using $B_{10}B$ to model that concept. Scores in columns 4 and 5 indicate that only two of the 15 student teachers were able to articulate the distributive property of multiplication, and model that property within $B_{10}B$. Scores in columns 6 and 7 show that none of the participants activated or attempted to model the commutative property. This characteristic of multiplication could easily be displayed by arranging the blocks in rectangular arrays.

Table 1: Properties of multiplication

| Participant | Place value | PVM | Distr | DM | Comm | CM |
|---|---|---|---|---|---|---|
| 1 | 3 | 3 | 0 | 0 | 0 | 0 |
| 2 | 3 | 3 | 0 | 0 | 0 | 0 |
| 3 | 1 | 1 | 3 | 3 | 0 | 0 |
| 4 | 3 | 3 | 0 | 0 | 0 | 0 |
| 5 | 3 | 3 | 0 | 0 | 0 | 0 |
| 6 | 3 | 3 | 0 | 0 | 0 | 0 |
| 7 | 3 | 3 | 0 | 0 | 0 | 0 |
| 8 | 3 | 3 | 0 | 0 | 0 | 0 |
| 9 | 3 | 3 | 0 | 0 | 0 | 0 |
| 10 | 3 | 3 | 1 | 1 | 0 | 0 |
| 11 | 3 | 3 | 0 | 0 | 0 | 0 |
| 12 | 3 | 3 | 0 | 0 | 0 | 0 |
| 13 | 3 | 3 | 0 | 0 | 0 | 0 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 14 | 3 | 3 | 0 | 0 | 0 | 0 |
| 15 | 3 | 3 | 0 | 0 | 0 | 0 |

PVM - modelling of place value; Distr - Distributive property; DM - Modelling of Distributive property; Comm - Commutative property; CM - Modelling of Commutative property

Further analysis of data on multiplication concepts and their modelling is presented in Table 2. The results here focus on representing multiplication as repeated addition/arrays, and identification of connections that could be established among these representations. In scoring for 'links' and their modelling I took into consideration all the concepts that appear in Tables 1 and 2. Table 2 shows that most of the participants viewed multiplication as repeated addition and demonstrated competency in modelling it within $B_{10}B$. About 50% of the student teachers attempted to show that multiplication could be conceptualized in the form of arrays or area of rectangles. However, only two of these prospective teachers succeeded in modelling this representation for multiplication.

Analysis of connections (columns 6 and 7) indicate that 8 out of the 15 participants attempted to construct or show how any one conceptualization and/or modelling could be related to the others. Further, almost all of their relational statements involved the concept of place value, and how to use $B_{10}B$ to highlight place value to children by using the different blocks in the program.

Table 2: Modelling of representations

| Participants | RA | MRA | ARR | MARR | Links | MLinks |
|---|---|---|---|---|---|---|
| 1 | 3 | 3 | 3 | 1 | 0 | 0 |
| 2 | 3 | 2 | 2 | 1 | 2 | 1 |
| 3 | 3 | 3 | 0 | 0 | 0 | 0 |
| 4 | 3 | 3 | 1 | 1 | 1 | 1 |
| 5 | 2 | 2 | 0 | 0 | 1 | 1 |
| 6 | 3 | 3 | 1 | 1 | 1 | 1 |
| 7 | 3 | 2 | 1 | 1 | 2 | 2 |
| 8 | 3 | 3 | 0 | 0 | 1 | 1 |
| 9 | 3 | 3 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 0 | 0 |
| 11 | 3 | 3 | 0 | 0 | 0 | 0 |
| 12 | 3 | 3 | 0 | 0 | 0 | 0 |
| 13 | 1 | 1 | 0 | 0 | 0 | 0 |
| 14 | 3 | 3 | 0 | 0 | 1 | 0 |
| 15 | 3 | 2 | 0 | 0 | 2 | 2 |

RA – Repeated addition, MRA - Modelling of Repeated Addition, ARR - Array; MARR - Modelling of Arrays; Links - Connections, Mlinks - Modelling of Links

Figure 3 shows a summary of the actions of one student teacher (Participant 3) in her attempt to model the distributive property of multiplication. In this instance, she attempted to show the following relation: 12 x 5 = (10 +2) x 5 = (10 x 5) + (2 x 5) = 50 + 10 = 60. This student teacher made appropriate use of not only the blocks but also the base-10 chart on the left, which generated the *x*- and *y*- axis on the panel. She showed 12 as 10 plus 2 on the y-axis by using one *long* and two units. Likewise, she placed 5 units on the *x*-axis. She also explained that the product could be grouped by using the lasso into 5 *longs* and 2 sets of 5 units (one *long*), forming 60 (6 *longs*). This student teacher also commented that it would be useful to be able to write the numbers along side the blocks thus indicating a desire to integrate symbolic representation within her model.

