In this paper, we present a novel method for fringe pattern profilometry based on the inverse function analysis. The method is particularly useful for measuring the height of objects with smooth surfaces. The key idea is to establish a relationship between the phase information obtained from the fringe pattern and the height of the object. This relationship is then used to reconstruct the 3D surface profile of the object.

The method involves two main steps: (1) extracting phase information from the fringe pattern, and (2) using the extracted phase information to calculate the height. The phase information is obtained through a Fourier transform of the fringe pattern. The height is then calculated using a function that maps the phase to the height. The function is learned from a set of training data that consists of fringe patterns and their corresponding heights.

We demonstrate the effectiveness of our method through simulations and experiments. The results show that our method is capable of accurately reconstructing the 3D surface profile of objects with smooth surfaces. The method is also robust to noise and can handle significant deviations from the assumed model.

The proposed method has potential applications in various fields such as medicine, materials science, and manufacturing. For example, in medicine, the method can be used to measure the thickness of tissues or the curvature of the retina. In materials science, the method can be used to monitor the deformation of materials during testing. In manufacturing, the method can be used to inspect the quality of manufactured parts.
Inverse function analysis method for fringe pattern profilometry

Abstract
In this paper, we present a mathematical model that describes a general relationship between the projected signal and the deformed signal in fringe pattern profilometry (FPP) systems. The derived mathematical model proves that in theory any kind of fringe pattern could be utilized for profilometry. Based on the derived mathematical model, this paper also proposes a new algorithm, referred to as inverse function analysis (IFA) method, to reconstruct 3-D surfaces using the FPP technique. Compared with traditional methods, our algorithm has neither the requirement for the structure of projected fringe patterns nor the prior knowledge of the distortion characteristics of projection systems. The correctness of the proposed mathematical model and IFA method has been confirmed by simulation results, which are provided to demonstrate that compared with the conventional analysis methods, the measurement accuracy has been significantly improved by the IFA method, particularly when the expected sinusoidal fringe patterns are distorted by unknown factors.

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Inverse Function Analysis Method for Fringe Pattern Profilometry

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Abstract—In this paper, we present a mathematical model that describes a general relationship between the projected signal and the deformed signal in fringe pattern profilometry (FPP) systems. The derived mathematical model proves that in theory any kind of fringe pattern could be utilized for profilometry. Based on the derived mathematical model, this paper also proposes a new algorithm, referred to as inverse function analysis (IFA) method, to reconstruct 3-D surfaces using the FPP technique. Compared with traditional methods, our algorithm has neither the requirement for the structure of projected fringe patterns nor the prior knowledge of the distortion characteristics of projection systems. The correctness of the proposed mathematical model and IFA method has been confirmed by simulation results, which are provided to demonstrate that compared with the conventional analysis methods, the measurement accuracy has been significantly improved by the IFA method, particularly when the expected sinusoidal fringe patterns are distorted by unknown factors.

Index Terms—Fringe pattern analysis, fringe pattern profilometry (FPP), inverse function analysis (IFA), phase shifting profilometry, 3-D profile reconstruction.

I. INTRODUCTION

FRINGE pattern profilometry (FPP) has been one of the most popular noncontact methods for measuring the 3-D surface of an object in recent years. With FPP, a Ronchi grating or sinusoidal grating is used to project fringe patterns onto a 3-D diffuse surface, which results in deformed grating images. The deformed grating images are captured by a charge-coupled device (CCD) camera and are processed to yield the shape of the object. To reconstruct the 3-D surface information from the pattern images, a number of fringe analysis methods have been developed, including Fourier transform profilometry (FTP) [1]–[4], phase shifting profilometry (PSP) [5]–[8], spatial phase detection [9], phase-locked loops [10], and other analysis methods [11], [12].

There are various methods of generating fringe patterns. In recent years, digital projectors have been widely used to obtain fringe patterns [13], [14]. The advantage of utilizing digital projectors for FPP is its simplicity and controllability. However, it is very difficult to obtain a purely sinusoidal fringe pattern by digital projectors due to the existence of geometrical and color distortions. Meanwhile, we can note that for all the methods mentioned above, it has been assumed that fringe patterns are purely sinusoidal or can be filtered to be sinusoidal by using digital filters to pick up the fundamental frequency component while eliminating the higher harmonics. However, in most practical cases, the filter cannot be ideal, and the results are still not purely sinusoidal. Therefore, errors will arise if the measurement is based on a pure sinusoidal assumption. This problem motivates us to look for a new method of reconstructing the 3-D profile based on nonsinusoidal fringe patterns.

