1981

A representation approach to the tower of Hanoi problem

M. C. Er
University of Wollongong

A REPRESENTATION APPROACH TO THE TOWER OF HANOI PROBLEM

M. C. ER

Department of Computing Science,
The University of Wollongong,
Post Office Box 1144,
Wollongong, N.S.W. 2500
Australia

ABSTRACT

By making the moving direction of each disc explicit in the representation, a bit-string so constructed can be used to drive the Tower of Hanoi algorithm. The behaviour of disc moves is further analyzed based on the bit-string representation. It has been shown that the bit-string for moving n discs can be used to generate successively the Gray codes of n bits.

Keywords and phases: Tower of Hanoi, representation approach, Gray codes, combinatorial algorithms.

CR Categories: 5.30, 5.39

A REPRESENTATION APPROACH TO THE TOWER OF HANOI PROBLEM

M. C. ER

Department of Computing Science,
The University of Wollongong,
Post Office Box 1144,
Wollongong, N.S.W. 2500
Australia.

1. INTRODUCTION

Algorithms for the Tower of Hanoi problem are often used in the introductory texts on computer programming for demonstrating the power of recursion (Hayes, 1977; Dijkstra, 1971; Dromey, 1981). Interesting though these recursive algorithms are, beginners are not always convinced that these algorithms will work until they are run. Given such recursive algorithms, it is not obvious how to move discs around until one actually steps through the programs.

Hayes (1977), Buneman and Levy (1980) present two iterative algorithms for solving the Tower of Hanoi problem, hoping that they are less mysterious than the recursive solutions. It is however not clear why the smallest disc is always moved in the cyclic order. At any rate, they did not argue that their solutions are the optimal ones; namely, to move the tower of discs from one peg to the other peg in the minimum number of steps. An analysis is clearly necessary.

It is a common experience in the Articifical Intelligence research (Korf, 1980) that a suitable representation may lead to an efficient and transparent algorithm. It is the theme of this paper to examine the Tower of Hanoi problem in this light. By encoding the disc moves into a bit-string, we show that a straight forward iteractive algorithm can be constructed. More important, the bit-string lends a hand to an analysis of the behaviour of the algorithm.

2. THE PROBLEM

The Tower of Hanoi problem involves three pegs (P1, P2 and P3) and n discs (D_1, D_2, D_3 ... D_n) such that $D_1 < D_2 < D_3 < \ldots < D_n$, where D_1 is the smallest disc. Initially, all the n discs are placed on a peg Pi as a pyramid with D_1 on the top. The task is to move these n discs from Pi to Pj such that $i \neq j$, subject to the following constraints:

C1 : Only the top disc of a tower may be moved from one peg to the other;

C2 : No disc may rest upon a smaller disc at any time;

C3 : Only one disc may be moved at a time.

3. OTHER ITERATIVE SOLUTIONS

Hayes (1977) gives reasons to believe that the smallest-disc moves must alternate with other disc moves. However, he did not explain why the smallest disc must move cyclically. Nor did Buneman and Levy (1980) give reasons to support their algorithm that the smallest disc must always move in the clockwise direction. Clearly, to move n discs from peg 1 (P1) to peg 3 (P3), one needs not have to move n discs from P1 to peg 2 (P2), then move them to P3. Buneman and Levy's algorithm therefore does not provide an optimal solution under certain circumstances.

Consider the general case. Suppose the smallest disc (D_1) is on top of P1, and the other two discs D_i and D_j are on top of P2 and P3 respectively. Clearly $D_1 < D_i$ and $D_1 < D_j$. Suppose we have just moved D_1, either D_i or D_j will be moved in the next move. As $D_i \neq D_j$, the next move is a unique solution depending on which of D_i and D_j is smaller. Suppose $D_i < D_j$, the next move is to move D_i on top of D_j. After that, we have to move D_1. Now we have two choices; either to move D_1 on top of D_i or to move D_1 to P2. It is under these circumstances that both Hayes (1977), and Buneman and Levy (1980) fail to satisfy us that one choice is better than the other. A deeper analysis is clearly called for.

