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Abstract

Although generation of correlated Rayleigh fading envelopes has been intensively considered in the literature, all conventional methods have their own shortcomings, which seriously impede their applicability. In this paper, a very general, straightforward algorithm for generation of an arbitrary number of Rayleigh envelopes with any desired, equal or unequal power, in wireless channels either with or without Doppler frequency shifts, is proposed. The proposed algorithm can be applied in case of spatial correlation, such as with antenna arrays in Multiple Input Multiple Output (MIMO) systems, or spectral correlation between the random processes like in Orthogonal Frequency Division Multiplexing (OFDM) systems. It can also be used for generating correlated Rayleigh fading envelopes in either discrete-time instants or a real-time scenario. Besides being more generalized, our proposed algorithm is more precise, while overcoming all shortcomings of the conventional methods.

1. Introduction

Generation of correlated Rayleigh fading envelopes has been intensively examined in the literature. However, all conventional methods have their own shortcomings, which seriously limit their applicability. To point this out more clearly, we first analyze the shortcomings of some interesting conventional methods.

In [1], the authors derived fading correlation properties in antenna arrays and, then, briefly mentioned the algorithm to generate complex Gaussian random variables (with Rayleigh envelopes) corresponding to a desired correlation coefficient matrix $\frac{\hat{\mathbf{R}}_k}{\sigma_k^2}$. The subscript k denotes the kth user (see Section III in [1] for more details). In this algorithm, the authors generated a vector $\mathbb{C}_k$ of $2M$ *real, correlated* Gaussian random variables, based on a vector $\mathbb{A}_k$ of $2M$ *real, independent* Gaussian random variables. This algorithm was proposed for generating *equal power* Rayleigh envelopes only, rather than *arbitrary* (*equal or unequal*) power Rayleigh envelopes. Moreover, the desired correlation coefficient matrix $\frac{\hat{\mathbf{R}}_k}{\sigma_k^2}$ must be *positive semi-definite* in order that the vector $\mathbb{C}_k$ is a *real* vector (see Eq. (16) and (17) in [1]). Otherwise, the vector $\mathbb{C}_k$ is *complex* and, consequently, the *resultant* correlation coefficient matrix of the resultant Gaussian random variables (with Rayleigh envelopes) is *not* equal to the *desired* correlation coefficient matrix $\frac{\hat{\mathbf{R}}_k}{\sigma_k^2}$. This means that the algorithm in [1] *fails* to generate Rayleigh envelopes corresponding the *desired* correlation coefficient matrix $\frac{\hat{\mathbf{R}}_k}{\sigma_k^2}$ if this matrix is *not* positive semi-definite.

In the papers [2] and [3], the authors proposed different methods for generating only N=2 *equal power* correlated Rayleigh envelopes. In [4], the authors generalized the method in [3] for $N \geq 2$. However, in this method, Cholesky decomposition is used, and consequently, the covariance matrix must be positive definite, which is not always realistic.

These methods are then more generalized in [5], where one can generate *any number* of Rayleigh envelopes corresponding to a desired covariance matrix and with *any power*, i.e., even with *unequal* power. However, again, the covariance matrix *must be positive definite* in order for Cholesky decomposition to be performable. In addition, the authors in [5] forced the covariances of the complex Gaussian random variables (with Rayleigh fading envelopes) to be *real* (see Eq. (8) in [5]). This limitation is *incorrect* in various cases because, in fact, the covariances of the complex Gaussian random variables are more likely to be complex.

In [6], the authors proposed a method for generating *any number* of Rayleigh envelopes with *equal power* only. In this method, a covariance matrix $\mathbf{R}_{xx}$, which is *not* positive definite, is approximated by a positive definite covariance matrix $\hat{\mathbf{R}}_{xx}$. Consequently, the positive definiteness of the covariance matrix, which is the unavoidable requirement in order for Cholesky decomposition to be performable, is relaxed. Although the method in [6] works well in various cases, it however *fails* to perform Cholesky decomposition for some complex covariance matrices $\hat{\mathbf{R}}_{xx}$ in MatLab due to the roundoff errors of Matlab. This shortcoming is overcome by some modifications mentioned later in our proposed algorithm. Interestingly, we realize that this shortcoming can be overcome by using the similar method mentioned in [1] to calculate the coloring matrix of the covariance matrix $\mathbf{R}_{xx}$.

More importantly, the authors in [6] generated Rayleigh envelopes corresponding to a desired covariance matrix when Doppler frequency shifts were considered, such as in *fast fading channels*, by associating their proposed method with the method proposed in [7] (see Section VI in [6]).

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The method proposed in [7] is aimed at generating an *individual* Rayleigh process corresponding to a given autocorrelation function. In the method proposed in [6], the authors took the outputs of the method in [7] and *simply* input them into step 6 in their method.

However, the step 6 in the method proposed in [6] was proposed for generating complex Gaussian random variables with *fixed* (unit) variances. Meanwhile, passing Gaussian random variables with variances assumed to be unit (for simplicity of explanation) through a Doppler filter changes remarkably the variances of those variables. It means that the variances σ_g^2 of the variables at the outputs of Doppler filters are *not* unit any more. In general, they may have *arbitrary* values depending on the variance of the variables at the inputs of the filters as well as the characteristics of those filters.

The authors in [6] did not realize this variance-changing effect caused by Doppler filters, and consequently, the method proposed in [6] *fails* to generate Rayleigh fading envelopes corresponding to a desired covariance matrix in a real-time scenario *where Doppler frequency shifts are considered*, although the authors claimed otherwise in their paper. We will return this issue later in this paper.