In this paper, we propose a new approach for measuring object surface based on projected fringe patterns with unknown nonlinear distortions. The difference between original fringe pattern signal and deformed fringe pattern signal contains the 3-D information of the object, irrespective of whether they are sinusoidal or not. Therefore, we derive a generalized model to mathematically describe the relationship between original fringe pattern signal and deformed fringe pattern signal. Based on the model, a new fringe pattern analysis approach, called the inverse function analysis (IFA) method is proposed.

This paper is organized as follows. In Section II, a mathematical model for the FPP system is derived and briefly compared with the traditional analytical model. In Section III, the IFA method is proposed to reconstruct the 3-D surface based on our model. To compare the measurement accuracy using IFA with that using traditional methods, in Section IV, we take PSP as an example to theoretically analyze the measurement errors caused by nonlinear distortions and estimate the errors using IFA. Analytical results show that IFA provides much better performance than PSP in terms of reconstruction accuracy. In Section V, we give our simulation results, which confirm the theoretical analysis. Section VI concludes this paper.

II. SYSTEM MODEL

A schematic diagram of a typical FPP system is shown in Fig. 1. A projector creates a set of fringe patterns projected onto a pure plane surface, called reference plane, and is picked up by a CCD camera. Then, by removing the reference plane, the same fringe pattern is projected onto a object surface, yielding a deformed fringe pattern in the CCD camera. The profile of the object can be obtained based on the relationship between the projected and the deformed fringe patterns.

For the sake of simplicity, we assume that the distance between the camera and the reference plane is long enough.
and is much greater than the height range of the object profile so that the reflected light beams captured by CCD camera from the reference plane and object can be approximately regarded as being parallel [5]. Meanwhile, because of the long distance, those parallel light beams reflected to the camera can be regarded as being vertical to the reference plane in practice [5] (see [15] for error analysis in detail). The fringe pattern projected onto the reference plane is denoted by \( s(x, y) \), which is the intensity of the signal at the location of \((x, y)\), where \( x, y \), and a typical fringe pattern are depicted in Fig. 1. Similarly, the deformed fringe pattern is denoted by \( d(x, y) \).

As shown in Fig. 1, the projected fringe pattern on the reference plane is designed to exhibit a constant intensity along the \( y \)-direction. Hence, we can simply consider the signal intensity along the \( x \)-direction for any given \( y \)-coordinate, and the results can be extended to other values of \( y \). Therefore, instead of \( s(x, y) \) and \( d(x, y) \), we simply use \( s(x) \) and \( d(x) \) to denote the projected signal on the reference plane and the deformed signal, respectively. Similarly, the height distribution function \( h(x, y) \) of the object surface can also be represented as \( h(x) \), which is a function with only one independent variable \( x \), when we are considering the sections of object surfaces.

To establish the relationship between \( s(x) \) and \( d(x) \), we consider a beam of light corresponding to a pixel of the fringe pattern, which is denoted as \( E_p \), \( CH \) in Fig. 1. It is seen that the light beam is projected at point \( C \) and is reflected back to the camera if the reference plane exists. When the reference plane is removed, the same beam will be projected onto point \( H \) on the surface of the object, which is reflected to the camera via point \( D \). This implies that the \( x \)-coordinate of point \( H \) in the image of object surface equals to the \( x \)-coordinate of point \( D \) in the image of reference plane, because points \( D \) and \( H \) are on the same reflected beam from point \( H \) to the camera. Assuming that the object surface and the reference plane have the same reflective characteristics, \( s(x) \) at location \( C \) should exhibit the same intensity as \( d(x) \) does at location \( D \) because they originate from the same point of the fringe pattern created by the projector. Hence, we have

\[
d(x_d) = s(x_c).
\]

We use \( u \) to denote the distance from \( C \) to \( D \), i.e.,

\[
u = x_d - x_c
\]

where \( x_c \) and \( x_d \) are the coordinate locations of points \( C \) and \( D \), respectively. From (1) and (2), we have

\[
d(x_d) = s(x_d - u).
\]

Obviously, \( u \) varies with the height of point \( H \) on the object surface.

Meanwhile, because triangles \( E_pHE_c \) and \( CHD \) are similar, we have

\[
x_c - x_d = \frac{d_0}{l_0 - h(x_h)}\]

where \( x_h \) is the \( x \)-coordinate of point \( H \), \( l_0 \) is the distance between the camera and the reference plane, and \( d_0 \) is the distance between the camera and the projector.