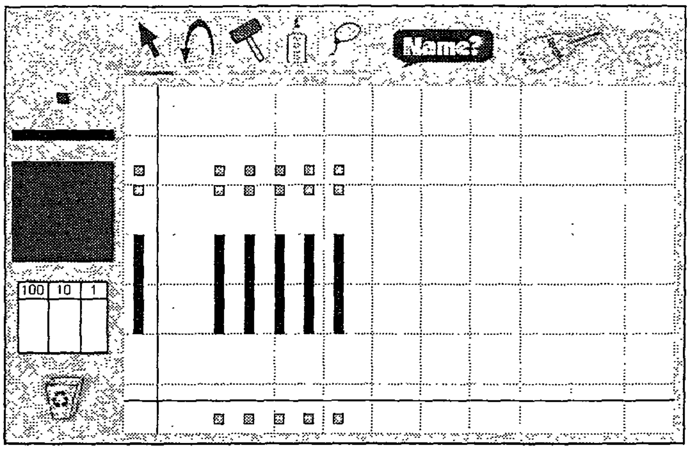


Figure 3: Modelling of distributive property by Participant 3

## Discussion

This study explored two components of the knowledge base of pre-service teachers: understanding of multiplication and its modelling with the aid of an ICT resource, $B_{10}B$. It was underpinned by the assumption that multiplication situations are complex in nature and that appropriate modelling of this concept could be an effective strategy in helping young children acquire a meaningful understanding of not only the concept but also examine connections to other operations involving whole numbers. It was hypothesized that student teachers with a greater repertoire of representations of multiplications and situations involving multiplications would also be more adept at using the ICT resource. More importantly, it was expected that this resource will be exploited to depict multiplication in ways that would develop links between children's implicitly held understandings and the more formal understandings that are expected in the mathematics curriculum.

Reactions from participating pre-service teachers indicated that they used the software in different ways to perceive and teach about multiplication. The most frequent response from the prospective teachers involved the use of $B_{10}B$ to show repeated addition and place value of numbers that are involved in multiplication. All the participants could draw on the place-value chart and made appropriate use of the *units* and *longs* to create numbers that were involved in the multiplicative process. As expected, none of the participants used the *flat* as the interview questions asked for multiplication of single and double-digit numbers. However, they could have used the *flats* to demonstrate properties of multiplication. These results suggest that the student teachers were more focused on showing that multiplication of two whole numbers could produce a third number that was larger than the initial two numbers. The increase in the size of the product was also portrayed well in the models that were constructed by the participants. Repeated addition and its modelling indicated that student teachers viewed multiplication as a form of addition. By grouping and gluing these blocks, and placing them on the place-value chart, the teachers also showed competency in demonstrating increase in the size of the product. The above approach could provide children with an opportunity to 'see' the connection between numeration and the computational process that was considered to be pivotal in understanding numbers and operations (Hiebert & Wearne, 1992).

The strategy of modeling multiplication as repeated addition in the $B_{10}B$ environment also addresses two key learning issues raised by Schwartz (1988) in relation to difficulties that could be experienced by young children when they shift from dealings with addition to multiplication. Unlike addition and subtraction, in multiplication situations children are expected to work with composite units as opposed to single units. Additionally, multiplication may involve either like or unlike quantities to produce a third quantity (the product). The use of $B_{10}B$ to model repeated addition appears to be an effective way to help children make the transition from their earlier experiences with addition and subtraction to multiplication. The results indicate that the participants did not activate much knowledge about the distributive and commutative properties of multiplication both of which were discussed in the tutorials that were held prior to the interviews. As argued elsewhere in this report, $B_{10}B$ provides an effective way to visualize and illustrate both these properties. It would appear the student teachers did not regard these as an integral aspect of teaching multiplication.

While there was evidence of generating array model of multiplication, a significant proportion of the student teachers did not exploit $B_{10}B$ for this purpose. This could be due to a lack of knowledge about this form of representation of multiplication. The array modelling could also be used effectively to demonstrate commutative properties. It was also expected that the participants would relate the numbers that were involved in the operation with real-life situations. For example, problems such as 'Five children have 10 balloons each. How many do they have altogether?' can be solved by showing the one-to-one relationship between balloons and children with blocks in $B_{10}B$. However, contrary to expectation, none of the participants saw the need to links $B_{10}B$ with real-life multiplication problems.

The failure of prospective teachers in the present study to provide more varied and potentially richer experiences with $B_{10}B$, and the modelling of multiplication could be due to a number of reasons. Firstly, the teachers did not have sufficient knowledge about multiplicative process. It was also possible that the focus questions employed in the present study did not provide sufficient prompts to encourage them search a wider knowledge base than that revealed by the results reported here. Future studies need to examine this aspect of the study more closely and develop more sensitive interview questions.

Pollard and Duke (2001) identified two groups of programmes that can be used in the primary mathematics classroom: process- and concept-oriented software. They argued that the process-oriented programmes help children perform operations without providing any information about the underlying reasoning and concepts that support the technique. $B_{10}B$ has features that are both process- and concept-oriented. Modelling of multiplication as repeated addition shows the relationship between process and concept. One limitation of the program is the lack of features to show symbolic representations of numbers that are involved in the operation. Teachers need to be aware of this shortcoming and modify learning activities appropriately so that they cater for individual differences.

It has been argued that ICT-based mathematics teaching offers advantages to children's learning (Wiest, 2001). However, the extent of pre-service teachers' use of ICT in the classroom depends on a range of factors such as their beliefs about mathematics, competency with ICT and their perceptions about the efficacy of technologically driven lessons in fostering learning among their children. The results reported here indicate that student teachers who had built different levels of understandings about multiplication tended to show greater proclivity towards not only modelling them but also exploiting the ICT with a view to engaging children in the modelling process.

In general, responses from the participants showed that interactions with the software had increased their awareness of the shortcomings in their own understanding of the subject matter. This situation could be attributed to the many representations of multiplication that could be generated by the software. There was also a consensus that the visual features of $B_{10}B$ would be significant in introducing the fun element in learning about multiplication.

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