For the aforementioned reasons, a *more generalized* algorithm, which can be used to generate *any number* of Rayleigh fading envelopes with *any power* (*equal or unequal* power) corresponding to *any desired* covariance matrix $\mathcal{K}$ and is applicable in both *time instant* scenario and *real-time* scenario, is required to be considered. The algorithm is also expected to overcome roundoff errors which may cause the interruption of Matlab programs. The algorithm should work well, regardless of the positive definiteness of the covariance matrices. Further, the algorithm should provide a straightforward method for the generation of complex Gaussian random variables (with Rayleigh envelopes) with correlation properties as functions of *time delay* and *frequency separation* (as in Orthogonal Frequency Division Multiplexing (OFDM) systems), or *spatial separation* between transmission antennas like with antenna arrays in Multiple Input Multiple Output (MIMO) systems. Finally, this method should allow us to start generation with *either* the desired variances (powers) of Rayleigh envelopes *or* with the desired variances (powers) of complex Gaussian random variables with Rayleigh envelopes, depending on our actual requirement. This paper proposes one of such algorithms.

The paper is organized as follows. In Sections 2 and 3, we shortly review the discussions on the correlation property between the transmitted signals as functions of time delay and frequency separation, such as in OFDM systems, and as functions of spatial separation between transmission antennas, such as in MIMO systems, respectively. In Section 4, we propose a very general, straightforward algorithm to generate correlated Rayleigh fading envelopes. Section 5 derives an algorithm to generate correlated Rayleigh fading envelopes in a real-time scenario. Simulation results are presented in Section 6. The paper is concluded by Section 7.

2. Fading correlation as functions of time delay and frequency separation

In [8], Jakes considered the scenario where all complex Gaussian random processes with Rayleigh envelopes have *equal* powers σ^2 and derived the correlation properties between random processes as functions of both time delay and frequency separation, such as in OFDM systems. Let $z_k(t)$ and $z_j(t)$ be the two zero-mean complex Gaussian random processes at time instant t, corresponding to frequencies f_k and f_j, respectively. Denote:

$$x_k \triangleq Re(z_k(t)); \qquad y_k \triangleq Im(z_k(t))$$
$$x_j \triangleq Re(z_j(t+\tau_{k,j})); \qquad y_j \triangleq Im(z_j(t+\tau_{k,j}))$$

where $\tau_{k,j}$ is the arrival time delay between two signals at the considered time instant and $Re()$, $Im()$ are the real and imaginary parts of the argument, respectively. By definition, the covariances between the real and imaginary parts of $z_k(t)$ and $z_j(t+\tau_{k,j})$ are:

$$R_{xx_{k,j}} \triangleq E(x_k x_j); \qquad R_{yy_{k,j}} \triangleq E(y_k y_j) \quad (1)$$
$$R_{xy_{k,j}} \triangleq E(x_k y_j); \qquad R_{yx_{k,j}} \triangleq E(y_k x_j) \quad (2)$$

Then, those covariances have been derived in [8] (see Eq. (1.5-20)) as:

$$R_{xx_{k,j}} = R_{yy_{k,j}} = \frac{\sigma^2 J_0(2\pi F_m \tau_{k,j})}{2[1+(\Delta\omega_{k,j}\sigma_\tau)^2]} \quad (3)$$
$$R_{xy_{k,j}} = -R_{yx_{k,j}} = -\Delta\omega_{k,j}\sigma_\tau R_{xx_{k,j}} \quad (4)$$

where

σ^2 is the variance (power) of the complex Gaussian random processes ($\frac{\sigma^2}{2}$ is the variance per each dimension).

J_0 is the zeroth-order Bessel function of the first kind.

F_m is the maximum Doppler frequency $F_m = \frac{v}{\lambda} = \frac{v f_c}{c}$. In this formulas, λ is the wavelength of the carrier, f_c the carrier frequency, c the speed of light and v the mobile speed.

$\tau_{k,j}$ is the arrival time delay between two signals

$\Delta\omega_{k,j} = 2\pi(f_k - f_j)$ is the angular frequency separation between the two complex Gaussian processes with Rayleigh envelopes at frequencies f_k and f_j.

σ_τ is the rms delay spread of the wireless channel.

It should be emphasized that, the equalities (3) and (4) hold only when the set of *multi-path channel coefficients*, which were denoted as C_{nm} and derived in Eq. (1.5-1) and (1.5-2) in [8], as well as the *powers* are assumed to be the *same* for different random processes (with different frequencies). Readers may refer to [8] (pp. 46–49) for an explicit exposition.

3. Fading correlation as functions of spatial separation in antenna arrays

The fading correlation properties between wireless channels as functions of antenna spacing in antenna arrays have been mentioned in [1]. Fig. 1 presents a typical model of the channel where all signals from a receiver are assumed to arrive at T_x transmit antennas within $\pm\Delta$ at angle Φ ($|\Phi| \leq \pi$). Let λ be the wavelength, D the distance between the two adjacent transmit antennas, and $z = 2\pi\frac{D}{\lambda}$. In [1], it is assumed that fades corresponding to different receivers are independent of one another. This is reasonable if receivers are not on top of each other within some wavelengths and they are surrounded by their own scatterers. Consequently, we only need to calculate the correlation properties for a typical receiver. The fade in the channel between a given k^{th} transmit antenna and the receiver may be considered as a com-

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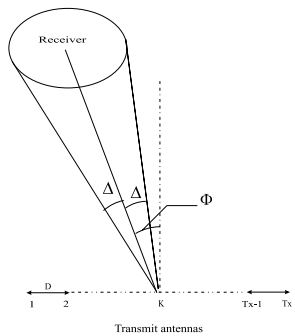

Figure 1. Model to examine the spatial correlation between transmit antennas.