As mentioned above, point \( H \) has the same \( x \)-coordinate as point \( D \) does in captured images, which implies that \( x_h = x_d \). So (4) can be rewritten as

\[
x_c - x_d = \frac{d_0}{l_0 - h(x_d)}\]

As defined in (2), (5) can be expressed as

\[
-u = \frac{d_0}{l_0 - h(x_d)}\]

As the height distribution \( h(x) \) is a function of \( x_d \), \( u \) should be also a function of \( x_d \). Then, we have

\[
u(x_d) = \frac{d_0 h(x_d)}{l_0 - h(x_d)}\]

An equivalent representation is

\[
h(x_d) = \frac{l_0 u(x_d)}{d_0 + u(x_d)}\]

Therefore, (3) can be expressed as

\[
d(x_d) = s(x_d - u(x_d))\]

where \( u(x_d) \) is given by (7).

To simplify (9), and considering the model only mathematically, we let \( x_d = x \), we can derive a general mathematical model from (8) and (9) as follows:

\[
d(x) = s(x - u(x))\]

\[
h(x) = \frac{l_0 u(x)}{d_0 + u(x)}\]

Equation (10) reveals that the deformed signal \( d(x) \) is a shifted version of \( s(x) \), and the shift function \( u(x) \) varies with the height of the object.

Obviously, by using (10) and (11), as long as we can obtain the function \( u(x) \), the surface reconstruction can be achieved. Equation (10) expresses the general relationship between the deformed signal and the projected signal. Note that the projected signal \( s(x) \) can be of any form and, hence, is not necessarily required to be sinusoidal. Clearly, if we use periodic
fringe patterns, the projected signal can be expressed in the form of the following Fourier expansion:

\[ s(x) = \sum_{k=0}^{+\infty} A_k \cos(2\pi kf_0 x + \phi_k) \]  

where \( f_0 \) is the fundamental frequency of the fringe pattern, and \( A_k \) is the amplitude of intensity of the \( k \)th-order harmonic. \( \phi_k \) is the initial phase of the \( k \)th-order harmonic. Thus, according to the (10), the deformed signal is

\[ d(x) = s(x - u(x)) \]

\[ = \sum_{k=0}^{+\infty} A_k \cos \left( 2\pi kf_0 (x - u(x)) + \phi_k \right) \]

\[ = \sum_{k=0}^{+\infty} A_k \cos \left( 2\pi kf_0 x + k\phi(x) + \phi_k \right) \]  

where \( \phi(x) = -2\pi f_0 u(x) \).

As expected, for the case where a periodic fringe pattern is used, the deformed signal \( d(x) \) is a phase-modulated signal. Hence, conventional analysis methods attempt to demodulate the deformed signals and retrieve the phase map \( \phi(x) \) first. Then, by using the relationship between the phase shift function \( \phi(x) \) and the height distribution \( h(x) \), the shapes of objects can be reconstructed. Actually, when a periodic fringe pattern is used, we can easily derive the relationship between \( \phi(x) \) and \( h(x) \) by our proposed model. Note in (13) that \( u(x) = -\phi(x)/2\pi f_0 \). By substituting this equation into (11), we can derive the following well-known equation:

\[ h(x) = \frac{l_0 \phi(x)}{\phi(x) - 2\pi f_0 d_0}. \]  

An equivalent form of (14) is

\[ \phi(x) = \frac{2\pi f_0 d_0 h(x)}{h(x) - l_0}. \]  

Equations (14) and (15) appear as classical formulas in most of the articles on FPP (e.g., [1], [2], and [4]). Essentially, our proposed model is consistent with the conventional mathematical model and has a form that is identical with the conventional one when a purely sinusoidal fringe pattern is used for profilometry. However, in practical cases, particularly when we are using a digital projector to generate fringe patterns, the challenge we have to face is when the projected signal could not be a purely sinusoidal signal. Equations (10) and (11) reveal that the shift function \( u(x) \) contains all the 3-D information of the object surface. Hence, in theory, we should be able to obtain the profile by projecting any signal. In Section III, we will present an IFA method to retrieve the shift function \( u(x) \).

III. CALCULATION OF SHIFT FUNCTION

A straightforward method to calculate the shift function is to use inverse function. We assume that the projected signal function \( r = s(x) \) is a monotonic function, or that it is monotonic in intervals of \( x \), in which \( s(x) \) has a unique inverse function. By denoting the inverse function of \( s(x) \) as \( s^{-1}(v) \), we have

\[ s^{-1}(s(x)) = x. \]  

Therefore, if we apply the inverse function \( s^{-1}(v) \) to the deformed signal \( d(x) \), we will have

\[ s^{-1}(d(x)) = s^{-1}\{s\{x - u(x)\}\} = x - u(x) \]  

which means that we can obtain the shift function \( u(x) \) by

\[ u(x) = x - s^{-1}(d(x)). \]  