4. TREE AND BIT-STRING REPRESENTATION

Suppose the three pegs are arranged in a circle as shown in figure 1.

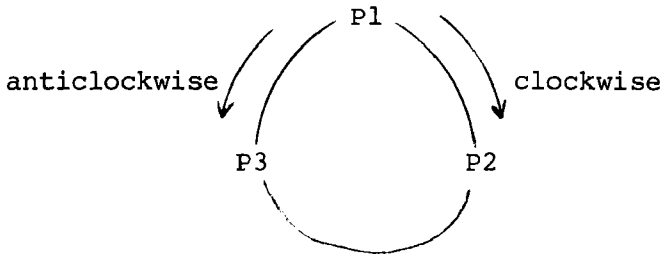


Figure 1 Arrangement of Pegs

Define the clockwise direction as P1 → P2 → P3 → P1, and the anticlockwise direction as P1 → P3 → P2 → P1. Any top disc can be moved to its neighbouring peg in either clockwise or anticlockwise direction at any movement. We can view the clockwise and anticlockwise moves as traversal of left and right branches of a binary tree respectively. It is then possible to represent the directions of disc moves as a binary tree. In other words, we make explicit the directions of disc moves in the representation.

Suppose we are asked to move n discs from a peg to its neighbouring peg in the clockwise direction, we can assume that the source is P1, and the destination is P2 without loss of generality. This can be expressed formally as follows:

$$D_1, D_2, \ldots, D_n : P1 \downarrow P2.$$

When n = 1, only disc D_1 can be moved from P1 to P2 in the clockwise direction in one step:

$$D_1 : P1 \downarrow P2.$$

When N = 2, the smallest disc D_1 should be moved in the anticlockwise direction from P1 to P3. Then the larger disc D_2 will be moved from P1 to P2 in the clockwise direction. After that, D_1 will be moved from P3 to P1 in the anticlockwise direction thus completing the task. Namely,

$$D_1, D_2 : P1 \downarrow P2 \quad \begin{cases} D_1 : P1 \swarrow P3 \\ D_2 : P1 \downarrow P2 \\ D_1 : P3 \swarrow P2 \end{cases}$$

The tree representations of disc moving directions are shown in figure 2.

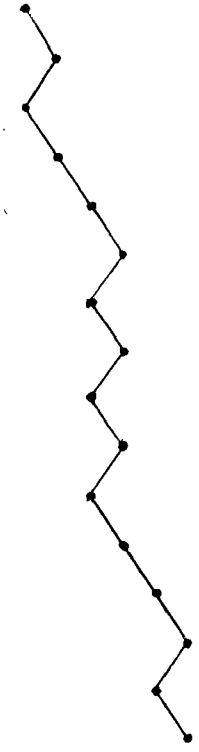
disc	tree representation	bit-string representation
n = 1		0
n = 2		101
n = 3		0100010
n = 4		101110101011101

Figure 2 Moving towers clockwise

When $n = 3$, the steps are as follows:

$$D_1, D_2, D_3 : P1 \downarrow P2 \begin{cases} D_1, D_2 : P1 \downarrow P3 \\ D_3 : P1 \downarrow P2 \\ D_1, D_2 : P3 \downarrow P2 \end{cases}$$

To move D_1 and D_2 in the anticlockwise direction, the steps can be detailed as follows:

$$D_1, D_2 : P1 \downarrow P3 \begin{cases} D_1 : P1 \downarrow P2 \\ D_2 : P1 \downarrow P3 \\ D_1 : P2 \downarrow P3 \end{cases}$$

From the tree representation, it is apparent that the tree for moving two discs anticlockwise is the mirror image of the tree for moving them clockwise as shown in figure 3.