plex, zero-mean Gaussian random variable, which is presented as $b^{(k)}$=$x^{(k)}$+$iy^{(k)}$. Denote the covariances between the real parts as well as imaginary parts themselves of fades corresponding to the kth and j^{th} antennas[1] be $R_{xx_{k,j}}$ and $R_{yy_{k,j}}$, while those terms between the real and imaginary parts of fades be $R_{xy_{k,j}}$ and $R_{yx_{k,j}}$ (similar to (1) and (2)). Then, it has been proved that the closed-form expressions of the above covariances normalized by the variance per each dimension (real and imaginary) are (see Eq. (A. 19) and (A. 20) in [1]):

$$\begin{aligned} \tilde{R_{xx_{k,j}}} &= \tilde{R_{yy_{k,j}}} = J_0(z(k-j)) \\ &+ 2\sum_{m=1}^{\infty} J_{2m}(z(k-j))cos(2m\Phi)\frac{sin(2m\Delta)}{2m\Delta} \end{aligned} \quad (5)$$

$$\begin{aligned} \tilde{R_{xy_{k,j}}} &= -\tilde{R_{yx_{k,j}}} = 2\sum_{m=0}^{\infty}\Big[J_{2m+1}(z(k-j)) \\ &\times sin[(2m+1)\Phi]\frac{sin[(2m+1)\Delta]}{(2m+1)\Delta}\Big] \end{aligned} \quad (6)$$

where $\tilde{R}_{k,j} = \frac{2R_{k,j}}{\sigma^2}$. In other words, we have:

$$R_{k,j} = \frac{\sigma^2\tilde{R}_{k,j}}{2} \quad (7)$$

J_q is the first-kind Bessel function of the integer order q, and $\sigma^2/2$ is the variance of the received signal at each transmit antenna per each dimension, i.e., it is assumed in [1] that the signals received from all transmit antennas have *equal* variances σ^2.

Similarly to Section 2, the equalities (5) and (6) hold only when the set of *multi-path channel coefficients*, which were denoted as g_n and derived in Eq. (A-1) in [1], and the *powers* are assumed to be the *same* for different random processes. Readers may refer to [1] (pp. 1054–1056) for an explicit exposition.

1 Note that k and j here are antenna indices, while they are frequency indices in the previous section.

4. Generalized algorithm to generate correlated Rayleigh fading envelopes

4.1. Covariance matrix of complex Gaussian random variables with Rayleigh fading envelopes

It is known that Rayleigh fading envelopes can be generated from the zero-mean complex Gaussian random variables. We consider here a column vector Z of N zero-mean complex Gaussian random variables with variances $\sigma_{g_j}^2$, j=1...N. Denote Z=$(z_1,\ldots,z_N)^T$, where z_j (j=1...N) is regarded as z_j=$r_je^{i\theta_j}$=x_j+iy_j. It is assumed that the phases θ_js are independent, identically uniformly distributed random variables. As a result, the real and imaginary parts of each z_j are independent (but z_js are not necessarily independent), i.e., the covariances $E(x_jy_j)$=0 for $\forall j$ and therefore, r_js are *Rayleigh envelopes*. Let $\sigma_{g_{xj}}^2$ and $\sigma_{g_{yj}}^2$ be the variances per dimension (real and imaginary), i.e., $\sigma_{g_{xj}}^2$=$E(x_j^2)$, $\sigma_{g_{yj}}^2$=$E(y_j^2)$. Clearly, $\sigma_{g_j}^2$=$\sigma_{g_{xj}}^2$+$\sigma_{g_{yj}}^2$. If $\sigma_{g_{xj}}^2$=$\sigma_{g_{yj}}^2$, then $\sigma_{g_{xj}}^2$=$\sigma_{g_{yj}}^2$=$\frac{\sigma_{g_j}^2}{2}$. Note that we consider a very general scenario where the variances (powers) of the real parts are not necessarily equal to those of the imaginary parts. Also, the powers of Rayleigh envelopes are not necessarily equal to one another. Therefore, the scenario where the variances of those envelopes are equal to one another and the powers of real parts are equal to those of imaginary parts, such as the one mentioned in either Section 2 or Section 3, is considered as a particular case. For $k\neq j$, define the covariances $R_{xx_{k,j}}$, $R_{yy_{k,j}}$, $R_{xy_{k,j}}$ and $R_{yx_{k,j}}$ between the real as well as imaginary parts of z_k and z_j similarly to those mentioned in (1) and (2). By definition, the covariance matrix $\mathcal{K}$ of Z is:

$$\mathcal{K} = E(ZZ^H) \triangleq [\mu_{k,j}]_{N\times N}$$

where H denotes the Hermitian transposition operation and

$$\mu_{k,j} = \begin{cases} \sigma_{g_j}^2 & \text{if } k \equiv j \\ (R_{xx_{k,j}}+R_{yy_{k,j}})-i(R_{xy_{k,j}}-R_{yx_{k,j}}) & \text{if } k \neq j \end{cases}$$

4.2. Forced positive semi-definiteness of the covariance matrix

First, we need to define the *coloring matrix* $\mathcal{L}$ corresponding to a covariance matrix $\mathcal{K}$. The *coloring matrix* $\mathcal{L}$ is defined to be the $N \times N$ matrix satisfying:

$$\mathcal{L}\mathcal{L}^H = \mathcal{K}$$

It is noted that the coloring matrix is *not* necessarily a lower triangular matrix. Particularly, to determine the coloring matrix $\mathcal{L}$ corresponding to a covariance matrix $\mathcal{K}$, we can use *either* Cholesky decomposition as mentioned in a number of papers, which have been reviewed in Section 1 of this paper, *or* eigen decomposition which is mentioned in the next section of this paper. The former yields a lower triangular coloring matrix, while the later yields a square coloring matrix.

Unlike Cholesky decomposition, where the covariance matrix $\mathcal{K}$ must be *positive definite*, eigen decomposition requires that $\mathcal{K}$ is at least *positive semi-definite*, i.e. the eigenvalues of $\mathcal{K}$ are either zeros or positive. We will explain later why the covariance matrix must be positive semi-definite

even in the case where eigen decomposition is used to calculate the coloring matrix. The covariance matrix $\mathcal{K}$, in fact, may *not* be positive semi-definite.

To overcome this obstacle, similarly to (but not exactly as) the method in [6], we approximate the covariance matrix being *not* positive semi-definite by a positive semi-definite one. This procedure is presented as follows.