It is obvious that from (18), the shift function can be calculated based on the fringe pattern projected on the reference plane and the deformed fringe pattern on the surface of the object. The key to calculating the shift function is to obtain the inverse function \( s^{-1}(v) \). A possible way is to employ polynomial curve fitting, which will consequentially introduce fitting errors. We use the mean square error to evaluate the curve-fitting error, which is defined as

\[ e_f = E \left[ (y_f(x) - y(x))^2 \right] \]  

where \( E(w) \) is the operation of calculating the mean value of \( w \), \( y(x) \) represents the data to be fitted, and \( y_f(x) \) represents the values of the curve-fitting results calculated by the approximate polynomial. The fitting error \( e_f \) will decrease with an increase in the polynomial degree. Therefore, to determine the degree of polynomial used for curve fitting, we set up an upper bound of \( e_f \) in advance, and then, we find out the minimum degree of polynomial that makes the curve-fitting error \( e_f \) less than the upper bound that we have set up. Hence, the procedure of surface reconstruction is given as follows.

Step 1) Set an upper bound of the curve-fitting error \( e_m \), and initialize \( k \), which is the degree of the polynomial used for curve fitting. The initial value of \( k \) is equal to 1.

Step 2) Based on the captured fringe pattern on the reference plane \( s(x) \), work out \( j_k \), which is a polynomial of degree \( k \), to approximate the inverse function \( s^{-1}(v) \) in the least squares sense. In more detail, at first, we take the straight line \( r = x \), where \( r \) and \( x \) are the vertical and horizontal coordinates of the signal \( s(x) \), respectively, as a symmetry axis to obtain a symmetrical curve of \( s(x) \) in each monotonic interval, which is actually the curve of the inverse function \( s^{-1}(v) \). Then, we carry out curve fitting to the obtained symmetrical curve and obtain the curve-fitting result \( j_k \). This process is equivalent to directly fitting the inverse function \( s^{-1}(v) \) by regarding the value of \( s(x) \) as the variable and \( x \) as the value of the inverse function rather than obtaining an approximate polynomial of the original function \( s(x) \) before fitting the inverse function \( s^{-1}(v) \).

Step 3) By (19), calculate the curve-fitting error when using \( j_k \) to approximate \( s^{-1}(v) \). If the error is less than \( e_m \),
continue to do Step 4); otherwise, set \( k = k + 1 \), and return to Step 2).

Step 4) Based on the curve-fitting result \( s^{-1}(v) \approx j_k \) and the values of the deformed signal \( d(x) \), we calculate the shift function \( u(x) \) using (18).

Note that it has been proven that \( j_k \), which satisfies \( E[(j_k(v) - s^{-1}(v))^2] < e_m \), always exists \([16]\).

The IFA method presented above is advantageous in two aspects. First, there is no restriction on the structure of the projected fringe pattern. Second, the approach is directly based on the projected fringe patterns, which implies that this approach always works, even if the signal on the reference plane has been distorted by unknown nonlinear factors. Meanwhile, the approach can still be used for sinusoidal fringe patterns, in which case there is no need to use bandpass filter to eliminate the higher order harmonic distortion.

IV. ERROR ANALYSIS AND COMPARISON

To demonstrate the advantages of the proposed inverse function approach, we will first study the error associated with the PSP approach due to the nonlinear distortion of the projected fringe pattern and then compare the results with those of using the proposed approach.

We only consider the effect of the second-order harmonic, which is reasonable, as the second-order harmonic has the highest power than other higher harmonic components after the signals have been filtered by bandpass filter. We assume that in the ideal situation, the captured fringe pattern on the object surface is a pure sinusoid, which is given by

\[
g_n(x) = a(x) + b(x) \cos (2 \pi f_0 x + \phi(x) + 2 \pi n/N) \quad \text{for } n = 0, 1, 2, \ldots, N - 1 \tag{20}
\]

where \( N \) is the number of phase shifting steps, \( g_n(x) \) is the intensity of the image at point \( x \), \( a(x) \) is a slowly varying function representing the background illumination, \( b(x) \) is a slowly varying function representing the contrast between bright and dark fringes captured by the CCD, \( f_0 \) is the spatial carrier fringe frequency, and \( \phi(x) \) is the phase shift caused by the object surface and the angle of projection. For PSP method, the phase map \( \phi(x) \) can be obtained by following formula \([5]\):

\[
\tan (2 \pi f_0 x + \phi(x)) = - \frac{\sum_{n=0}^{N-1} g_n(x) \sin(2 \pi n/N)}{\sum_{n=0}^{N-1} g_n(x) \cos(2 \pi n/N)}. \tag{21}
\]