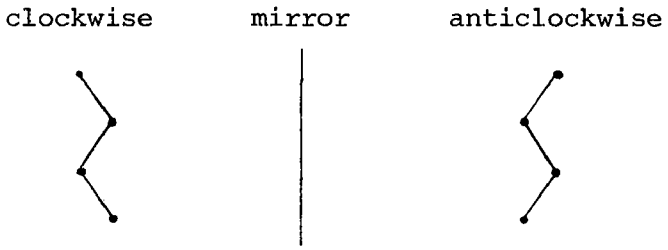


Figure 3 clockwise and anticlockwise trees

Indeed, the clockwise and anticlockwise trees are mirror images of each other is generally true for $n \geq 1$.

Hence, it appears that to construct the tree for $D_1, D_2, \ldots, D_{n-1}, D_n$: P1 ↓ P2, we simply prepend and append the mirror image of the tree for $D_1, D_2 \ldots, D_{n-1}$: P1 ↓ P2 to the root and leaf of the following tree:

If we now encode the binary trees by using Huffman's (1952) method (namely left and right branches are represented by 0 and 1 respectively), the binary trees are collapsed into bit-strings as shown in figure 2. By virtue of the way the binary trees are constructed, we can generate the bit-strings without referring to the trees. Let BS(n ↓) denotes the bit-string for moving n discs clockwise.

It is obvious that

$$BS(1 \downarrow) = 1 \qquad (1)$$

Let C(bs) be the one's complement of the bit-string bs. One can easily show that the mirror image of a binary tree is precisely the one's complement of its bit-string. Therefore,

$$BS(n \downarrow) = C(BS(n-1 \downarrow)) \; 1 \; C(BS(n-1 \downarrow)) \qquad (2)$$

For moving n discs anticlockwise, D_1, D_2, ..., D_n : Pl $\downarrow$ P3, one can derive that the tree is precisely the mirror image of the clockwise case. The results are shown in figure 4.

Let $BS(n \downarrow)$ be the bit-string for moving n disks anticlockwise. By the nature of binary digits, one can easily verify that

$$BS(n \downarrow) = C(BS(n \downarrow))$$
$$\text{or } BS(n \downarrow) = C(BS(n \downarrow)) \qquad (3)$$

Finally, we establish a property of the bit-strings so generated.

Property 0 : The bit-string for moving n discs is symmetric with respect to the centre bit.

Proof : This property readily follows from (1) and (2) by induction.

[QED]

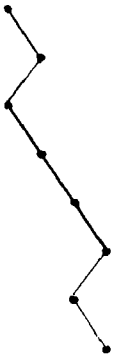
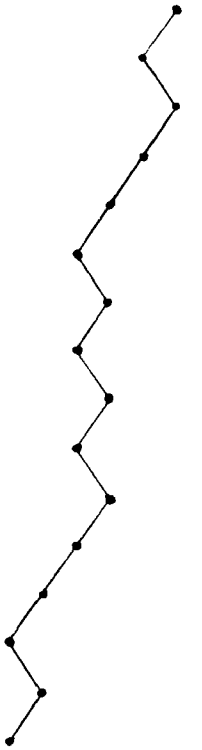
disc	tree representation	bit-string representation
n = 1		1
n = 2		010
n = 3		1011101
n = 4		010001010100010

Figure 4 Moving towers anticlockwise

5. HOW TO MAKE MOVES

Given a bit-string for n discs, we need to be able to interpret it in order to guide the disc moves. First of all, let us establish some properties.

Property 1 : D_1 is the first and last disc to move.

Proof : When n = 1, this is trivially true.

When n = 2, $BS(2\downarrow) = C(BS(1\downarrow)) \; 1 \; C(BS(1\downarrow))$

$= BS(1\downarrow) \; 1 \; BS(1\downarrow)$

where $BS(1\downarrow)$ is D_1's move.

Likewise for $BS(2\downarrow)$.