Assuming that $\mathcal{K}$ is the desired covariance matrix, which is *not* positive semi-definite. Perform the *eigen decomposition* $\mathcal{K} = \mathbf{V}\mathbf{G}\mathbf{V}^H$, where $\mathbf{V}$ is the matrix of eigenvectors and $\mathbf{G}$ is a diagonal matrix of eigenvalues of the matrix $\mathcal{K}$. Let $\mathbf{G} = diag(\lambda_1, \ldots, \lambda_N)$. Calculate the approximate matrix $\mathbf{\Lambda} \triangleq diag(\hat{\lambda}_1, \ldots, \hat{\lambda}_N)$, where:

$$\hat{\lambda}_j = \begin{cases} \lambda_j & \text{if } \lambda_j \geq 0 \\ 0 & \text{if } \lambda_j < 0 \end{cases}$$

We now compare our approximation procedure to the approximation procedure mentioned in [6]. The authors in [6] used the following approximation:

$$\hat{\lambda}_j = \begin{cases} \lambda_j & \text{if } \lambda_j > 0 \\ \varepsilon & \text{if } \lambda_j \leq 0 \end{cases}$$

where ε is a small, positive real number.

Clearly, besides overcoming the disadvantage of Cholesky decomposition, our approximation procedure is *more precise* than the one mentioned in [6] since the matrix $\mathbf{\Lambda}$ in our algorithm approximates to the matrix $\mathbf{G}$ better than the one mentioned in [6]. Therefore, the desired covariance matrix $\mathcal{K}$ is well approximated by the positive semi-definite matrix $\mathbf{K} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$ from Frobenius point of view [6].

4.3. Determine the coloring matrix using eigen decomposition

In most of the conventional methods, except the one in [1], Cholesky decomposition was used to determine the coloring matrix. It has been well known that, Cholesky decomposition may not work for the covariance matrix which has eigenvalues being equal or close to zeros.

To overcome this disadvantage, similarly to (but not exactly as) the method proposed in [1], we use eigen decomposition, instead of Cholesky decomposition, to calculate the coloring matrix. It is important to note that estimating and comparing the computational efforts between the two methods (eigen decomposition versus Cholesky decomposition) are *not* our targets in this paper. The coloring matrix is calculated as follow:

At this stage, we have the forced positive semi-definite covariance matrix $\mathbf{K}$, which is equal to the desired covariance matrix $\mathcal{K}$ if $\mathcal{K}$ is positive semi-definite, or approximates to $\mathcal{K}$ otherwise. Further, as mentioned earlier, we have $\mathbf{K} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$, where $\mathbf{\Lambda} = diag(\hat{\lambda}_1, \ldots, \hat{\lambda}_N)$ is the matrix of eigenvalues of $\mathbf{K}$. Since $\mathbf{K}$ is a positive semi-definite matrix, it follows that $\{\hat{\lambda}_j\}_{j=1}^N$ are *real* and *non-negative*.

We now calculate a new matrix $\bar{\mathbf{\Lambda}}$ as:

$$\bar{\mathbf{\Lambda}} = \sqrt{\mathbf{\Lambda}} = diag\left(\sqrt{\hat{\lambda}_1}, \ldots, \sqrt{\hat{\lambda}_N}\right) \tag{8}$$

Clearly, $\bar{\mathbf{\Lambda}}$ is a *real*, *diagonal* matrix that results in:

$$\bar{\mathbf{\Lambda}}\bar{\mathbf{\Lambda}}^H = \bar{\mathbf{\Lambda}}\bar{\mathbf{\Lambda}} = \mathbf{\Lambda} \tag{9}$$

If we denote $\mathbf{L} \triangleq \mathbf{V}\bar{\mathbf{\Lambda}}$, then it follows that:

$$\mathbf{L}\mathbf{L}^H = (\mathbf{V}\bar{\mathbf{\Lambda}})(\mathbf{V}\bar{\mathbf{\Lambda}})^H = \mathbf{V}\bar{\mathbf{\Lambda}}\bar{\mathbf{\Lambda}}^H\mathbf{V}^H = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H = \mathbf{K} \tag{10}$$

It means that the coloring matrix $\mathbf{L}$ corresponding to the covariance matrix $\mathbf{K}$ can be computed *without* using Cholesky decomposition. Thereby, the shortcoming of the paper [6], which is related to round-off errors in Matlab caused by Cholesky decomposition can be overcome.

We now explain why the covariance matrix must be positive semi-definite even when eigen decomposition is used to compute the coloring matrix. It is easy to realize that, if $\mathbf{K}$ is *not* positive semi-definite covariance matrix, then $\bar{\mathbf{\Lambda}}$ calculated by (8) is a *complex* matrix. As a result, (9) and (10) are not satisfied.

4.4. Proposed algorithm

From the above observations, the authors propose here a generalized algorithm to generate N correlated Rayleigh envelopes as given below:

1. In a general case, the desired variances (powers) $\{\sigma_{g_j}^2\}_{j=1}^N$ of complex Gaussian random variables with Rayleigh envelopes must be known. Specially, if one wants to generate Rayleigh envelopes corresponding to the desired variances (powers) $\{\sigma_{r_j}^2\}_{j=1}^N$, then $\{\sigma_{g_j}^2\}_{j=1}^N$ are calculated as follows:

$$\sigma_{g_j}^2 = \frac{\sigma_{r_j}^2}{(1 - \frac{\pi}{4})} \qquad \forall j = 1 \ldots N \tag{11}$$

2. From the desired correlation properties of correlated complex Gaussian random variables with Rayleigh envelopes, determine the covariances $R_{xx_{k,j}}$, $R_{yy_{k,j}}$, $R_{xy_{k,j}}$ and $R_{yx_{k,j}}$, for $k, j = 1, \ldots, N$ and $k \neq j$. In other words, in a general case, those covariances must be known.

   Specially, in the case where the powers of all random processes are *equal* and other conditions hold as mentioned in Section 2 and 3, we can follow equations (3) and (4) in the case of time delay and frequency separation, such as in OFDM systems, or equations (5), (6) and (7) in the case of spatial separation like with antenna arrays in MIMO systems to calculate the covariances $R_{xx_{k,j}}$, $R_{yy_{k,j}}$, $R_{xy_{k,j}}$ and $R_{yx_{k,j}}$.