However, in practice, there is always nonlinear distortion with the projected fringe pattern, which can be described as harmonic components. For simplicity, we only consider the second-order harmonic distortion. Hence, the actual fringe pattern projected on the surface of the object can be expressed as follows:

\[
\tilde{g}_n(x) = a(x) + b(x) \cos (2 \pi f_0 x + \phi(x) + 2 \pi n/N) \\
+ c(x) \cos (4 \pi f_0 x + 2 \phi(x) + 4 \pi n/N) \quad \text{for } n = 0, 1, 2, \ldots, N - 1. \tag{22}
\]

Then, if we still use (21) to calculate \( \phi(x) \), errors will be introduced. As three-step PSP requires the least number of captured pattern images and consequently provides the shortest measurement time, let us consider the situation when \( N = 3 \), and let us denote the error of \( \phi(x) \) as \( \delta \). We will have

\[
\tan (2 \pi f_0 x + \phi(x) + \delta) = - \frac{\sum_{n=0}^{N-1} g_n(x) \sin(2 \pi n/N)}{\sum_{n=0}^{N-1} g_n(x) \cos(2 \pi n/N)}. \tag{23}
\]

Considering (22) and (23) together, when \( N = 3 \), we will have

\[
\begin{align*}
tan (2 \pi f_0 x + \phi(x) + \delta) &= \frac{3b \sin (2 \pi f_0 x + \phi(x)) - 3c \sin (4 \pi f_0 x + 2 \phi(x))}{2b \cos (2 \pi f_0 x + \phi(x)) + 2c \cos (4 \pi f_0 x + 2 \phi(x))} \\
&= \frac{3b \sin (2 \pi f_0 x + \phi(x)) - 3c \sin (2 \cdot 2 \pi f_0 x)}{2b \cos (2 \pi f_0 x) + 2c \cos (2 \cdot 2 \pi f_0 x)}. \tag{24}
\end{align*}
\]

where \( b \) and \( c \) denote \( b(x) \) and \( c(x) \), respectively, for simplicity of expression. Let \( \theta \) denote \( 2 \pi f_0 x + \phi(x) \). Then, we have

\[
\tan(\theta + \delta) = \frac{b \sin(\theta - c \sin(20))}{b \cos(\theta) + c \cos(20)}. \tag{25}
\]

where \( \delta \) is also a function of \( \theta \) and can be expressed as \( \delta(\theta) \).

\( \phi_0 \), which is the phase map of the distorted fringe pattern on the reference plane can be derived by (23) when \( \phi(x) = 0 \), i.e.,

\[
\tan(\phi_0) = \frac{b \sin (2 \pi f_0 x) - c \sin (2 \cdot 2 \pi f_0 x)}{b \cos (2 \pi f_0 x) + c \cos (2 \cdot 2 \pi f_0 x)}. \tag{26}
\]

As \( \theta = 2 \pi f_0 x + \phi(x) \), the equation above can be expressed as

\[
\hat{\phi}_0 = \arctan \left( \frac{b \sin(\theta - \phi) - c \sin(2(\theta - \phi))}{b \cos(\theta - \phi) + c \cos(2(\theta - \phi))} \right). \tag{27}
\]

In PSP, we will retrieve \( \hat{\phi} \), an estimation of \( \phi \), which is the phase shifted by object profiles using the following equation:

\[
\hat{\phi} = \arctan \left( \frac{b \sin(\theta - c \sin(20))}{b \cos(\theta) + c \cos(20)} \right) - \phi_0 \\
= \arctan \left( \frac{b \sin(\theta - c \sin(20))}{b \cos(\theta) + c \cos(20)} \right) - \arctan \left( \frac{b \sin(\theta - \phi) - c \sin((2 \theta - \phi))}{b \cos(\theta - \phi) + c \cos(2(\theta - \phi))} \right) \\
= [\theta + \delta(\theta)] - [(\theta - \phi) + \delta(\theta - \phi)] \\
= \phi + \delta(\theta) - \delta(\theta - \phi). \tag{28}
\]

Therefore, the phase measurement error \( \varepsilon \) can be expressed as

\[
\varepsilon = \hat{\phi} - \phi = \delta(\theta) - \delta(\theta - \phi). \tag{29}
\]

Hence, the maximum measurement error is

\[
\varepsilon_{\max} = \delta_{\max}(\theta) - \delta_{\min}(\theta) \tag{30}
\]

where \( \delta_{\max}(\theta) \) and \( \delta_{\min}(\theta) \) represent the maximum and minimum value of function, respectively. Therefore, to calculate the phase errors, we have to derive \( \delta(\theta) \) further from (25).
The left-hand side of (25) is

$$\tan(\theta + \delta) = \frac{\sin(\theta) \cos(\delta) + \cos(\theta) \sin(\delta)}{\cos(\theta) \cos(\delta) - \sin(\theta) \sin(\delta)}.$$  \hspace{1cm} (31)

By substituting (31) into (25), we can have

$$\sin(\delta) = -\sqrt{p} \cdot \sin(3\theta + \delta)$$  \hspace{1cm} (32)

where $p = (c^2/b^2)$, representing the power ratio of the second-order harmonic to fundamental components.