Suppose $BS(n\downarrow)$ and $BS(n\downarrow)$ preserve this property.

Then, by (2)

$BS(n+1\downarrow) = C(BS(n\downarrow)) \; 1 \; C(BS(n\downarrow))$

We have proved by induction.

Likewise for $BS(n+1\downarrow)$.

[QED]

Property 2 : D_1's moves always alternate with the moves of other discs.

Proof : When n = 1, it is trivial.

When n = 2, $BS(2\downarrow) = BS(1\downarrow) \; 1 \; BS(1\downarrow)$

where $BS(1\downarrow)$ is D_1's move.

This property is obviously true.

Similarly for $BS(2\downarrow)$.

Suppose this property holds for n discs.

Combine with property 1, the D_1's moves must occur at the odd positions.

Now we can prove that this property also holds for (n+1) discs.

By (2)

$BS(n+1\downarrow) = C(BS(n\downarrow)) \; 1 \; C(BS(n\downarrow))$.

Note that one's complement does not change the position of its bits indicating the D_1's moves. By virtue of the fact that $BS(n\downarrow)$ has D_1's moves as first and last moves, as well as at odd positions, $C(BS(n\downarrow)) \; 1 \; C(BS(n\downarrow))$ therefore preserves property 2. Likewise for $BS(n+1\downarrow)$.

[QED]

Now, we are in a position to interpret the bit-string. Let $\{b_1, b_2, \ldots, b_m\}$ be a bit-string. For a bit b_i, i is odd, D_1 is moved according to this parity : 0 and 1 are clockwise and anticlockwise respectively. For b_i, i is even, other disc other than D_1 is moved. Suppose D_1 rests on P1 and $b_i = 0$, we cannot move the top disc from P3 to P1 due to constraint C2. Therefore, we have a unique solution, namely to move the top disc from P2 to P3. Similar argument applies to $b_i = 1$. Suppose MoveDisc (P_i, P_j) is to move the top disc from P_i to P_j. The algorithm for moving D_i, such that $i \neq 1$ is summarized in figure 5.

```
Switchon D1 into
{
    Case  P1 : If  bi = 0
                   then  MoveDisc  (P2, P3)
                   else  MoveDisc  (P3, P2)
               endcase

    Case P2  : If  bi = 0
                   then  MoveDisc  (P3, P1)
                   else  MoveDisc  (P1, P3)
               endcase

    Case P3  : If  bi = 0
                   then  MoveDisc  (P1, P2)
                   else  MoveDisc  (P2, P1)
               endcase
}
```

Figure 5 Moving other disc

6. FURTHER ANALYSIS

We now further analyze the bit-string for moving n discs to reveal the inherent properties of the Tower of Hanoi problem.

Property 3 : The smallest disc always moves in a cyclic order.

Proof : When n = 1, this is trivial.

Suppose this property holds for (n-1) discs.
Namely, all the odd position bits are having the same parity.
By properties 1 and 2 and (2), the odd position bits of the bit-string for moving n discs again have the same parity.

[QED]

Property 4 : The solution offered by the bit-string for moving n discs is optimal.

Proof : When n = 1, BS(1 ↓) for D_1 : P1 ↓ P2 or
BS(1 ↓) for D_1 : P1 ↓ P2
is obviously optimal.

Suppose BS(n-1 ↓) and BS(n-1 ↓) are optimal.
We now show that BS(n ↓) is optimal too.
To move n discs from P1 to P2, we need to move the top (n-1) discs from P1 to P3, then move D_n from P1 to P2, and finally move the (n-1) discs again from P3 to P2. That is,

$$D_1, D_2, \ldots, D_n : P1 \downarrow P2 \quad \begin{cases} D_1, D_2, \ldots, D_{n-1} : P1 \downarrow P3 \\ D_n : P1 \downarrow P2 \\ D_1, D_2, \ldots, D_{n-1} : P3 \downarrow P2 \end{cases}$$

As BS(n-1 ↓) is optimal for $D_1, D_2, \ldots, D_{n-1}$: Pi ↓ Pj, where $i \neq j$. It follows that BS(n ↓) for the composite solution of moving n discs, $D_1, D_2, \ldots, D_n$: P1 ↓ P2, is optimal. Similar argument holds for BS(n ↓).