   The values $\{\sigma_{g_j}^2\}_{j=1}^N$, $R_{xx_{k,j}}$, $R_{yy_{k,j}}$, $R_{xy_{k,j}}$ and $R_{yx_{k,j}}$ ($k, j = 1, \ldots, N$; $k \neq j$) are the input data of our proposed algorithm.

3. Create the $N \times N$-sized covariance matrix $\mathcal{K}$:

$$\mathcal{K} = [\mu_{k,j}]_{N \times N} \tag{12}$$

where

$$\mu_{k,j} = \begin{cases} \sigma_{g_j}^2 & \text{if } k \equiv j \\ (R_{xx_{k,j}} + R_{yy_{k,j}}) - i(R_{xy_{k,j}} - R_{yx_{k,j}}) & \text{if } k \neq j \end{cases} \tag{13}$$

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The covariance matrix of complex Gaussian random variables is considered here, as opposed to the covariance matrix of Rayleigh fading envelopes like in the conventional methods.

4. Perform the eigen decomposition $\mathcal{K} = \mathbf{V}\mathbf{G}\mathbf{V}^H$. Denote $\mathbf{G} \triangleq diag(\lambda_1, \ldots, \lambda_N)$. Then, calculate a new diagonal matrix $\mathbf{\Lambda} = diag(\hat{\lambda}_1, \ldots, \hat{\lambda}_N)$, where:

$$\hat{\lambda}_j = \begin{cases} \lambda_j & \text{if } \lambda_j \geq 0 \\ 0 & \text{if } \lambda_j < 0 \end{cases} \quad j = 1, \ldots, N.$$

Thereby, we have a diagonal matrix $\mathbf{\Lambda}$ with all elements in the main diagonal being *real* and definitely *non-negative*.

5. Determine a new matrix $\bar{\mathbf{\Lambda}} = \sqrt{\mathbf{\Lambda}}$ and calculate the coloring matrix $\mathbf{L}$ by setting $\mathbf{L} = \mathbf{V}\bar{\mathbf{\Lambda}}$.
6. Generate a column vector $\mathbb{W}$ of N *independent* complex Gaussian random samples with zero means and *arbitrary, equal* variances σ_g^2:

$$\mathbb{W} = (u_1, \ldots, u_N)^T$$

7. Generate a column vector $\mathbb{Z}$ of N *correlated* complex Gaussian random samples as follows:

$$\mathbb{Z} = \frac{\mathbf{L}\mathbb{W}}{\sigma_g} \triangleq (z_1 \ldots, z_N)^T$$

As shown later in the next section, the elements $\{z_j\}_{j=1}^N$ are zero-mean, (*correlated*) complex Gaussian random variables with variances $\{{\sigma_g}_j^2\}_{j=1}^N$. The N moduli $\{r_j\}_{j=1}^N$ of the Gaussian samples in $\mathbb{Z}$ are the *desired* Rayleigh fading envelopes.

4.5. Statistical properties of the resultant envelopes

In this section, we check the covariance matrix and the variances (powers) of the resultant correlated complex Gaussian random samples as well as the variances (powers) of the resultant Rayleigh fading envelopes.

It is easy to check that $E(\mathbb{W}\mathbb{W}^H) = \sigma_g^2 \mathbf{I}_N$, and therefore:

$$E(\mathbb{Z}\mathbb{Z}^H) = E\left(\frac{\mathbf{L}\mathbb{W}\mathbb{W}^H\mathbf{L}^H}{\sigma_g^2}\right) = E(\mathbf{L}\mathbf{L}^H) = \mathbf{K}$$

It means that the generated Rayleigh envelopes are corresponding to the forced positive semi-definite covariance matrix $\mathbf{K}$, which is, in turn, *equal* to the desired covariance matrix $\mathcal{K}$ in case $\mathcal{K}$ is *positive semi-definite*, or *well approximates* to $\mathcal{K}$ otherwise. In other words, the desired covariance matrix $\mathcal{K}$ of complex Gaussian random variables (with Rayleigh fading envelopes) is achieved.

In addition, note that the variance of the jth Gaussian random variable in $\mathbb{Z}$ is the jth element on the main diagonal of $\mathbf{K}$. Because $\mathbf{K}$ approximates to $\mathcal{K}$, the elements on the main diagonal of $\mathbf{K}$ are thus equal (or close) to ${\sigma_g}_j^2$s (see Eq. (12) and (13)). As a result, the resultant complex Gaussian random variables $\{z_j\}_{j=1}^N$ in $\mathbb{Z}$ have zero means and variances (powers) $\{{\sigma_g}_j^2\}_{j=1}^N$.

It is known that the means and the variances of Rayleigh envelopes $\{r_j\}_{j=1}^N$ have the relation with the variances of the corresponding complex Gaussian random variables $\{z_j\}_{j=1}^N$ in $\mathbb{Z}$ as given below (see (5.51), (5.52) in [9] and (2.1-131) in [10]):

$$E\{r_j\} = \sigma_{g_j}\frac{\sqrt{\pi}}{2} = 0.8862\sigma_{g_j} \quad (14)$$

$$Var\{r_j\} = {\sigma_g}_j^2\left(1 - \frac{\pi}{4}\right) = 0.2146{\sigma_g}_j^2 \quad (15)$$

From (11), (14) and (15), it is clear that:

$$E\{r_j\} = \sigma_{rj}\sqrt{\frac{\pi}{4-\pi}}$$

$$Var\{r_j\} = \sigma_{r_j}^2$$

Therefore, the *desired* variances (powers) $\{{\sigma_r}_j^2\}_{j=1}^N$ of Rayleigh envelopes are achieved.

5. Generation of correlated rayleigh envelopes in a real-time scenario

In Section 4.4, we have proposed the algorithm for generating N correlated Rayleigh fading envelopes in multipath, flat fading channels in a *single time instant*. We can repeat steps 6 and 7 of this algorithm to generate Rayleigh envelopes in the *continuous time interval*. It is noted that, the discrete-time samples of each Rayleigh fading process generated by this algorithm in *different* time instants are *independent* of one another.