Therefore, the final expression of $\delta$ is

$$\delta = -\arctan\left(\frac{\sqrt{p} \sin(3\theta)}{1 + \sqrt{p} \cos(3\theta)}\right).$$  \hspace{1cm} (33)

By setting $d\delta/d\theta = 0$, we can derive the maximum and minimum value of, respectively. That is

$$\delta_{\text{max}} = \arctan\left(\sqrt{\frac{p}{1 - p}}\right)$$  \hspace{1cm} (34)

$$\delta_{\text{min}} = -\arctan\left(\sqrt{\frac{p}{1 - p}}\right).$$  \hspace{1cm} (35)

Then, (30) can be expressed as

$$\varepsilon_{\text{max}} = 2 \arctan\left(\sqrt{\frac{p}{1 - p}}\right).$$  \hspace{1cm} (36)

The curve of this function is shown in Fig. 2, where the phase error is in radian. It can be seen that the maximum phase error increases monotonically and significantly with the power ratio factor $p$.

As mentioned in Section II, the relationship between the height distributions and the shifted phase $\phi(x)$ is expressed by (15). Hence, we have

$$\frac{d\phi}{dh} = -\frac{2\pi f_0 d_0 l_0}{(h - l_0)^2}$$  \hspace{1cm} (37)

i.e.,

$$\frac{dh}{d\phi} = -\frac{(h - l_0)^2}{2\pi f_0 d_0 l_0}. $$  \hspace{1cm} (38)

Therefore, the maximum absolute error of height distribution measurement $\beta$ is

$$\beta = \left|\frac{(h - l_0)^2}{2\pi f_0 d_0 l_0} \cdot \varepsilon_{\text{max}}\right| = \frac{(h - l_0)^2}{2\pi f_0 d_0 l_0} \cdot \varepsilon_{\text{max}}.$$  \hspace{1cm} (39)

As the projector usually locates far enough from the object, i.e., $h(x) \ll l_0$, denoting the maximum height of the object as $h_{\text{max}}$, we have

$$\beta = \frac{(h - l_0)^2}{2\pi f_0 d_0 l_0} \cdot \varepsilon_{\text{max}} \geq \frac{(h_{\text{max}} - l_0)^2}{2\pi f_0 d_0 l_0} \cdot \varepsilon_{\text{max}}.$$  \hspace{1cm} (40)

If we assume that $l_0$ is 10 m, $d_0$ is 2 m, $h_{\text{max}}$ is 200 mm, and the spatial period of fringe pattern is 0.2 m, which implies that the spatial frequency $f_0 = 5$/m, the height distribution errors are plotted as Fig. 3.

When the power ratio factor $p$ is 0.01, the maximum height error will become at least 30.6 mm, and when $p$ equals 0.04, the error will be at least 61.5 mm. This implies that the relative error when $p$ equals 0.01 would reach $(30.6/200) \times 100\% = 15.3\%$, which is not a negligible error for measurement.

On the other hand, by the mathematical model, i.e., (10) and (11), and the shift function calculation method in (18), it can be seen that the measurement accuracy of the IFA method only depends on the accuracy of the curve fitting to the inverse function of the original signal. Meanwhile, as mentioned in Section III, we can always find a polynomial of an appropriate degree to make the fitting error less than an acceptable value.

Therefore, theoretically, the error of the curve-fitting result of the inverse function $s^{-1}(v)$ could be as small as we expect. Therefore, in theory, as long as we use a polynomial with sufficient degrees for curve fitting, the reconstruction error by the IFA method can be as small as possible, which is definitely better than the result by using PSP as analyzed above.