[QED]

Property 5 : The optimal solution takes $2^n - 1$ steps.

Proof : Let S_n be the number of bits in $BS(n\downarrow)$.
From (2), $S_n = 2 * S_{n-1} + 1$.
As $S_1 = 1$ by (1). Therefore

$$S_n = \sum_{i=0}^{n-1} 2^i$$

$$= 2^n - 1$$

As a bit corresponds to a step, thus the optimal solution takes $2^n - 1$ steps.

[QED]

Property 6 : All D_i, i = odd, move in the same direction. Whereas, all D_j, j = even, move in the opposite direction.

Proof : From (2), $BS(n\downarrow) = C(BS(n-1\downarrow))\ 1\ C(BS(n-1\downarrow))$
where $C(BS(n-1\downarrow)) = C(C(BS(n-2\downarrow))\ 1\ C(BS(n-2\downarrow)))$
$= C(C(BS(n-2\downarrow)))\ 0\ C(C(BS(n-2\downarrow)))$
As the centre bit of $BS(n\downarrow)$ is an indication of the moving direction of D_n, so the centre bit of $C(BS(n-1\downarrow))$ indicates the moving direction of D_{n-1}.

As the centre bits of $BS(n\downarrow)$ and $C(BS(n-1\downarrow))$ are of different parity, we have proved that D_i and D_j, such that $|i-j| = 1$, move in opposite directions.

Furthermore,

$$C(C(BS(n-2\downarrow))) = BS(n-2\downarrow)$$
$$= C(BS(n-3\downarrow))\ 1\ C(BS(n-3\downarrow)).$$

As the centre bits of $BS(n\downarrow)$ and $C(C(BS(n-2\downarrow)))$ are having the same parity, we thus prove that D_n and D_{n-2} are moving in the same direction.

By induction, the property holds.

[QED]

Property 7 : Let M_i be the number of steps taken by D_i to get to the destination in the optimal solution. Then $M_i = 2^{n-i}$.

Proof : As we know, the centre bit of $BS(n\downarrow)$ indicates the moving direction of D_n, and that is the only bit to do so. Hence, $M_n = 1$.

Suppose, $M_i = 2^{n-i}$ for $0 < i \leq n$.

By (2), $BS(i\downarrow) = C(BS(i-1\downarrow))\ 1\ C(BS(i-1\downarrow))$.

Thus, $M_{i-1} = 2 * 2^{n-i}$

$= 2^{n-(i-1)}$

[QED]

7. IMPLEMENTATIONS

Now we are in a position to implement the algorithm. From property 5, we know that it takes $2^n - 1$ steps to move n discs from a peg to a target peg. So the control loop can be implemented as a for-loop. Further, we know from property 2 that the moves of D_1 alternate with other disc moves. Therefore, the body of the for-loop comprises two move-disc instructions; one for moving D_1, another for moving other discs. A program based on these ideas has been written using C, and is included in Appendix A for reference. Notice that the generating function, Generate, successively generates the bit-string from 1 disc up to n discs based on (2).

A moment's reflection would convince us that to generate the bit-string for n discs, it is not necessary to generate all the bit-strings for 1 disc up to n discs. We can indeed generate the bit-string for n discs straight away, by taking the advantage of property 6. Before we spell out the details of the direction generating function, we prove a property first.

Property 8 : All bit positions $2^{i-1} + 2^i * j \leq 2^n - 1$ where $j=0,1,2...$ and i=even, are occupied by bits of same parity. Whereas, other bit positions, $P \neq 2^{i-1} + 2^i * j$, are occupied by bits of opposite parity.