It has been known that the discrete-time samples of each *realistic* Rayleigh fading process may have *autocorrelation* properties, which are the functions of the Doppler frequency corresponding to the motion of receivers as well as other factors such as the sampling frequency of transmitted signals. It is because the band-limited communication channels not only limit the bandwidth of transmitted signals, but also limit the bandwidth of fading. This filtering effect limits the rate of changes of fading in time domain, and consequently, results in the autocorrelation properties of fading. Therefore, the algorithm generating Rayleigh fading envelopes in *realistic* conditions must consider the autocorrelation properties of Rayleigh fading envelopes.

To simulate a multipath fading channel, Doppler filters are normally used [9]. The analysis of Doppler spectrum spread was first derived by Gans [11], based on Clarke's model [12]. Motivated by these works, Smith [13] developed a computer-assisted model generating an *individual* Rayleigh fading envelope in flat fading channels corresponding to a given *normalized autocorrelation* function. This model was then modified by Young [14], [7] to provide more accurate channel realization.

It should be emphasized that, in all works mentioned above, the models are aimed at generating an *individual* Rayleigh envelope corresponding to a certain normalized *autocorrelation* function of itself, rather than generating different Rayleigh envelopes corresponding to a desired covariance matrix (*autocorrelation* and *cross-correlation* properties between those envelopes).

Therefore, the model for generating N correlated Rayleigh fading envelopes in realistic fading channels (each individual envelope is corresponding to a desired normalized autocorrelation property) can be created by associating the model proposed in [7] with our algorithm mentioned in

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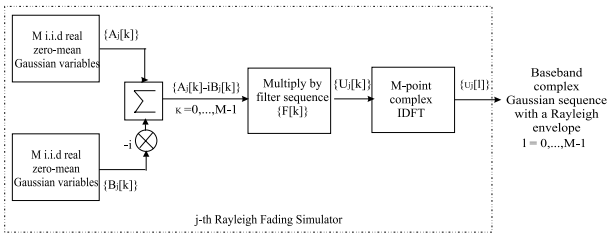


Figure 2. Model of a Rayleigh generator for an individual Rayleigh envelope corresponding to a desired *normalized* autocorrelation function.

Section 4.4 in such a way that, the resultant Rayleigh fading envelopes are corresponding to the desired covariance matrix.

This combination must overcome the main shortcoming of the method proposed in [6] as analyzed in Section 1. This is an easy task in our algorithm. The key for the success of this task is the modification in steps 6 and 7 of our algorithm (see Section 4.4), where the variances of N complex Gaussian random variables are *not fixed* as in [6], but can be *arbitrary* in our algorithm. Again, besides being more generalized, our modification in these steps allows the *accurate* combination of the method proposed in [7] and our algorithm, i.e. guaranteeing that the generated Rayleigh envelopes are exactly corresponding to the desired covariance matrix.

The model of a Rayleigh fading generator for generating an *individual* baseband Rayleigh fading envelope proposed in [14], [7] is shown in Fig. 2. This model generates a Rayleigh fading envelope using Inverse Discrete Fourier Transform (IDFT), based on *independent* zero-mean Gaussian random variables weighted by appropriate Doppler filter coefficients. The sequence $\{u_j[l]\}_{l=0}^{M-1}$ of the complex Gaussian random samples at the output of the jth Rayleigh generator (Fig. 2) can be expressed as:

$$u_j[l] = \frac{1}{M}\sum_{k=0}^{M-1} U_j[k]e^{i\frac{2\pi kl}{M}}$$

where

M denotes the number of points with which the IDFT is carried out;

l is the discrete-time sample index ($l = 0, \ldots, M-1$);

$U_j[k]$=$F[k]A_j[k] - iF[k]B_j[k]$;

$\{F[k]\}$ are the Doppler filter coefficients.

For brevity, we omit the subscript j in the expressions, except when this subscript is necessary to emphasize. If we denote $u[l]$=$u_R[l] + iu_I[l]$, then it has been proved that, the *autocorrelation* property between the real parts $u_R[l]$ and $u_R[m]$ as well as that between the imaginary parts $u_I[l]$ and $u_I[m]$ at different discrete-time instants l and m is as given below (see Eq. (7) in [7]):

$$\begin{aligned} r_{RR}[l,m] &= r_{II}[l,m] = r_{RR}[d] = r_{II}[d] \\ &= E\{u_R[l]u_R[m]\} = \frac{\sigma^2_{orig}}{M} Re\{g[d]\} \end{aligned} \quad (16)$$

where $d \triangleq l - m$ is the sample lag, σ^2_{orig} is the variance of the *real*, independent zero-mean Gaussian random sequences $\{A[k]\}$ and $\{B[k]\}$ at the inputs of Doppler filters, and the sequence $\{g[d]\}$ is the IDFT of $\{F[k]^2\}$, i.e.:

$$g[d] = \frac{1}{M}\sum_{k=0}^{M-1} F[k]^2 e^{i\frac{2\pi kd}{M}} \quad (17)$$

Similarly, the correlation property between the real part $u_R[l]$ and the imaginary part $u_I[m]$ is calculated as (see Eq. (8) in [7]):

$$r_{RI}[d] = E\{u_R[l]u_I[m]\} = \frac{\sigma^2_{orig}}{M} Im\{g[d]\} \quad (18)$$

The mean value of the output sequence $\{u[l]\}$ has been proved to be zero (see Appendix A in [7]).

If d=0 and $\{F[k]\}$ are real, from (16), (17) and (18), we have:

$$\begin{aligned} r_{RR}[0] &= r_{II}[0] = E\{u_R[l]u_R[l]\} = \frac{\sigma^2_{orig}}{M^2}\sum_{k=0}^{M-1} F[k]^2 \\ r_{RI}[0] &= E\{u_R[l]u_I[l]\} = 0 \end{aligned}$$

Therefore, by definition, the variance of the sequence $\{u[l]\}$ at the output of the Rayleigh generator is:

$$\begin{aligned} \sigma^2_g &\triangleq E\{u[l]u[l]^*\} \\ &= 2E\{u_R[l]u_R[l]\} = \frac{2\sigma^2_{orig}}{M^2}\sum_{k=0}^{M-1} F[k]^2 \end{aligned} \quad (19)$$

where * denotes the complex conjugate operation.