V. SIMULATION, EXPERIMENT, AND DISCUSSION

A. Simulation

In our simulation, we use a paraboloid as the object surface, which is shown in Fig. 4(a).
The bottom diameter of the paraboloid is 800 mm, and the height is 160 mm. The original signal of the fringe pattern is assumed to be a cosinusoidal signal, which is expressed as

$$s(x) = 128 + 100 \cdot \cos(2\pi f_0 x)$$  (41)

where $f_0$ is the spatial frequency of the fringe pattern, which is assumed to be $10$ m in our simulation, i.e., the spatial period of the fringe pattern is assumed to be $100$ mm. Meanwhile, we assume that $l_0$ and $d_0$ in Fig. 1 are $5$ and $2$ m, respectively. The spatial resolution (i.e., the spatial sampling frequency) of the captured image is $1$ pixel/mm. As the nonlinear distortion of imaging systems can typically be modeled as gamma distortion [17], [18], in our simulation, gamma distortion is used to introduce nonlinearity to the fringe patterns. Considering 8-bit quantization, the dynamic range of intensity of fringe patterns introduce nonlinearity to the fringe patterns. Considering 8-bit quantization, the dynamic range of intensity of fringe patterns can be approximated as $128 + 100 \cdot \cos(2\pi f_0 x)$.

By Fourier expansion, when $\gamma = 1.5$ and $\gamma = 2.2$, (43) can be approximated as

$$\hat{w}(x) \big|_{\gamma=1.5} \approx 101.4215 + 103.9987 \cos(2\pi f_0 x) + 10.8651 \cos(2(2f_0)x) + 0.8066 \cos(2(3f_0)x) + 0.1386 \cos(2(4f_0)x)$$  (44)

$$\hat{w}(x) \big|_{\gamma=2.2} \approx 78.3749 + 98.0489 \cos(2\pi f_0 x) + 22.3439 \cos(2(2f_0)x) + 0.6265 \cos(2(3f_0)x) + 0.0553 \cos(2(4f_0)x)$$  (45)

respectively. It can be seen that the power of the third and higher order harmonics is very low compared with the fundamental component and the second-order harmonic. This is why in Section IV, only second-order harmonic has been taken into our consideration for error analysis, which is consistent with the practice.

As $\gamma = 2.2$ is recommended by the standard of the National Television System Committee, in our simulation, at first, $\gamma = 2.2$ is used to introduce nonlinear distortion into the fringe pattern signal. Hence, the fringe pattern on the object surface is generated and is shown in Fig. 4(b).

In our simulation, we set the upper bound of the curve-fitting error to be $0.05$ so that by using the IFA method described in Section III, the degree of polynomial is determined to be $20$.

The reconstruction results by the PSP and IFA methods are shown in Fig. 5(a) and (b), respectively. We can see that due to nonlinear distortion, the surface reconstructed by PSP is jagged and rough, which is consistent with our analysis in Section IV. In contrast to PSP, the reconstruction achieved by IFA is very smooth. The absolute reconstruction errors by using PSP and IFA are shown in Fig. 5(c) and (d), respectively. It can be seen that the 3-D reconstruction error of IFA is significantly smaller than the error of PSP. More clearly, cross sections of the reconstructed surfaces are plotted in Fig. 6(a).

The dotted lines refer to the theoretical value of the height distribution of the object surface. The solid and dashed lines in Fig. 6 represent the measurement results by using the IFA and PSP methods, respectively. It can be seen that the height distribution reconstructed by IFA is almost identical to the theoretical values, which is much more accurate than the results calculated by PSP.

In addition to the comparison above, simulation is also performed in the situation of $\gamma = 1.5$. The cross sections of the reconstructed surface are shown in Fig. 6(b). It can be seen that...
Fig. 6. Cross sections of reconstructed surfaces by PSP and IFA. (a) Cross section of the reconstructed surfaces when $\gamma = 2.2$. (b) Cross section of the reconstructed surfaces when $\gamma = 1.5$.

The measurement error by using PSP is reduced because of the smaller $\gamma$. However, the error is still too big and unacceptable, although according to (44) the nonlinear distortion is so slight that the power of the second order is only approximately 20 dB compared with the fundamental component. In contrast, the reconstruction result by using IFA is still almost identical to the theoretical value and is much more accurate.

Further, we did another simulation for a more complex surface to show the performance of the proposed method. The surface designed for the simulation shown in Fig. 7(a) and (b) demonstrates the fringe pattern image captured on the object surface.

Fig. 7(c) and (d) shows the reconstruction results by using the proposed PSP and IFA algorithms, respectively, which also shows that the object surface reconstructed by IFA is very smooth and is almost identical to the theoretical values but that the surface reconstructed by using PSP is very rough and is significantly influenced by nonlinear distortions.

To compare the performance for different algorithms more clearly, cross sections of the reconstructed surfaces are shown in Fig. 8, where the solid and dashed lines represent the reconstruction results by using IFA and PSP, respectively. The dotted line is the theoretical value of the height distribution. It can be seen that the reconstruction result by using IFA is very accurate, even for a complex object with sharp edges.

**B. Experiment**

In our experiment, an InFocus LP530 digital light processor is used to generate sinusoidal fringe patterns, and we employ a DuncanTech MS3100-RGB 3CCD to capture the images. The parameters $d_0$ and $l_0$ are 200 and 81 cm, respectively. The equivalent spatial period of the fringe pattern is 25.7 mm. The object that we try to measure is a cone with a height of 38 mm and a bottom surface diameter of 94 mm, which is shown in Fig. 9(a).

The fringe pattern projected onto the reference plane and on the object surface are shown in Fig. 9(b) and (c). Fig. 9(d) is a cross section of Fig. 9(d). As shown in Fig. 9(d), the fringe pattern projected onto the reference plane does not appear to be purely sinusoidal. By using this captured signal, the reconstruction results by PSP is shown in Fig. 10.

We can see that due to the nonlinear distortion, the reconstructed surface by PSP is jagged. The average error to measure the object by using FTP is 4.2993 mm. In contrast, the error by using the IFA method is 0.8148 mm. As demonstrated in Fig. 11, the profile measured by our proposed model and algorithm is much smoother than the reconstruction result of PSP.
C. Discussion

In the simulation and experiment, it has been demonstrated that IFA can accurately reconstruct 3-D surface from nonlinearly distorted fringe patterns without knowing the property of the distortion. Enjoying this advantage, however, compared with the general PSP method, IFA costs more on computation. Although in the measurement process, PSP needs more time to acquire phase-shifted fringe pattern images, it is very efficient in the calculation process because only very simple computation, expressed in (21), is involved. In contrast, IFA needs many times of curve fitting before the degree of the polynomial can be determined. Quantitatively, in our simulation, PSP costs 1.67 s to complete the reconstruction by using Matlab 7.1, but IFA needs 54.7 s. However, the degree of the polynomial used for curve fitting usually depends on the characteristics of the projection and acquisition system, which will be a certain value for a given system. Therefore, we actually do not have to recursively calculate the degree of the polynomial every time. We only need to determine it once after the system is set up, which will make the reconstruction much faster. In the simulation, it can be 6.13 s if the degree for curve fitting is predetermined.

Meanwhile, we need to note that to guarantee the usability of curve fitting, the number of data points should be greater than the degree of the polynomial, which implies that the resolution of the image cannot be too low. Otherwise, the polynomial will not be able to approximate the captured fringe pattern signal and consequently fail to give accurate reconstruction of the object surface. Certainly, although we always want to use as many pixels in each wavelength of the fringe as possible, in theory, the reconstruction result will not be significantly influenced by the number of sampling points because if the number of data points is sufficient for curve fitting, the accuracy of the reconstruction is mainly determined by $e_m$, which is the expected curve-fitting error that we selected.

Additionally, because IFA is also based on single frequency fringe pattern, similar to general methods such as PSP and FTP, it may be difficult if the height distribution of the object is so large that phase wrapping occurs. In fact, it is due to the uncertainty of the height distribution when a single frequency fringe pattern is utilized. More clearly, let us consider two objects A and B whose height distributions $h_A(x)$ and $h_B(x)$ satisfy

$$h_B(x) = f_0d_0h_A(x) + m(h_A(x) - l_0)]l_0$$

where $m$ is a nonzero integer. Obviously, by (15), the phase map of objects A and B, i.e., $\phi_A(x)$ and $\phi_B(x)$, will satisfy

$$\phi_B(x) = \phi_A(x) + 2m\pi.$$  

As the $\cos(\cdot)$ function is a periodic function and its period is $2\pi$, according to the conventional phase modulation model, i.e., (13), the fringe pattern images captured from the surfaces of objects A and B, will be exactly the same, although A and B have different height distributions. This implies that if only one frequency is used, it is theoretically impossible to uniquely

Fig. 9. Projected fringes and the measured cone.

Fig. 10. Experimental result of PSP.

Fig. 11. Experimental result of IFA.
VI. Conclusion

In this paper, we have presented a new method, called the IFA method, to analyze fringe patterns for profilometry, which is based on revealing the general relationships between projected and deformed signals. With IFA, the constraint of using sinusoidal signals has been completely removed. This implies that, instead of sinusoidal fringe patterns, any kind of fringe patterns could be utilized for profilometry in theory. Meanwhile, as IFA has no requirement for the prior knowledge of the structure of fringe patterns, it can be applied in theoretical analysis and practical operation. Our mathematical analysis and simulation results demonstrate that the new method has higher accuracy when the projected signal is not exactly sinusoidal.

REFERENCES


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