Let $J_0(2\pi f_m d)$ be the function:

$$J_0(2\pi f_m d) = \frac{r_{RR}[d]}{\sigma^2_g} = \frac{r_{II}[d]}{\sigma^2_g} \quad (20)$$

i.e. $J_0(2\pi f_m d)$ be the autocorrelation function in (16) *normalized* by the variance σ^2_g in (19). In the above equation, J_0 denotes the zero-order Bessel function of the first kind and f_m is the maximum Doppler frequency F_m normalized by the sampling frequency F_s of the transmitted signals, i.e. $f_m = \frac{F_m}{F_s}$. $J_0(2\pi f_m d)$ is called the *normalized autocorrelation* function.

To achieve a desired *normalized autocorrelation* function $J_0(2\pi f_m d)$, the Doppler filter $\{F[k]\}$ is determined in Young's model [14], [7] as given below (see Eq. (21) in [7]).

$$F[k] = \begin{cases} 0 & k = 0 \\ \sqrt{\frac{1}{2\sqrt{1-(\frac{k}{Mf_m})^2}}} & k = 1,\ldots,k_m\text{-}1 \\ \sqrt{\frac{k_m}{2}\left[\frac{\pi}{2} - arctan\left(\frac{k_m-1}{\sqrt{2k_m-1}}\right)\right]} & k = k_m \\ 0 & k = k_m\text{+}1,\ldots,\text{M - }k_m\text{-}1 \\ \sqrt{\frac{k_m}{2}\left[\frac{\pi}{2} - arctan\left(\frac{k_m-1}{\sqrt{2k_m-1}}\right)\right]} & k = \text{M - }k_m \\ \sqrt{\frac{1}{2\sqrt{1-(\frac{M-k}{Mf_m})^2}}} & k = \text{M - }k_m\text{+}1,\ldots,\text{M - }1 \end{cases} \quad (21)$$

In (21), $k_m \triangleq \lfloor f_m M \rfloor$, where $\lfloor . \rfloor$ indicates the biggest rounded integer being less or equal to the argument.

It has been proved in [7] that the (real) filter coefficients in (21) will produce a complex Gaussian sequence with the

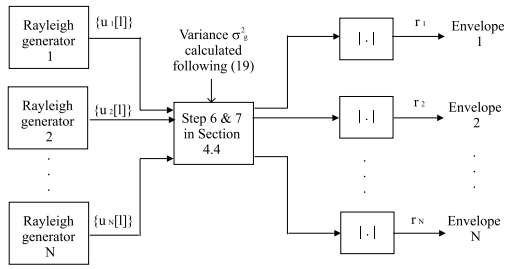


Figure 3. Model for generating N Rayleigh envelopes corresponding to a desired *normalized* autocorrelation function in a real-time scenario.

normalized autocorrelation function $J_0(2\pi f_m d)$, where d is the sample lag, and with the *expected independence* between the real and imaginary parts of Gaussian samples, i.e., the correlation property in (18) is zero. The zero-correlation property between the real and imaginary parts is necessary in order that the resultant envelopes are Rayleigh distributed.

Our model for generating N correlated Rayleigh fading envelopes corresponding to a desired covariance matrix in a real-time scenario where Doppler frequency shifts are considered is presented in Fig. 3. In this model, N Rayleigh generators, each of which is presented in Fig. 2, are simultaneously used. To generate N *correlated* Rayleigh envelopes corresponding to a desired covariance matrix at *an observed discrete-time instant* l ($l = 0, \ldots, M-1$), similarly to the method in [6], we take the output $u_j[l]$ of the jth Rayleigh simulator, for $j = 1, \ldots, N$, and input it as the element u_j into step 6 of our algorithm proposed in Section 4.4. However, as opposed to the method in [6], the variance σ_g^2 of complex Gaussian samples u_j in step 6 of our method is calculated following (19). This value is used as the input parameter for steps 6 and 7 of our algorithm (see Fig. 3). Thereby, the variance-changing effect caused by Doppler filters is taken into consideration in our algorithm, and consequently, our proposed algorithm overcomes the main shortcoming of the method in [6].

The algorithm for generating N correlated Rayleigh envelopes (when Doppler frequency shifts are considered) *at a discrete-time instant* l, for $l = 0, \ldots, M-1$, can be summarized as:

1. Perform the steps 1 to 5 mentioned in Section 4.4.
2. From the *desired autocorrelation* properties (16) and (20) of each of the complex Gaussian random sequences (with Rayleigh fading envelopes), determine the desired value M. The value of M is also the number of points with which IDFT is carried out.
3. For each Rayleigh generator presented in Fig. 2, generate M identically independently distributed (i.i.d.), real, zero-mean Gaussian random samples $\{A[k]\}$ with an arbitrary variance σ_{orig}^2 and, independently, generate M i.i.d., real, zero-mean Gaussian samples $\{B[k]\}$ with the distribution (0,σ_{orig}^2). From $\{A[k]\}$ and $\{B[k]\}$, generate M i.i.d complex Gaussian random variables $\{A[k] - iB[k]\}$. N Rayleigh generators are simultaneously used to generate N Rayleigh envelopes as presented in Fig. 3.
4. Multiply complex Gaussian samples $\{A[k] - iB[k]\}$, for $k = 1, \ldots, M$, with the corresponding filter coefficient $F[k]$ given in (21).
5. Perform M-point IDFT of the resultant samples.
6. Calculate the variance σ_g^2 of the output $\{u[l]\}$ following (19). It is noted that σ_g^2 is the same for N Rayleigh generators. We also emphasize that, by this calculation, the variance-changing effect of Doppler filters are taken into account.
7. Generate a column vector $\mathbb{W} = (u_1, \ldots, u_N)^T$ of N i.i.d. complex Gaussian random samples with the distribution (0,σ_g^2) where the element u_j, for $j = 1, \ldots, N$, is the output $u_j[l]$ of the jth Rayleigh generator and σ_g^2 has been calculated in step 6.
8. Continue the step 7 mentioned in Section 4.4. The N envelopes of elements in the column vector $\mathbb{Z}$ are the desired Rayleigh envelopes *at the considered time instant* l.

Steps 7 and 8 are repeated for different time instants l ($l = 0, \ldots, M-1$), and therefore, the algorithm can be used for a real-time scenario.

6. Simulation results

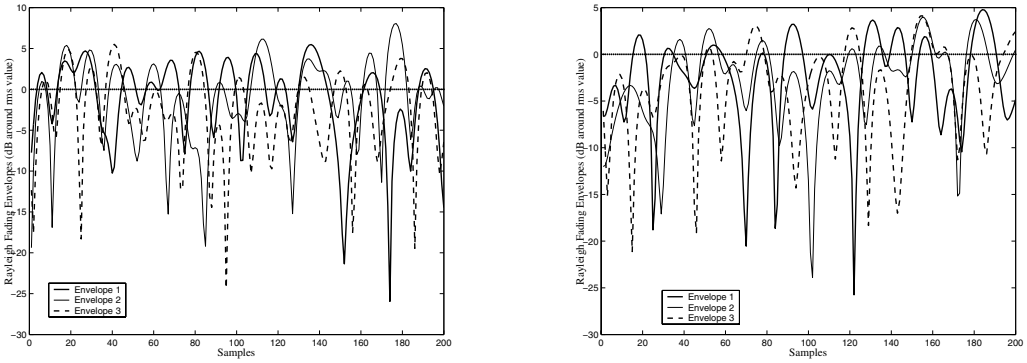
In this section, first, we simulate N=3 *frequency-correlated* Rayleigh fading envelopes corresponding to the complex Gaussian random variables with equal powers ${\sigma_g}_j^2 = 1$ ($j = 1, \ldots, 3$) in the flat fading channels. Parameters considered here include $M = 4096$ (the number of IDFT points), $\sigma_{orig}^2 = 1/2$ (variances per dimension in Young's model), $F_s = 1$kHz, $F_m = 50$Hz (corresponding to a carrier frequency 900 MHz and a mobile speed $v = 60$ km/hr). Therefore, we have $f_m = 0.05$, $k_m = 204$. Frequency separation between two adjacent carrier frequencies considered here is 200 kHz (e.g in GSM 900) and we assume that $f_1 > f_2 > f_3$. Also, we consider the rms delay spread $\sigma_\tau = 1\mu s$ and time delays between three envelopes are $\tau_{1,2} = 1ms$, $\tau_{2,3} = 3ms$, $\tau_{1,3} = 4ms$.

From (3), (4), (12) and (13), we have the *desired* covariance matrix $\mathcal{K}$ as given below:

$$\begin{bmatrix} 1 & 0.3782 + 0.4753i & 0.0878 + 0.2207i \\ 0.3782 - 0.4753i & 1 & 0.3063 + 0.3849i \\ 0.0878 - 0.2207i & 0.3063 - 0.3849i & 1 \end{bmatrix} \quad (22)$$

It is easy to check that $\mathcal{K}$ in (22) is positive definite. Using the proposed algorithm in Section 5, we have the simulation result presented in Fig. 4(a).

Next, we simulate N=3 *spatially-correlated* Rayleigh fading envelopes. We consider an antenna array comprising three transmitter antennas, which are equally separated by a distance D. Assume that $\frac{D}{\lambda} = 1$, i.e., $D = 33.3$ cm for GSM 900. Additionally, we assume that $\Delta = \pi/18$ rad (or $\Delta = 10°$) and $\Phi = 0$ rad. The parameters ${\sigma_g}_j^2$ and σ_{orig}^2 are the same as the previous case.



(a) Spectral correlation

(b) Spatial correlation

Figure 4. Examples of three equal power correlated Rayleigh fading envelopes.

From (5), (6), (7), (12) and (13), we have the following *desired* covariance matrix:

$$\begin{bmatrix} 1 & 0.8123 & 0.3730 \\ 0.8123 & 1 & 0.8123 \\ 0.3730 & 0.8123 & 1 \end{bmatrix} \tag{23}$$

Since $\Phi = 0$ rad, the covariances $R_{xy_{k,j}}$ and $R_{yx_{k,j}}$ between the real and imaginary components of any pair of the complex Gaussian random processes (with Rayleigh fading envelopes) are zeros, and consequently, $\mathcal{K}$ is a *real* matrix. Readers may refer to (6) and (7) for more details. It is easy to realize that $\mathcal{K}$ in (23) is positive definite. The simulation result is presented in Fig. 4(b).

7. Conclusion

In this paper, we have derived a more generalized algorithm to generate correlated Rayleigh fading envelopes. Using the presented algorithm, one can generate an *arbitrary number* N of *either* Rayleigh envelopes with any desired power $\sigma_{r_j}^2$, $j = 1, \ldots, N$, *or* those envelopes corresponding to any desired power $\sigma_{g_j}^2$ of Gaussian random variables. This algorithm also facilitates to generate *equal* as well as *unequal* power Rayleigh envelopes. It is applicable for both scenarios of *spatial correlation* and *spectral correlation* between the random processes. The coloring matrix is determined by a positive semi-definiteness forcing procedure and an eigen decomposition procedure without using Cholesky decomposition. Consequently, the restriction on the positive definiteness of the covariance matrix is relaxed and the algorithm works well without being impeded by the round-off errors of Matlab. The proposed algorithm can be used to generate Rayleigh envelopes corresponding to any desired covariance matrix, no matter whether or not it is positive definite. Besides being *more generalized*, our proposed algorithm (with or without Doppler spectrum spread) is *more precise*, while overcoming all shortcomings of the conventional methods.

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